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EMPIRICAL SAMPLING INVESTIGATION OF A GLOBAL MEASURE OF FIT OF PROBABILITY DENSITY FUNCTIONS

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THESIS

EMPIRICAL SAMPLING INVESTIGATION OF A GLOBAL
MEASURE OF FIT OF PROBABILITY DENSITY FUNCTIONS

by

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Advisor: Peter A. W. Lewis

March 1974

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Empirical Sampling Investigation of a Global

Measure of Fit of Probability Density Functions

by

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requirements for the degree of

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from the

NAVAL POSTGRADUATE SCHOOL
March 1974

ABSTRACT

The distribution of a measure of distance between a probability density function and its estimate is examined through Monte Carlo methods. The estimate of the density function is that proposed by Rosenblatt using sums of weight functions centered at the observed values of the random variables. The weight function used in all cases was triangular, but both uniform and Cauchy densities were tried for different sample sizes and bandwidths. The simulated distributions appear in all cases to be close to Gamma distributions, but it has not been possible to relate the parameters to the population characteristics or to the window shape and bandwidth.

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I. INTRODUCTION

There are several recently proposed classes of empirical probability density function [Refs. 1-5] all generally considered to be superior to the classical histogram estimates. The class considered in this paper is based on independent observations, i.e., $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with continuous unknown density function $f(x)$. The method used to estimate $f(x)$ is that proposed by Rosenblatt [4] and is as follows. Denote the estimator of $f(x)$ by $f_n(x)$. Then

$$f_n(x) = \frac{1}{nb(n)} \sum_{j=1}^{n} W\left\{\frac{x - X_j}{b(n)}\right\} ,$$

where $W(u)$ is a bounded integrable weight function with

$$\int_{-\infty}^{\infty} W(u)\, du = 1$$

and $b(n)$ is a bandwidth that tends to zero as $n \to \infty$, but is such that $o(b(n)) = n^{-1}$. i.e. it converges at a rate slower than $1/n$. Thus we might have $b(n) \sim 1/n^{1/2}$. Note that all estimates of this form for given observations are themselves density functions; that is

$$f_n(x) \geq 0 ,$$

$$\int_{-\infty}^{\infty} f_n(x)\, dx = 1 .$$

Since the X_j's are random variables, $f_n(x)$ is a continuous parameter stochastic process.

The local properties of such estimates satisfying relatively mild conditions can be shown to be biased [3]. Our object in this thesis is to investigate a global measure of how good $f_n(x)$ is as an estimate of $f(x)$. This measure was proposed by Bickel and Rosenblatt [2]. The particular global measure suggested for investigation is the statistic:

$$\beta(n) = \int \frac{[f_n(x) - f(x)]^2}{f(x)} \, dx ,$$

whose value will vary with each realization of $X_1, \ldots, X_n$, i.e. it is a statistic or function of the n random variables.

Since an exact distribution of this statistic is not mathematically tractable, we have examined some representative cases by simulation.

The conjecture made by Bickel and Rosenblatt [2] is that the statistic $\beta(n)$ is asymptotically normally distributed regardless of the underlying distribution $f(x)$. With this conjecture in mind, we hoped to establish a criterion that can be used to perform goodness of fit testing.

A problem with this type of criterion is that, unlike the Kolmogorov-Smirnov or Cramer-von-Mises statistics for goodness of fit of an empirical distribution function to the unknown distribution, $\beta(n)$ is not distribution free and its distribution for finite n is unknown for any case. However, it is believed that $\beta(n)$ is fairly robust with rapid convergence to the asymptotic distribution; another object of the thesis is to examine this hypothesis.

Going beyond asymptotic normality of $\beta(n)$, the following result for a quadratic functional is also of some interest [2]. The function $a(x)$ is assumed to be a bounded piece-wise smooth integrable function. If $b(n) = 0(n^{-2/9})$ as $n \to \infty$,

$$\{b(n)\}^{-1/2}[nb(n)\int[f_n(x)- f(x)]^2 \, a(x)dx - \int f(x)a(x)dx \int W(z)^2 dz]$$

is asymptotically normally distributed with mean zero and variance

$$2W^{(4)}(0) \int a(x)^2 \, f(x)^2 \, dx$$

as $n \to \infty$, where $W^{(4)}$ is the 4-th convolution of W with itself.

II. SIMULATION

The primary object of the simulation is to investigate the distribution of the statistic $\beta(n)$:

$$\beta(n) = \int \frac{[f_n(x) - f(x)]^2}{f(x)} \, dx ,$$

over a suitable range of integration. We have broken the simulation into the following cases:

A. UNIFORM (0,1) CASE

Since

$$f(x) = \begin{cases} 1 , & 0 \leq x \leq 1 \\ 0 , & \text{otherwise} \end{cases}$$

$\beta(n)$ becomes

$$\beta(n) = \int_{b(n)}^{1-b(n)} [f_n(x)-1]^2 \, dx .$$

We integrate from b(n) to 1-b(n) instead of from 0 to 1, simply because we want to avoid the marked bias near 0 and 1. It can easily be shown that $f_n(x)$ is biased in the end regions, but not otherwise.

Thus the mean and variance of $f_n(x)$ in this case is:

$$E[f_n(x)] = \frac{1}{b(n)} \int_0^1 W\left\{\frac{x-y}{b(n)}\right\} dy$$

$$= \frac{1}{b(n)} \int_{x-b(n)}^{x+b(n)} \left(1-\frac{|x-y|}{b(n)}\right) dy$$

$$= \frac{1}{b(n)} \left[\int_{x-b(n)}^{x+b(n)} dy - \int_{x-b(n)}^{x} \frac{(x-y)}{b(n)} dy - \int_{x}^{x+b(n)} \frac{(y-x)}{b(n)} dy\right]$$

$$= \frac{1}{b(n)} \left[2b(n) - \frac{1}{b(n)}\frac{(b(n))^2}{2} - \frac{1}{b(n)}\frac{(b(n))^2}{2}\right] = 1 .$$

This is only true for $b(n) \le x \le 1-b(n)$. Also

$$\mathrm{Var}\left[\frac{1}{nb(n)} \sum_{j=1}^{n} W\left\{\frac{x-X_{(j)}}{b(n)}\right\}\right]$$

$$= \frac{1}{n^2 b^2(n)} \sum_{j=1}^{n} \mathrm{Var}\left[W\left\{\frac{x-X_{(j)}}{b(n)}\right\}\right]$$

$$= \frac{1}{nb^2(n)} \mathrm{Var}\left[W\left\{\frac{x-X_{(j)}}{b(n)}\right\}\right]$$

$$= \frac{1}{nb^2(n)} \left[\int_0^1 W^2\left\{\frac{x-y}{b(n)}\right\} dy - \left(\int_0^1 W\left\{\frac{x-y}{b(n)}\right\} dy\right)^2\right].$$

The integration was performed in the simulation using Simpson's rule, dividing the interval of integration into 100 equally spaced sub-intervals. The results were found to be satisfactory in the sense that the value of the integral for 100 sub-intervals differs little from those with more sub-intervals.

To evaluate $f_n(x)$, use the following formula:

$$f_n(x) = \frac{1}{nb(n)} \sum_{j=1}^{n} W\left\{\frac{x-X_{(j)}}{b(n)}\right\},$$

where the weight function W(u) is triangular, i.e.:

$$W(u) = \begin{cases} 1-|u|, & \text{if } |u| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

X_j, j=1, ..., n are independent and identically distributed random variables. In this first case, they are uniform (0,1) random variables. Also b(n) is the bandwidth. We employed three bandwidths in the uniform (0,1) case, namely: $3/\sqrt{n}$, $1/\sqrt{n}$, 1/n, one at a time. For each bandwidth, we used five different sample sizes. i.e. 100, 200, 500, 1000, 1500, and for each sample size, we used five different random number seeds, i.e. simulated five times. The reason for this is to have the ability to assess the variability of the simulation results from the sections. The random number seeds are taken from the immediate previous "1500 sample size" output, except the first random number seed which is arbitrary. So, there are five sets of random numbers which are entirely different from one another. The following table should help to clarify the scheme:

TABLE I

Bandwidth	Sample size, n	No. replications	# Sections
$\frac{3}{n^{1/2}}$	100	2000	5
	200	2000	5
	500	2000	5
	1000	2000	5
	1500	2000	5
$\frac{1}{n^{1/2}}$	100	2000	5
	200	2000	5
	500	2000	5
	1000	2000	5
	1500	2000	5
$\frac{1}{n}$	100	2000	5
	200	2000	5
	500	2000	5
	1000	2000	5
	1500	2000	5

For each sample size, n, repeat five times using different random number seeds.

For the uniform (0,1) case, there are 75 outputs altogether. In each single simulation (one computer output), we evaluated 400 $\beta(n)$'s, say $\beta_j(n)$, j=1, ..., 400. The output includes the following:

1. A bar histogram of the 400 $\beta(n)$'s.

2. SNAP/IEDA graphs:

   a. A refined solid histogram.

   b. Graph of empirical log-survivor function, $\ell n(1-j/(N+1))$ vs $\beta_{(j)}(n)$, where $\beta_{(j)}(n)$ is an ordered value.

c. Empirical CDF graph $i/(n+1)$ vs $\beta_{(j)}(n)$, where $\beta_{(j)}(n)$ denotes the ordered observed $\beta_j(n)$'s.

d. Normal probability distribution with estimated mean and variance. Norm vs e.c.d.f. of $\beta_j(n)$, where Norm is computed as follows: Let

$$\hat{\mu} = \frac{\Sigma\ \beta_j(n)}{n}$$

and $S^2(\beta(n))$ = sample variance of the $\beta_j(n)$. Then

$$\text{Norm} = \Phi\left[\frac{\beta_j(n) - \hat{\mu}}{S(\beta(n))}\right]$$

e. Normal probability plot Norm vs $i/(N+1)$.

A sample output is included in the "Computer Output" section (page 30 - 35).

For each bandwidth and each sample size, there are five computer outputs, or 5 x 400 = 2000 $\beta_j(n)$'s. With this 2000 sample size, we constructed a histogram again. Also we used the Rosenblatt smoothing technique to get a smooth density function estimate for $\beta(n)$. After a number of trials, we found that the bandwidth needed in the Rosenblatt estimator has very much to do with the smoothing and how "good" the density looks. Fortunately it is robust to a certain extent. This graph can give us a better picture of the true distribution than the classical histogram. The last graph of this complete output for sample size 2000 is the empirical log survival function;

$$\ln(1 - i/(N+1)) \quad \text{vs} \quad \beta_{(j)}(n)\ .$$

There are 15 "2000 sample size" computer outputs in the uniform (0,1) case. (Since each bandwidth has five "2000 sample size" outputs.)

The complete computer outputs for each bandwidth and each sample size in the uniform (0,1) case are included in the "Computer Output" section (page 42 - 86).

B. THE CAUCHY DISTRIBUTION

The next case investigated is the Cauchy distribution, again with a triangular weight function and with the range of integration (-3,3):

$$\beta(n) = \int_{-3}^{+3} \frac{[f_n(x) - f(x)]^2}{f(x)} dx .$$

The interval of integration comprises 80% of the probability mass for this distribution. The integration is carried out, in this case, using Simpson's rule with 600 sub-intervals. This grid was used after examining the 100, 300, 600, 900 sub-interval cases to ensure that a fine enough grid is used in the numerical integration.

The Cauchy density function is

$$f(x) = \frac{1}{\pi(1 + x^2)} .$$

The density function estimator is the same as in the uniform case:

$$f_n(x) = \frac{1}{nb(n)} \sum_{j=1}^{n} W\left\{\frac{x-X_j}{b(n)}\right\} .$$

The weight function W(u) is again triangular. Thus the only difference in this formula from the previous case is that X_j, j=1, ..., n

are i.i.d. Cauchy random variables. Note also that for finite n the estimator will have a bias component, this bias component usually decreasing, for given n, as bandwidth decreases. In that case though the (pointwise) variance of $f_n(x)$ increases and possibly also the variance of $\beta(n)$. Note that $\beta(n)$ and its distribution is a function of both sample size and bandwidth.

The bandwidth b(n)'s used here are $1/\sqrt{n}$, $3/\sqrt{n}$, $20/\sqrt{n}$, the last one representing a case in which bias in the estimate $f_n(x)$ plays a major role in the distribution of $\beta(n)$.

The rest of the simulation scheme is exactly the same as for the uniform (0,1) case.

A sample output for the 400 $\beta_j(n)$ case is included in the "Computer Output" section (page 36 - 41). Also the complete computer outputs (2000 replications) for each bandwidth and each sample size in the Cauchy case are included in the "Computer Output" section (page 87 -131).

C. HAND PLOT GRAPHS

Besides the computer work, there are six hand plot graphs. These are the $\beta(n)$ quantiles versus the standard normal quantiles. They are included in the Appendix, to illustrate the rate of convergence of the distribution of $\beta(n)$ to its asymptotic normal distribution.

III. RESULTS

The asymptotic result, conjectured by Rosenblatt, is that by using a triangular weight function when estimating the uniform density function on (0,1)

$$\{b(n)\}^{-1/2}\left\{nb(n)\int_{b(n)}^{1-b(n)} |f(x)-1|^2 dx-(1-2b(n))\int W(u)^2 du\right\}$$

is asymptotically normally distributed with mean zero and variance

$$2W^{(4)}(0)(1-2b(n))$$

as $n \to \infty$ if $nb(n) \to \infty$ and $b(n) = o(n^{-2/9})$. Here n is the sample size and b(n) the bandwidth. For the triangular weight function

$$\int_{-\infty}^{\infty} W(u)^2 du = \frac{2}{3}$$

and $W^{(4)}(0)$ the 4th convolution of W with itself at zero, is 302/630.

From the above expressions, we get:

$$E(\beta(n)) = E\left[\int_{b(n)}^{1-b(n)} |f_n(x)-1|^2 dx\right] \sim \frac{2}{3}\,\frac{(1-2b(n))}{nb(n)} ,$$

$$\text{Var }(\beta(n)) = \text{Var}\left[\int_{b(n)}^{1-b(n)} |f_n(x)-1|^2 dx\right] \sim \frac{2W^{(4)}(0)(1-2b(n))}{n^2 b(n)} .$$

A comparison of the simulated values with that of the conjectured is tabulated below for the different cases investigated when f(x) is a uniform density:

TABLE II

Comparison of estimated mean values and asymptotic mean values of $\beta(n)$ for different bandwidths and sample sizes.

n	$b(n)=3/\sqrt{n}$	$E(\beta(n))$	$E(\beta(n))/(1-2b(n))$ Conjectured	Computer output
100	.3000	.0089	.0222	.0127
200	.2121	.0090	.0157	.0109
500	.1342	.0073	.0099	.0075
1000	.0949	.0057	.0070	.0058
1500	.0775	.0048	.0057	.0051
n	$b(n)=1/\sqrt{n}$	$E(\beta(n))$	$E(\beta(n))/(1-2b(n))$ Conjectured	Computer output
100	.1000	.0533	.0667	.0583
200	.0707	.0405	.0471	.0415
500	.0447	.0271	.0298	.0269
1000	.0316	.0197	.0211	.0197
1500	.0258	.0163	.0172	.0168

TABLE III

Comparison of estimated standard deviation values and asymptotic standard deviation values of $\beta(n)$ for different bandwidths and sample sizes.

n	$b(n)=3/\sqrt{n}$	$\sigma(\beta(n))$	$\sigma(\beta(n))/(1-2b(n))$ Conjectured	Computer output
100	.3000	.0113	.0283	.0115
200	.2121	.0081	.0141	.0088
500	.1342	.0046	.0063	.0047
1000	.0949	.0029	.0036	.0030
1500	.0775	.0022	.0026	.0023
n	$b(n)=1/\sqrt{n}$	$\sigma(\beta(n))$	$\sigma(\beta(n))/(1-2b(n))$ Conjectured	Computer output
100	.1000	.0277	.0346	.0315
200	.0707	.0171	.0199	.0189
500	.0447	.0088	.0097	.0092
1000	.0316	.0053	.0057	.0056
1500	.0258	.0040	.0042	.0043

Especially for small bandwidth the agreement between the asymptotic and simulated variances is very good even for small n(n=100). The same is true for the expected value, although convergence is slower than for σ and again slower for large bandwidth.

The simulation reveals that the distribution of the $\beta(n)$ statistic converges to normality very slowly (cf. the six quantile plots in the Appendix). For sample sizes 100, 200, 500, 1000, 1500, the histograms are all skewed to the right; the empirical log-survivor function plots are concave downward; and none of the normal probability plots is approximately straight lines (refer to "Computer Output" section). It is very suggestive that the $\beta(n)$ statistic is approximately Gamma (θ,k) distributed where the density of the Gamma distribution is:

$$f(x) = \frac{(k/\theta)^k e^{-kx/\theta} x^{k-1}}{(k-1)!}$$

and the mean and variance are:

$$E(X) = \theta \ ; \quad Var(X) = k\theta^2 .$$

Therefore, for each underlying distribution, each bandwidth, and each sample size, we used the 2000 evaluated $\beta(n)$'s to estimate the θ's and k's, and tabulate them as follows:

TABLE IV

Estimated parameters for fitted Gamma distribution for $\beta(n)$ in the case where the densities are uniform (0,1) and Cauchy, with a triangular weight function.

(1) Bandwidth = $1/n$ Underlying distribution: Uniform (0,1)

Sample sizes and estimated θ's and k's

	100	200	500	1000	1500 (std. dev.)
$\tilde{\theta}$	.6540	.6638	.6612	.6621	.6645 ± .0045
$\tilde{k}$	39.48	38.46	32.95	30.86	31.06 ± 2.91

(2) Bandwidth = $1/\sqrt{n}$ Underlying distribution: Uniform (0,1)

Sample sizes and estimated θ's and k's

	100	200	500	1000	1500 (std. dev.)
$\tilde{\theta}$	.0565	.0420	.0279	.0201	.0166 ± .0003
$\tilde{k}$	3.62	5.56	8.63	12.61	16.92 ± 1.45

Underlying distribution: Cauchy

Sample sizes and estimated θ's and k's

	100	200	500	1000	1500 (std. dev.)
$\tilde{\theta}$	.3921	.2800	.1764	.1258	.1029 ± .0004
$\tilde{k}$	21.02	31.49	59.34	79.19	100.83 ± 4.37

(3) Bandwidth = $3/\sqrt{n}$ Underlying distribution: Uniform (0,1)

Sample sizes and estimated θ's and k's

	100	200	500	1000	1500 (std. dev.)
$\tilde{\theta}$	.0123	.0107	.0079	.0060	.0051 ± .0001
$\tilde{k}$	1.11	1.59	2.43	3.65	5.03 ± 0.37

Underlying distribution: Cauchy

Sample sizes and estimated θ's and k's

	100	200	500	1000	1500 (std. dev.)
$\tilde{\theta}$	.1247	.0911	.0576	.0412	.0340 ± .0004
$\tilde{k}$	8.62	12.30	20.32	29.14	33.98 ± 1.55

(4) Bandwidth = $20/\sqrt{n}$ Underlying distribution: Cauchy

Sample sizes and estimated θ's and k's

	100 (std.dev.)	200	500	1000	1500 (std. dev.)
$\tilde{\theta}$	.0558 ± .0008	.0262	.0110	.0066	.0052 ± .0001
$\tilde{k}$	6.71 ± 0.78	4.03	3.21	4.10	5.24 ± 0.16

where $\tilde{\theta}$ = mean ,

$\tilde{k} \approx 1/(\text{estimated coefficient of variation})^2$

The standard deviation estimates for the parameter estimates are obtained from the five identical simulations using different random number seeds, each generating a sample size of 400, as detailed in Table I.

IV. CONCLUSIONS

Although the conjecture proposed by Rosenblatt is that the distribution of the statistic $\beta(n)$ is asymptotically normal, this is certainly not the case for moderate sample sizes such as those used in the simulations. Based on this fact, the conjecture has little practical value, but the fact that all cases examined give rise to an approximate Gamma distribution gives some hope for stability in the results.

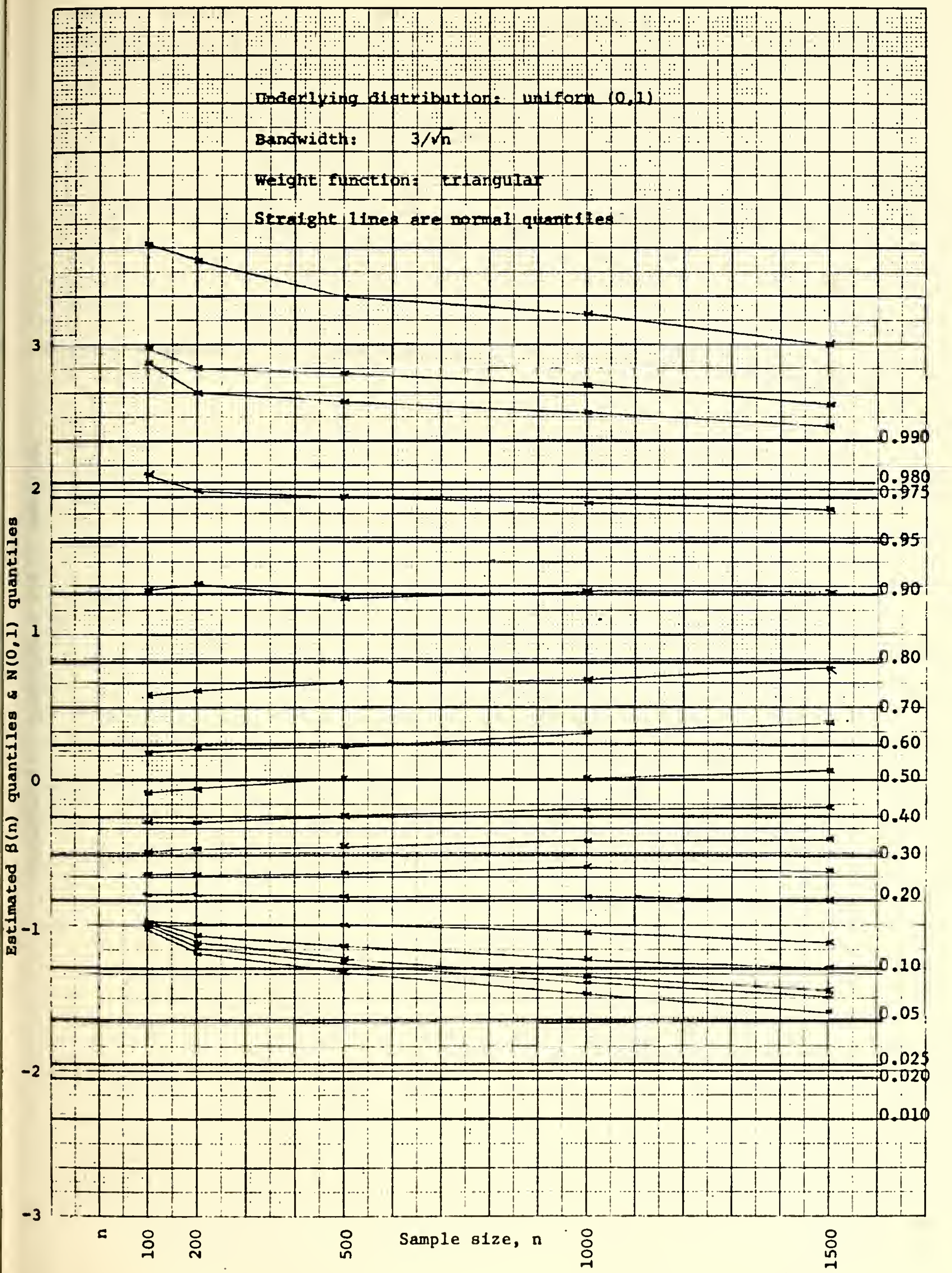
However, the method proposed by Rosenblatt for estimating density functions is no doubt plausible as can be seen from the Rosenblatt type estimates used in this study. More effort in this area is needed, particularly to relate the Gamma distribution parameters to underlying factors. Since we realize that for moderate sample sizes, the statistic $\beta(n)$ is not distribution free, and the bandwith $b(n)$ (a function of sample size) plays a crucial role in estimating the density function and also the shape of the weight function $W(u)$ is a major factor for evaluating $f_n(x)$, the future work may be to find the best combination of the above three factors by simulating the distribution of the statistic $\beta(n)$ for different underlying distributions, bandwidths, sample sizes and weight functions and tabulate the resulting quantiles. Then we may be able to put the statistic $\beta(n)$ to actual use.

