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# NAVAL POSTGRADUATE SCHOOL

## Monterey, California



A NOTE ON THE MAXIMUM SIZE OF  
A RECTILINEAR MAZE

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Kim A.S. Hefner

September 1989

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# A note on the maximum size of a rectilinear maze

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## ABSTRACT

In this paper, we study the problem of searching through an unknown maze by a robot and show that the size of the largest rectilinear maze the robot can explore in at most  $k$  steps is bounded by  $2k^2 + 2k + 1$  for mazes with circuits, and is bounded by  $\frac{4}{3}k^2 + \frac{8}{3}k + 1$  for mazes without circuits. Furthermore, we show that the bounds are tight.

## 1. Introduction

One of the central problem-solving techniques in AI is the use of *search*. In this paper, we focus our attention on the problem of searching through an unknown terrain by a physical agent such as a robot. The unknown terrain over which the robot will travel is represented by a rectilinear graph or grid-graph as shown in Figure 1.

Nodes are arranged in a rectangular fashion with arcs connecting nodes immediately above and below and to the right and left (north, south, east and west respectively). Each node (except those on the boundary) then has four possible immediate successors which are distinguished from one another by their  $x$  and  $y$  coordinates.

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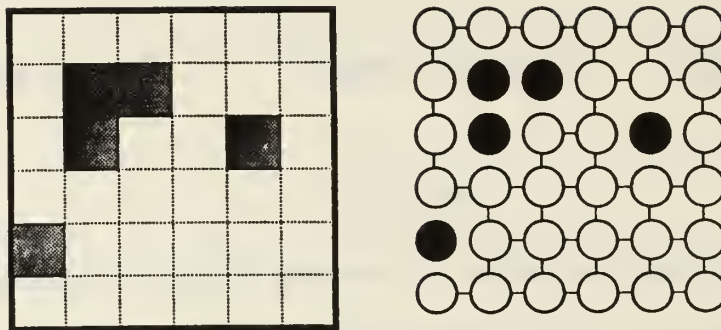


Figure 1

Locations within the maze are either passable or impassable. Impassable locations in the maze are represented by disconnected nodes in the graph, making them unreachable from any direction.

The robot begins with no knowledge of the graph. It can learn about the graph only as it moves from node to node via the edges connecting them. The robot's ability to sense its surroundings is limited to determining whether the immediate successors of the current state are passable or impassable. In effect, the robot can judge whether it can move to the north, south, east or west. Since the terrain is a grid-graph, the robot is only able to move and survey in the four cardinal directions.

Additionally, the robot knows its position relative to the goal. Several different means could be used to accomplish this. If the robot initially knows the goal's position relative to its own, it can maintain this knowledge through dead reckoning as it moves about the maze from its initial starting point. Using this information the robot can determine its rectilinear distance from the goal as well as the distance from the goal to any immediate successor state. Rectilinear distance is a lower bound for the number of steps necessary to move from the current position to the goal. The following formula computes rectilinear distance from the node's  $X$  and  $Y$  coordinates:

$$|X_r - X_g| + |Y_r - Y_g|$$

The subscript  $r$  is used to indicate the robot's present coordinates while the subscript  $g$  indicates the goal's.

Using a different method, either an active or passive guidance system could also be used to direct the robot to the goal (e.g., either a form of radar or a beaconing system). While only lines of bearing are required to choose which immediate successor is closer to the goal, an exact location for the goal is required to totally order the search space. If the goal is northeast of the current location, then the immediate successor states to the north and the east are closer to the goal and have the same rectilinear distance to the goal. However, when comparing widely separated locations, a ranking for the locations based on their closeness to the goal cannot be determined without an exact location for the goal. If location  $P$  is north of the goal and location  $Q$  is south of the goal, there is no static way to tell which one is closer merely by their direction to the goal. However, by triangulating lines of bearing taken from different locations, an exact location for the goal can be determined.

## 2. The maximum size of an unknown rectilinear maze

In analyzing the complexity of the algorithms for solving the above maze searching problem, it is essential to estimate the size of the area that the robot can explore in at most  $k$  steps.

For the mazes with circuits, the area reachable by the robot from the starting node in at most  $k$  steps corresponds to a complete grid-graph  $G$  with diameter  $2k+1$ , as illustrated in Figure 2 for  $k = 7$ . Let us number each node by the length of the shortest path from the node to the center. It is easy to see that there is one node with label 0 and  $4i$  nodes with label  $i$  for  $1 \leq i \leq k$ . Hence, the number of nodes in a complete grid-graph with diameter  $2k+1 = 1 + \sum_{i=1}^k 4i = 2k^2 + 2k + 1$ .



