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Mathematics Tables

Naval Postgraduate School, Monterey, CA

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MATHEMATICS TABLES
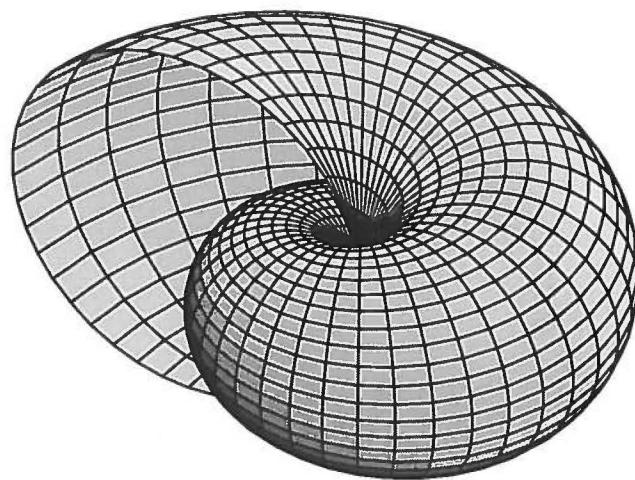
Mathematics Department
Naval Postgraduate School

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Cover: in spherical coordinates (ρ, ϕ, θ), shell surface is $\rho(\phi, \theta) = e^{k\theta} \sin \phi$, where $k = \frac{1}{\pi} \ln G$ and $G = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio.

FOREWORD

This booklet provides convenient access to formulas and other data that are frequently used in mathematics courses. If a more comprehensive reference is needed, see, for example, the STANDARD MATHEMATICAL TABLES published by the Chemical Rubber Company, Cleveland, Ohio.

DIFFERENTIALS AND DERIVATIVES

A. Letters u and v denote independent variables or functions of an independent variable; letters a and n denote constants.

B. To obtain a derivative, divide both members of the given formula for the differential by du or by dx.

1. $d(a) = 0$

2. $d(au) = a\,du$

3. $d(u + v) = du + dv$

4. $d(uv) = u\,dv + v\,du$

5. $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$

6. $d(u^n) = n\,u^{n-1}\,du,$

7. $d(e^u) = e^u\,du$

8. $d(a^u) = a^u \ln a\,du$

9. $d(\ln u) = \dfrac{du}{u}$

10. $d(\log_a u) = \dfrac{du}{u \ln a}$

11. $d(\sin u) = \cos u\,du$

12. $d(\cos u) = -\sin u\,du$

13. $d(\tan u) = \sec^2 u\,du$

14. $d(\cot u) = -\csc^2 u\,du$

15. $d(\sec u) = \sec u \tan u\,du$

16. $d(\csc u) = -\csc u \cot u\,du$

17. $d(\arcsin u) = \dfrac{du}{\sqrt{1 - u^2}}$

18. $d(\arccos u) = -\dfrac{du}{\sqrt{1 - u^2}}$

19. $d(\arctan u) = \dfrac{du}{1 + u^2}$

20. $d(\sinh u) = \cosh u\,du$

21. $d(\cosh u) = \sinh u\,du$

22. $d(\operatorname{arcsinh} u) = \dfrac{du}{\sqrt{u^2 + 1}}$

23. $d(\operatorname{arccosh} u) = \dfrac{du}{\sqrt{u^2 - 1}}, \quad u > 1$

24. (Differentiation of Integrals) If f is continuous, then

$$d\left(\int_a^u f(t)\,dt\right) = f(u)\,du.$$

25. (Chain Rule) If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

INTEGRALS

A. Letters u and v denote functions of an independent variable such as x; letters a, b, m, and n denote constants.

B. Generally, each formula gives only one antiderivative. To find an expression for all antiderivatives, add a constant of integration.

ELEMENTARY FORMS

1. $\displaystyle\int a\,du = a\,u$

2. $\displaystyle\int a\,f(u)\,du = a\int f(u)\,du$

3. $\displaystyle\int (f(u) + g(u))\,du = \int f(u)\,du + \int g(u)\,du$

4. $\displaystyle\int u\,dv = u\,v - \int v\,du$ \qquad (Integration by parts)

5. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1}, \qquad n \neq -1$

6. $\displaystyle\int \frac{du}{u} = \ln|u|$

7. $\displaystyle\int e^u\,du = e^u$

8. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a}, \qquad a > 0$

9. $\displaystyle\int \cos u\,du = \sin u$

10. $\displaystyle\int \sin u\,du = -\cos u$

11. $\displaystyle\int \sec^2 u\,du = \tan u$

12. $\displaystyle\int \csc^2 u\,du = -\cot u$

13. $\displaystyle\int \sec u \tan u\,du = \sec u$

14. $\displaystyle\int \csc u \cot u\,du = -\csc u$

15. $\displaystyle\int \tan u\,du = -\ln|\cos u|$

16. $\displaystyle\int \cot u\,du = \ln|\sin u|$

17. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u|$

18. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u|$

19. $\displaystyle\int \frac{du}{u^2 + a^2} = \frac{1}{a}\arctan\frac{u}{a}$

20. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a}$

21. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{u}{a}$

INTEGRALS INVOLVING $au + b$

22. $\displaystyle\int (au + b)^n\,du = \frac{1}{a}\frac{(au + b)^{n+1}}{n + 1}, \qquad n \neq -1$

23. $\displaystyle\int \frac{du}{au + b} = \frac{1}{a}\ln|au + b|$

24. $\displaystyle\int \frac{u\,du}{au + b} = \frac{u}{a} - \frac{b}{a^2}\ln|au + b|$

25. $\displaystyle\int \frac{u\,du}{(au + b)^2} = \frac{b}{a^2(au + b)} + \frac{1}{a^2}\ln|au + b|$

26. $\displaystyle\int \frac{du}{u(au + b)} = \frac{1}{b}\ln\left|\frac{u}{au + b}\right|$

27. $\displaystyle\int u\sqrt{au + b}\,du = \frac{2(3au - 2b)}{15a^2}(au + b)^{\frac{3}{2}}$

28. $\displaystyle\int \frac{u\,du}{\sqrt{au + b}} = \frac{2(au - 2b)}{3a^2}\sqrt{au + b}$

29. $\displaystyle\int u^2\sqrt{au + b}\,du = \frac{2(8b^2 - 12abu + 15a^2u^2)}{105a^3}(au + b)^{\frac{3}{2}}$

30. $\displaystyle\int \frac{u^2\,du}{\sqrt{au + b}} = \frac{2(8b^2 - 4abu + 3a^2u^2)}{15a^3}\sqrt{au + b}$

INTEGRALS INVOLVING $u^2 \pm a^2$

31. $\displaystyle \int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \frac{u}{a}$ [See 19]

32. $\displaystyle \int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right|$

33. $\displaystyle \int \frac{u \, du}{u^2 \pm a^2} = \frac{1}{2} \ln |u^2 \pm a^2|$

34. $\displaystyle \int \frac{u^2 \, du}{u^2 - a^2} = u + \frac{a}{2} \ln \left| \frac{u - a}{u + a} \right|$

35. $\displaystyle \int \frac{u^2 \, du}{u^2 + a^2} = u - a \arctan \frac{u}{a}$

36. $\displaystyle \int \frac{du}{u(u^2 \pm a^2)} = \pm \frac{1}{2a^2} \ln \left| \frac{u^2}{u^2 \pm a^2} \right|$

37. $\displaystyle \int \frac{u \, du}{(u^2 \pm a^2)^{n+1}} = -\frac{1}{2n(u^2 \pm a^2)^n}, \quad n \neq 0$

INTEGRALS INVOLVING $\sqrt{u^2 \pm a^2}, \quad a > 0$

38. $\displaystyle \int \frac{u \, du}{\sqrt{u^2 \pm a^2}} = \sqrt{u^2 \pm a^2}$ [See 5]

39. $\displaystyle \int u\sqrt{u^2 \pm a^2} \, du = \frac{1}{3}(u^2 \pm a^2)^{\frac{3}{2}}$ [See 5]

40. $\displaystyle \int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln |u + \sqrt{u^2 \pm a^2}|$

41. $\displaystyle \int \sqrt{u^2 \pm a^2} \, du = \frac{u}{2} \sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \ln |u + \sqrt{u^2 \pm a^2}|$

42. $\displaystyle \int \frac{du}{u\sqrt{u^2 + a^2}} = \frac{1}{a} \ln \left| \frac{u}{a + \sqrt{u^2 + a^2}} \right|$

43. $\displaystyle \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{u}{a}$ [See 21]

44. $\displaystyle \int \frac{\sqrt{u^2 - a^2}}{u} \, du = \sqrt{u^2 - a^2} - a \operatorname{arcsec} \frac{u}{a}$

45. $\displaystyle \int \frac{\sqrt{u^2 + a^2}}{u} \, du = \sqrt{u^2 + a^2} + a \ln \left| \frac{u}{a + \sqrt{u^2 + a^2}} \right|$

INTEGRALS INVOLVING $\sqrt{u^2 \pm a^2}$, $\quad a > 0$ $\quad$ (Continued)

46. $\displaystyle \int \frac{u^2\,du}{\sqrt{u^2 \pm a^2}} = \frac{u}{2}\sqrt{u^2 \pm a^2} \mp \frac{a^2}{2}\ln\left|u + \sqrt{u^2 \pm a^2}\right|$

47. $\displaystyle \int u^2\sqrt{u^2 \pm a^2}\,du = \frac{u}{4}(u^2 \pm a^2)^{\frac{3}{2}} \mp \frac{a^2 u}{8}\sqrt{u^2 \pm a^2} - $
$$\frac{a^4}{8}\ln\left|u + \sqrt{u^2 \pm a^2}\right|$$

INTEGRALS INVOLVING $\sqrt{a^2 - u^2}$, $\quad a > 0$

48. $\displaystyle \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} \qquad$ [See 20]

49. $\displaystyle \int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin\frac{u}{a}$