APPENDIX

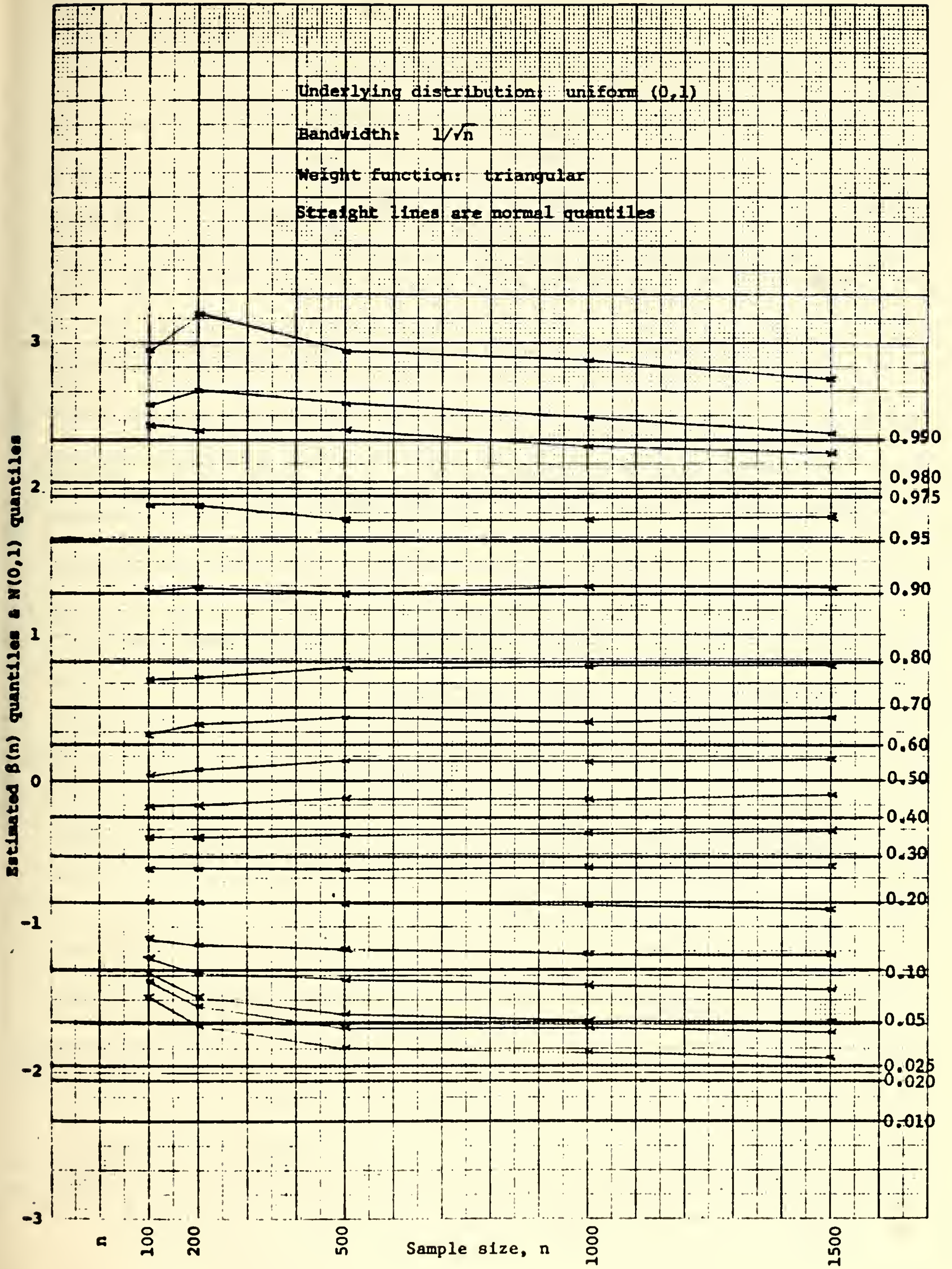
Included in this Appendix are six plots of sample size versus estimated quantiles of the statistic $\beta(n)$, normalized by subtracting the estimated mean and dividing by the estimated standard deviation. Also plotted as straight lines are the corresponding quantiles of the unit normal distribution.

The 17 quantiles, extracted from each 2000-$\beta(n)$ sample, are 0.010, 0.020, 0.025, 0.05, 0.10, 0.20, 0.30, 0.40, 0.50, 0.60, 0.70, 0.80, 0.90, 0.95, 0.975, 0.980, 0.990.

Other data for each plot are self-explanatory.



Underlying distribution: uniform (0,1)
Bandwidth: 3/√n
Weight function: triangular
Straight lines are normal quantiles
Estimated B(n) quantiles & N(0,1) quantiles
3
2
1
0
-1
-2
-3
0.990
0.980
0.975
0.95
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.05
0.025
0.020
0.010
n
100
200
500
Sample size, n
1000
1500



Underlying distribution: uniform (0,1)
Bandwidth: 1/√n
Weight function: triangular
Straight lines are normal quantiles
Estimated β(n) quantiles & N(0,1) quantiles
3
2
1
0
-1
-2
-3
0.990
0.980
0.975
0.95
0.90
0.80
0.70
0.60
0.50
0.40
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Sample size, n
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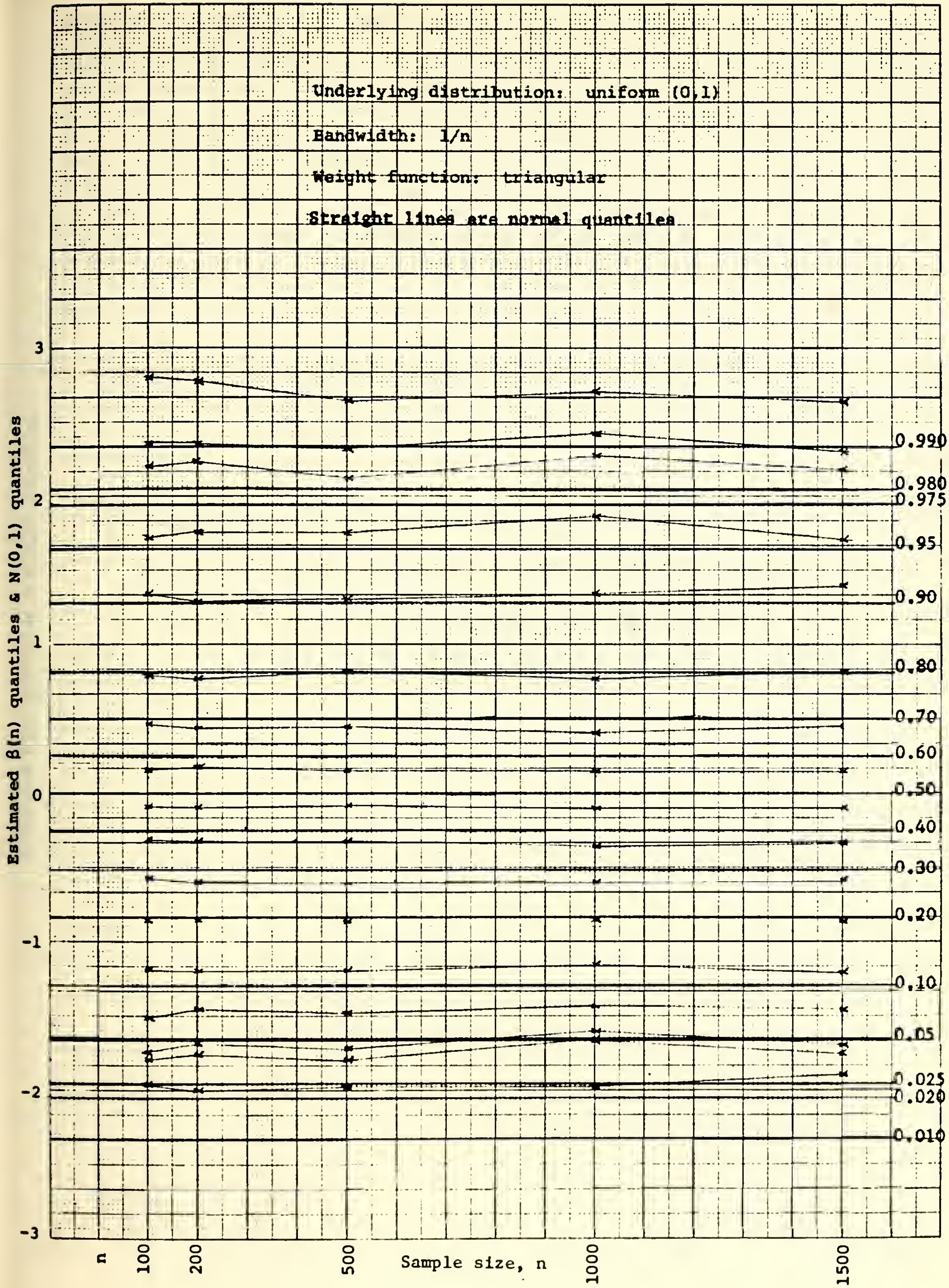
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Sample size, n
1000
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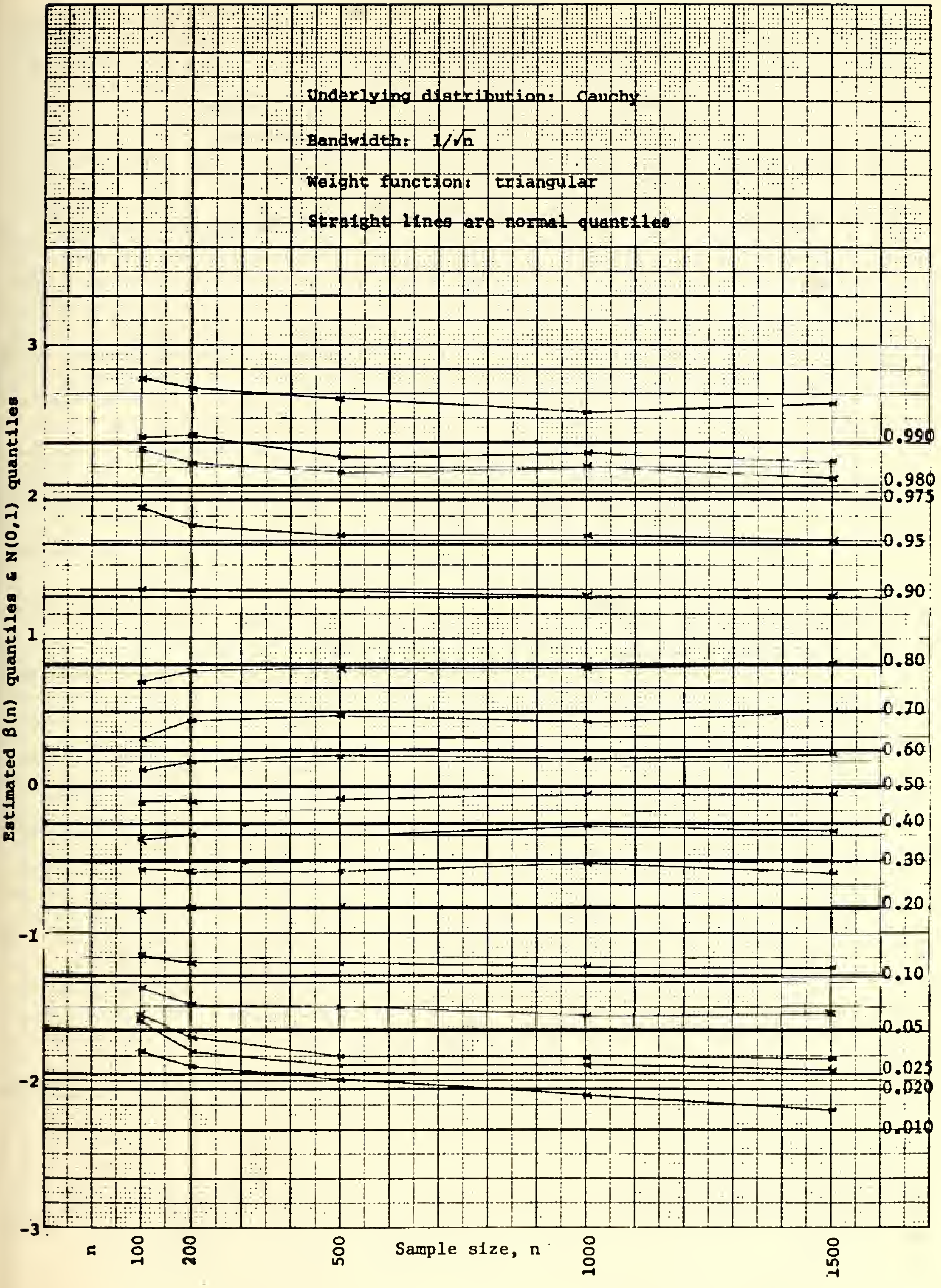

Underlying distribution: Cauchy
Bandwidth: 1/√n
Weight function: triangular
Straight lines are normal quantiles
Estimated β(n) quantiles & N(0,1) quantiles
3
2
1
0
-1
-2
-3
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0.980
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0.40
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Sample size, n
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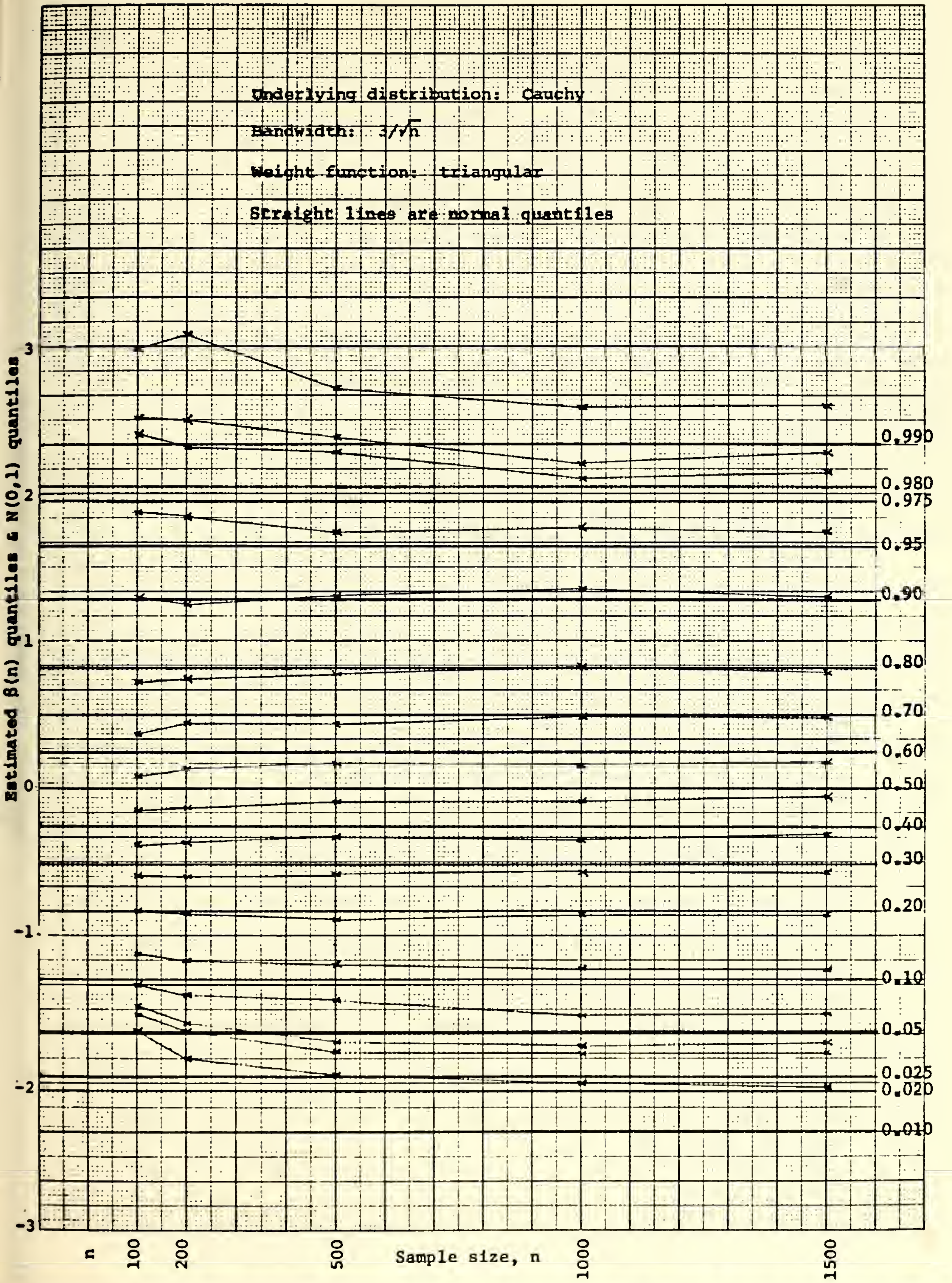
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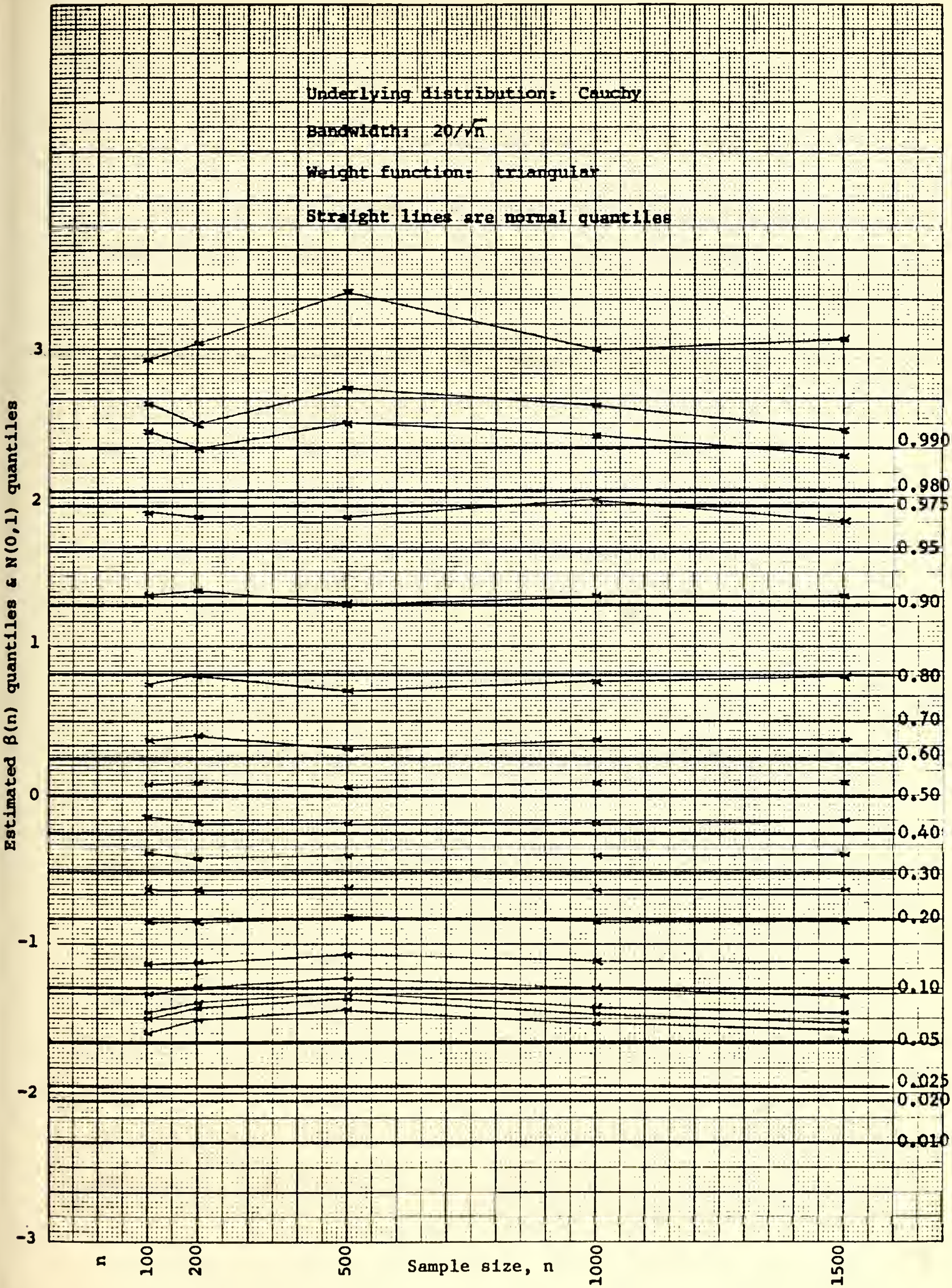
500
Sample size, n
1000
1500



Underlying distribution: Cauchy
Bandwidth: 20/√n
Weight function: triangular
Straight lines are normal quantiles
Estimated β(n) quantiles & N(0,1) quantiles
3
2
1
0
-1
-2
-3
0.990
0.980
0.975
0.95
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.05
0.025
0.020
0.010
n
100
200
500
1000
1500
Sample size, n

COMPUTER OUTPUT

There are altogether 102 sheets of graphs. All of them are self-explanatory, except that the letter "X" in the graphs represents the statistic $\beta(n)$.

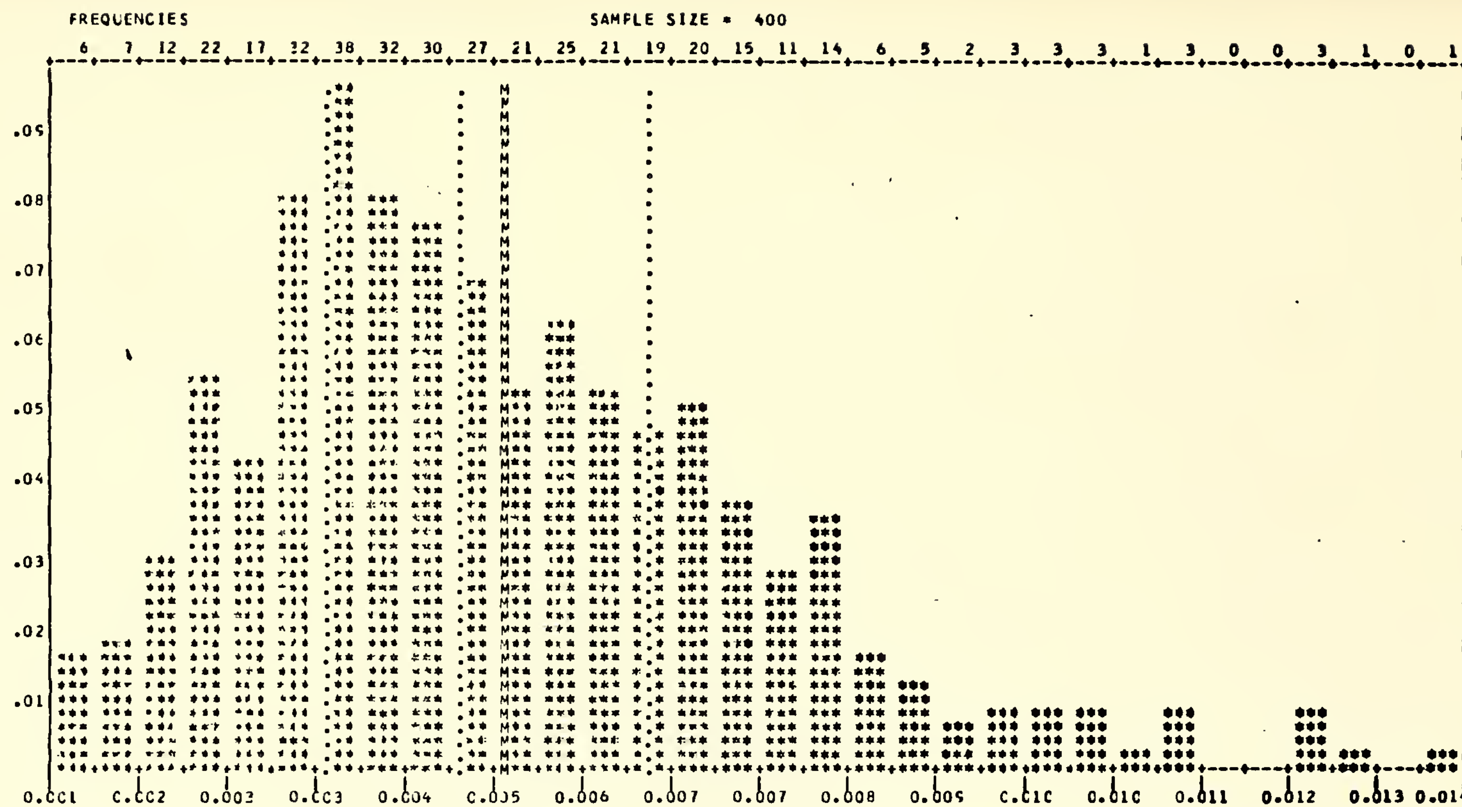
The following is a list of index of the graphs.

Page	Description
30 - 35	sample computer output for the case: underlying distribution-uniform (0,1); underlying distribution sample size n=1500; bandwidth $3/\sqrt{n}$; statistic $\beta(n)$ sample size m=400.
36 - 41	sample computer output for the case: underlying distribution-Cauchy; underlying distribution sample size n=100; bandwidth $1/\sqrt{n}$; statistic $\beta(n)$ sample size m=400.