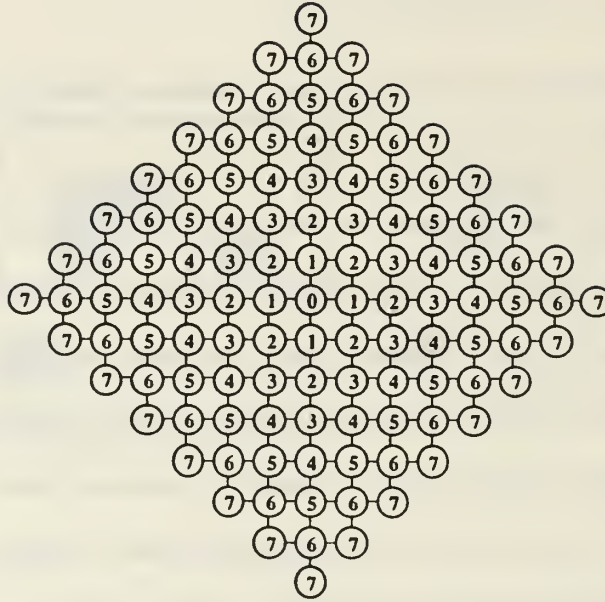


Figure 2

For the mazes without circuits, the analysis becomes more complicated. Let  $T(k)$  be the maximum number of nodes reachable by a robot from the starting node in any tree-maze, i.e. maze without circuits, in at most  $k$  steps.

**Theorem 1.**  $T(k) \leq \frac{4}{3} k^2 + \frac{8}{3} k + 1.$

**Proof.** The largest tree-maze that the robot can explore in  $k$  steps must be a subgraph of the complete grid-graph  $G$  with diameter  $2k+1$ . A necessary condition for any subgraph of  $G$  to be a tree is that

$$\text{the number of nodes in the subgraph} = \quad (*)$$

$$1 + \text{the number of edges in the subgraph}$$

Hence, it suffices to show that at least  $(2k^2 - 2k)/3$  nodes and their adjacent edges must be removed from  $G$  to form a tree-maze, since the number of nodes in any tree-maze will be less than or equal to  $(2k^2 + 2k + 1) - \frac{1}{3}(2k^2 - 2k) =$

$$\frac{4}{3} k^2 + \frac{8}{3} k + 1.$$

Let  $V$  and  $E$  be the set of nodes and edges in  $G$  respectively. We can partition  $V$  into three disjoint sets  $V_1, V_2$  and  $V_3$  as follows:

$V_1 =$  the set of all nodes with degree one in  $G$

$V_2 =$  the set of all nodes with degree two in  $G$

$V_3 =$  the set of all nodes with degree four in  $G$ .

We have

$$|V_1| = 4 \tag{1}$$

$$|V_2| = 4k - 4 \tag{2}$$

$$|V_3| = 2k^2 + 2k + 1 - 4k = 2k^2 - 2k + 1 \tag{3}$$

$$|E| = \frac{1}{2} \left[ 4 + 2(4k - 4) + 4(2k^2 - 2k + 1) \right] = 4k^2 \tag{4}$$

Let  $x$  be the minimum number of nodes that have to be removed (together with their adjacent edges) from  $G$  to obtain a subgraph satisfying the condition (\*). It is easy to see that  $x$  places a lower bound on the number of nodes that have to be removed to obtain the largest tree-maze embedded in  $G$ . In order to minimize  $x$ , the best we can do is to reduce the number of edges by four for every node removed from  $G$ . Hence, we want to find  $x$  such that

$$2k^2 + 2k + 1 - x = 1 + 4k^2 - 4x \tag{5}$$

Solving the above equation, we have  $x = (2k^2 - 2k) / 3$ .

□

**Theorem 2.** There exists a tree-maze with  $\frac{4}{3}k^2 + \frac{8}{3}k + 1$  nodes for  $k = 2^i - 1$ ,  $i \geq 0$ .

**Proof.**

The theorem is obviously true for  $i = 0$ . For  $i \geq 1$  and  $k = 2^i - 1$ , we can construct a maze with  $\frac{4}{3} k^2 + \frac{8}{3} k + 1$  nodes as follows.

Start with a single node at the center of the complete grid-graph  $G$ . Allow the maze to fully expand level by level as long as it does not result in any circuits, as illustrated in Figure 3 for  $k = 7$ .

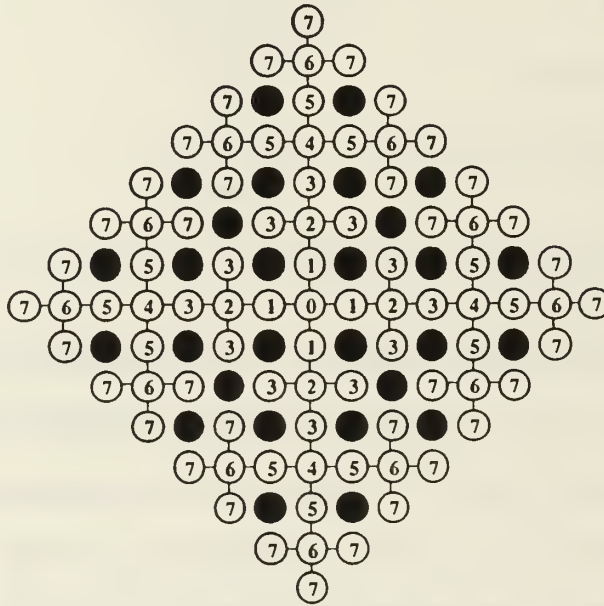


Figure 3

The resultant maze corresponds to a tree rooted at the center of the complete grid-graph with diameter  $2k+1$ . The tree has four identical subtrees hanging from the root, each of which in turn corresponds to the tree-maze generated by the above procedure for a complete grid-graph of diameter  $k$ .