50. $\displaystyle \int \frac{u\,du}{\sqrt{a^2 - u^2}} = -\sqrt{a^2 - u^2} \qquad$ [See 5]

51. $\displaystyle \int u\sqrt{a^2 - u^2}\,du = -\frac{1}{3}(a^2 - u^2)^{\frac{3}{2}} \qquad$ [See 5]

52. $\displaystyle \int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} + \ln\left|\frac{u}{a + \sqrt{a^2 - u^2}}\right|$

53. $\displaystyle \int \frac{du}{u\sqrt{a^2 - u^2}} = \frac{1}{a}\ln\left|\frac{u}{a + \sqrt{a^2 - u^2}}\right|$

54. $\displaystyle \int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u}$

55. $\displaystyle \int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{\sqrt{a^2 - u^2}}{u} - \arcsin\frac{u}{a}$

56. $\displaystyle \int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin\frac{u}{a}$

57. $\displaystyle \int u^2\sqrt{a^2 - u^2}\,du = -\frac{u}{4}(a^2 - u^2)^{3/2} + \frac{a^2 u}{8}\sqrt{a^2 - u^2} + $
$$\frac{a^4}{8}\arcsin\frac{u}{a}$$

INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

58. $\displaystyle\int \sin^2 au\,du = \frac{u}{2} - \frac{\sin 2au}{4a}$

59. $\displaystyle\int \cos^2 au\,du = \frac{u}{2} + \frac{\sin 2au}{4a}$

60. $\displaystyle\int \sin^3 au\,du = \frac{\cos^3 au}{3a} - \frac{\cos au}{a}$

61. $\displaystyle\int \cos^3 au\,du = \frac{\sin au}{a} - \frac{\sin^3 au}{3a}$

62. $\displaystyle\int \sin^n au\,du = -\frac{\sin^{n-1} au \cos au}{na} + \frac{n-1}{n}\int \sin^{n-2} au\,du,$

63. $\displaystyle\int \cos^n au\,du = \frac{\cos^{n-1} au \sin au}{na} + \frac{n-1}{n}\int \cos^{n-2} au\,du,$

64. $\displaystyle\int \sin^2 au \cos^2 au\,du = \frac{u}{8} - \frac{\sin 4au}{32a}$

65. $\displaystyle\int \frac{du}{\sin^2 au} = -\frac{1}{a}\cot au$

66. $\displaystyle\int \frac{du}{\cos^2 au} = \frac{1}{a}\tan au$

67. $\displaystyle\int \tan^2 au\,du = \frac{1}{a}\tan au - u$

68. $\displaystyle\int \cot^2 au\,du = -\frac{1}{a}\cot au - u$

69. $\displaystyle\int \sec^3 au\,du = \frac{1}{2a}\sec au \tan au + \frac{1}{2a}\ln|\sec au + \tan au|$

70. $\displaystyle\int \csc^3 au\,du = -\frac{1}{2a}\csc au \cot au + \frac{1}{2a}\ln|\csc au - \cot au|$

71. $\displaystyle\int u \sin au\,du = \frac{1}{a^2}\sin au - \frac{u}{a}\cos au$

72. $\displaystyle\int u \cos au\,du = \frac{1}{a^2}\cos au + \frac{u}{a}\sin au$

73. $\displaystyle\int u^2 \sin au\,du = \frac{2u}{a^2}\sin au - \frac{a^2 u^2 - 2}{a^3}\cos au$

74. $\displaystyle\int u^2 \cos au\,du = \frac{2u}{a^2}\cos au + \frac{a^2 u^2 - 2}{a^3}\sin au$

INTEGRALS INVOLVING EXPONENTIAL FUNCTIONS

75. $\displaystyle\int ue^{au}\,du = \frac{e^{au}}{a^2}(au - 1)$

76. $\displaystyle\int u^2 e^{au}\,du = \frac{e^{au}}{a^3}(a^2u^2 - 2au + 2)$

77. $\displaystyle\int u^n e^{au}\,du = \frac{u^n e^{au}}{a} - \frac{n}{a}\int u^{n-1}e^{au}\,du, \quad n > 1$

78. $\displaystyle\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a\sin bu - b\cos bu)$

79. $\displaystyle\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a\cos bu + b\sin bu)$

80. $\displaystyle\int \frac{du}{1 + e^{au}} = u - \frac{1}{a}\ln(1 + e^{au})$

MISCELLANEOUS INTEGRALS

81. $\displaystyle\int \arcsin au\,du = u\arcsin au + \frac{\sqrt{1 - a^2u^2}}{a}$

82. $\displaystyle\int \arccos au\,du = u\arccos au - \frac{\sqrt{1 - a^2u^2}}{a}$

83. $\displaystyle\int \arctan au\,du = u\arctan au - \frac{1}{2a}\ln(1 + a^2u^2)$

84. $\displaystyle\int \text{arccot}\,au\,du = u\,\text{arccot}\,au + \frac{1}{2a}\ln(1 + a^2u^2)$

85. $\displaystyle\int \ln au\,du = u\ln au - u$

86. $\displaystyle\int u^n \ln au\,du = u^{n+1}\left[\frac{\ln au}{n + 1} - \frac{1}{(n + 1)^2}\right]$

87. $\displaystyle\int_0^\infty e^{-a^2u^2}\,du = \frac{\sqrt{\pi}}{2a}$

WALLIS' FORMULAS

88. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^m u \, du = \begin{cases} \dfrac{(m-1)(m-3)\cdots(3)(1)}{(m)(m-2)\cdots(4)(2)} \dfrac{\pi}{2}, & m \text{ even}; \\[2ex] \dfrac{(m-1)(m-3)\cdots(4)(2)}{(m)(m-2)\cdots(5)(3)}, & m \text{ odd} \end{cases}$

89. $\displaystyle\int_0^{\frac{\pi}{2}} \cos^m u \, du = \int_0^{\frac{\pi}{2}} \sin^m u \, du$

90. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^m u \cos^n u \, du =$

$$\begin{cases} \dfrac{(m-1)(m-3)\cdots(3)(1)(n-1)(n-3)\cdots(4)(2)}{(m+n)(m+n-2)\cdots(3)(1)}, & m \text{ even, } n \text{ odd}; \\[2ex] \dfrac{(m-1)(m-3)\cdots(4)(2)(n-1)(n-3)\cdots(3)(1)}{(m+n)(m+n-2)\cdots(3)(1)}, & m \text{ odd, } n \text{ even}; \\[2ex] \dfrac{(m-1)(m-3)\cdots(4)(2)(n-1)(n-3)\cdots(4)(2)}{(m+n)(m+n-2)\cdots(4)(2)}, & m \text{ odd, } n \text{ odd}; \\[2ex] \dfrac{(m-1)(m-3)\cdots(3)(1)(n-1)(n-3)\cdots(3)(1)}{(m+n)(m+n-2)\cdots(4)(2)} \dfrac{\pi}{2}, & m \text{ even, } n \text{ even} \end{cases}$$

GAMMA FUNCTION

Definition :

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt, \qquad x > 0$$

Shift Property :

$$\Gamma(x+1) = x\Gamma(x), \qquad x > 0$$

Definition :

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \qquad x < 0, \ x \neq -1, -2, \ldots$$

Special Values :

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(n) = (n-1)!, \qquad n = 1, 2, \ldots$$

LAPLACE TRANSFORMS

1.	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\, dt$
2.	$af(t) + bg(t)$	$aF(s) + bG(s)$
3.	$f'(t)$	$sF(s) - f(0)$
4.	$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
5.	$t^n f(t), \quad n = 1, 2, 3, \ldots$	$(-1)^n F^{(n)}(s)$
6.	$e^{at} f(t)$	$F(s - a)$
7.	$f(t + P) \equiv f(t)$	$\dfrac{\int_0^P e^{-st} f(t)\, dt}{1 - e^{-sP}}$
8.	$f(t) H(t - a)$	$e^{-as} L\{f(t + a)\}$
9.	$f(t - a) H(t - a)$	$e^{-as} F(s)$
10.	$\int_0^t f(u)\, du$	$\dfrac{F(s)}{s}$
11.	$\int_0^t f(u) g(t - u)\, du$	$F(s) G(s)$
12.	$\dfrac{f(t)}{t}$	$\int_s^\infty F(v)\, dv$
13.	$1, \ H(t)^*, \ u(t)$	$\dfrac{1}{s}$
14.	$t^n, \quad n = 1, 2, 3, \ldots$	$\dfrac{n!}{s^{n+1}}$
15.	$t^a, \quad a > -1$	$\dfrac{\Gamma(a+1)}{s^{a+1}}$
16.	e^{at}	$\dfrac{1}{s - a}$
17.	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
18.	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
19.	$\dfrac{1}{a}(e^{at} - 1)$	$\dfrac{1}{s(s - a)}$
20.	$\dfrac{1}{a - b}(e^{at} - e^{bt})$	$\dfrac{1}{(s - a)(s - b)}, \quad a \neq b$
21.	te^{at}	$\dfrac{1}{(s - a)^2}$
22.	$t^n e^{at}, \quad n = 1, 2, 3, \ldots$	$\dfrac{n!}{(s - a)^{n+1}}$
23.	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
24.	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
25.	$\dfrac{1}{a^2}(1 - \cos at)$	$\dfrac{1}{s(s^2 + a^2)}$
26.	$\dfrac{1}{a^3}(at - \sin at)$	$\dfrac{1}{s^2(s^2 + a^2)}$

* The Heaviside step function H is defined by

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0. \end{cases}$$

LAPLACE TRANSFORMS (Continued)