The rest of the computer graphs in this section are all for statistic $\beta(n)$ sample size m=2000. The page index is as follows.

Page	Underlying distribution	Underlying distribution sample size	Bandwidth
42 - 44	U(0,1)	100	$3/\sqrt{n}$
45 - 47	U(0,1)	200	$3/\sqrt{n}$
48 - 50	U(0,1)	500	$3/\sqrt{n}$
51 - 53	U(0,1)	1000	$3/\sqrt{n}$
54 - 56	U(0,1)	1500	$3/\sqrt{n}$
57 - 59	U(0,1)	100	$1/\sqrt{n}$
60 - 62	U(0,1)	200	$1/\sqrt{n}$

Page	Underlying distribution	Underlying distribution sample size	Bandwidth
63 - 65	U(0,1)	500	$1/\sqrt{n}$
66 - 68	U(0,1)	1000	$1/\sqrt{n}$
69 - 71	U(0,1)	1500	$1/\sqrt{n}$
72 - 74	U(0,1)	100	1/n
75 - 77	U(0,1)	200	1/n
78 - 80	U(0,1)	500	1/n
81 - 83	U(0,1)	1000	1/n
84 - 86	U(0,1)	1500	1/n
87 - 89	Cauchy	100	$1/\sqrt{n}$
90 - 92	Cauchy	200	$1/\sqrt{n}$
93 - 95	Cauchy	500	$1/\sqrt{n}$
96 - 98	Cauchy	1000	$1/\sqrt{n}$
99 - 101	Cauchy	1500	$1/\sqrt{n}$
102- 104	Cauchy	100	$3/\sqrt{n}$
105- 107	Cauchy	200	$3/\sqrt{n}$
108- 110	Cauchy	500	$3/\sqrt{n}$
111- 113	Cauchy	1000	$3/\sqrt{n}$
114- 116	Cauchy	1500	$3/\sqrt{n}$
117- 119	Cauchy	100	$20/\sqrt{n}$
120- 122	Cauchy	200	$20/\sqrt{n}$
123- 125	Cauchy	500	$20/\sqrt{n}$
126- 128	Cauchy	1000	$20/\sqrt{n}$
129- 131	Cauchy	1500	$20/\sqrt{n}$



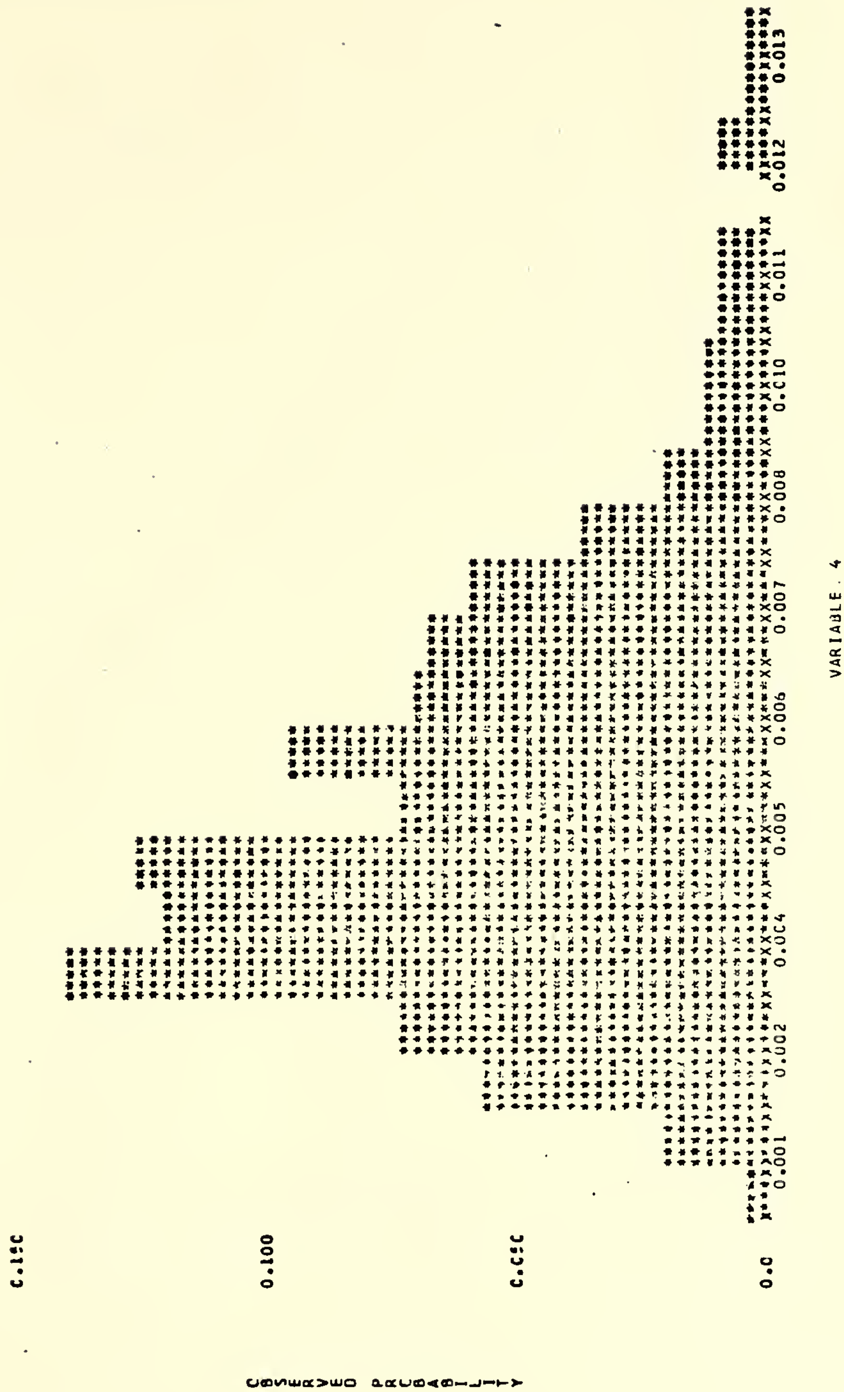
CENTRAL TENDENCY		SPREAD		HIGHER MOMENTS		DISTRIBUTION	
MEAN	5.052119E-03	VARIANCE	4.703124E-06	M3	9.107605E-09	MINIMUM	1.134356E-03
MEDIAN	4.641008E-03	STD DEV	2.168669E-03	M4	8.755627E-11	.10 QUANTILE	2.512589E-03
TRIMEAN	4.774030E-03	COEF VAR	4.292591E-01	SKEWNESS	8.929450E-01	.25 QUANTILE (HINGE)	3.490532E-03
MIDMEAN	4.753117E-03	RANGE	1.241564E-02	KURTOSIS	9.583540E-01	.50 QUANTILE (MEDIAN)	4.641008E-03
GEOM MEAN	4.604988E-03	MIDSPREAD	2.833055E-03			.75 QUANTILE (HINGE)	6.323587E-03
HARM MEAN	4.155193E-03					.90 QUANTILE	7.822972E-03
						MAXIMUM	1.354999E-02

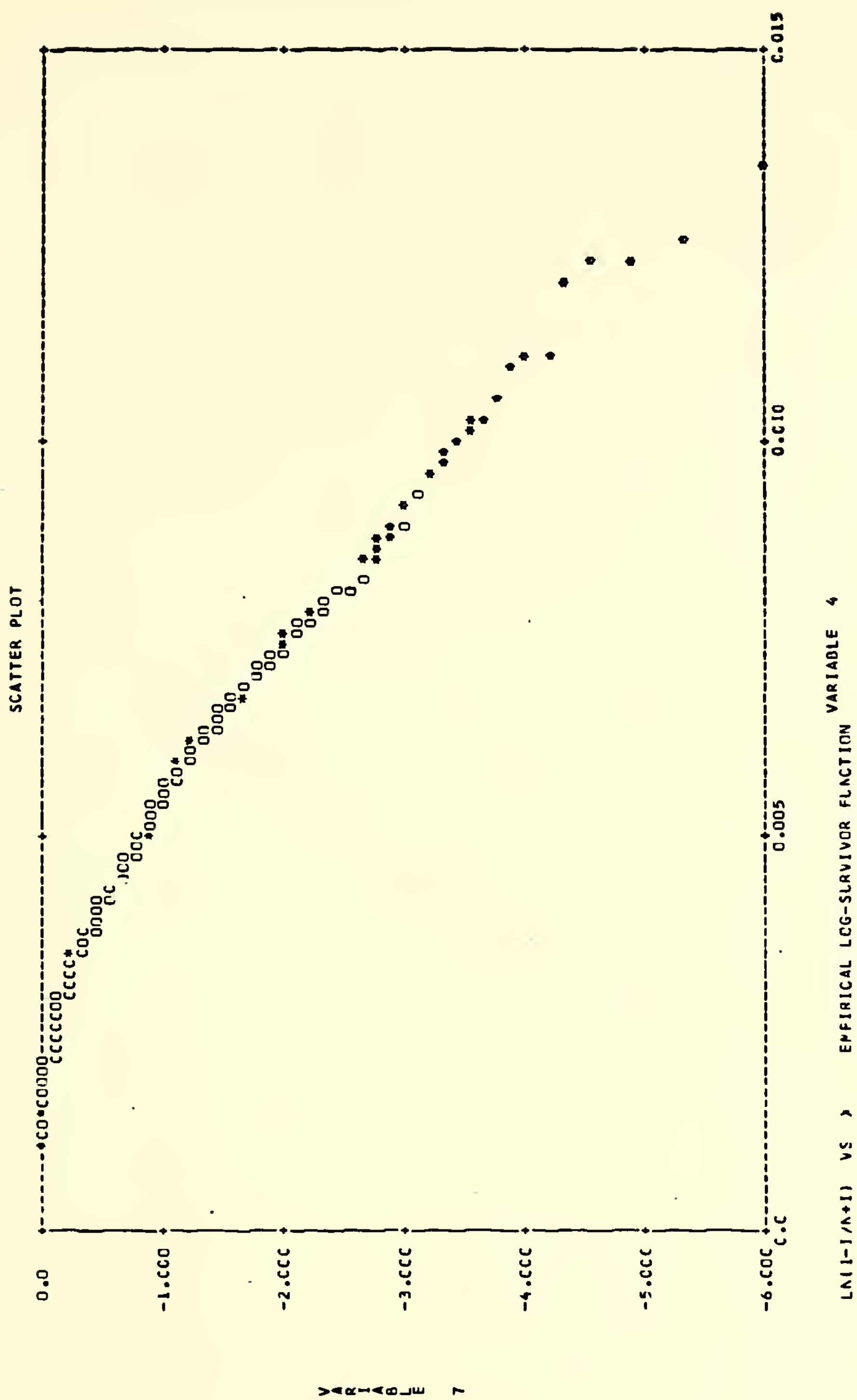
INTEGRAL SQUARE NORM SAMPLE SIZE M = 400

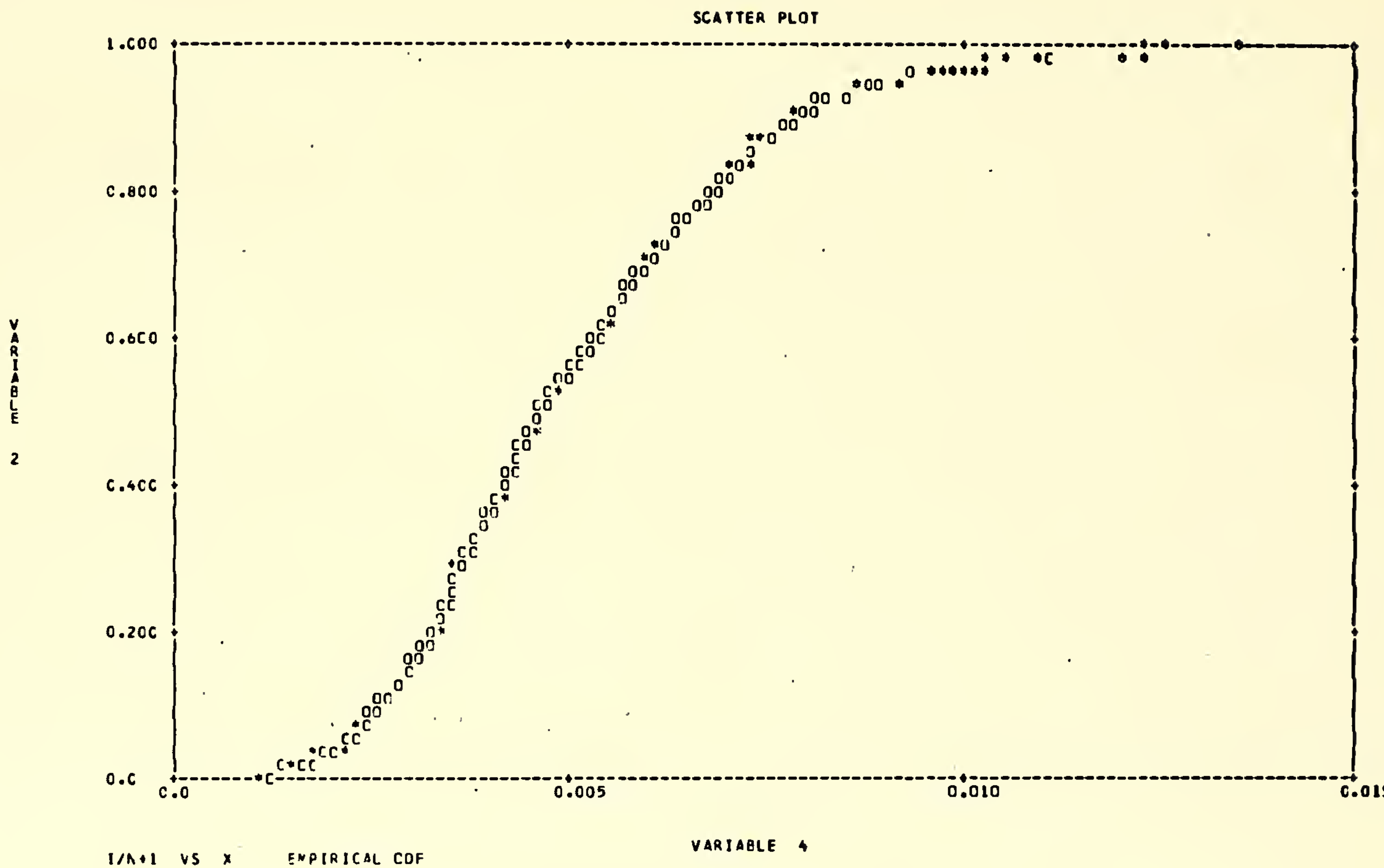
UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 1500

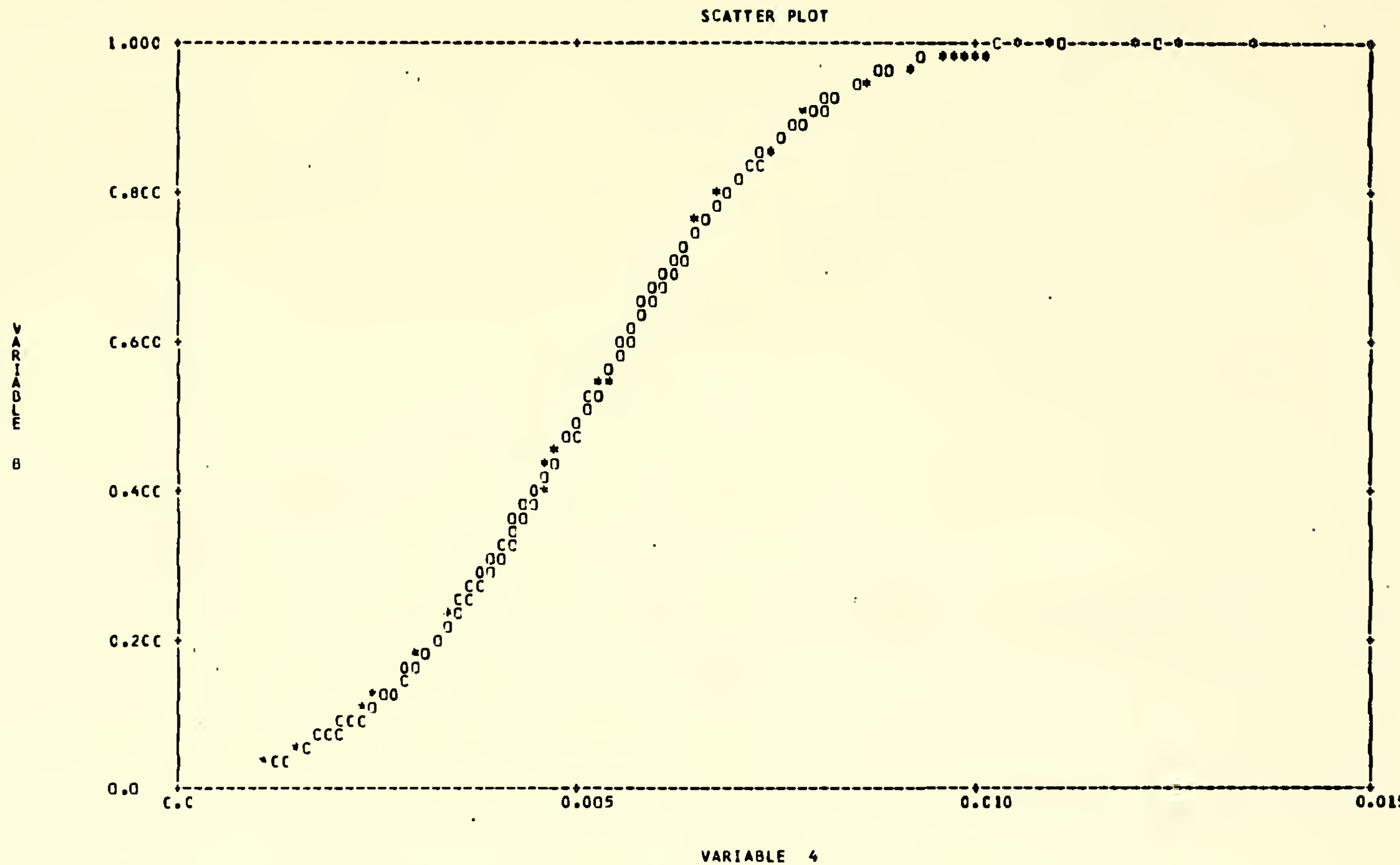
TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)

LAST RANDOM NUMBER SEED = 933898713

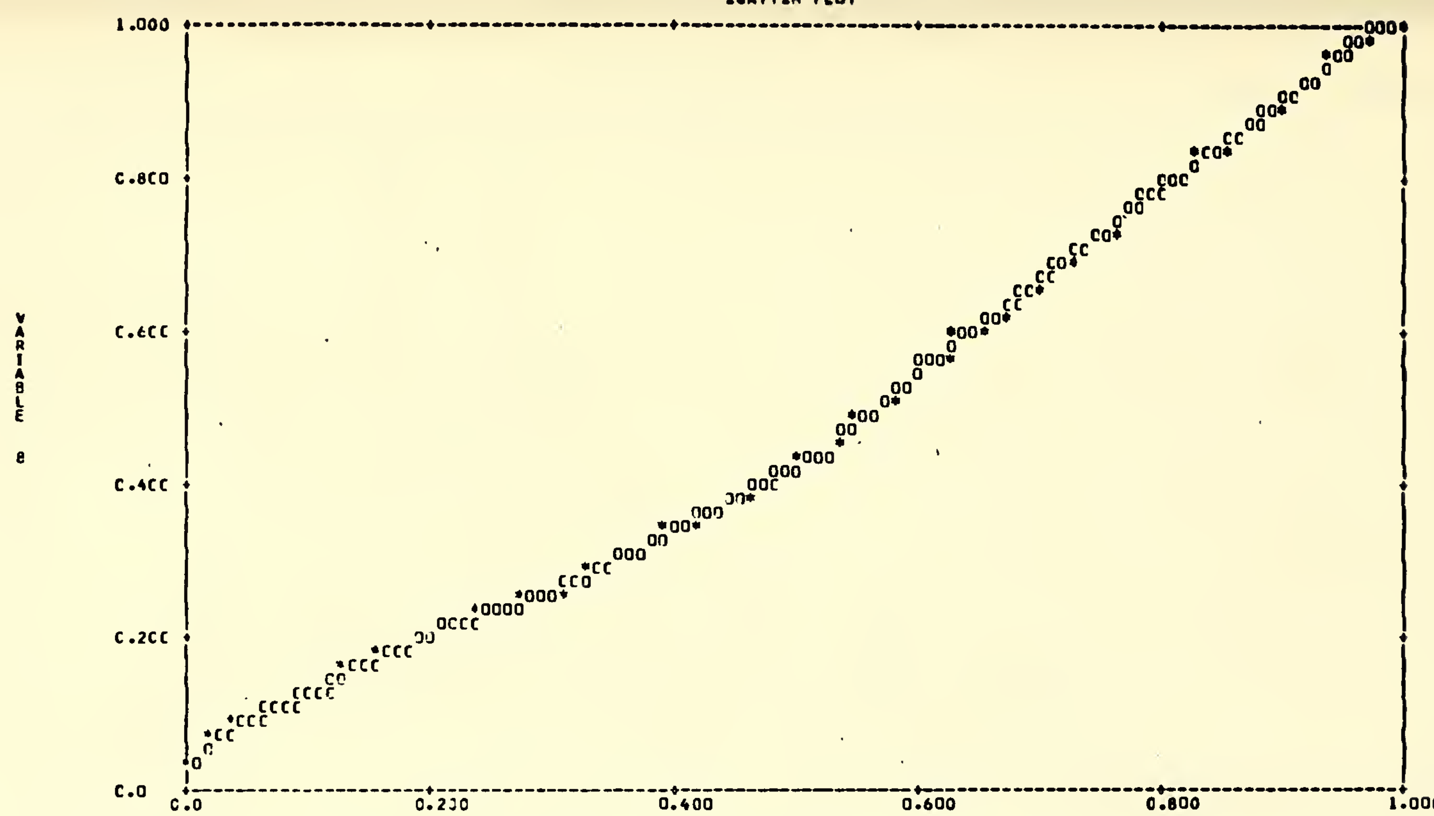

0.150
0.100
0.050
0.0
OBSERVED PROBABILITY
0.001
0.002
0.004
0.005
0.006
0.007
0.008
0.010
0.011
0.012
0.013
VARIABLE 4


SCATTER PLOT
LN(1-I/(N+1)) VS EMPIRICAL LOG-SURVIVOR FUNCTION VARIABLE 4
0.0
-1.000
-2.000
-3.000
-4.000
-5.000
-6.000
0.0
0.005
0.010
0.015
VARIABLE 7