Let  $M(k)$  be the number of nodes in the tree-maze generated by the above procedure on a complete grid-graph of diameter  $2k+1$ . We can express  $M(k)$  recursively as follows:

$$M(k) = 4 M((k-1)/2) + 1 \quad \text{where } k = 2^i - 1 \text{ for } i > 0 \quad (6)$$

$$M(0) = 1$$

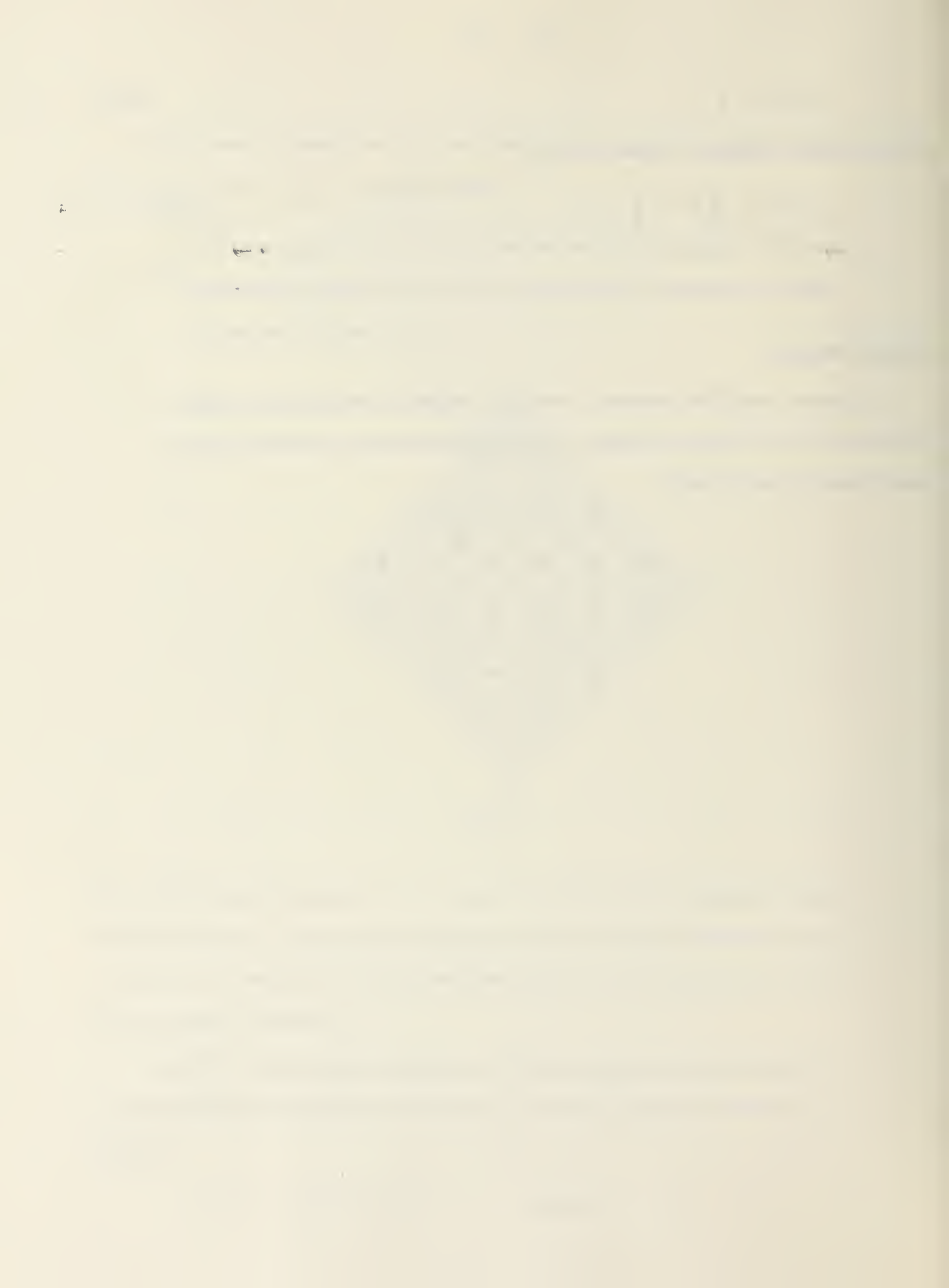
Solving the above recurrence relation, we have

$$M(k) = \frac{4}{3} k^2 + \frac{8}{3} k + 1. \quad (7)$$

□

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