27.	$\frac{1}{2a^3}\left(\sin at - at\cos at\right)$	$\frac{1}{(s^2+a^2)^2}$
28.	$\frac{1}{2a}t\sin at$	$\frac{s}{(s^2+a^2)^2}$
29.	$\frac{1}{2a}\left(\sin at + at\cos at\right)$	$\frac{s^2}{(s^2+a^2)^2}$
30.	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
31.	$\frac{b\sin at - a\sin bt}{ab(b^2-a^2)}$	$\frac{1}{(s^2+a^2)(s^2+b^2)},\quad a\neq b$
32.	$\frac{\cos at - \cos bt}{(b^2-a^2)}$	$\frac{s}{(s^2+a^2)(s^2+b^2)},\quad a\neq b$
33.	$\frac{a\sin at - b\sin bt}{(a^2-b^2)}$	$\frac{s^2}{(s^2+a^2)(s^2+b^2)},\quad a\neq b$
34.	$e^{-bt}\sin\omega t$	$\frac{\omega}{(s+b)^2+\omega^2}$
35.	$e^{-bt}\cos\omega t$	$\frac{s+b}{(s+b)^2+\omega^2}$
36.	$1-\cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
37.	$\delta(t-a)$	e^{-sa}
38.	$\frac{\sqrt{(a-b)^2+\omega^2}}{\omega}e^{-bt}\sin(\omega t+\Psi)$ where $$\Psi=\arctan\frac{\omega}{a-b}$$	$\frac{s+a}{(s+b)^2+\omega^2}$
39.	$1+\frac{\sqrt{b^2+\omega^2}}{\omega}e^{-bt}\sin(\omega t-\Psi)$ where $$\Psi=\arctan\frac{\omega}{-b}$$	$\frac{b^2+\omega^2}{s[(s+b)^2+\omega^2]}$
40.	$1+\frac{e^{-act}}{\sqrt{1-c^2}}\sin(\sqrt{1-c^2}\,at-\Psi)$ where $\Psi=\arctan\frac{\sqrt{1-c^2}}{-c},\quad -1<c<1$	$\frac{a^2}{s[s^2+2acs+a^2]}$
41.	$\frac{a}{c^2}+\frac{\sqrt{(a-b)^2+\omega^2}}{c\omega}e^{-bt}\sin(\omega t-\Psi)$ where $\Psi=\arctan\frac{a\omega}{c^2-ab},\quad c^2=b^2+\omega^2$	$\frac{s+a}{s[(s+b)^2+\omega^2]}$

PROBABILITY AND STATISTICS

Formulas for counting permutations and combinations:

(a) $\ _nP_k = \dfrac{n!}{(n-k)!}$
(b) $\ _nC_k = \dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Let A and B be subsets (events) of the same sample space.

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(b) The conditional probability of B given A, $P(B|A)$, is defined by
$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \qquad P(A) \neq 0.$$

(c) A and B are statistically independent if and only if
$$P(A \cap B) = P(A)P(B).$$

Let X be a discrete random variable with probability function $f(x)$.

(a) $P(X = x) = f(x)$, where $x \in S \equiv$ range of X

(b) $E(X) = \sum_S x f(x) = \mu =$ mean of X

(c) $\mathrm{var}(X) = E((X - \mu)^2) = E(X^2) - \mu^2 = \sum_S x^2 f(x) - \mu^2 = \sigma^2$

$=$ variance of X

(d) $\sigma = \sqrt{\mathrm{var}(X)} =$ standard deviation of X.

Let X be a continuous random variable with and probability density $f(x)$.

(a) $P(a \leq X \leq b) = \displaystyle\int_a^b f(x)\,dx$.

(b) $E(X) = \displaystyle\int_S x f(x)\,dx = \mu =$ mean of X

(c) $var(X) = E((X - \mu)^2) = E(X^2) - \mu^2 = \displaystyle\int_S x^2 f(x)\,dx - \mu^2 = \sigma^2$

$=$ variance of X

(d) $\sigma = \sqrt{\mathrm{var}(X)} =$ standard deviation of X.

Let $X_1, X_2, \ldots, X_n$ be n statistically independent random variables, each having the same expected value, μ, and the same variance, σ^2. Let
$$Y = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

Then Y is a random variable for which $E(Y) = \mu$ and $\mathrm{var}(Y) = \frac{\sigma^2}{n}$.

PROBABILITY FUNCTIONS (Discrete Random Variables)

Binomial

$$f(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad (x = 0, 1, 2, \ldots, n),$$

$$\mu = np, \qquad \sigma^2 = np(1-p).$$

Poisson

$$f(x; a) = \frac{e^{-a} a^x}{x!}, \qquad (x = 0, 1, 2, \ldots),$$

$$\mu = \sigma^2 = a.$$

STANDARD NORMAL CUMULATIVE DISTRIBUTION FUNCTION

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} \, dt$$
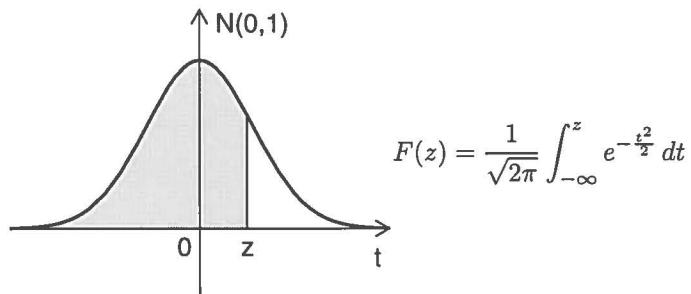
TABLE OF NORMAL AREAS

z	$F(z)$	z	$F(z)$	z	$F(z)$	z	$F(z)$
0.0	0.500	0.8	0.788	1.6	0.945	2.4	0.992
0.1	0.540	0.9	0.816	1.7	0.955	2.5	0.994
0.2	0.579	1.0	0.841	1.8	0.964	2.6	0.995
0.3	0.618	1.1	0.864	1.9	0.971	2.7	0.997
0.4	0.655	1.2	0.885	2.0	0.977	2.8	0.997
0.5	0.691	1.3	0.903	2.1	0.982	2.9	0.998
0.6	0.726	1.4	0.919	2.2	0.986	3.0	0.999
0.7	0.758	1.5	0.933	2.3	0.989		

PROBABILITY FUNCTIONS (Continuous Random Variables)

Uniform

$$f(x; a, b) = \frac{1}{b-a} \qquad (a \le x \le b).$$

$$\mu = \frac{a+b}{2}; \qquad \sigma^2 = \frac{(b-a)^2}{12}.$$

Exponential

$$f(x; b) = be^{-bx} \qquad (x \ge 0, \ b > 0).$$

$$\mu = \frac{1}{b}; \qquad \sigma^2 = \frac{1}{b^2}.$$

Normal

$$f(x; a, b) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2b^2}} \qquad (-\infty < x < \infty).$$

$$\mu = a; \qquad \sigma^2 = b^2.$$

FOURIER SERIES

The Fourier series expansion of a function $f(t)$ is defined by:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right),$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(t)\, dt$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L}\, dt, \quad n \ge 1$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L}\, dt, \quad n \ge 1$$

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FOURIER SERIES (Continued)

<u>Examples of Fourier Series</u>

$$f(t) = \begin{cases} 1, & 0 < t < L; \\ -1, & -L < t < 0. \end{cases}$$
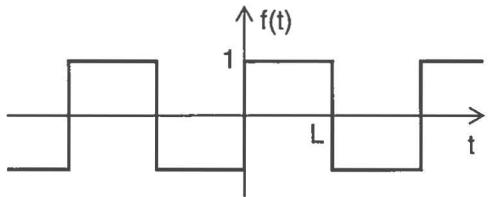
$$\frac{4}{\pi}\left(\sin\frac{\pi t}{L} + \frac{1}{3}\sin\frac{3\pi t}{L} + \frac{1}{5}\sin\frac{5\pi t}{L} + \cdots\right)$$

$$f(t) = \begin{cases} 0, & \frac{L}{2} < |t| < L \\ 1, & 0 < |t| < \frac{L}{2}. \end{cases}$$
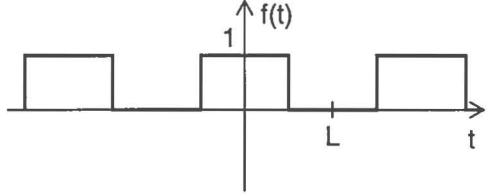
$$\frac{1}{2} + \frac{2}{\pi}\left(\cos\frac{\pi t}{L} - \frac{1}{3}\cos\frac{3\pi t}{L} + \frac{1}{5}\cos\frac{5\pi t}{L} - +\cdots\right)$$

$$f(t) = t, \qquad -L < t < L$$
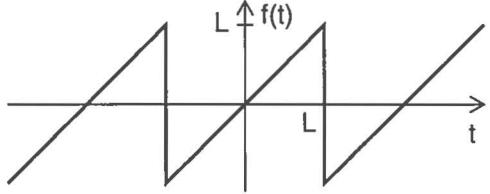
$$\frac{2L}{\pi}\left(\sin\frac{\pi t}{L} - \frac{1}{2}\sin\frac{2\pi t}{L} + \frac{1}{3}\sin\frac{3\pi t}{L} - +\cdots\right)$$

$$f(t) = |t|, \qquad -L < t < L$$
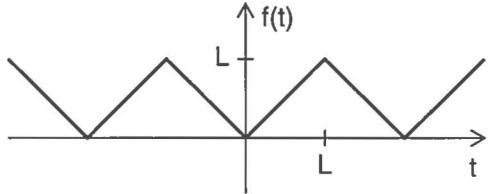
$$\frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi t}{L} + \frac{1}{3^2}\cos\frac{3\pi t}{L} + \frac{1}{5^2}\cos\frac{5\pi t}{L} + \cdots\right)$$

SEPARATION OF VARIABLES

Eigenvalues and Eigenfunctions for the differential equation

$$\varphi'' + \lambda\varphi = 0.$$

1. $\varphi(0) = \varphi(L) = 0$

   $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \varphi_n = \sin\frac{n\pi}{L}x, \quad n = 1, 2, \ldots$

2. $\varphi'(0) = \varphi(L) = 0$

   $\lambda_n = \left[(n - \frac{1}{2})\frac{\pi}{L}\right]^2, \quad \varphi_n = \cos(n - \frac{1}{2})\frac{\pi}{L}x, \quad n = 1, 2, \ldots$