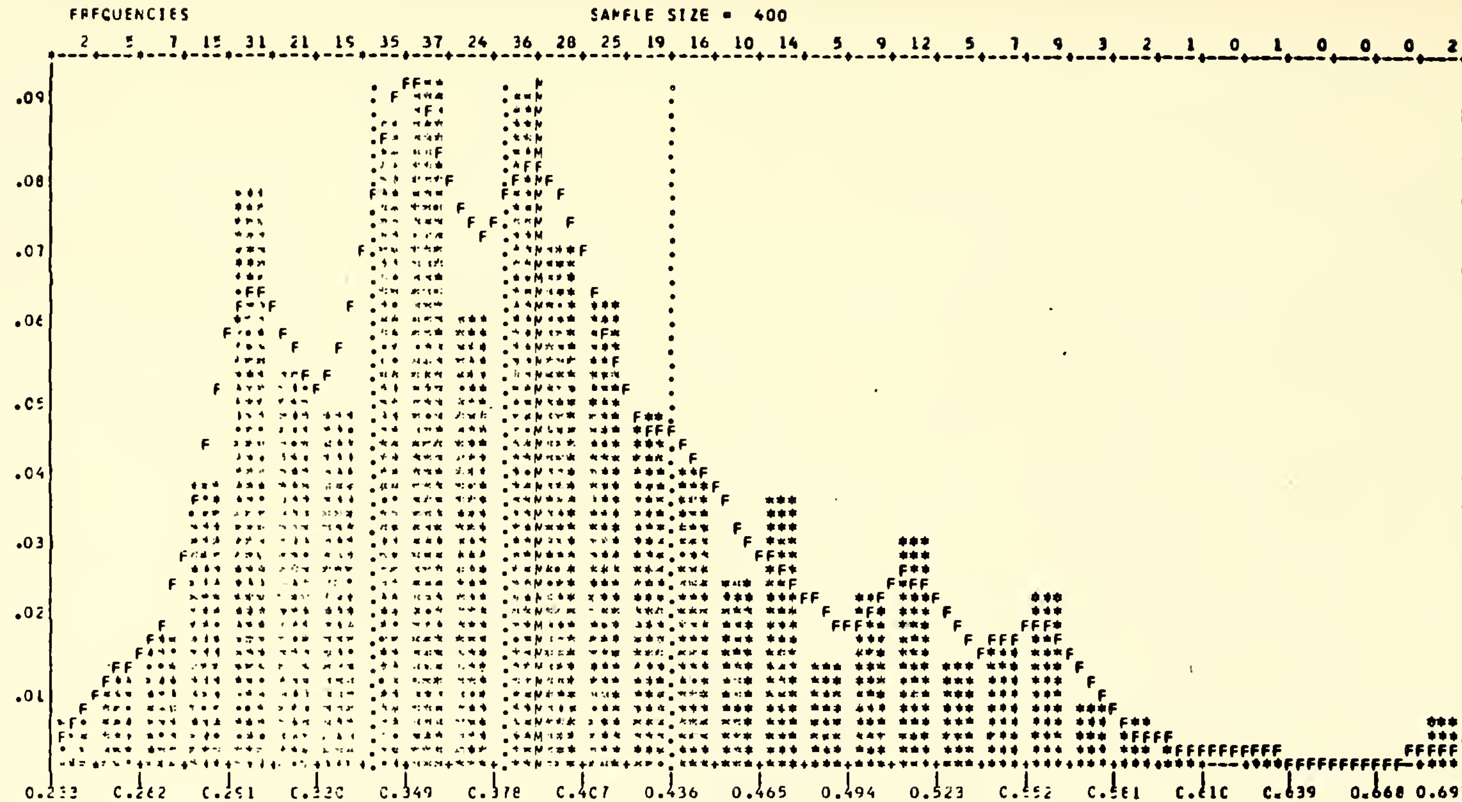

SCATTER PLOT
1.000
0.800
0.600
0.400
0.200
0.0
VARIABLE 2
0.0
0.005
0.010
0.015
VARIABLE 4
I/N+1 VS X EMPIRICAL CDF



NCRM VS X NCRMAL PRCBABILITY DISTRIBUTICN WITH ESTIMATED MEAN & VARIANCE



NCRM VS I/N+1 NORMAL PROBABILITY PLOT



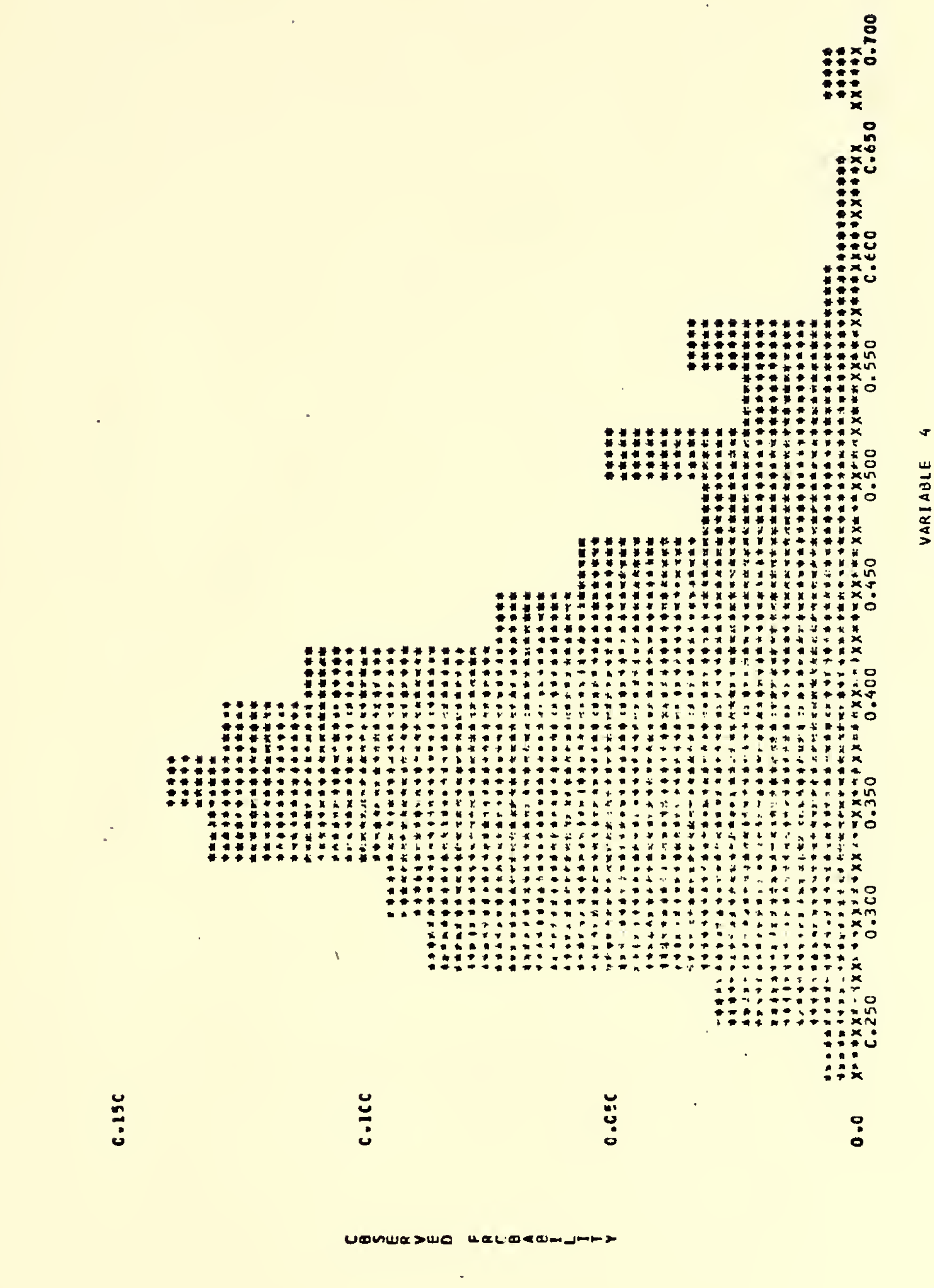
CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	3.858916E-01	VARIANCE	6.415933E-03	M3	4.018054E-04	MINIMUM	2.330768E-01
MEDIAN	3.805658E-01	STD DEV	8.009952E-02	M4	1.438781E-04	.10 QUANTILE	2.552183E-01
TRIMEAN	3.826540E-01	COEF VAR	2.075405E-01	SKEWNESS	7.818547E-01	.25 QUANTILE (HINGE)	3.347112E-01
MIDMEAN	3.791556E-01	RANGE	4.642918E-01	KURTOSIS	4.952250E-01	.50 QUANTILE (MEDIAN)	3.809658E-01
GEOM MEAN	3.821529E-01	MIDSPREAD	1.002232E-01			.75 QUANTILE (HINGE)	4.349344E-01
HARM MEAN	3.748313E-01					.90 QUANTILE	5.108541E-01
						MAXIMUM	6.973686E-01

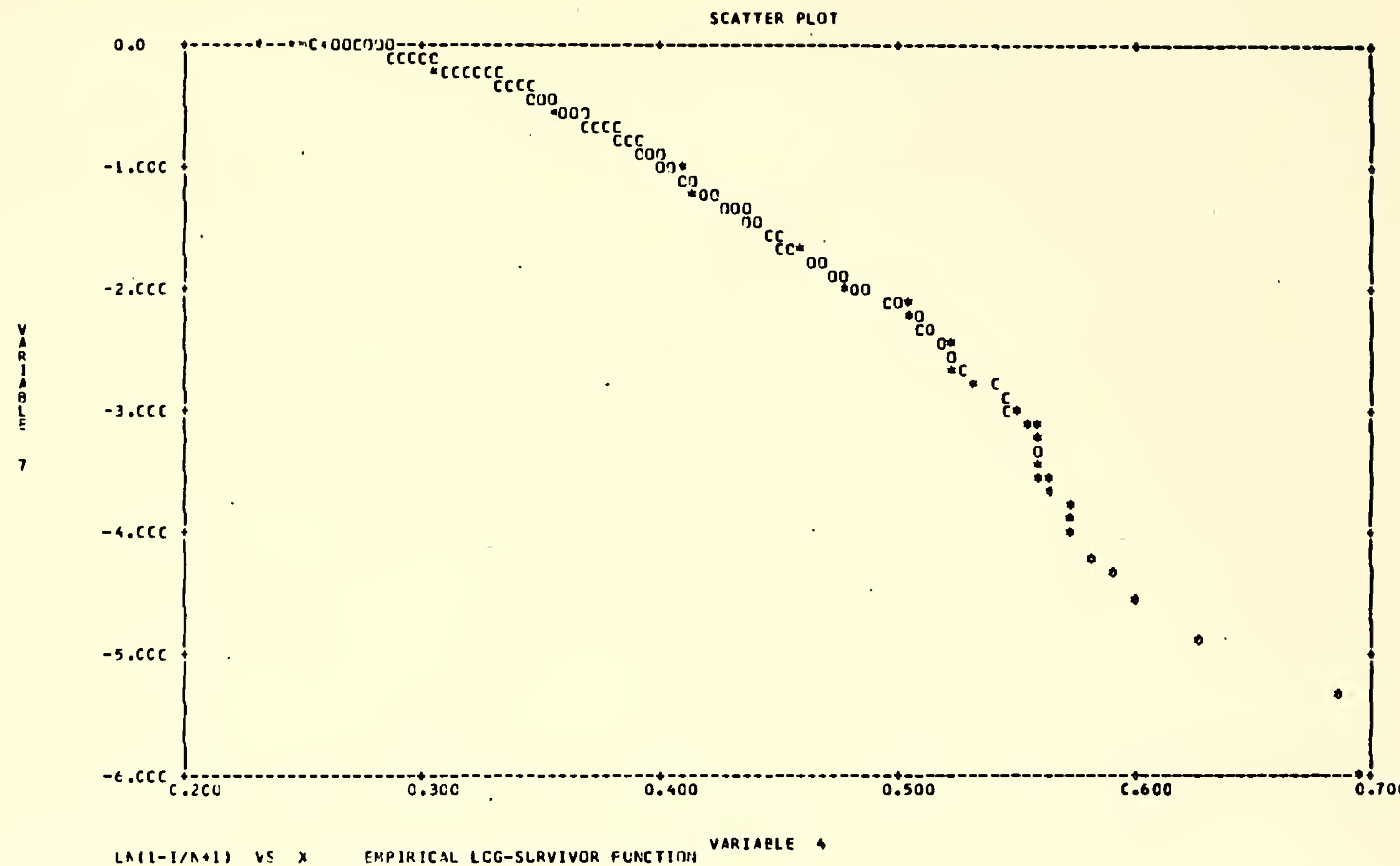
INTEGRAL SQUARE NORM SAMPLE SIZE M = 400

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 100

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)

LAST RANDOM NUMBER SEEDS IXA = 323656143 IXB = 1983041623

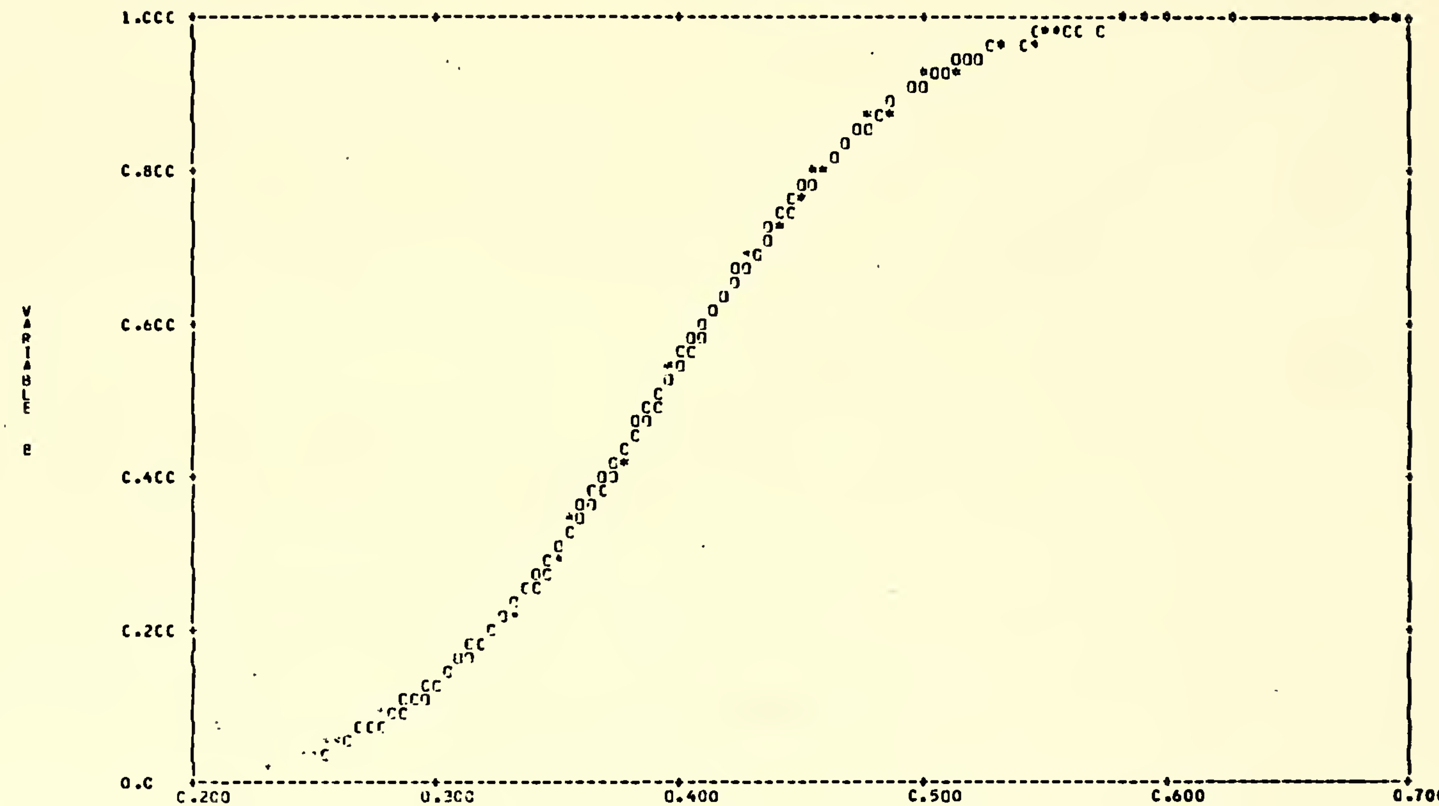

OBSERVED PROBABILITY
0.150
0.100
0.050
0.0
0.250
0.300
0.350
0.400
0.450
0.500
0.550
0.600
0.650
0.700
VARIABLE 4



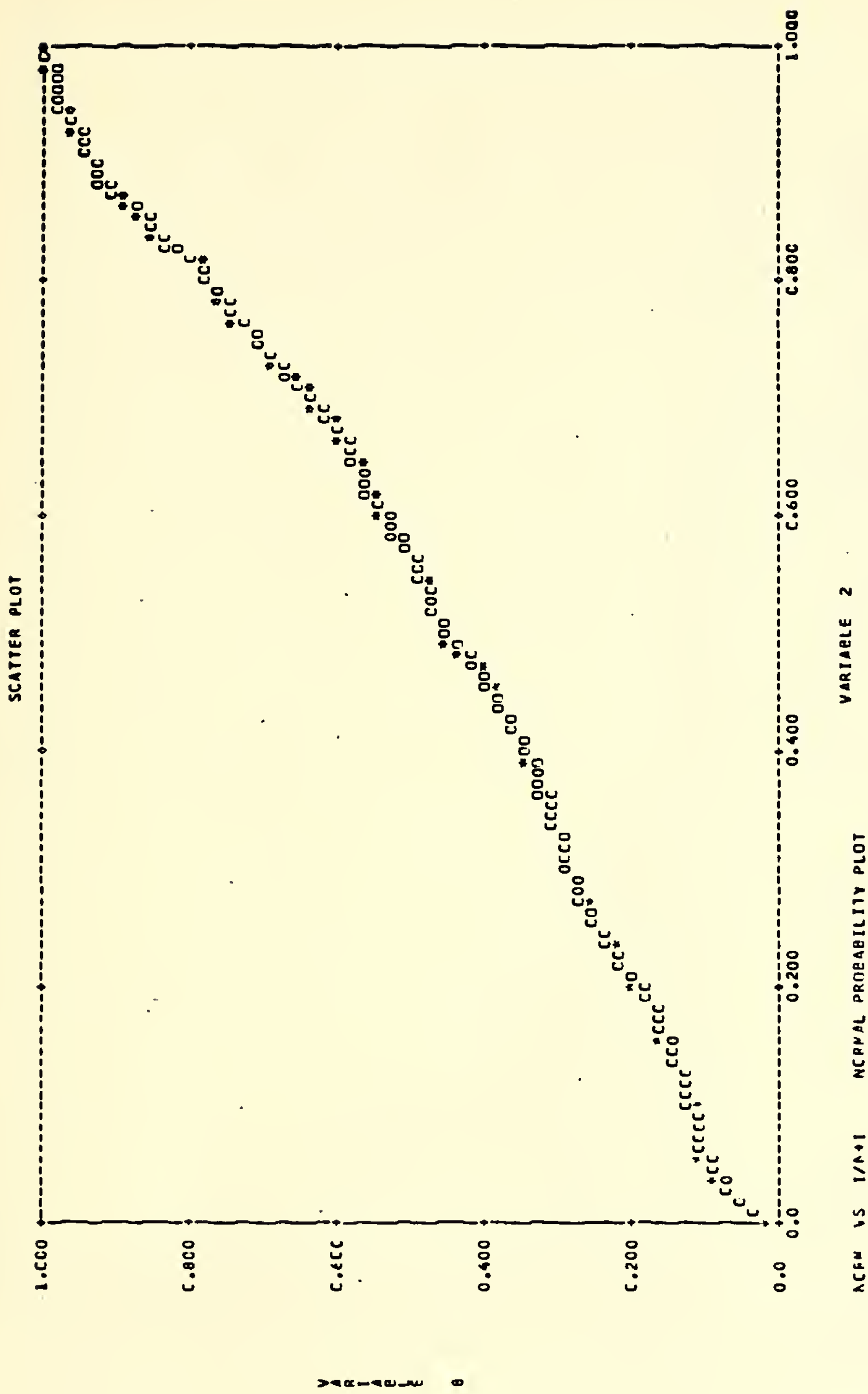
SCATTER PLOT
0.0
-1.000
-2.000
-3.000
-4.000
-5.000
-6.000
0.200
0.300
0.400
0.500
0.600
0.700
VARIABLE 7
VARIABLE 4
LN(1-I/(N+1)) VS X EMPIRICAL LOG-SURVIVOR FUNCTION

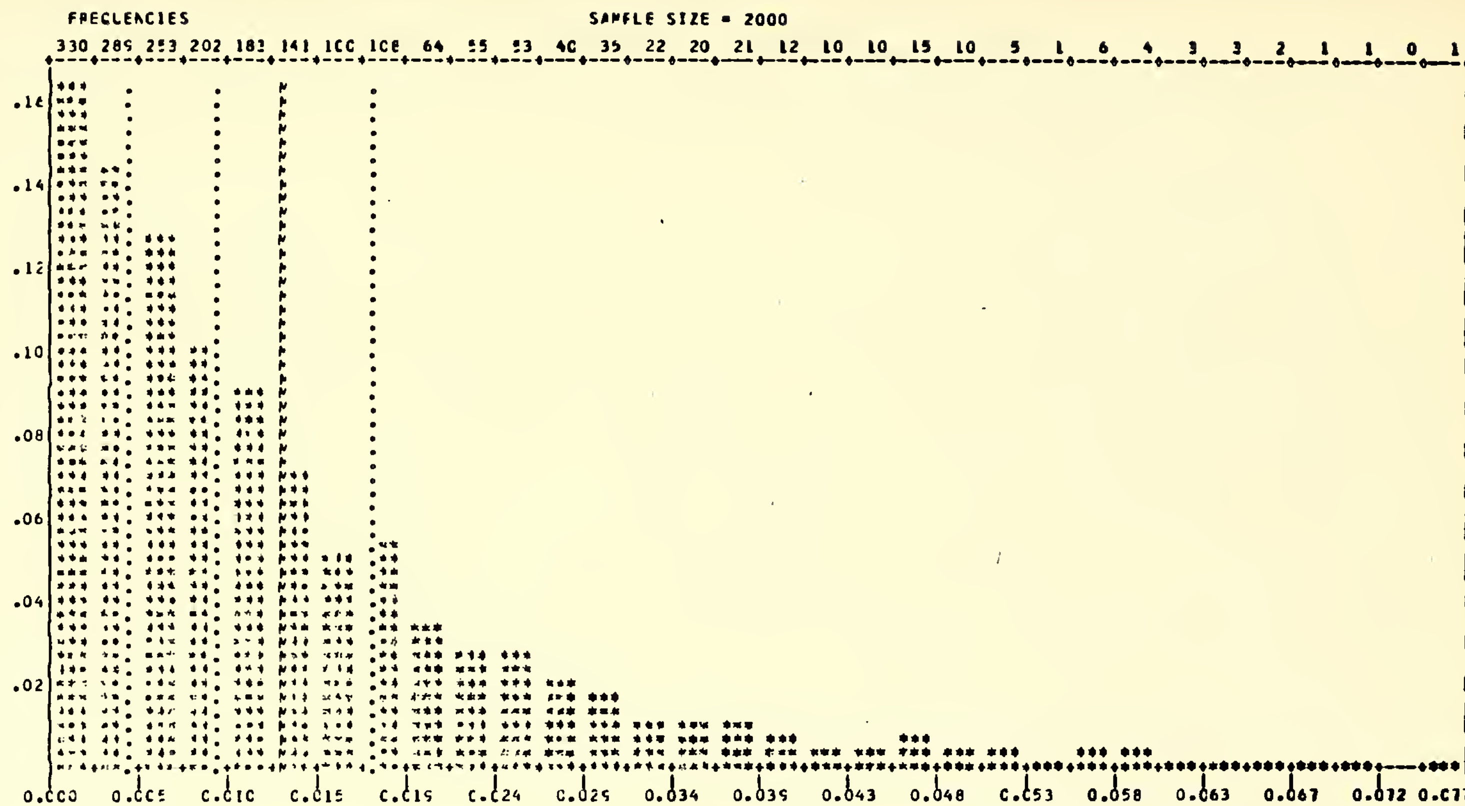
SCATTER PLOT
VARIABLE 4
VARIABLE 2
EMPIRICAL CDF
I/N+1 VS N
1.000
0.400
0.200
0.0
0.400
0.500
0.700

SCATTER PLOT



NORM VS X NORMAL PROBABILITY DISTRIBUTION WITH ESTIMATED MEAN & VARIANCE


SCATTER PLOT
1.CCO
C.8CO
C.6CC
0.400
C.200
0.0
VARIABLE 8
0.0
0.200
0.400
C.600
C.80C
1.000
VARIABLE 2
XCFM VS I/N+1 NCRMAL PROBABILITY PLOT