3. $\varphi(0) = \varphi'(L) = 0$

   $\lambda_n = \left[(n - \frac{1}{2})\frac{\pi}{L}\right]^2, \quad \varphi_n = \sin(n - \frac{1}{2})\frac{\pi}{L}x, \quad n = 1, 2, \ldots$

4. $\varphi'(0) = \varphi'(L) = 0$

   $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \varphi_n = \cos\frac{n\pi}{L}x, \quad n = 0, 1, 2, \ldots$

5. $\varphi(-L) = \varphi(L), \quad \varphi'(-L) = \varphi'(L)$

   $\lambda_0 = 0, \quad \varphi_0 = 1,$

   $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \varphi_n = \sin\frac{n\pi}{L}x \quad \text{and} \quad \cos\frac{n\pi}{L}x, \quad n = 1, 2, \ldots$

Solution of inhomogeneous ordinary differential equations

1. $u_n' + k\lambda_n u_n = h_n(t)$

   $u_n(t) = a_n e^{-k\lambda_n t} + \int_0^t h_n(\tau)e^{-k\lambda_n(t-\tau)}\,d\tau$

2. $u_n'' + \mu_n^2 u_n = h_n(t)$

   $u_n(t) = a_n\cos\mu_n t + b_n\sin\mu_n t + \frac{1}{\mu_n}\int_0^t h_n(\tau)\sin\mu_n(t-\tau)\,d\tau$

BESSEL FUNCTIONS

1. The differential equation

$$x^2 y'' + xy' + (\lambda^2 x^2 - n^2)y = 0$$

has solutions

$$y = y_n(\lambda x) = c_1 J_n(\lambda x) + c_2 Y_n(\lambda x),$$

where

$$J_n(t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^{n+2m}}{2^{n+2m} m! \Gamma(n+m+1)}.$$

2. General properties:

$$J_{-n}(t) = (-1)^n J_n(t); \qquad J_0(0) = 1; \qquad J_n(0) = 0, \qquad n = 1, 2, 3, \ldots;$$

for fixed n, $J_n(t) = 0$ has infinitely many solutions.

3. Identities:

(a) $\frac{d}{dt}[t^n J_n(t)] = t^n J_{n-1}(t).$

(b) $\frac{d}{dt}[t^{-n} J_n(t)] = -t^{-n} J_{n+1}(t); \qquad J_0'(t) = -J_1(t)$

(c) $J_n'(t) = \frac{1}{2}[J_{n-1}(t) - J_{n+1}(t)]$

$$= J_{n-1}(t) - \frac{n}{t} J_n(t) = \frac{n}{t} J_n(t) - J_{n+1}(t)$$

(d) $J_{n+1}(t) = \frac{2n}{t} J_n(t) - J_{n-1}(t)$ (recurrence).

4. Orthogonality:

Solutions $y_n(\lambda_0 x), y_n(\lambda_1 x), \ldots, y_n(\lambda_i x), \ldots$ of the differential eigensystem

$$x^2 y'' + xy' + (\lambda^2 x^2 - n^2)y = 0$$

$$A_k y_n(\lambda x_k) - B_k \left[\frac{d}{dx} y_n(\lambda x)\right]_{x=x_k} = 0, \qquad k = 1, 2$$

have the property

$$\int_{x_1}^{x_2} x y_n(\lambda_i x) y_n(\lambda_j x)\, dx = 0$$

for $i \neq j$, and

$$\int x y_n^2(\lambda_i x)\, dx = \frac{(\lambda_i^2 x^2 - n^2) y_n^2(\lambda_i x) + x^2 \left[\frac{d}{dx} y_n(\lambda_i x)\right]^2}{2\lambda_i^2}$$

for $i = j$.

5. Integration properties:

(a) $\quad \displaystyle\int t^\nu J_{\nu-1}(t)\,dt = t^\nu J_\nu(t) + C$

(b) $\quad \displaystyle\int t^{-\nu} J_{\nu+1}(t)\,dt = -t^{-\nu} J_\nu(t) + C$

(c) $\quad \displaystyle\int t J_0(t)\,dt = t J_1(t) + C$

(d) $\quad \displaystyle\int t^3 J_0(t)\,dt = (t^3 - 4t)J_1(t) + 2t^2 J_0(t) + C$

(e) $\quad \displaystyle\int t^2 J_1(t)\,dt = 2t J_1(t) - t^2 J_0(t) + C$

(f) $\quad \displaystyle\int t^4 J_1(t)\,dt = (4t^3 - 16t)J_1(t) - (t^4 - 8t^2)J_0(t) + C$

LEGENDRE POLYNOMIALS

The differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

has solution $y = P_n(x)$, where

$$P_n(x) = \frac{1}{2^n}\sum_{m=0}^{[n/2]}(-1)^m \binom{n}{m}\binom{2n-2m}{n}x^{n-2m}$$

on the interval $[-1, 1]$. The Legendre polynomials can also be obtained iteratively from the first two,

$$P_0(x) = 1 \qquad \text{and} \qquad P_1(x) = x,$$

and the recurrence relation,

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

The Legendre polynomials are also given by Rodrigues' formula:

$$P_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x^2)^n].$$

Orthogonality:

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0, & n \neq m; \\ \frac{2}{2n+1}, & n = m. \end{cases}$$

Standardization:

$$P_n(1) = 1.$$

TRIGONOMETRIC FUNCTIONS IN A RIGHT TRIANGLE

If A, B, and C are the vertices (C the right angle), and a, b, and r the sides opposite, respectively, then

$$\text{sine } A = \sin A = \frac{a}{r}, \qquad \text{cosine } A = \cos A = \frac{b}{r},$$

$$\text{tangent } A = \tan A = \frac{a}{b}, \qquad \text{cotangent } A = \cot A = \frac{b}{a},$$

$$\text{secant } A = \sec A = \frac{r}{b}, \qquad \text{cosecant } A = \csc A = \frac{r}{a},$$

exsecant A = exsec A = $\sec A - 1$

versine A = vers A = $1 - \cos A$

coversine A = covers A = $1 - \sin A$

haversine A = hav A = $\frac{1}{2}$vers A
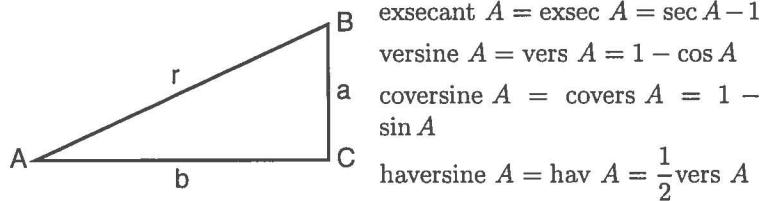
TRIGONOMETRIC FUNCTIONS OF SPECIAL ANGLES

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞
cot	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	∞	0
sec	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞	-1	∞
csc	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	∞	-1

COMPLEX EXPONENTIAL FORMS $(i^2 = -1)$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \qquad \csc x = \frac{2i}{e^{ix} - e^{-ix}}$$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \qquad \sec x = \frac{2}{e^{ix} + e^{-ix}}$$

$$\tan x = \frac{1}{i}\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} \qquad \cot x = i\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}$$

RELATIONS AMONG THE FUNCTIONS

$\sin x = \frac{1}{\csc x}$

$\cos x = \frac{1}{\sec x}$

$\tan x = \frac{1}{\cot x} = \frac{\sin x}{\cos x}$

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

* $\sin x = \pm\sqrt{1 - \cos^2 x}$

* $\tan x = \pm\sqrt{\sec^2 x - 1}$

* $\cot x = \pm\sqrt{\csc^2 x - 1}$

$\sin x = \cos(\frac{\pi}{2} - x) = \sin(\pi - x)$

$\tan x = \cot(\frac{\pi}{2} - x) = -\tan(\pi - x)$

$\csc x = \frac{1}{\sin x}$

$\sec x = \frac{1}{\cos x}$

$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$1 + \tan^2 x = \sec^2 x$

* $\cos x = \pm\sqrt{1 - \sin^2 x}$

* $\sec x = \pm\sqrt{1 + \tan^2 x}$

* $\csc x = \pm\sqrt{1 + \cot^2 x}$

$\cos x = \sin(\frac{\pi}{2} - x) = -\cos(\pi - x)$

$\cot x = \tan(\frac{\pi}{2} - x) = -\cot(\pi - x)$

SUMS AND MULTIPLES OF ANGLES

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\sin 3x = 3 \sin x - 4 \sin^3 x$

$\cos 3x = 4 \cos^3 x - 3 \cos x$

$\sin 4x = 8 \sin x \cos^3 x - 4 \sin x \cos x$

$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$

$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$ $\qquad$ $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$

$\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

* $\sin \dfrac{1}{2}x = \pm\sqrt{\dfrac{1 - \cos x}{2}}$ $\qquad$ * $\cos \dfrac{1}{2}x = \pm\sqrt{\dfrac{1 + \cos x}{2}}$

* $\tan \dfrac{1}{2}x = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}} = \dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$

* The choice of the sign in front of the radical depends on the quadrant in which x, regarded as an angle, falls.

MISCELLANEOUS RELATIONS

$\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y)$

$\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$

$\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$

$\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$

$\cos x \sin y = \frac{1}{2}[\sin(x + y) - \sin(x - y)]$

$\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$

$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$

$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}$ $\cot x \pm \cot y = \frac{\pm \sin(x \pm y)}{\sin x \sin y}$

$\frac{1 + \tan x}{1 - \tan x} = \tan(\frac{\pi}{4} + x)$ $\frac{\cot x + 1}{\cot x - 1} = \cot(\frac{\pi}{4} - x)$

$\frac{\sin x \pm \sin y}{\cos x + \cos y} = \tan \frac{1}{2}(x \pm y)$ $\frac{\sin x \pm \sin y}{\cos x - \cos y} = -\cot \frac{1}{2}(x \mp y)$

$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{\tan \frac{1}{2}(x + y)}{\tan \frac{1}{2}(x - y)}$

$\sin^2 x - \sin^2 y = \sin(x + y) \, \sin(x - y)$

$\cos^2 x - \cos^2 y = -\sin(x + y) \, \sin(x - y)$

$\cos^2 x - \sin^2 y = \cos(x + y) \, \cos(x - y)$

INVERSE TRIGONOMETRIC FUNCTIONS

The following table gives the domains of the inverse trigonometric functions.

Function	Interval containing principal value	
	x positive or zero	x negative
$y = \arcsin x$ and $y = \arctan x$	$0 \leq y \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq y \leq 0$
$y = \arccos x$ and $y = \operatorname{arccot} x$	$0 \leq y \leq \frac{\pi}{2}$	$\frac{\pi}{2} \leq y \leq \pi$
$y = \operatorname{arcsec} x$ and $y = \operatorname{arccsc} x$	$0 \leq y \leq \frac{\pi}{2}$	$-\pi \leq y \leq -\frac{\pi}{2}$

Two notation systems are in common use. For example, the inverse sine is often denoted by arcsin or $\sin^{-1}$. In many references, however, "principal value" is used for the function and is denoted by Arcsin or Sin^{-1}. Thus, in one book,

$$\operatorname{Arcsin} \frac{1}{2} = \operatorname{Sin}^{-1} x = \frac{\pi}{6} \quad \text{and} \quad \arcsin x = \frac{\pi}{6} + 2n\pi,$$

while in another, only lower case is used:

$$\arcsin \frac{1}{2} = \frac{\pi}{6}.$$

Special care is advisable in the use of reference materials.

SIDE-ANGLE RELATIONS IN PLANE TRIANGLES

Letters A, B, and C denote the angles of a plane triangle with opposite sides a, b, and c, respectively. The letter s denotes the semi-perimeter of the triangle, that is,

$$s = \frac{1}{2}(a + b + c).$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{ diameter of the circumscribed circle}$$

$$a^2 = b^2 + c^2 - 2\,b\,c\,\cos A$$

$$a = b\,\cos C + c\,\cos B$$

$$\tan\frac{A - B}{2} = \frac{a - b}{a + b}\cot\frac{C}{2}$$

$$\sin A = \frac{2}{bc}\sqrt{s(s - a)(s - b)(s - c)}$$

$$\text{area } = \sqrt{s(s - a)(s - b)(s - c)}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}$$

$$\frac{a + b}{a - b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan\frac{1}{2}(A + B)}{\tan\frac{1}{2}(A - B)} = \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A - B)}$$
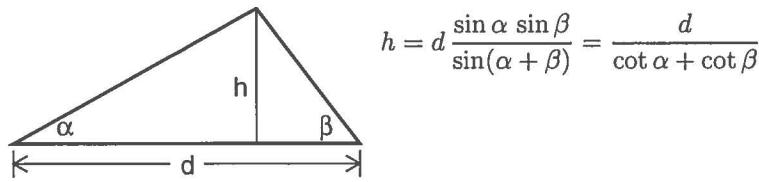
$$h = d\,\frac{\sin\alpha\,\sin\beta}{\sin(\alpha + \beta)} = \frac{d}{\cot\alpha + \cot\beta}$$
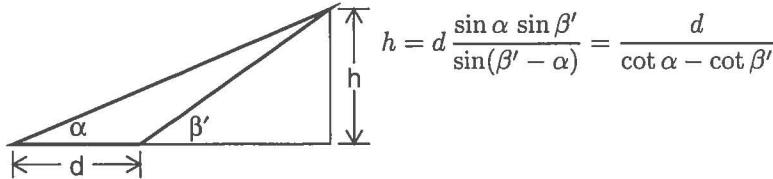
$$h = d\,\frac{\sin\alpha\,\sin\beta'}{\sin(\beta' - \alpha)} = \frac{d}{\cot\alpha - \cot\beta'}$$

HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \pm\frac{\sinh x}{\cosh x} = \frac{1}{\coth x}$$

$$\sinh x = \pm\sqrt{\cosh^2 x - 1} = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}} = \pm\frac{1}{\sqrt{\coth^2 x - 1}}$$

$$\cosh x = \sqrt{1 + \sinh^2 x} = \frac{1}{\sqrt{1 - \tanh^2 x}} = \pm\frac{\coth x}{\sqrt{\coth^2 x - 1}}$$

$$\tanh x = \frac{\sinh x}{\sqrt{1 + \sinh^2 x}} = \pm\frac{\sqrt{\cosh^2 x - 1}}{\cosh x} = \frac{1}{\coth x}$$

$$\coth x = \frac{\sqrt{1 + \sinh^2 x}}{\sinh x} = \pm\frac{\cosh x}{\sqrt{\cosh^2 x - 1}} = \frac{1}{\tanh x}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \qquad \coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}$$

$$\sinh 2x = 2\cosh x \sinh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \qquad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1 + x}{1 - x}\right| \qquad \coth^{-1} x = \frac{1}{2}\ln\left|\frac{x + 1}{x - 1}\right|$$

Complex hyperbolic functions involve the Euler relations

$$e^{iz} = \cos z + i\sin z; \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

$$\cos z = \cosh iz \qquad\qquad \cosh z = \cos iz$$

$$\sin z = -i\sinh iz \qquad\qquad \sinh z = -i\sin iz$$

$$\tan z = -i\tanh iz \qquad\qquad \tanh z = -i\tan iz$$

$$\cot z = i\coth iz \qquad\qquad \coth z = i\cot iz$$

$$\sinh(x \pm iy) = \sinh x \cos y \pm i\cosh x \sin y$$

$$\cosh(x \pm iy) = \cosh x \cos y \pm i\sinh x \sin y$$

$$\ln z = \ln|z| + i\arg z; \quad \text{so} \quad \ln ix = \ln|x| + \left(2n + \frac{1}{2}\right)\pi i$$

$$\text{and } \ln(-x) = \ln|x| + (2n + 1)\pi i \quad (n = 0, \pm1, \pm2, \dots)$$

FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

I. <u>Separable equations</u> : $N(y)\,dy = M(x)\,dx$. To solve, integrate both sides:

$$\int N(y)\,dy = \int M(x)\,dx + c.$$

II. <u>Exact equations</u> : $M(x,y)\,dx + N(x,y)\,dy = 0$, where $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. To solve,

integrate $\dfrac{\partial f}{\partial x} = M(x,y)$ and $\dfrac{\partial f}{\partial y} = N(x,y)$:

$$f(x,y) = \int M(x,y)\,dx + \phi(y) = \int N(x,y)\,dy + \psi(x) = c.$$

III. <u>Linear equations</u> :

$$\frac{dy}{dx} + p(x)\,y = q(x).$$

The solution is given by

$$y = \frac{1}{P(x)} \int P(x)q(x)\,dx + \frac{c}{P(x)}, \qquad \text{where} \qquad P(x) = e^{\int p(x)\,dx}.$$

IV. <u>Homogeneous equations</u> :

$$\frac{dy}{dx} = f\!\left(\frac{y}{x}\right) \quad \text{or} \quad \frac{dy}{dx} = f\!\left(\frac{x}{y}\right).$$

Substitute $y = ux$, $\dfrac{dy}{dx} = x\dfrac{du}{dx} + u$ $\;$ or $\;$ $x = vy$, $\dfrac{dx}{dy} = y\dfrac{dx}{dy} + v$,

integrate the resulting expression, then resubstitute for u or v.

V. <u>Equations in the rational form</u>

$$\frac{dy}{dx} = \frac{ax + by + c}{dx + ey + f},$$

where a, b, c, d, e, f are constants such that $ae \neq bd$. Substitute

$$x = X - h, \quad y = Y - k, \quad \text{where} \quad h = \frac{bf - ce}{ae - bd}, \quad k = \frac{cd - af}{ae - bd},$$

to obtain the homogeneous equation

$$\frac{dY}{dX} = \frac{aX + bY}{dX + eY}.$$

This equation may be solved as in item IV above.

LINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS
WITH CONSTANT COEFFICIENTS

I. <u>Homogeneous equation</u> $ay'' + by' + cy = 0$. Substitute $y = e^r x$ to find the characteristic equation, and write its roots r_1 and r_2 :

$$ar^2 + br + c = 0; \qquad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The solution, y_c, is given by the following table.

Case 1. $b^2 - 4ac > 0$, \qquad\qquad $y_c = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
real and distinct roots.

Case 2. $b^2 - 4ac = 0$, \qquad\qquad $y_c = (c_1 + c_2 x)e^{rx}$
two real equal roots.

Case 3. $b^2 - 4ac < 0$, \qquad\qquad $y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$,
two complex roots. \qquad\qquad\qquad where

$$\alpha = \frac{-b}{2a}, \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

II. <u>Undetermined Coefficients :</u> $ay'' + by' + cy = F(x)$.

1. Solve the homogeneous equation $ay'' + by' + cy = 0$ for y_c.

2. Assume a particular solution y_p for any term in $F(x)$ that is one of three special types, as follows.

A. For a polynomial of degree N, y_p is a polynomial of degree N.

B. For an exponential $ke^{p(x)}$, $y_p = Ae^{p(x)}$.

C. For $j \cos \omega x + k \sin \omega x$, $y_p = A \cos \omega x + B \sin \omega x$.

3. If any term in y_p found above appears in y_c, multiply the corresponding trial term by the power of x just high enough that the resulting product contains no term of y_c.

4. For any term in $F(x)$ that is a product of terms of the three types listed above, let y_p be the product of the corresponding trial solutions.

5. Substitute y_p into the given O.D.E. and evaluate the unknown coefficients by equating like terms.

6. The general solution of the given equation is

$$y = y_c + y_p.$$

Evaluate the two arbitrary constants c_1 and c_2 in the general solution y by applying boundary/initial conditions, as appropriate.