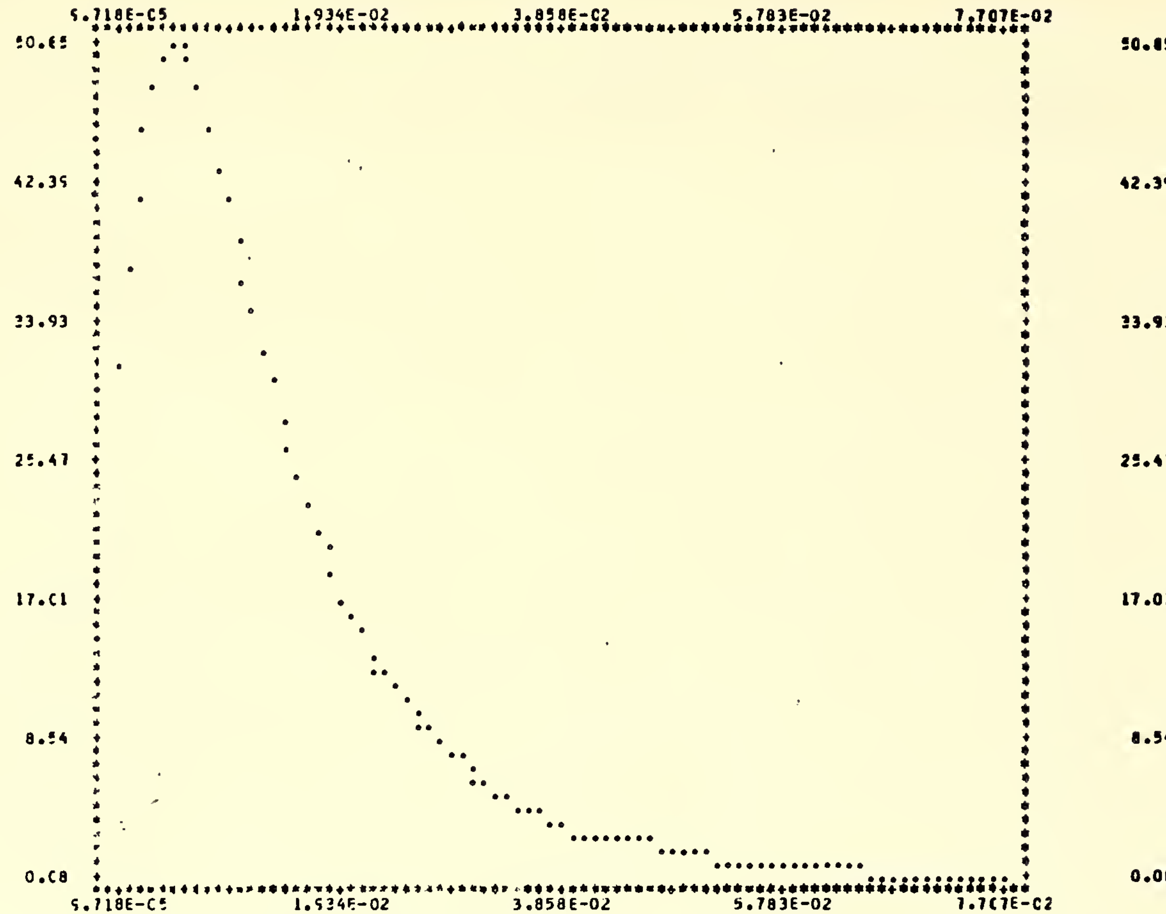


CENTRAL TENDENCY		SPREAD		HIGHER MOMENTS		DISTRIBUTION	
MEAN	1.230716E-02	VARIANCE	1.368225E-04	M3	2.819138E-06	MINIMUM	9.718350E-05
MEDIAN	8.952934E-03	STD DEV	1.169711E-02	M4	1.250644E-07	.10 QUANTILE	1.550523E-03
TRIMEAN	9.682409E-03	COEF VAR	9.504313E-01	SKEWNESS	1.761490E 00	.25 QUANTILE (HINGE)	3.867952E-03
MIDMEAN	9.279735E-03	RANGE	7.697374E-02	KURTOSIS	3.680655E 00	.50 QUANTILE (MEDIAN)	8.952934E-03
GEOM MEAN	7.544920E-03	MIDSPREAD	1.308787E-02			.75 QUANTILE (HINGE)	1.695583E-02
HARM MEAN	3.583454E-03					.90 QUANTILE	2.759558E-02
						MAXIMUM	7.707095E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 100

TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)



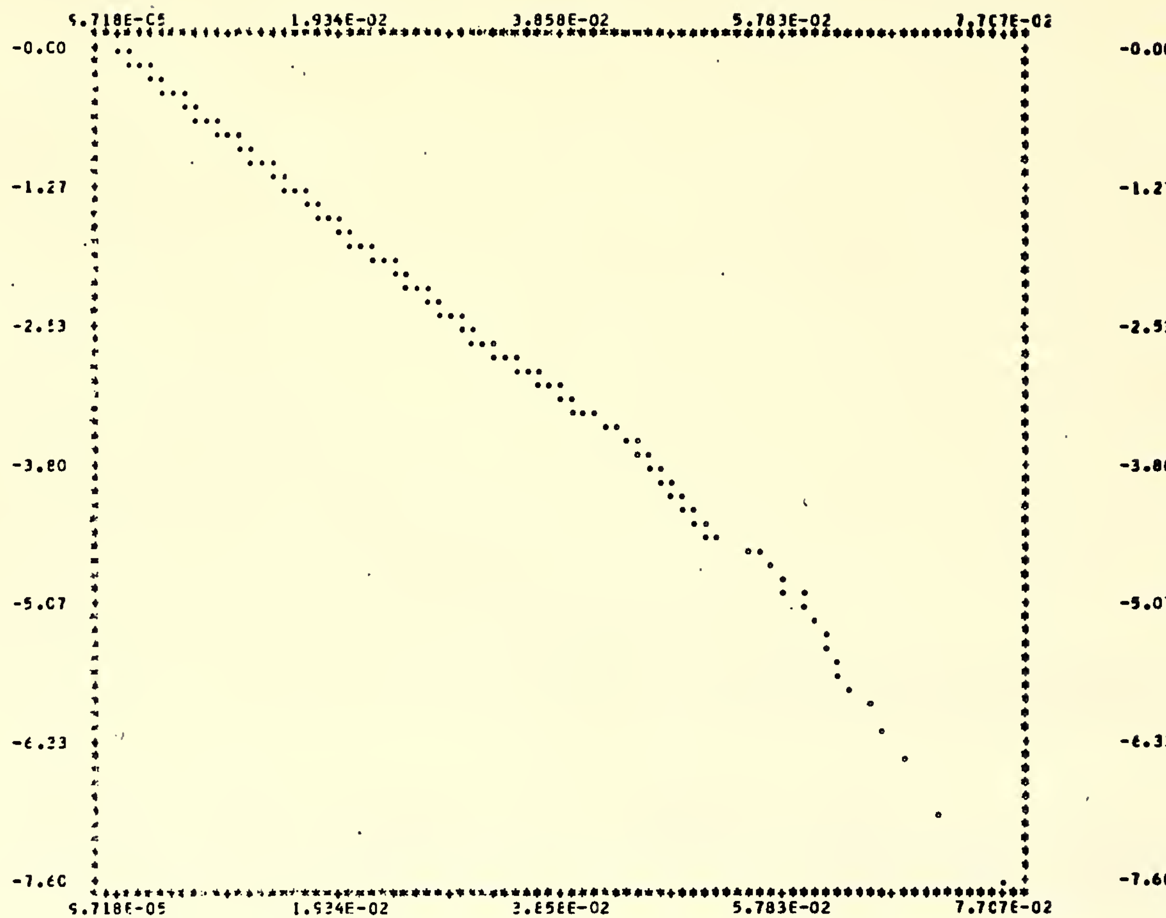
X-SCALE: "*"= 0.962E-03 UNITS

Y-SCALE: "*"= 0.846E 00 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE

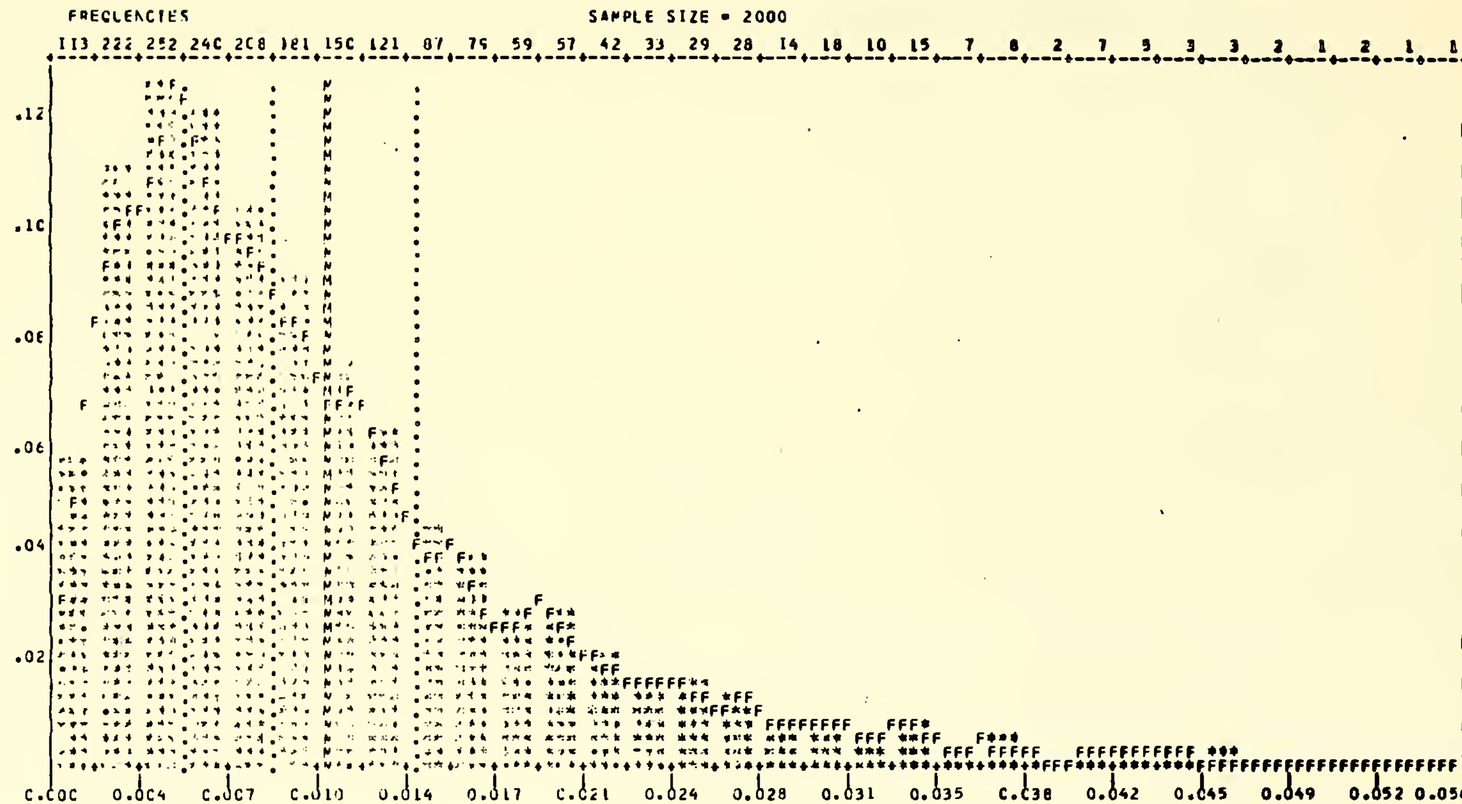
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10

ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: *=*= 0.962E-03 UNITS
Y-SCALE: *+*= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

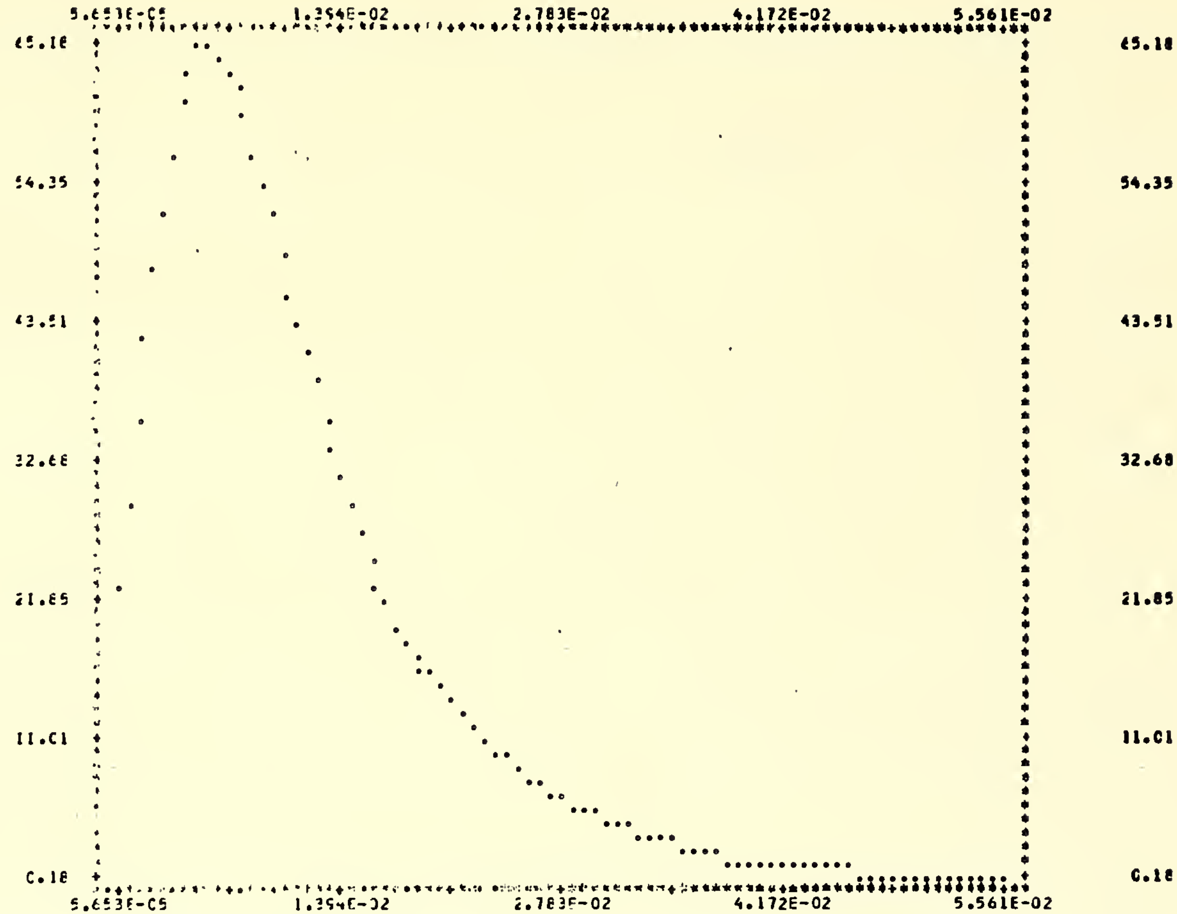


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION		
MEAN	1.074176E-02	VARIANCE	7.240816E-05	M3	9.966197E-07	MINIMUM		5.653262E-05
MEDIAN	8.374658E-03	STD DEV	8.509257E-03	M4	3.255729E-08	.10 QUANTILE		2.534249E-03
TRIMEAN	8.909554E-03	COEF VAR	7.921700E-01	SKEWNESS	1.617515E 00	.25 QUANTILE	(HINGE)	4.759349E-03
MIDMEAN	8.725718E-03	RANGE	5.555189E-02	KURTOSIS	3.209741E 00	.50 QUANTILE	(MEDIAN)	8.374658E-03
GEOM MEAN	7.616659E-03	MIDSPREAD	9.370283E-03			.75 QUANTILE	(HINGE)	1.412963E-02
HARM MEAN	4.645665E-03					.90 QUANTILE		2.220413E-02
						MAXIMUM		5.560843E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

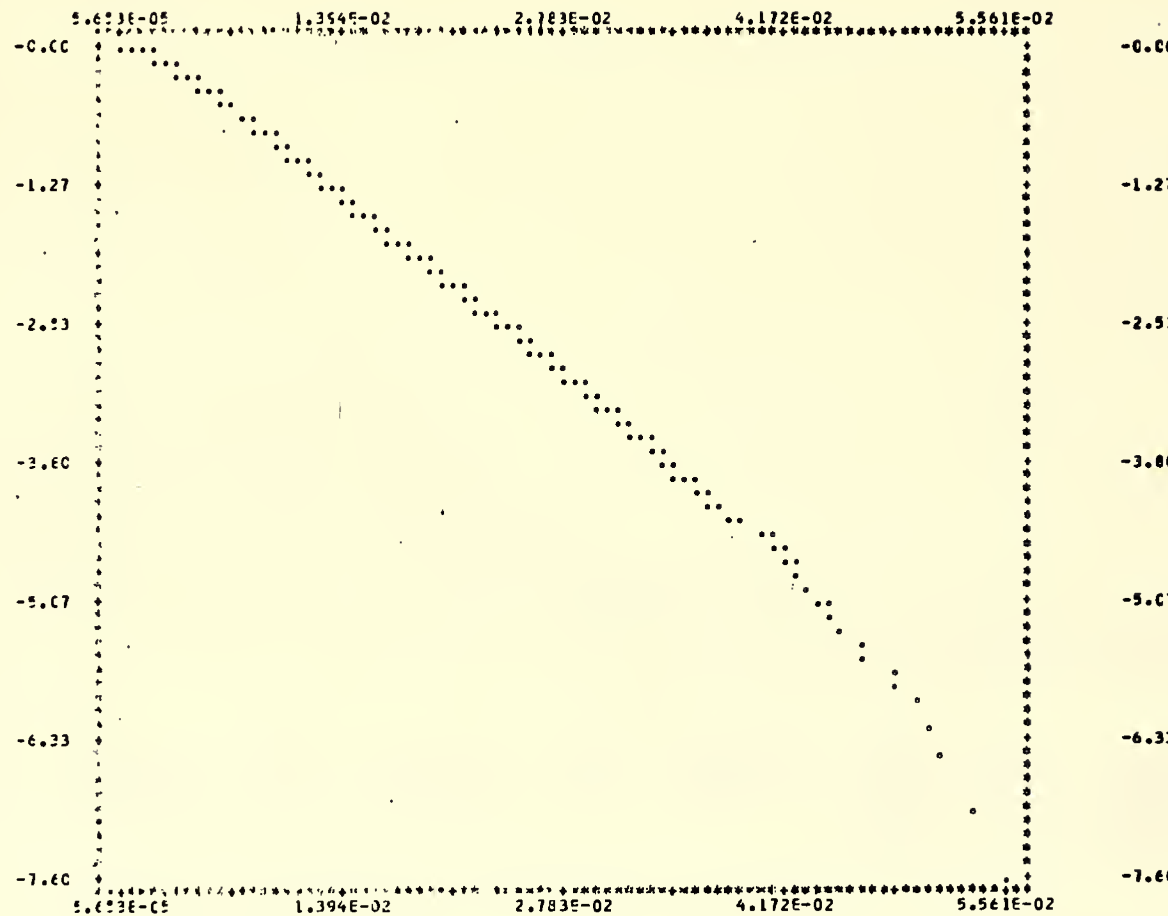
UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 200

TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)



X-SCALE: "*"= 0.6946-03 UNITS
Y-SCALE: "*"= 0.1083E 01 UNITS

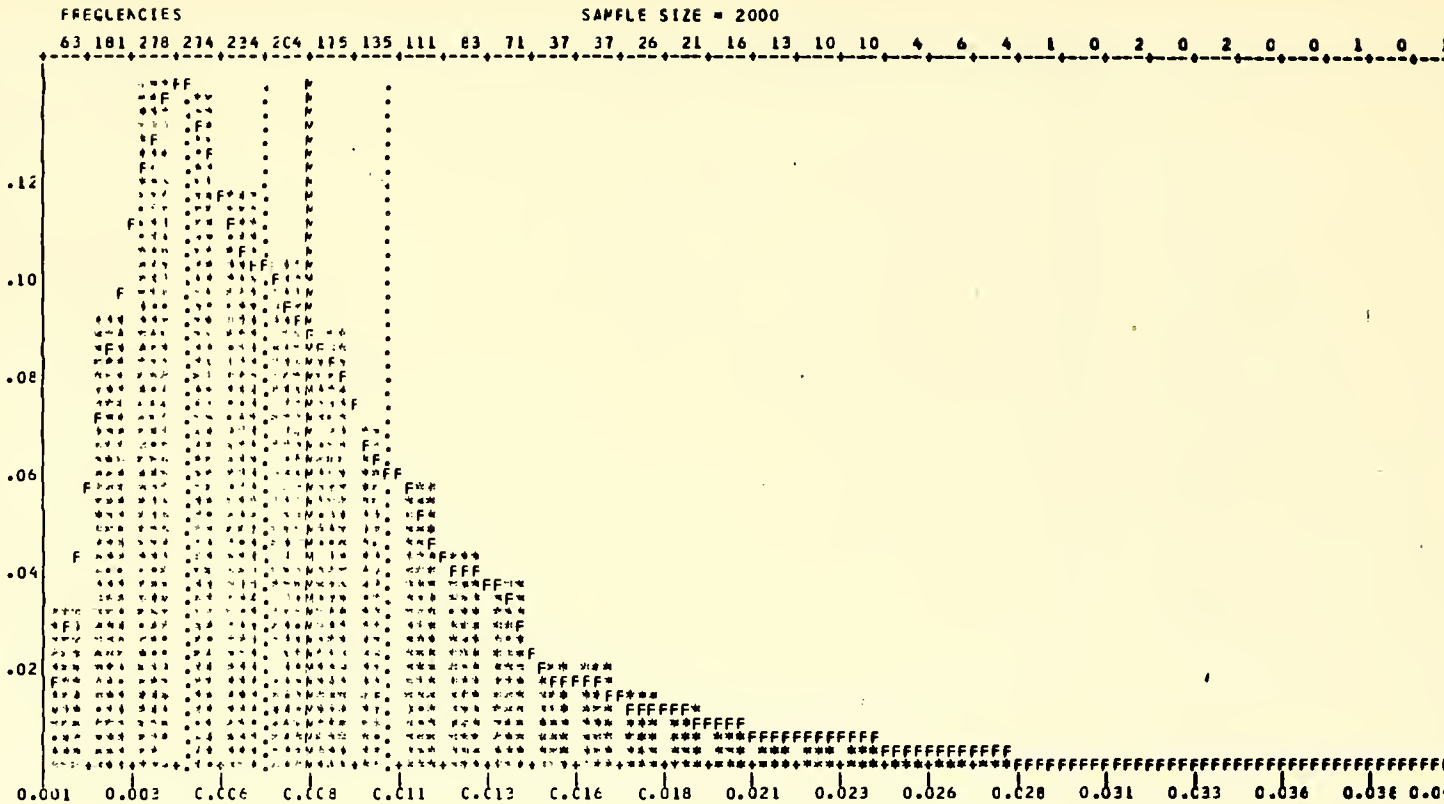
ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: ** = 0.694E-03 UNITS

Y-SCALE: ** = 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

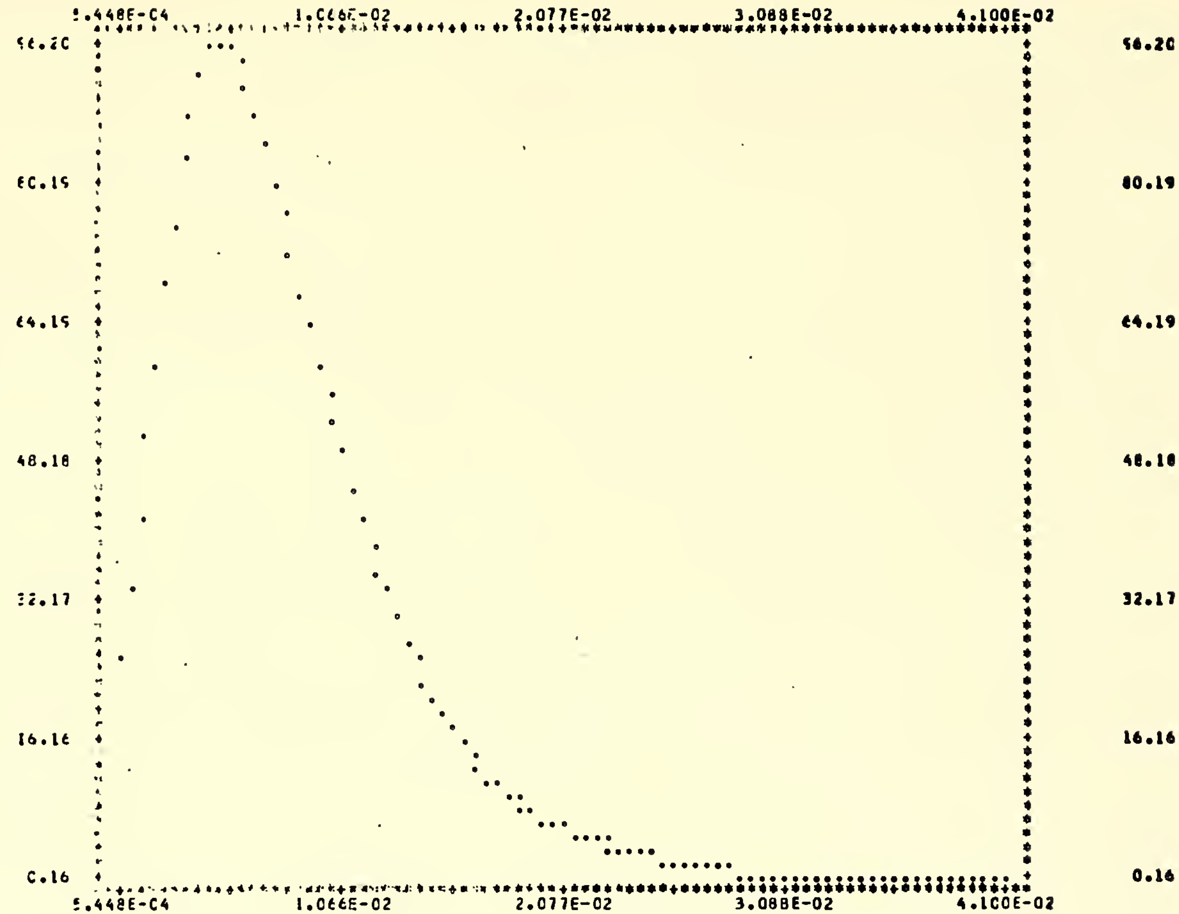


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION		
MEAN	7.874306E-03	VARIANCE	2.556866E-05	M3	2.013204E-07	MINIMUM		5.448014E-04
MEDIAN	6.673675E-03	STD DEV	5.056545E-03	M4	4.377604E-09	.10 QUANTILE		2.831203E-03
TRIMEAN	6.978244E-03	COEF VAR	6.421575E-01	SKEWNESS	1.557135E 00	.25 QUANTILE	(HINGE)	4.246000E-03
MIDMEAN	6.845247E-03	RANGE	4.045168E-02	KURTOSIS	3.696078E 00	.50 QUANTILE	(MEDIAN)	6.673675E-03
GEOM MEAN	6.475266E-03	MIDSPREAD	6.013635E-03			.75 QUANTILE	(HINGE)	1.031964E-02
HARM MEAN	5.184593E-03					.90 QUANTILE		1.420808E-02
						MAXIMUM		4.099648E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

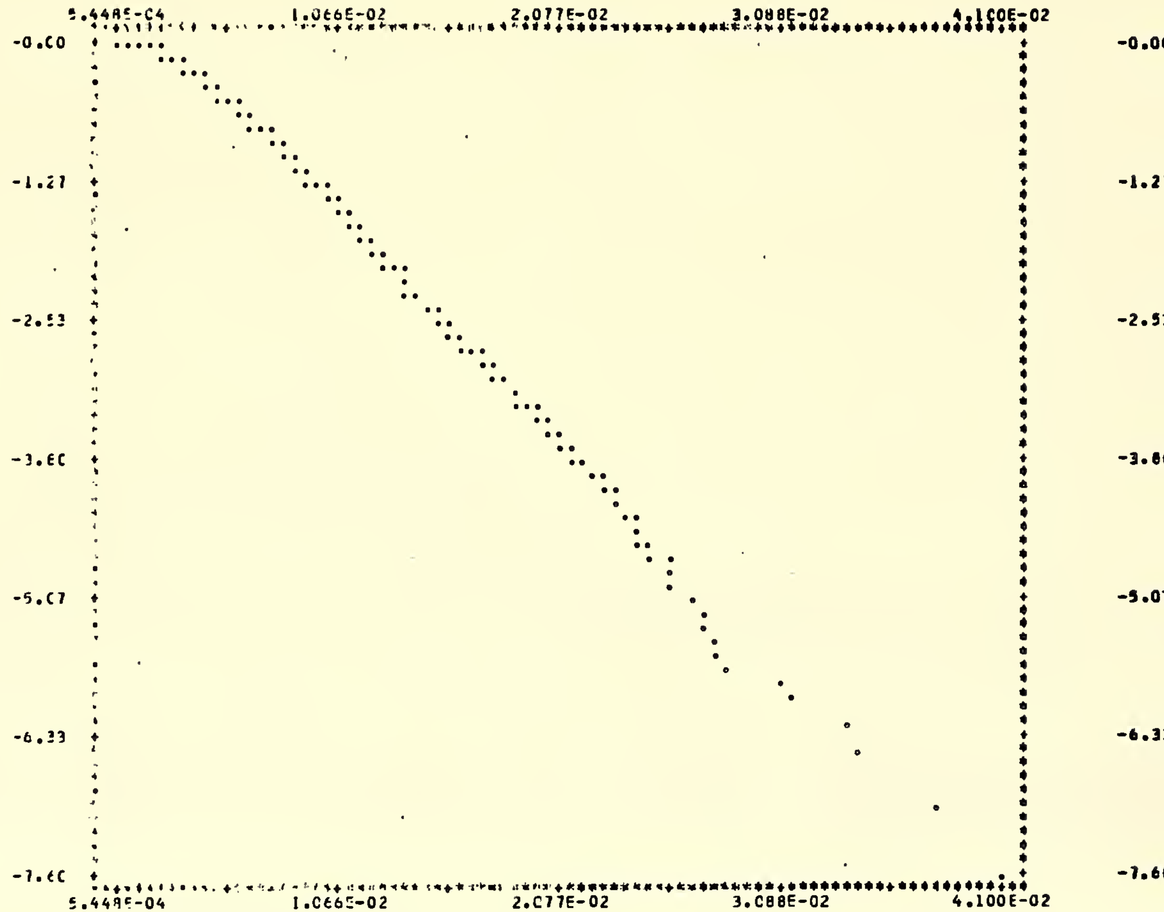
UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 500

TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)



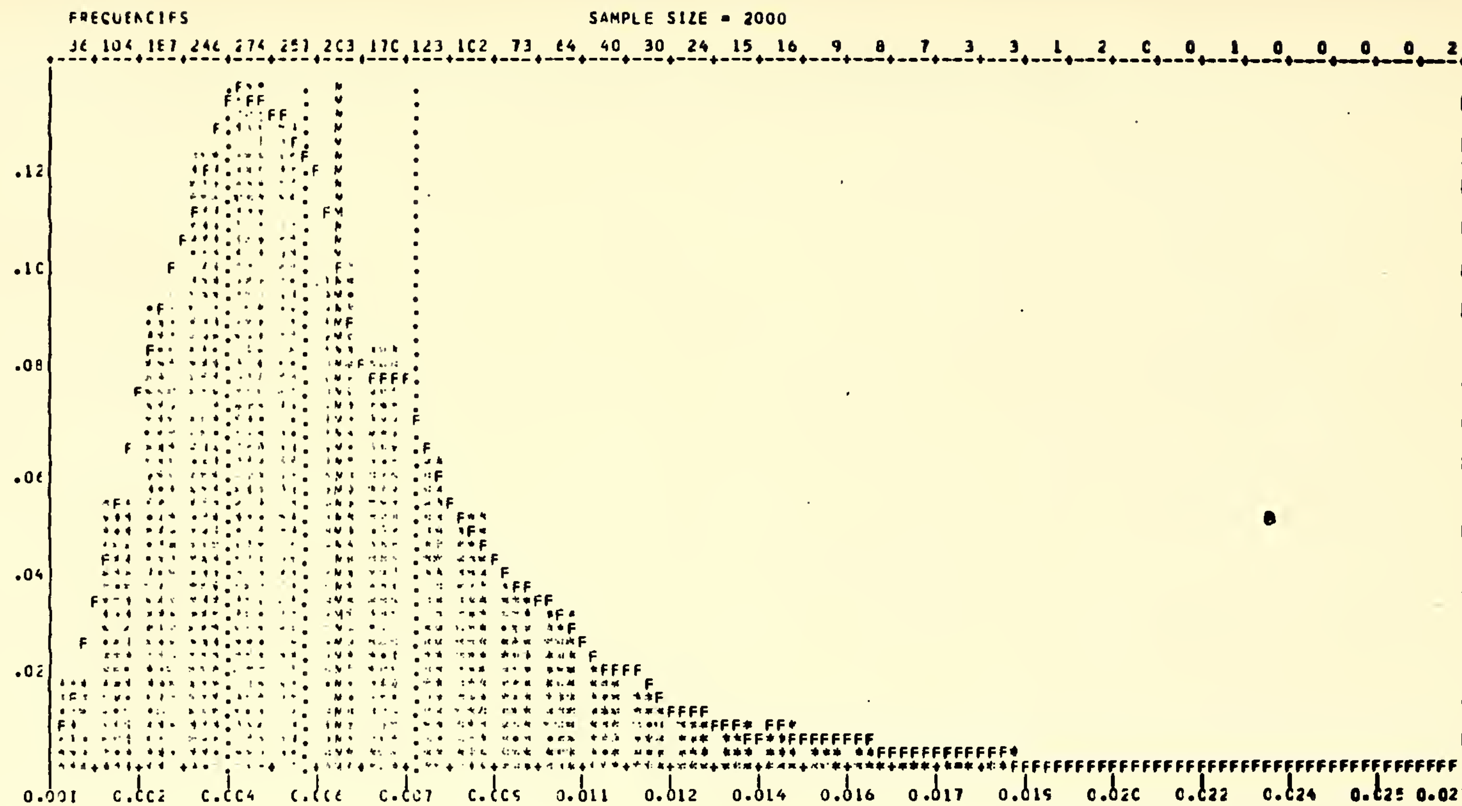
X-SCALE: *** 0.506E-03 UNITS
Y-SCALE: *** 0.160E 01 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "-"= 0.506E-03 UNITS
Y-SCALE: "+"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

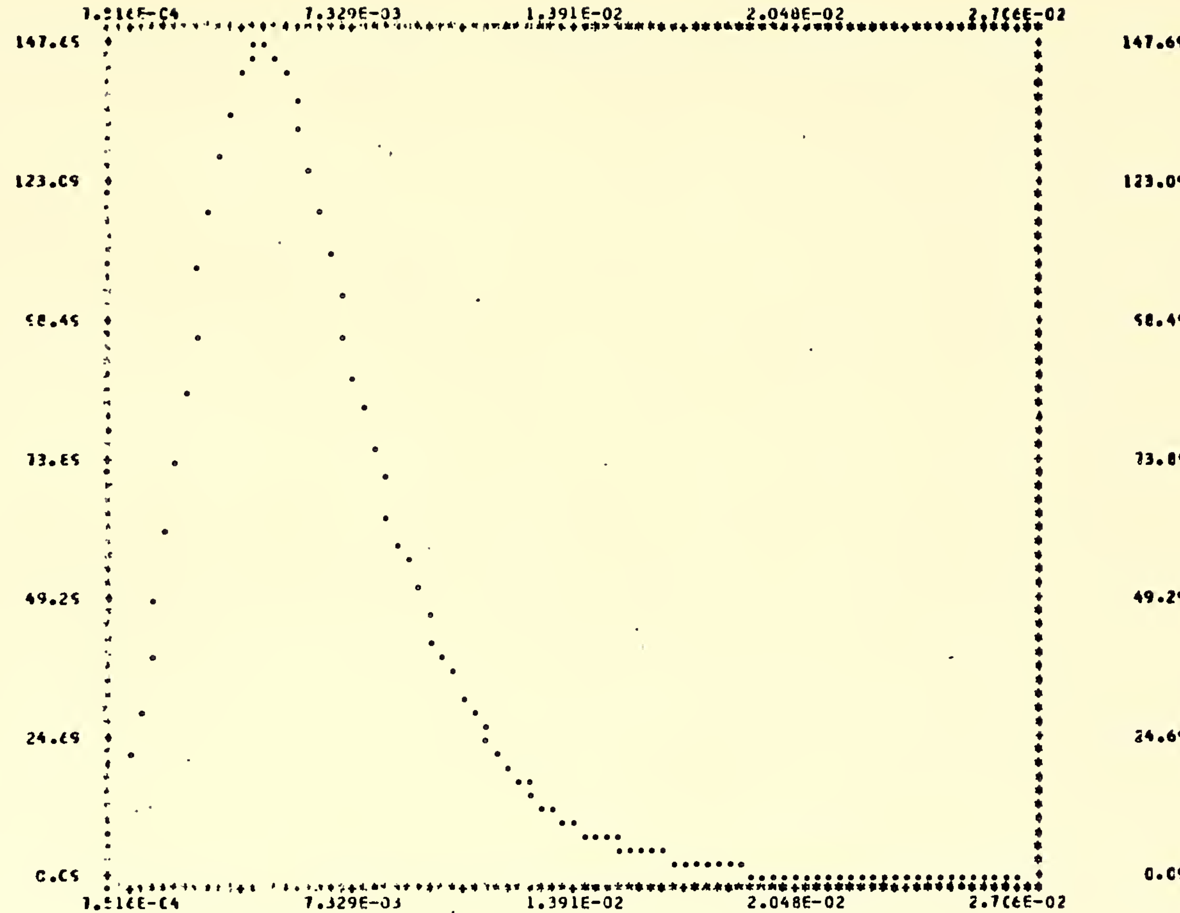


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION		
MEAN	5.585765E-03	VARIANCE	9.823831E-06	M3	4.418073E-08	MINIMUM		7.516197E-04
MEDIAN	5.330045E-03	STD DEV	3.134307E-03	M4	6.359899E-10	.10 QUANTILE		2.693556E-03
TRIMEAN	5.490068E-03	COEF VAR	5.236250E-01	SKEWNESS	1.434855E 00	.25 QUANTILE	(HINGE)	3.823010E-03
MIDMEAN	5.429175E-03	RANGE	2.630798E-02	KURTOSIS	3.589980E 00	.50 QUANTILE	(MEDIAN)	5.330045E-03
GEOM MEAN	5.261064E-03	MIDSPREAD	3.654173E-03			.75 QUANTILE	(HINGE)	7.477183E-03
HARM MEAN	4.575521E-03					.90 QUANTILE		1.009041E-02
						MAXIMUM		2.705960E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 1000

TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)



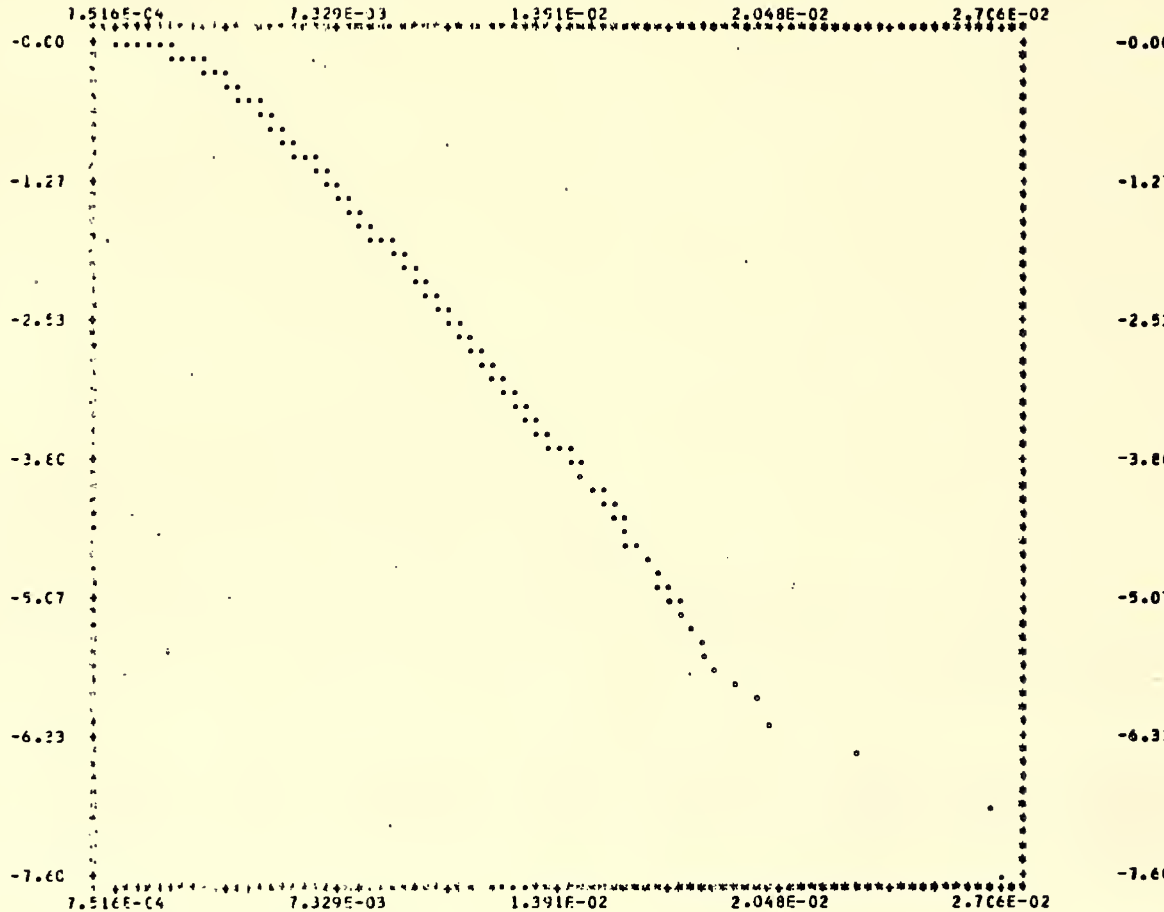
X-SCALE: "*"= 0.329E-03 UNITS

Y-SCALE: "*"= 0.246E 01 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE

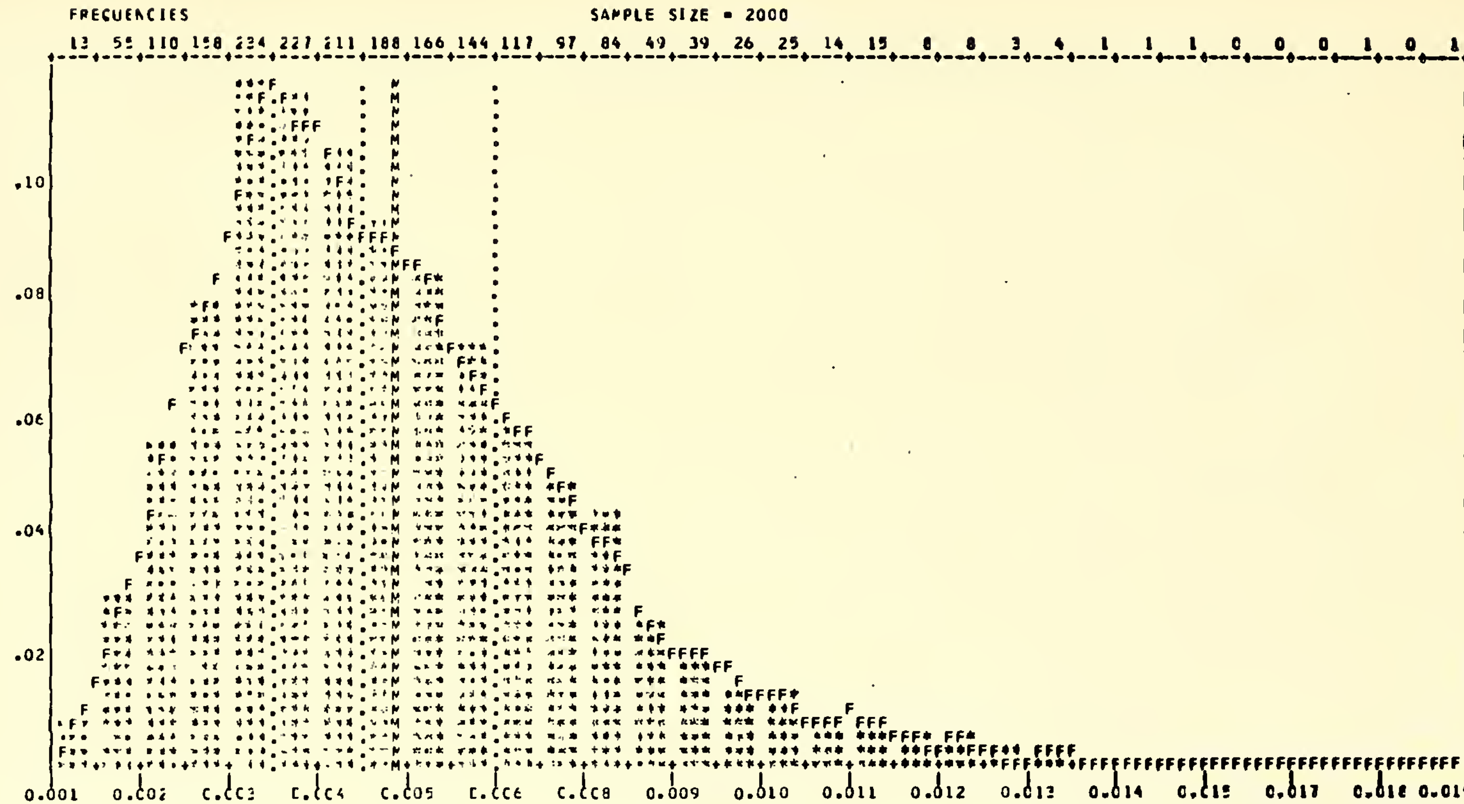
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10

ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: *-*= 0.329E-03 UNITS
Y-SCALE: *-*= 0.127E 00 UNITS

LN(1-I/N+1) VS X   EMPIRICAL LOG SURVIVAL FUNCTION



CENTRAL TENDENCY	
MEAN	5.110052E-03
MEDIAN	4.655513E-03
TRIMEAN	4.813515E-03
MIDMEAN	4.774112E-03
GEOM MEAN	4.632425E-03
HARM MEAN	4.166000E-03

SPREAD	
VARIANCE	5.192975E-06
STD DEV	2.278810E-03
COEF VAR	4.459429E-01
RANGE	1.801267E-02
MIDSPREAD	2.924502E-03

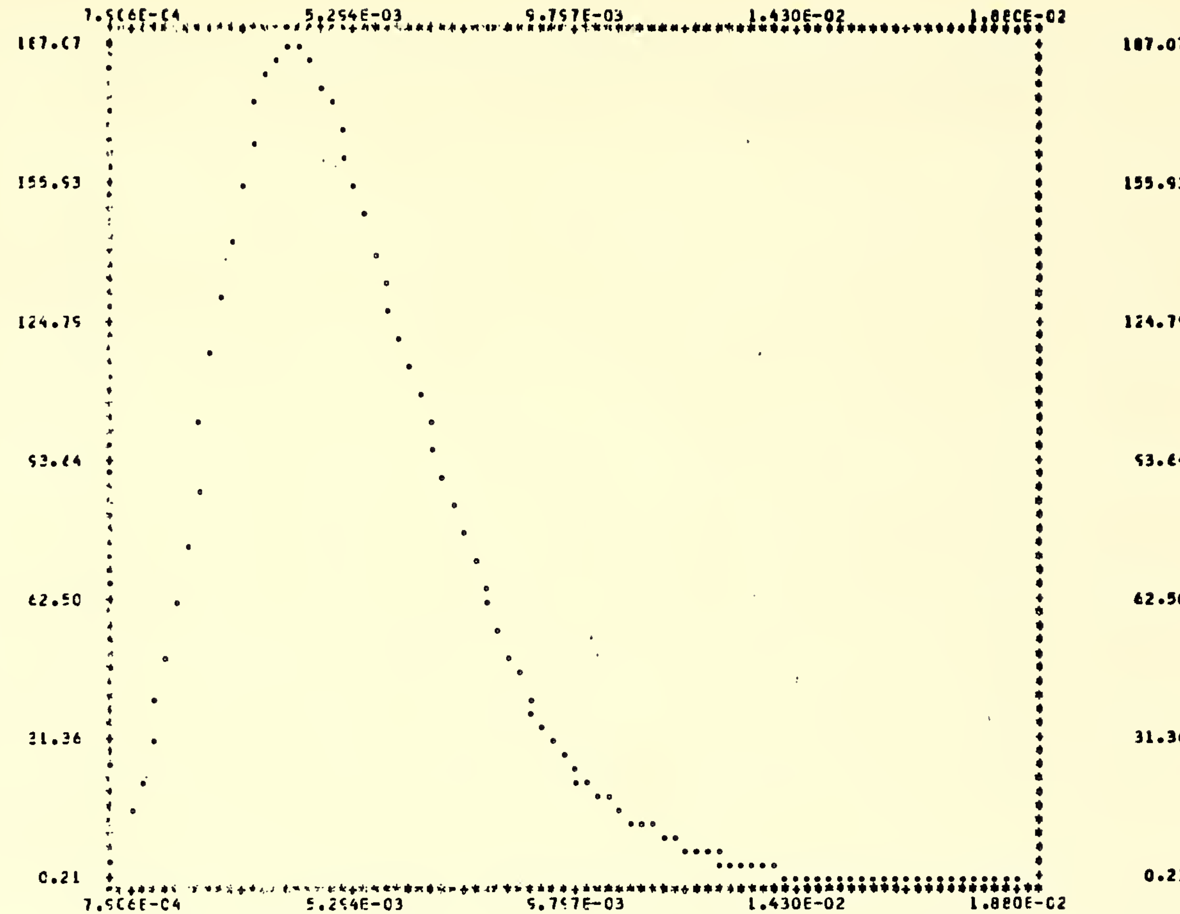
HIGHER CENTRAL MOMENTS	
M3	1.221252E-08
M4	1.262120E-10
SKEWNESS	1.032037E 00
KURTOSIS	1.680240E 00

DISTRIBUTION		
MINIMUM		7.906051E-04
.10 QUANTILE		2.556095E-03
.25 QUANTILE	(HINGE)	3.465066E-03
.50 QUANTILE	(MEDIAN)	4.655513E-03
.75 QUANTILE	(HINGE)	6.389568E-03
.90 QUANTILE		8.086053E-03
MAXIMUM		1.880328E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