III. <u>Variation of Parameters :</u> $ay'' + by' + cy = F(x)^*$.

1. Write the solution of the homogeneous equation $ay'' + by' + cy = 0$ in the form

$$y_c = c_1 y_1 + c_2 y_2.$$

2. Assume a particular solution y_p for any term in the form

$$y_p = u_1 y_1 + u_2 y_2,$$

where u_1 and u_2 are functions determined by the conditions

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = G(x) \text{ where } G(x) = \frac{F(x)}{a}.$$

3. The solution of this system of equations is given by

$$u_1' = -\frac{y_2 G}{W} \quad \text{and} \quad u_2' = \frac{y_1 G}{W},$$

where W denotes the Wronskian determinant:

$$W = W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0.$$

4. Integrate to find the unknown functions:

$$u_1(x) = -\int \frac{y_2(x) G(x)}{W(x)} \, dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x) G(x)}{W(x)} \, dx.$$

5. Form the particular solution $y_p = u_1 y_1 + u_2 y_2$.

6. The general solution of the given equation is

$$y = y_c + y_p.$$

Evaluate the two arbitrary constants c_1 and c_2 in the general solution y by applying boundary/initial conditions, as appropriate.

IV. <u>Laplace Transforms :</u> $ay'' + by' + cy = F(x)$, $y(0) = y_0$ and $y'(0) = y_0'$.

1. Take the Laplace transform (see pages 9-10) of both sides of the given equation, using the given initial conditions. The result is an algebraic equation for $F(s)$, the Laplace transform of the solution.

2. Solve the algebraic equation obtained in Step 1 to find $F(s)$ explicitly.

3. The inverse Laplace transform of $F(s)$ is the solution of the given initial value problem.

*The coefficients may also be functions, i.e., $b = b(x)$ and $c = c(x)$.

VECTOR ALGEBRA

1. <u>Scalar and vector products</u>. Given vectors

$$\mathbf{a} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$$
$$\mathbf{b} = \beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \beta_3 \mathbf{e}_3 = b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k},$$
$$\mathbf{c} = \gamma_1 \mathbf{e}_1 + \gamma_2 \mathbf{e}_2 + \gamma_3 \mathbf{e}_3 = c_x \mathbf{i} + c_y \mathbf{j} + c_z \mathbf{k},$$

let ϕ be the angle from $\mathbf{a}$ to $\mathbf{b}$. The <u>scalar (dot) product</u> of $\mathbf{a}$ and $\mathbf{b}$ is the scalar

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \phi = a_x b_x + a_y b_y + a_z b_z.$$

The <u>vector (cross) product</u> of $\mathbf{a}$ and $\mathbf{b}$ is the vector of magnitude

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \phi,$$

perpendicular to $\mathbf{a}$ and $\mathbf{b}$ and in the direction of the axial motion of a right-handed screw turning $\mathbf{a}$ into $\mathbf{b}$. In coordinate form,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_2 \times \mathbf{e}_3 & \alpha_1 & \beta_1 \\ \mathbf{e}_3 \times \mathbf{e}_1 & \alpha_2 & \beta_2 \\ \mathbf{e}_1 \times \mathbf{e}_2 & \alpha_3 & \beta_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & a_x & b_x \\ \mathbf{j} & a_y & b_y \\ \mathbf{k} & a_z & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \mathbf{i} + \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \mathbf{k}$$

$$= (a_y b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}.$$

The <u>box product</u> of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the scalar quantity given by

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \equiv [\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}] = -[\mathbf{bac}] = -[\mathbf{cba}] = -[\mathbf{acb}]$$

$$= \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} [\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3] = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix}.$$

The product of two box products is given by

$$[\mathbf{abc}][\mathbf{def}] = \begin{vmatrix} \mathbf{a} \cdot \mathbf{d} & \mathbf{a} \cdot \mathbf{e} & \mathbf{a} \cdot \mathbf{f} \\ \mathbf{b} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{e} & \mathbf{b} \cdot \mathbf{f} \\ \mathbf{c} \cdot \mathbf{d} & \mathbf{c} \cdot \mathbf{e} & \mathbf{c} \cdot \mathbf{f} \end{vmatrix}.$$

A special case gives <u>Gram's determinant :</u>

$$[\mathbf{abc}]^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix}$$

$$= [(\mathbf{a} \times \mathbf{b})(\mathbf{b} \times \mathbf{c})(\mathbf{c} \times \mathbf{a})]$$

$$= (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{c}) + 2(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c})^2 - (\mathbf{b} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})^2 - (\mathbf{c} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{b})^2.$$

VECTOR ALGEBRA (Continued)

The vector triple product of vectors **a**, **b**, and **c** is given by

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \begin{vmatrix} \mathbf{b} & \mathbf{c} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \end{vmatrix}.$$

Three additional formulas for scalar and vector products:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}.$$

$$(\mathbf{a} \times \mathbf{b})^2 = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{acd}]\mathbf{b} - [\mathbf{bcd}]\mathbf{a} = [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d}.$$

VECTOR ANALYSIS

Differential operators

$$\nabla \Phi(x, y, z) = \operatorname{grad} \Phi(x, y, z) \equiv \frac{\partial \Phi}{\partial x}\mathbf{i} + \frac{\partial \Phi}{\partial y}\mathbf{j} + \frac{\partial \Phi}{\partial z}\mathbf{k}$$

$$\nabla \cdot \mathbf{F}(x, y, z) = \operatorname{div} \mathbf{F}(x, y, z) \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \mathbf{F} = \operatorname{curl}\mathbf{F}(x, y, z)$$

$$\equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$(\mathbf{G} \cdot \nabla)\mathbf{F} = G_x\frac{\partial \mathbf{F}}{\partial x} + G_y\frac{\partial \mathbf{F}}{\partial y} + G_z\frac{\partial \mathbf{F}}{\partial z}$$

$$= (\mathbf{G} \cdot \nabla F_x)\mathbf{i} + (\mathbf{G} \cdot \nabla F_y)\mathbf{j} + (\mathbf{G} \cdot \nabla F_z)\mathbf{k}$$

$$\nabla^2 = (\nabla \cdot \nabla) \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}\right) \quad \text{(Laplace operator)}$$

$$\operatorname{div} \operatorname{grad} \Phi = \nabla \cdot (\nabla\Phi) = \nabla^2\Phi$$

$$\operatorname{grad} \operatorname{div} \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) = \nabla^2\mathbf{F} + \nabla \times (\nabla \times \mathbf{F})$$

$$\operatorname{curl} \operatorname{curl} \mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$$

$$\operatorname{curl} \operatorname{grad} \Phi = \nabla \times (\nabla\Phi) = 0$$

$$\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$$

$$\nabla(\Phi\Psi) = \Psi\nabla\Phi + \Phi\nabla\Psi$$

$$\nabla^2(\Phi\Psi) = \Psi\nabla^2\Phi + 2(\nabla\Phi) \cdot (\nabla\Psi) + \Phi\nabla^2\Psi$$

VECTOR ANALYSIS (Continued)

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$$

$$\nabla \cdot (\Phi\mathbf{F}) = \Phi\nabla \cdot \mathbf{F} + (\nabla\Phi) \cdot \mathbf{F}$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$$

$$(\mathbf{G} \cdot \nabla)(\Phi\mathbf{F}) = \mathbf{F}(\mathbf{G} \cdot \nabla\Phi) + \Phi(\mathbf{G} \cdot \nabla)\mathbf{F}$$

$$\nabla \times (\Phi\mathbf{F}) = \Phi\nabla \times \mathbf{F} + (\nabla\Phi) \times \mathbf{F}$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F})$$

Cylindrical : $x = r\cos\theta, y = r\sin\theta, z = z$

$$\text{grad } \Phi = \nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{k}}$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \frac{1}{r}\hat{\mathbf{r}} & \hat{\theta} & \frac{1}{r}\hat{\mathbf{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$$

$$\text{Laplacian } \Phi = \nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Spherical : $x = \rho\cos\theta\sin\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\phi$

$$\text{grad } \Phi = \nabla\Phi = \frac{\partial\Phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi} + \frac{1}{\rho\sin\phi}\frac{\partial\Phi}{\partial\theta}\hat{\theta}$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial(\rho^2 F_\rho)}{\partial\rho} + \frac{1}{\rho\sin\phi}\frac{\partial(\sin\phi F_\phi)}{\partial\phi} + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial\theta}$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \frac{1}{\rho^2\sin\phi}\hat{\rho} & \frac{1}{\rho\sin\phi}\hat{\phi} & \frac{1}{\rho}\hat{\theta} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta} \\ F_\rho & \rho F_\phi & \rho\sin\phi F_\theta \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\Phi}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2\Phi}{\partial\theta^2}$$

Integral Theorems

$\int_V \nabla \cdot \mathbf{F}(\mathbf{r})\,dV = \int_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{A}$ (Divergence theorem)

$\int_V \nabla\Phi \cdot \nabla\Psi\,dV + \int_V \Psi\nabla^2\Phi\,dV = \int_S(\Psi\nabla\Phi) \cdot d\mathbf{A}$ (Green's theorem)

$\int_V(\Psi\nabla^2\Phi - \Phi\nabla^2\Psi)\,dV = \int_S(\Psi\nabla\Phi - \Phi\nabla\Psi) \cdot d\mathbf{A}$ (symmetric form)

$\int_V \nabla^2\Phi\,dV = \int_S \nabla\Phi \cdot d\mathbf{A}$ (Gauss' theorem)

$\int_S[\nabla \times \mathbf{F}(\mathbf{r})] \cdot d\mathbf{A} = \oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ (Stokes' theorem)

FUNDAMENTALS OF SIGNALS AND NOISE

Complex numbers ($j^2 = -1$)

$$a + bj = Re^{j\theta}, \qquad \text{where}$$
$$R = \sqrt{a^2 + b^2},$$
$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right), & a > 0; \\ \arctan\left(\frac{b}{a}\right) \pm \pi, & a < 0; \\ \frac{\pi}{2}, & a = 0, b > 0; \\ -\frac{\pi}{2}, & a = 0, b < 0. \end{cases}$$

$$Ae^{j\theta} \cdot Be^{j\phi} = ABe^{j(\theta+\phi)}$$
$$e^{j\theta} = e^{j(\theta+2n\pi)}, \quad n = \pm 1, \pm 2, \ldots$$

Unit phasor

$$j = e^{j\frac{\pi}{2}}$$

$$e^{j\pi} = -1 \qquad 1 = e^{j\cdot 0} = e^{j2\pi}$$

$$-j = e^{-j\frac{\pi}{2}} = e^{j\frac{3\pi}{2}}$$
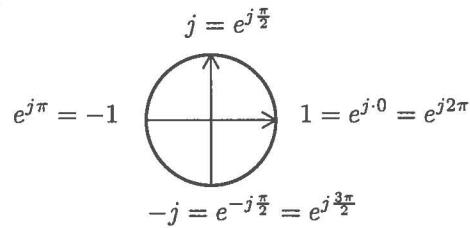
Euler Identity

$$e^{\pm jx} = \cos x \pm j \sin x; \qquad \begin{cases} \cos x & = \frac{e^{jx}+e^{-jx}}{2}; \\ \sin x & = \frac{e^{jx}-e^{-jx}}{2}. \end{cases}$$

Fourier series (Periodic signals. For the real number form, see p. 13.)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_1 t}, \qquad \text{where}$$
$$c_n = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} x(t) e^{-j2\pi n f_1 t}\, dt,$$
$$T_1 = \text{ period,}$$
$$f_1 = \frac{1}{T_1} = \text{fundamental frequency,}$$

(For real signals, $c_{-n} = c_n^*$ (complex conjugate).)