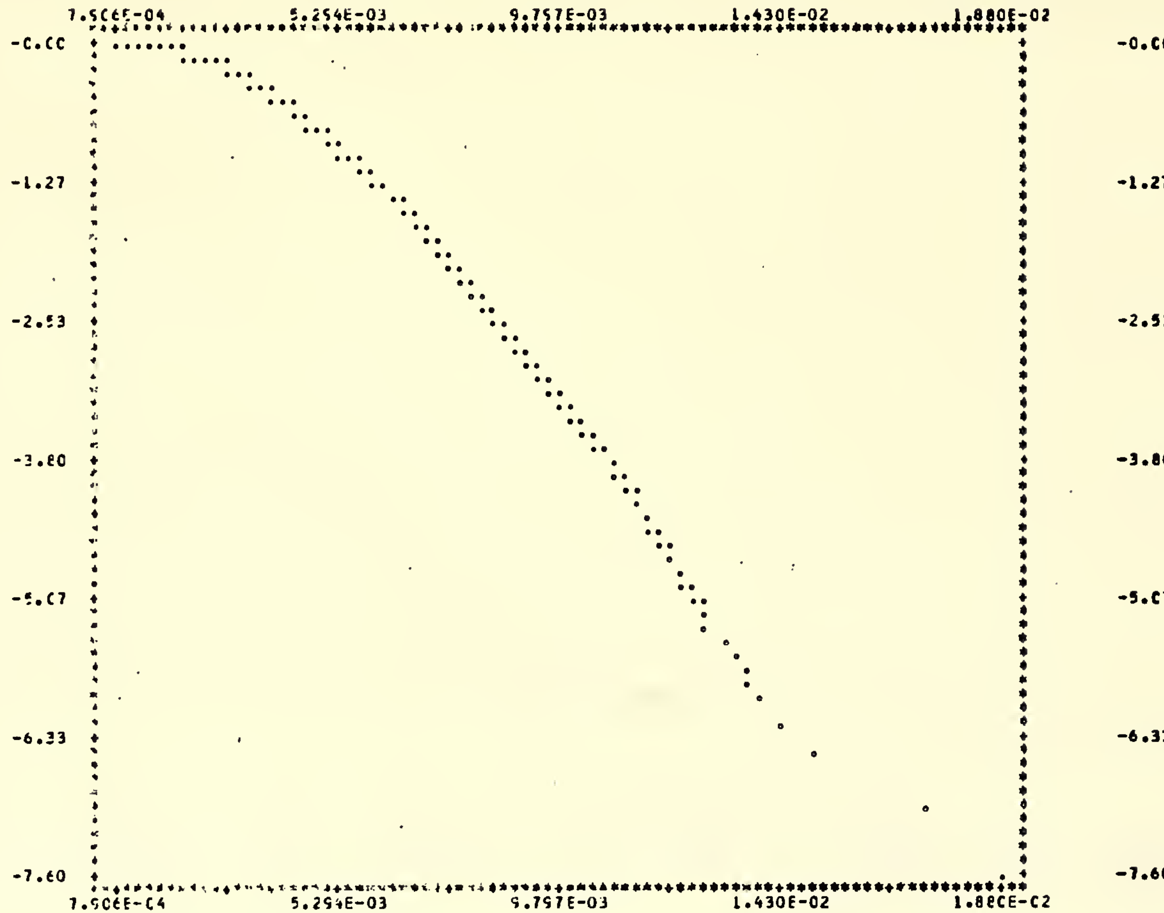
UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 1500

TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)



X-SCALE: "*"= 0.225E-03 UNITS
Y-SCALE: "*"= 0.311E 01 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



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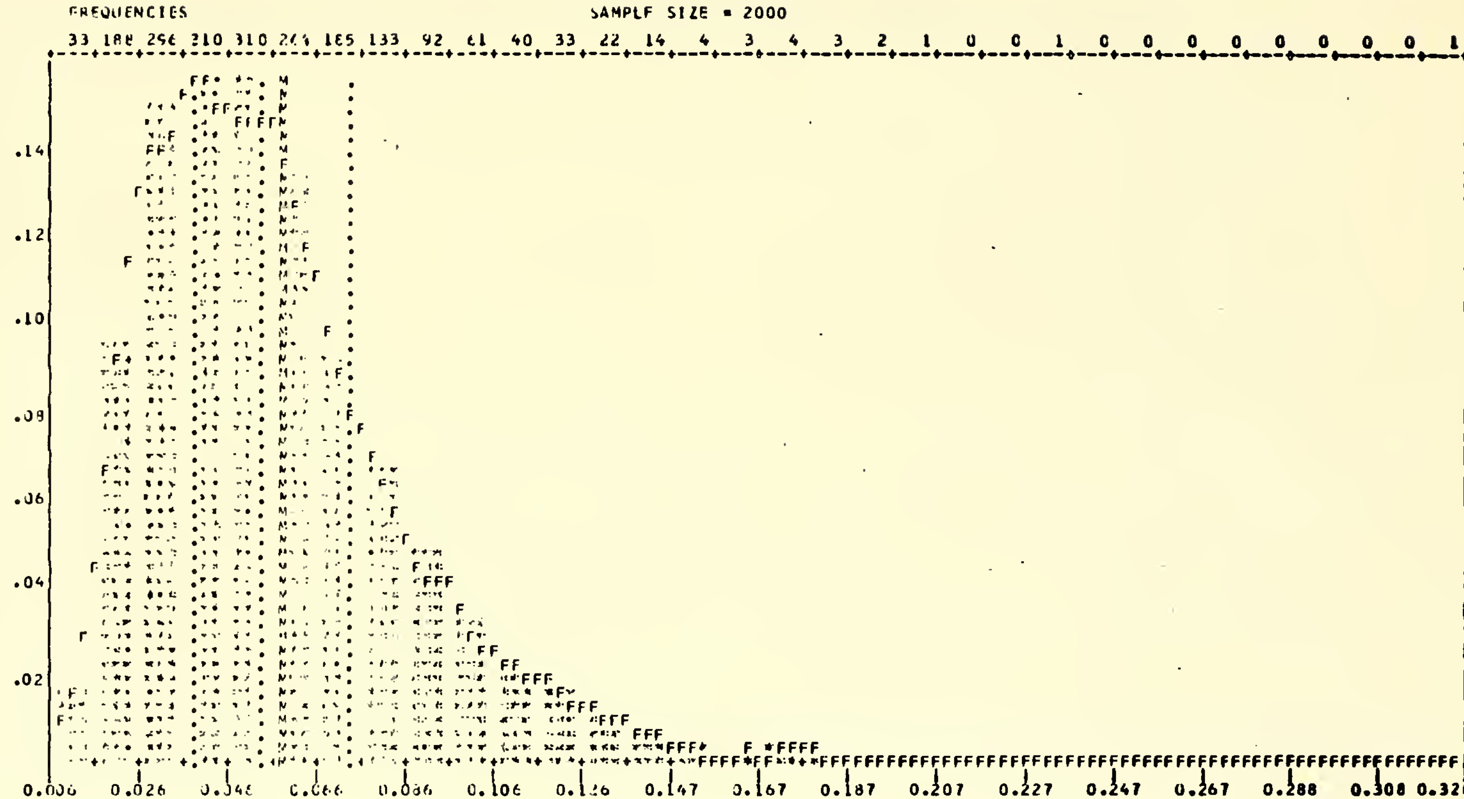
Y-SCALE: "+". 0.127E 00 UNITS

LN(1-I/N+1) VS X   EMPIRICAL LOG SURVIVAL FUNCTION

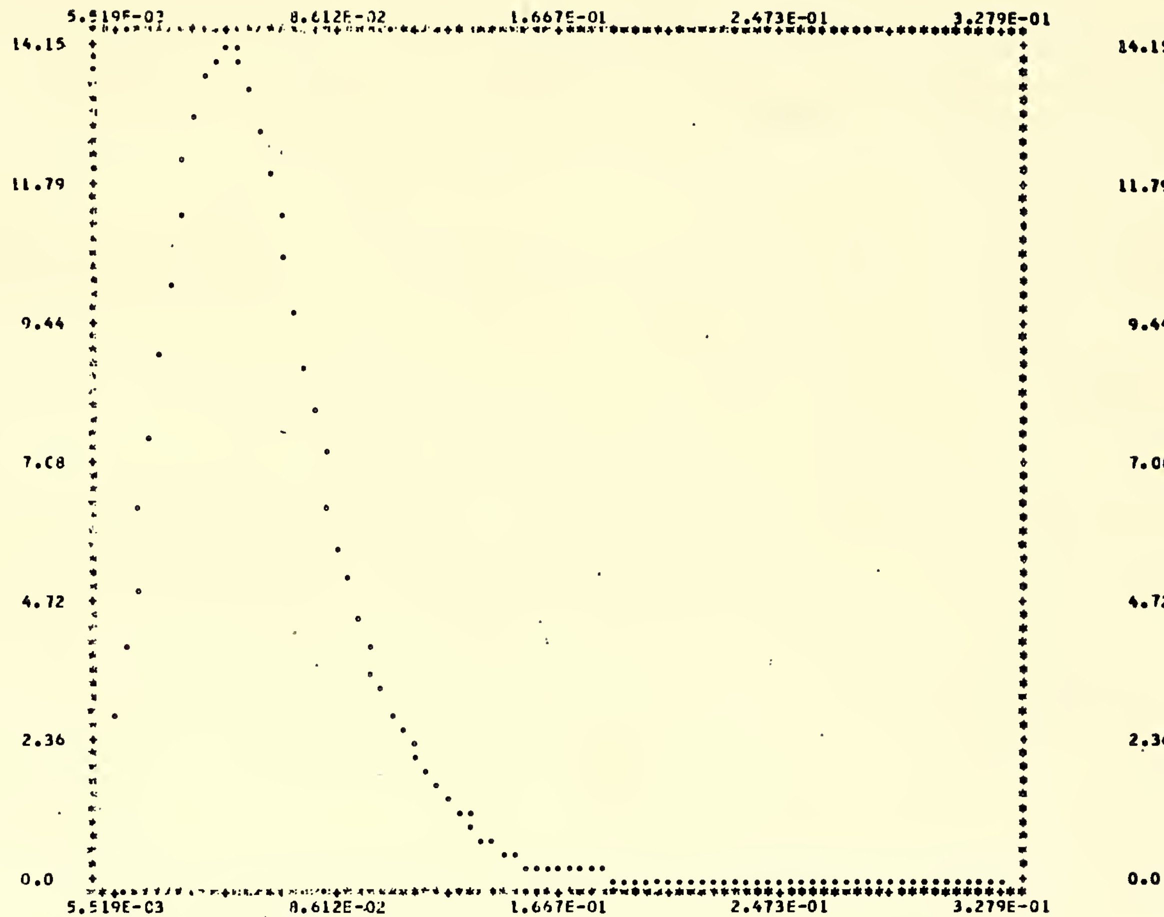


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	5.646861E-02	VARIANCE	8.604034E-04	M3	3.985895E-05	MINIMUM	5.519085E-03
MEDIAN	5.148327E-02	STD DEV	2.967155E-02	M4	6.508872E-06	.10 QUANTILE	2.480756E-02
TRIMEAN	5.227634E-02	COEF VAR	5.252668E-01	SKEWNESS	1.525819E 00	.25 QUANTILE (HINGE)	3.521070E-02
MIDMEAN	5.166065E-02	RANGE	3.223874E-01	KURTOSIS	5.397349E 00	.50 QUANTILE (MEDIAN)	5.148327E-02
GEOM MEAN	4.555042E-02	MIDSPREAD	3.571746E-02			.75 QUANTILE (HINGE)	7.092816E-02
HARM MEAN	4.285740E-02					.90 QUANTILE	9.518176E-02
						MAXIMUM	3.279065E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 100

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)



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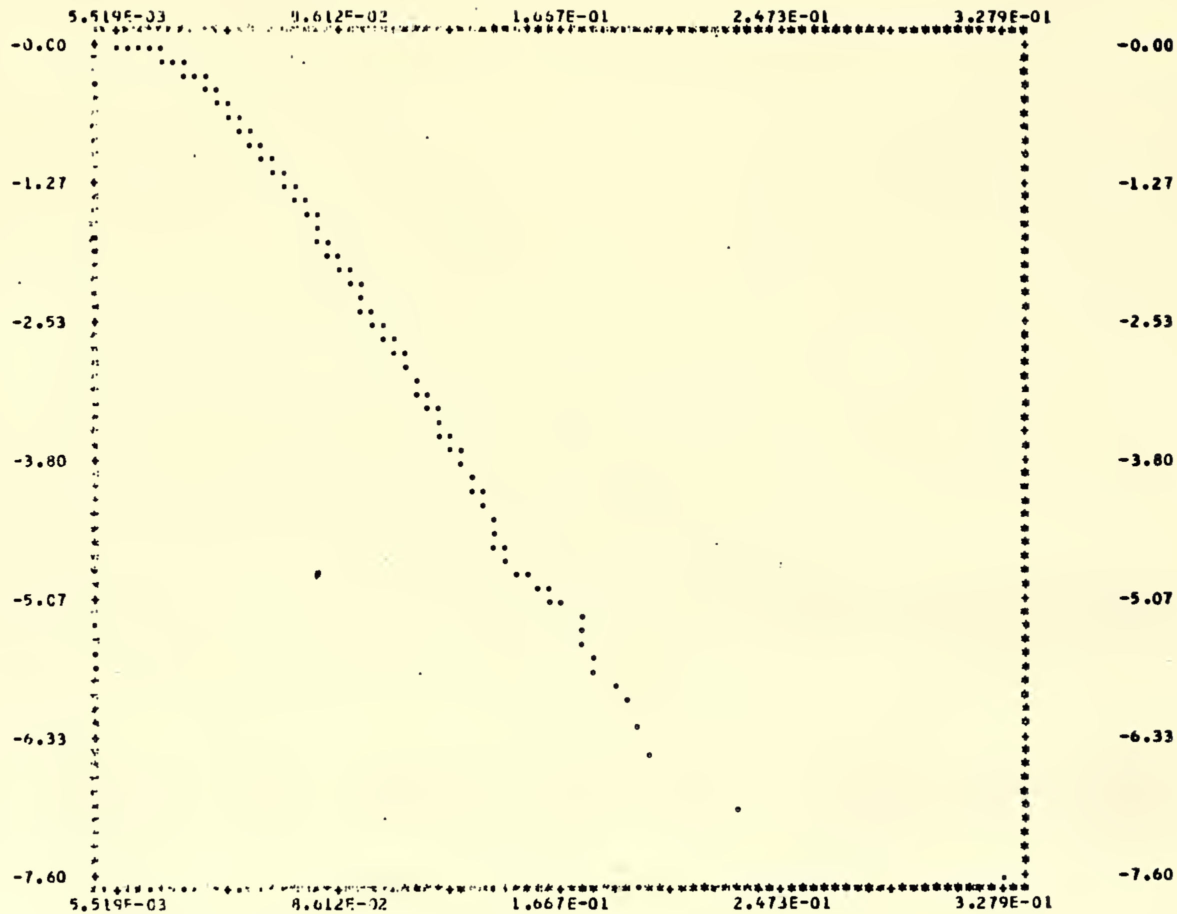
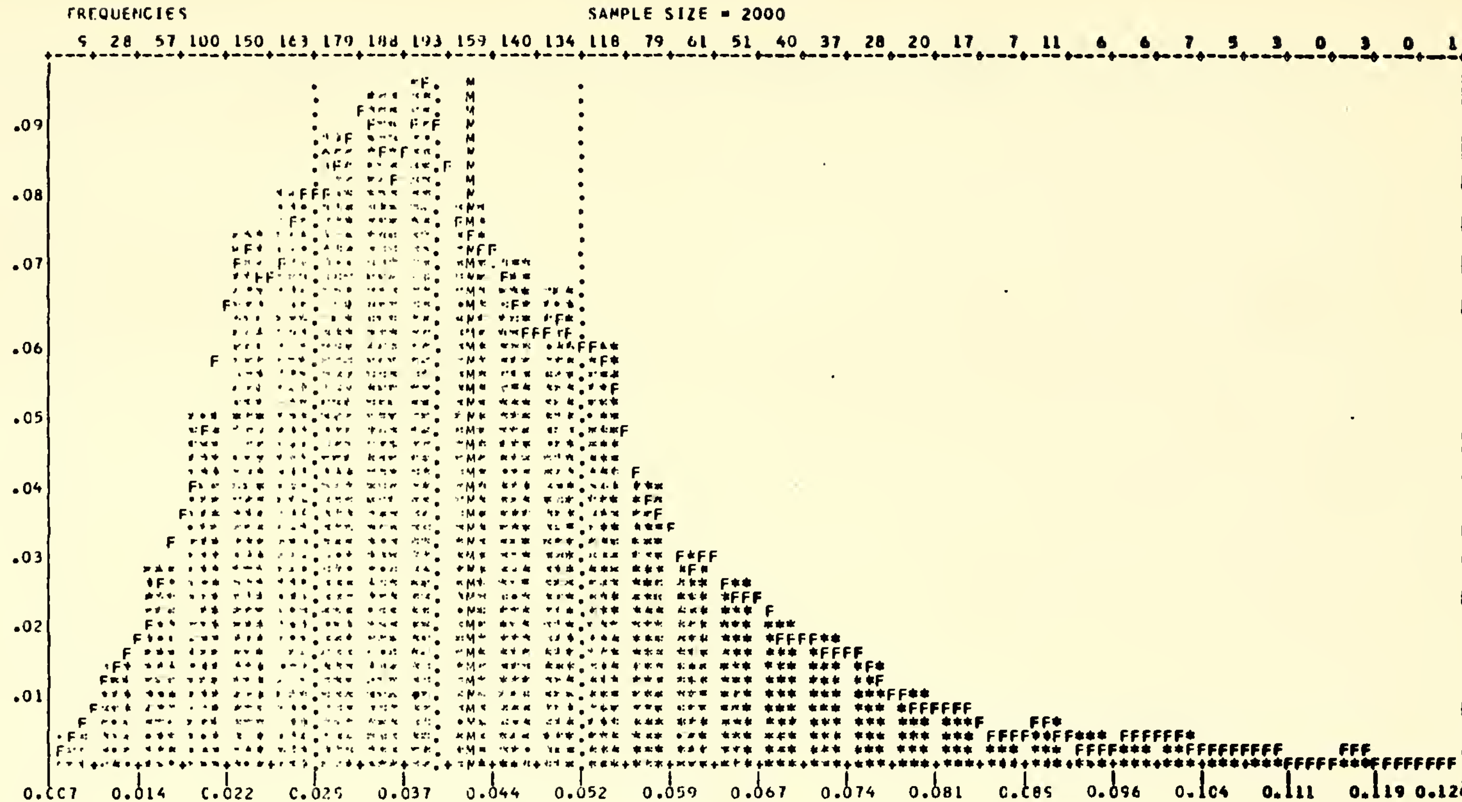
Y-SCALE: "*"= 0.236E 00 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "-"= 0.403E-02 UNITS
Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

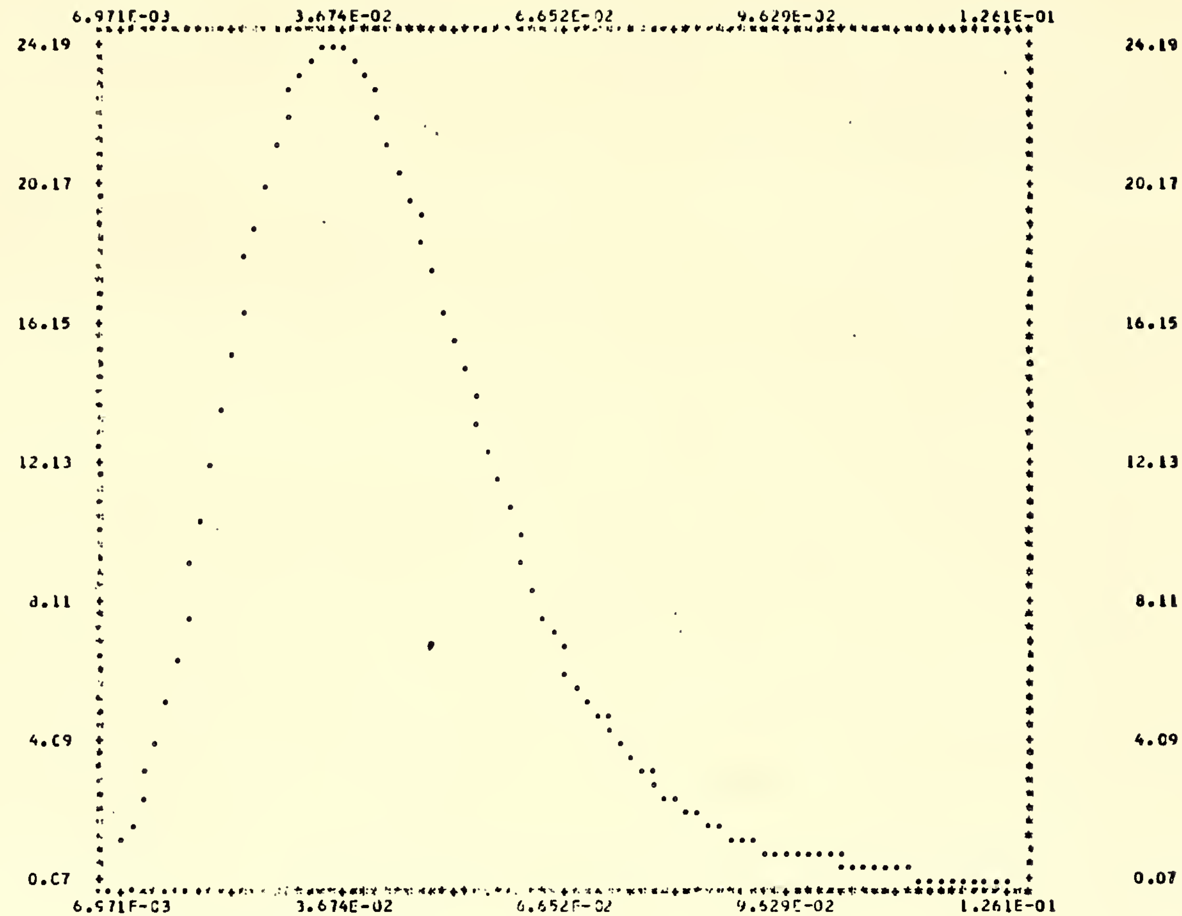


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	4.201795E-02	VARIANCE	3.174294E-04	M3	5.597397E-06	MINIMUM	6.971177E-03
MEDIAN	3.911397E-02	STD DEV	1.781655E-02	M4	4.394735E-07	.10 QUANTILE	2.206884E-02
TRIMEAN	3.975073E-02	COEFF VAR	4.240223E-01	SKEWNESS	9.897267E-01	.25 QUANTILE (HINGE)	2.913324E-02
MIDMEAN	3.963823E-02	RANGE	1.190948E-01	KURTOSIS	1.361526E 00	.50 QUANTILE (MEDIAN)	3.911397E-02
GEOM MEAN	3.644558E-02	MIDSPREAD	2.250852E-02			.75 QUANTILE (HINGE)	5.164176E-02
HARM MEAN	3.491189E-02					.90 QUANTILE	6.555706E-02
						MAXIMUM	1.260660E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 200

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)