Fourier transform (aperiodic signals)

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt; \qquad x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi tf}\, df.$$

The correspondence between $x(t)$ and $X(f)$ is denoted by $x(t) \leftrightarrow X(f)$.

Transform of Fourier Series

$$\text{If } x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_1 t}, \quad \text{then } X(f) = \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_1).$$

Convolution ($*$) and Correlation ($\star$)

$$x * y = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\, d\tau \equiv \int_{-\infty}^{\infty} y(\tau)x(t-\tau)\, d\tau = y * x$$

$$x \star y = \int_{-\infty}^{\infty} x(\tau)y(t+\tau)\, d\tau$$

Properties of the delta function

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)x(t)\, dt = x(t_0)$$

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

Parseval theorem (Average Power in a periodic signal)

$$\overline{P} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Rayleigh theorem (Total Energy in an aperiodic signal)

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

Fourier theorems

If $x(t) \leftrightarrow X(f)$ is a *transform pair*, then

LINEARITY $\qquad\qquad\qquad\qquad\qquad\qquad ax(t) + by(t) \leftrightarrow aX(f) + bY(f)$

SHIFT/MODULATION $\qquad\qquad\qquad\quad x(t - t_0) \leftrightarrow X(f)e^{-j2\pi t_0 f}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0)$

PRODUCT/CONVOLUTION $\qquad\qquad x(t)y(t) \leftrightarrow X * Y$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x * y \leftrightarrow X(f)Y(f)$

SCALE $\qquad\qquad\qquad\qquad\qquad\qquad\quad x(at) \leftrightarrow \dfrac{1}{|a|} X\left(\dfrac{f}{a}\right)$

DIFFERENTIATION $\qquad\qquad\qquad\quad \dfrac{d}{dt} x(t) \leftrightarrow j2\pi f X(f)$

INTEGRATION $\qquad\qquad\qquad\qquad \displaystyle\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \dfrac{X(f)}{j2\pi f}.$

Fourier pairs

Time Domain		Frequency Domain
$h(t) = 1 - u_{\frac{T_0}{2}}(|t|)$	$\leftrightarrow$	$H(f) = \dfrac{\sin(\pi T_0 f)}{\pi f}$

| $h(t) = \dfrac{\sin(2\pi f_0 t)}{2\pi f_0 t}$ | $\leftrightarrow$ | $H(f) = \dfrac{1}{2f_0}(1 - u_{f_0}(|f|))$ |

Fourier pairs Continued

Time Domain		Frequency Domain

$$h(t) = K \quad \leftrightarrow \quad H(f) = K\delta(f)$$

$$h(t) = K\delta(t) \quad \leftrightarrow \quad H(f) = K$$

$$h(t) = A\cos(2\pi f_0 t) \quad \leftrightarrow \quad H(f) = \frac{A}{2}[\delta(f + f_0) + \delta(f - f_0)]$$

$$h(t) = A\sin(2\pi f_0 t) \quad \leftrightarrow \quad H(f) = \frac{A}{2}j[\delta(f + f_0) - \delta(f - f_0)]$$

$$h(t) = \left[1 - \frac{|t|}{T_0}\right]\left[1 - u_{T_0}(|t|)\right] \quad \leftrightarrow \quad H(f) = \frac{\sin^2(\pi T_0 f)}{T_0 \pi^2 f^2}$$
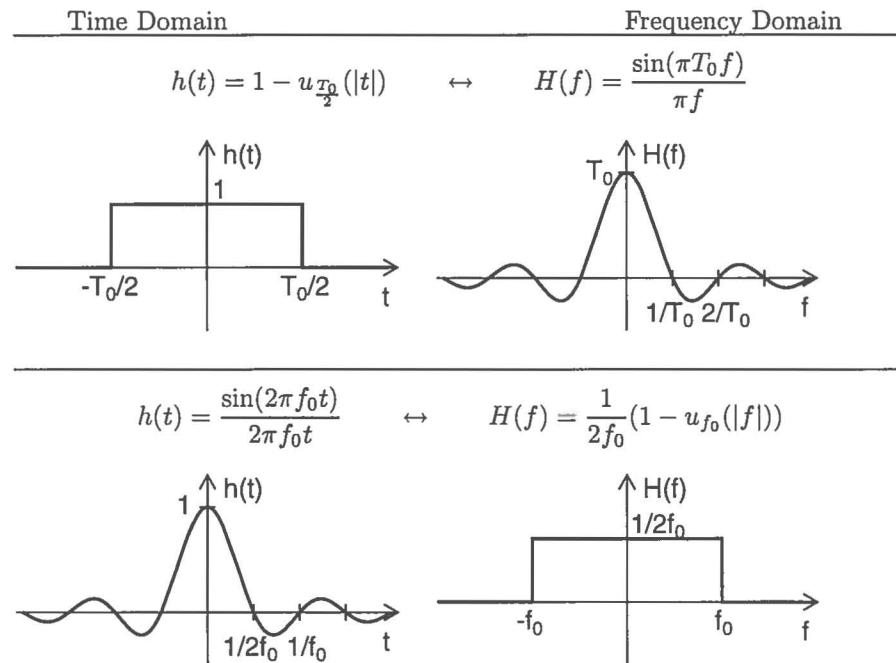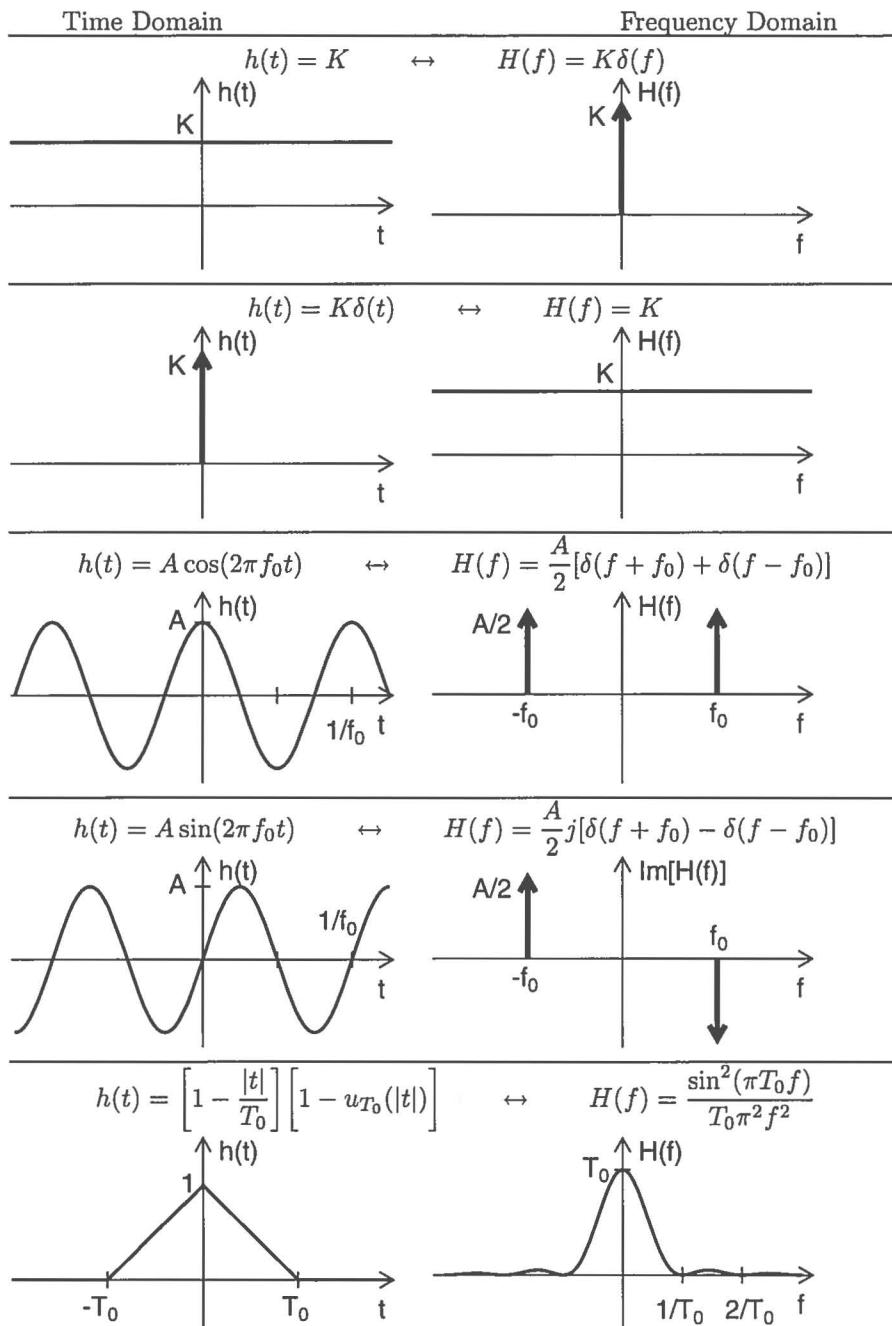
Fourier pairs Continued

Time Domain	Frequency Domain

$$h(t) = e^{-\alpha|t|} \qquad \leftrightarrow \qquad H(f) = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$
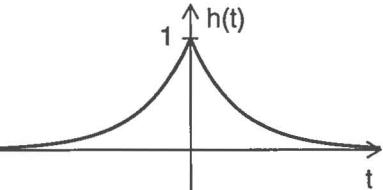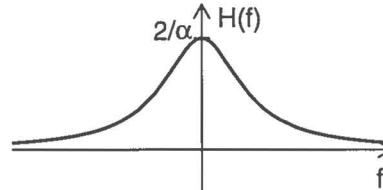
$$h(t) = \exp(-\alpha t^2) \qquad \leftrightarrow \qquad H(f) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{-\pi^2 f^2}{\alpha}\right)$$
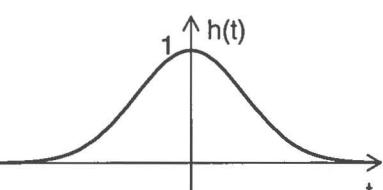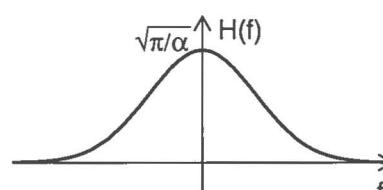
$$h(t) = \begin{cases} e^{-\alpha t}, & t > 0 \\ 0, & t \leq 0 \end{cases} \qquad \leftrightarrow \qquad H(f) = \frac{1}{\alpha + 2\pi j f}$$
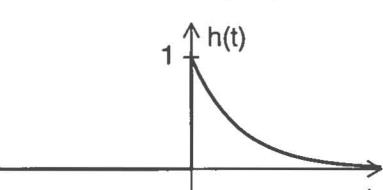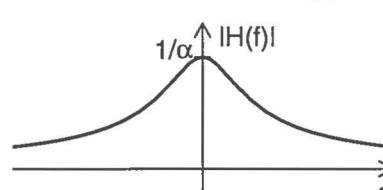
$$h(t) = \mathrm{sgn}(t) \qquad \leftrightarrow \qquad H(f) = \frac{-j}{\pi f}$$
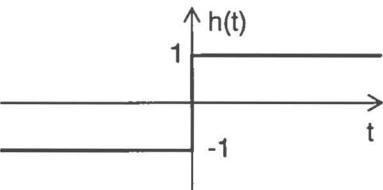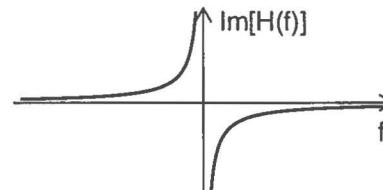
Linear time – invariant systems

Basic Properties

If

$$x(t) \quad \longrightarrow \boxed{} \longrightarrow \quad y(t)$$

$$g(t) \quad \longrightarrow \boxed{} \longrightarrow \quad q(t)$$

then

$$ax(t - t_0) + bg(t - t_0) \quad \longrightarrow \boxed{} \longrightarrow \quad ay(t - t_0) + bq(t - t_0)$$

Transform properties

	Input	Output
Time Domain	$\delta(t)$	$h(t)$
	$x(t)$	$y(t) = x(t) * h(t)$

$$\longrightarrow \boxed{\text{system}} \longrightarrow$$

Frequency Domain $\quad X(f) \qquad Y(f) = X(f) \cdot H(f)$

$h(t) \leftrightarrow H(f) \qquad h(t) \equiv$ impulse response; $\qquad H(f) \equiv$ transfer function

Random signals

$$\bar{x} = \int_{-\infty}^{\infty} x p(x)\, dx \qquad \text{Mean value}$$

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2 p(x)\, dx \qquad \text{Second moment}$$

$$<x, x> = \|x\|^2 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\, dt \qquad \text{Norm square}$$

$$<x> = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt \qquad \text{Time average}$$

For an ergodic process:
   Second moment = Norm square   and   Mean value = Time average.

Ideal filter (low − pass)

$$h(t) = \frac{\sin(2\pi f_0 t)}{2\pi f_0} \qquad \leftrightarrow \qquad H(f) = \frac{1 - u_{f_0}(|f|)}{2f_0}$$
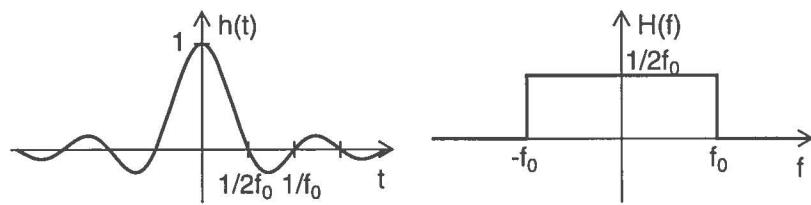

RC filter (low − pass)

$$V_C = h(t)*E(t) \qquad h(t) = \frac{1}{RC} e^{-t/RC} \qquad \leftrightarrow \qquad H(f) = \frac{1}{1 + 2\pi j f RC}$$
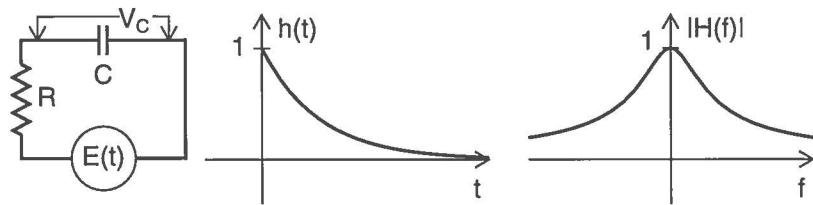
Ideal filter (high − pass)

$$h(t) = \delta(t) - \frac{\sin(2\pi f_0 t)}{\pi t} \qquad \leftrightarrow \qquad H(f) = u_{f_0}(|f|)$$
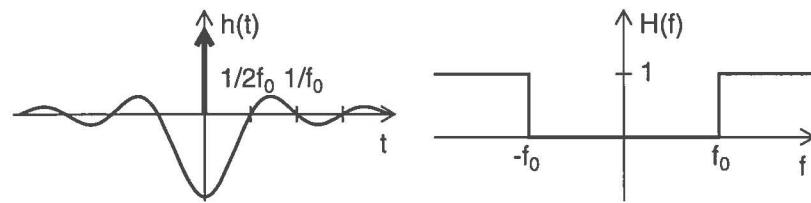

RC filter (high − pass)

$$V_R = h(t)*E(t) \qquad h(t) = \delta(t) - \frac{1}{RC} e^{-t/RC} \qquad \leftrightarrow \qquad H(f) = \frac{2\pi j f RC}{1 + 2\pi j f RC}$$
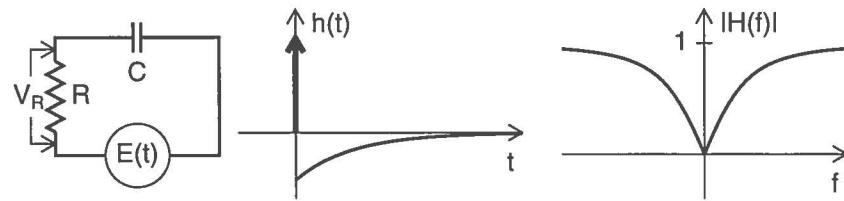
Ideal filter (band − pass)

$$h(t) = \frac{\sin(2\pi f_2 t)}{\pi t} - \frac{\sin(2\pi f_1 t)}{\pi t} \qquad \leftrightarrow \qquad H(f) = u_{f_2}(|f|) - u_{f_1}(|f|)$$
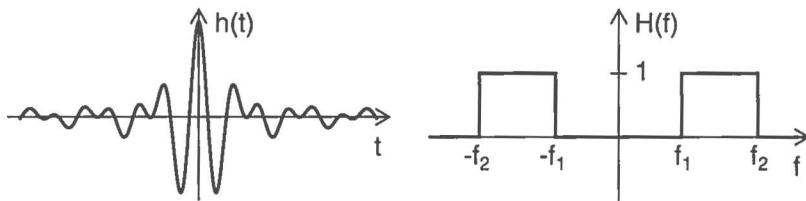
LRC filter (band − pass)

$$V_R = h(t) * E(t) \qquad h(t) \qquad \leftrightarrow \qquad H(f) = \frac{2\pi j f RC}{1 - (2\pi f)^2 LC + 2\pi j f RC}$$
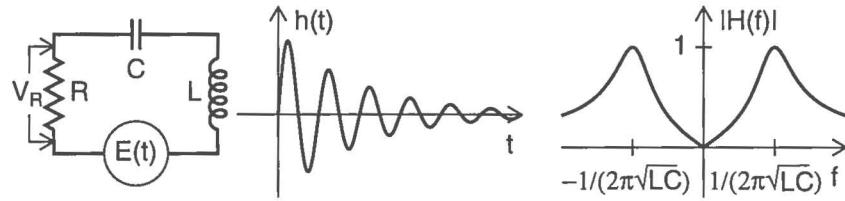
Ideal sampling

Sampled signal

$$x_\delta(t) = x(t) \cdot s_\delta(t) \qquad \text{where} \qquad s_\delta = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

Spectrum after sampling

$$X_\delta(f) = X(f) * S_\delta(f) \qquad \text{where} \qquad S_\delta(f) = f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$

Sampling theorem

If
$$T_s < \frac{1}{2W} \qquad \text{or} \qquad f_s > 2W \qquad (\text{``Nyquist rate''})$$

then the signal can be reconstructed with a filter of bandwidth B, given by:
$$W < B < f_s - W.$$