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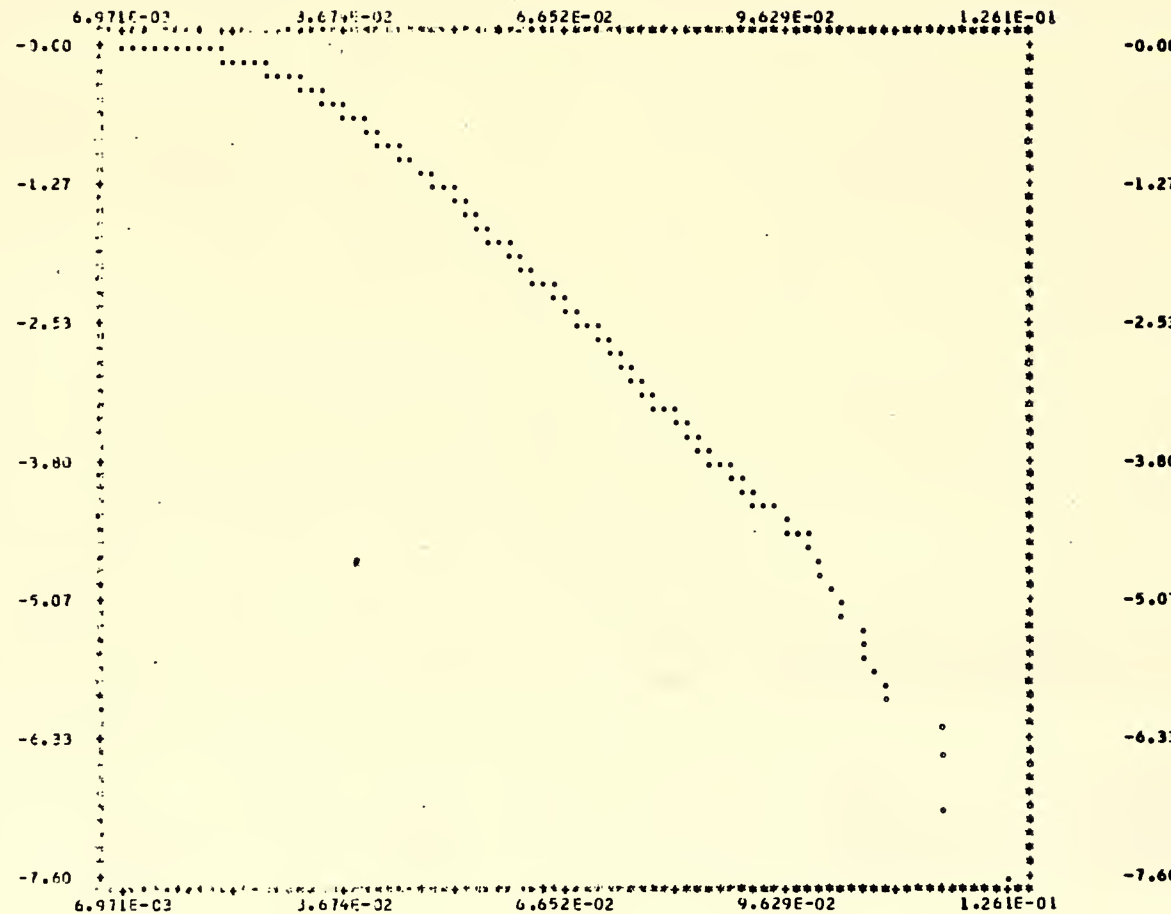
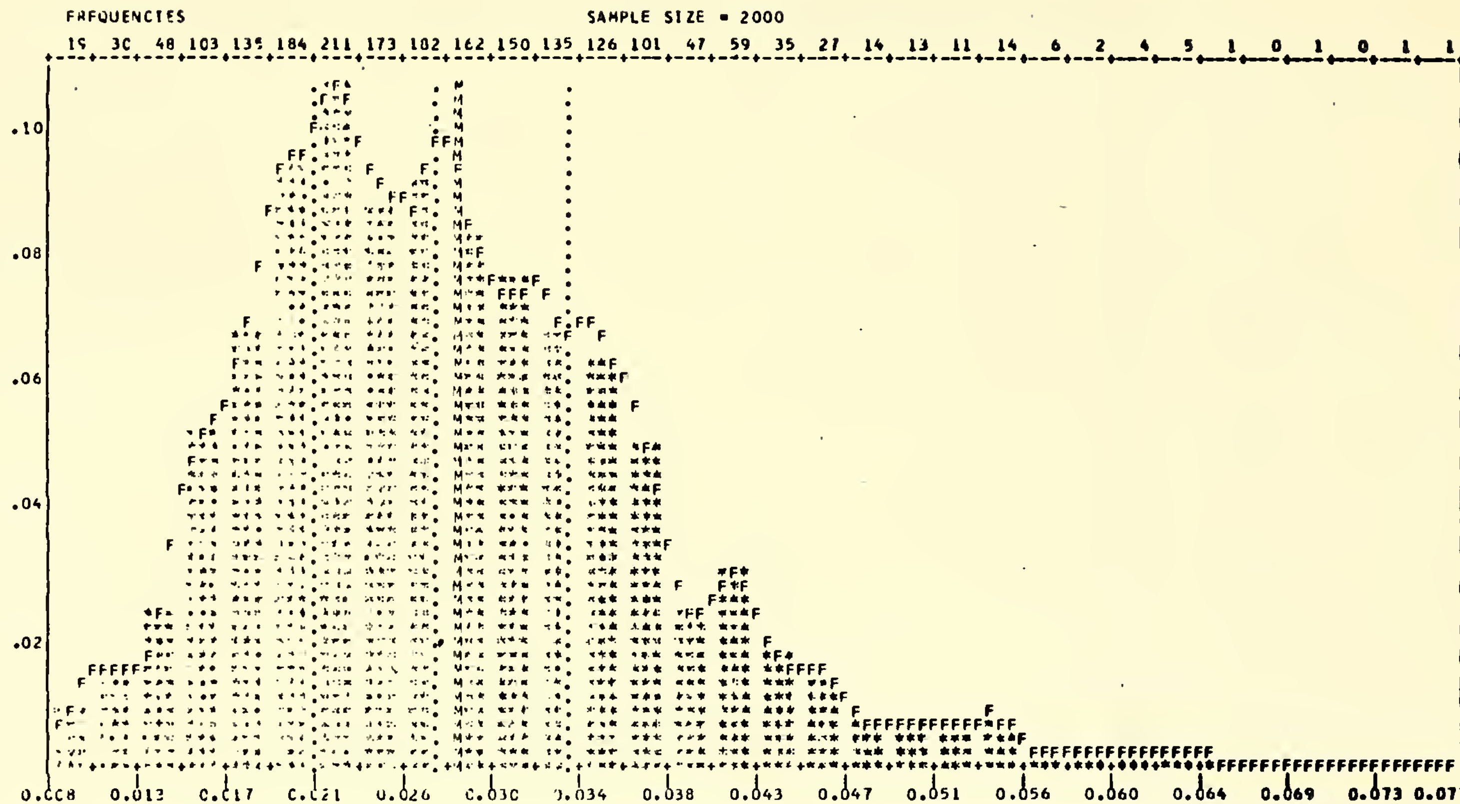
ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "-"= 0.149E-02 UNITS
Y-SCALE: "+"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

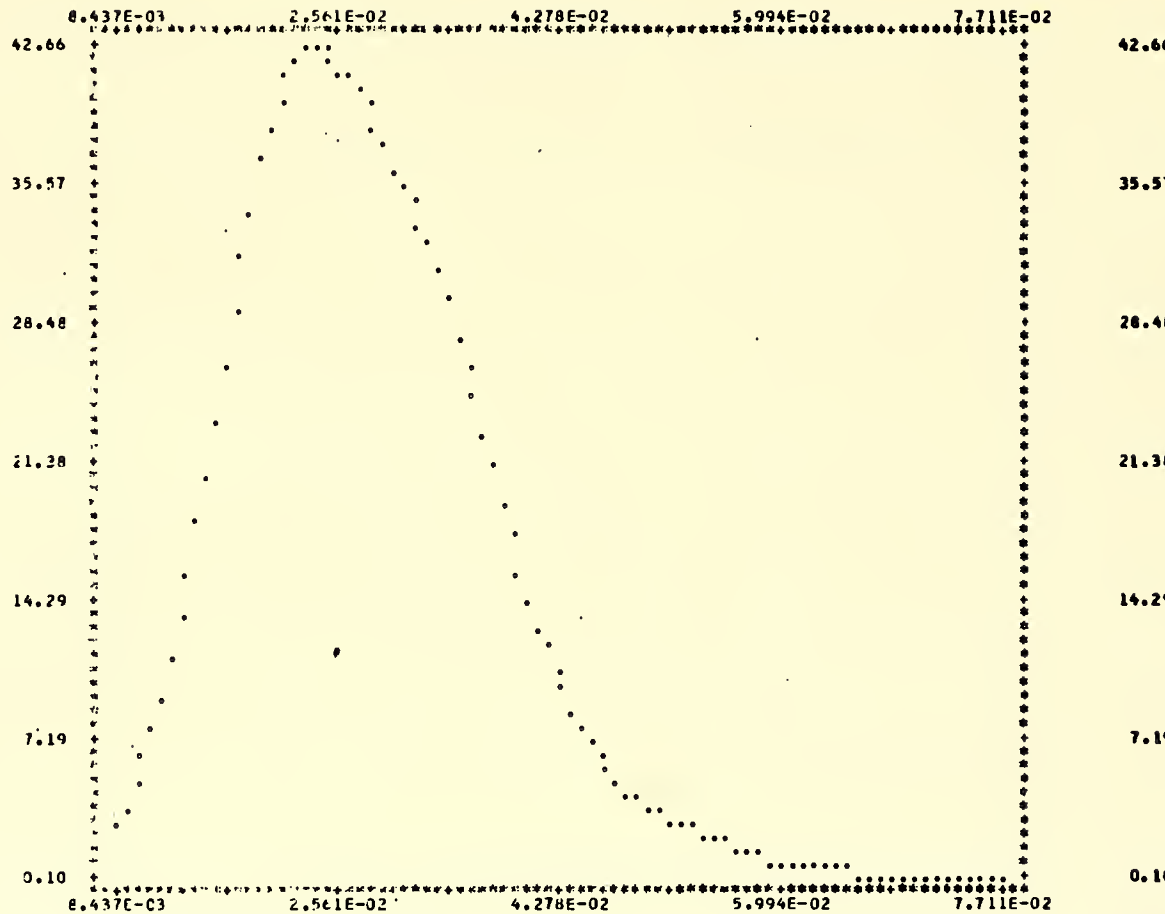


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION		
MEAN	2.793635E-02	VARIANCE	9.039621E-05	M3	7.264879E-07	MINIMUM		8.436870E-03
MEDIAN	2.684856E-02	STD DEV	9.507693E-03	M4	3.495943E-08	.10 QUANTILE		1.699141E-02
TRIMEAN	2.711935E-02	COEF VAR	3.403341E-01	SKEWNESS	8.452845E-01	.25 QUANTILE	(HINGE)	2.109230E-02
MIDMEAN	2.692206E-02	RANGE	6.867737E-02	KURTOSIS	1.278229E 00	.50 QUANTILE	(MEDIAN)	2.684856E-02
GEOM MEAN	2.638144E-02	MIDSPREAD	1.259571E-02			.75 QUANTILE	(HINGE)	3.368801E-02
HARM MEAN	2.483878E-02					.90 QUANTILE		4.011466E-02
						MAXIMUM		7.711428E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 500

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)



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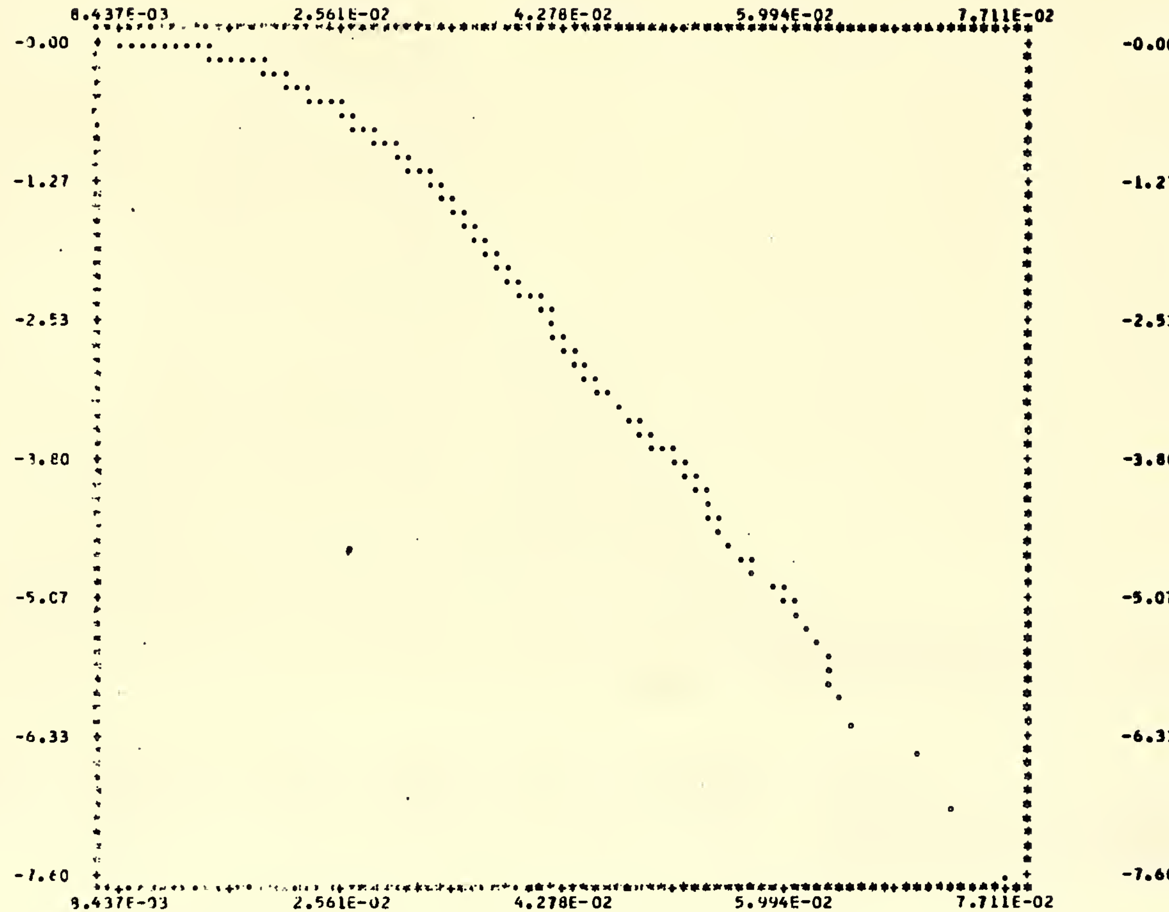
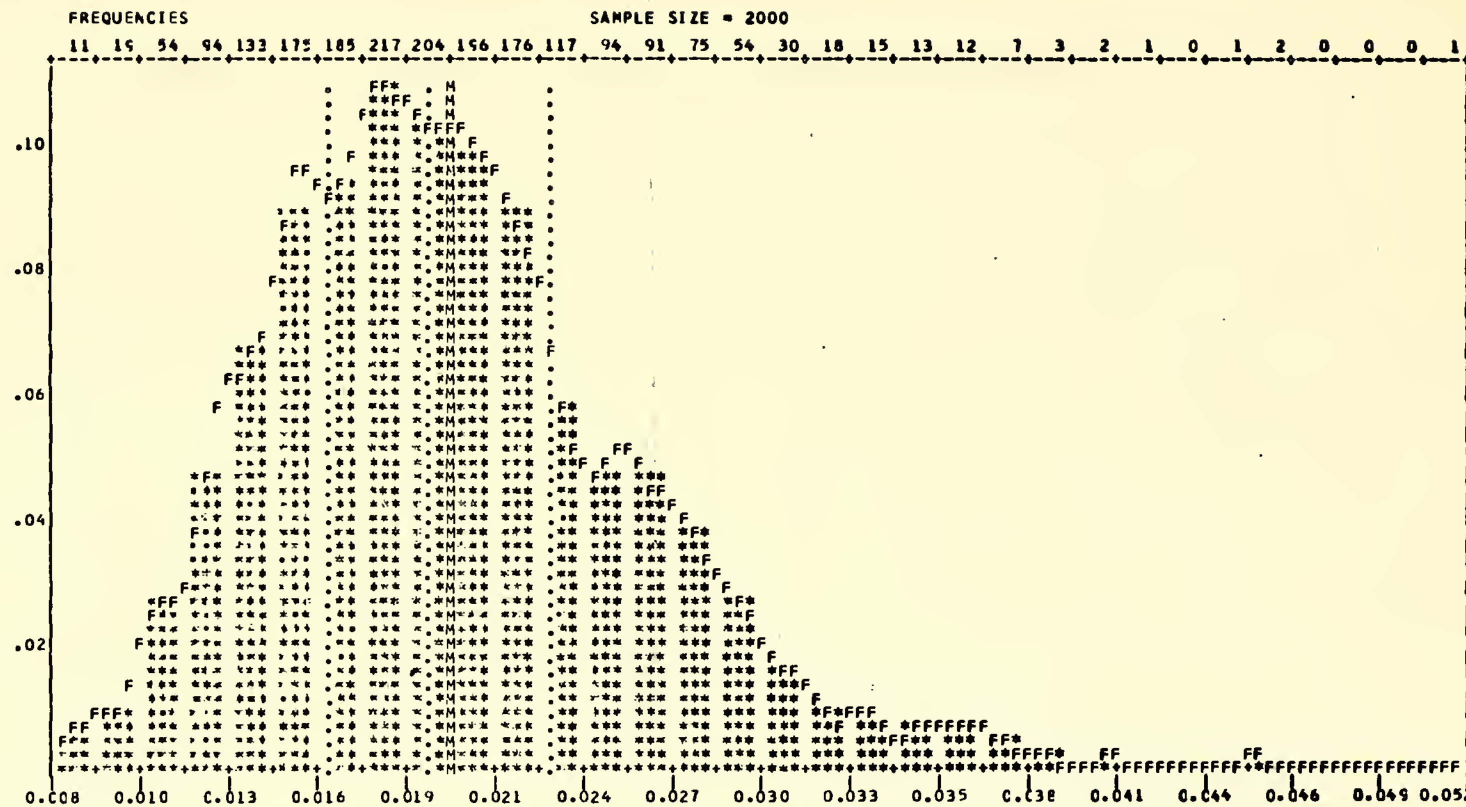
ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000   TRIANGULAR WINDOW   BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.858E-03 UNITS
Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

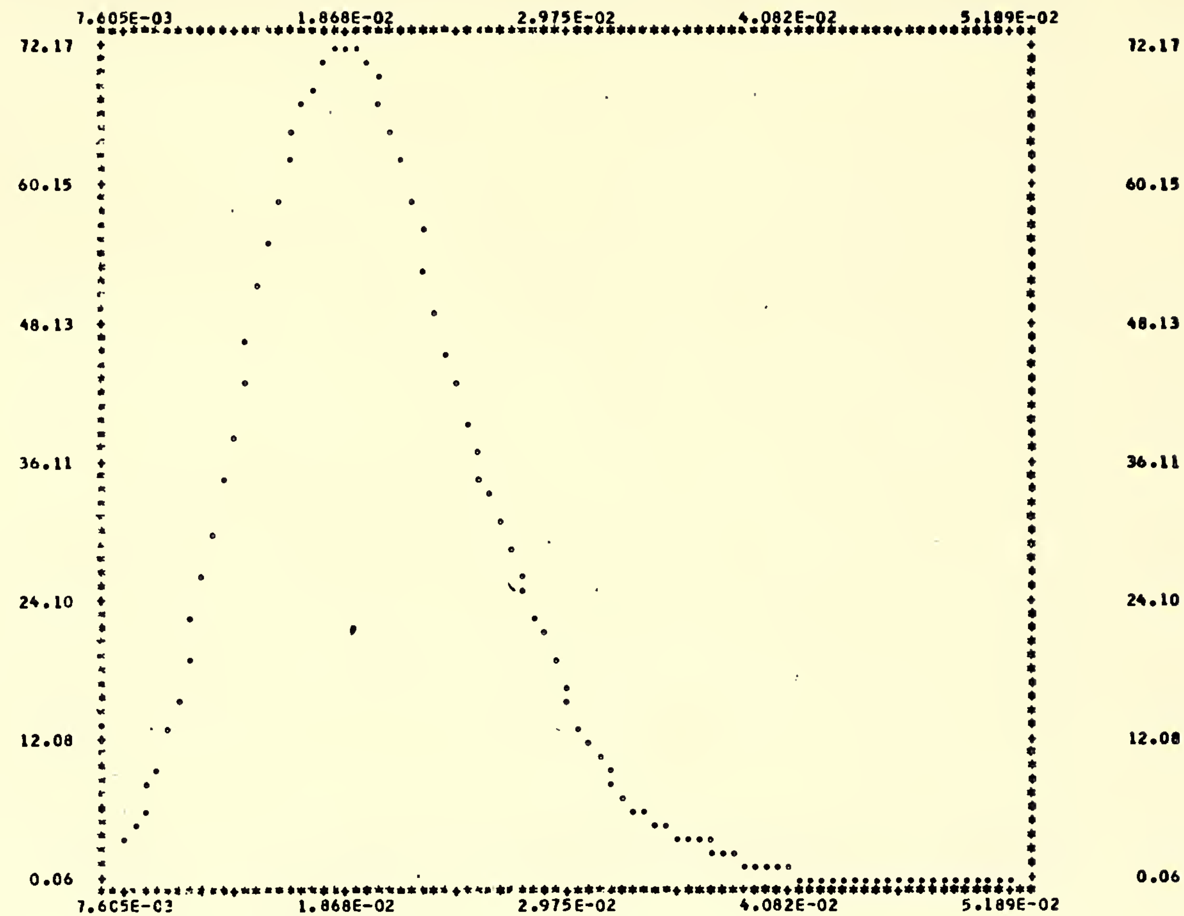


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	2.005229E-02	VARIANCE	3.188253E-05	M3	1.377122E-07	MINIMUM	7.605173E-03
MEDIAN	1.936814E-02	STD DEV	5.646463E-03	M4	4.195844E-09	.10 QUANTILE	1.335985E-02
TRIMEAN	1.946746E-02	COEF VAR	2.815869E-01	SKEWNESS	7.649671E-01	.25 QUANTILE (HINGE)	1.600368E-02
MIDMEAN	1.947337E-02	RANGE	4.428685E-02	KURTOSIS	1.127752E 00	.50 QUANTILE (MEDIAN)	1.936814E-02
GEOM MEAN	1.928757E-02	MIDSPREAD	7.126249E-03			.75 QUANTILE (HINGE)	2.312993E-02
HARM MEAN	1.853752E-02					.90 QUANTILE	2.750290E-02
						MAXIMUM	5.189202E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 1000

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)



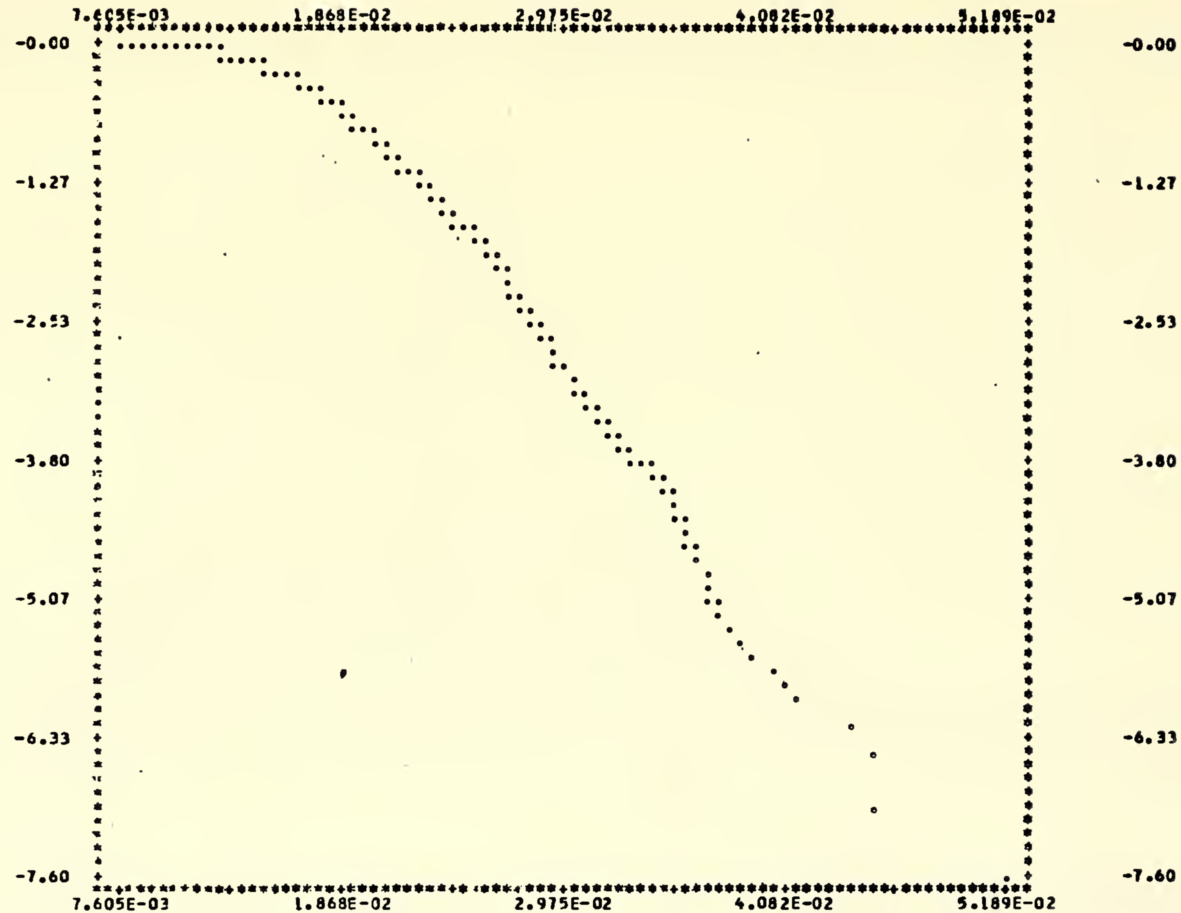
X-SCALE: "="= 0.554E-03 UNITS

Y-SCALE: "*"= C.120E 01 UNITS

ESTIMATEO CENSITY FUNCTION VS INTEGRAL SQUARE NCRM RANCOM VARIABLE

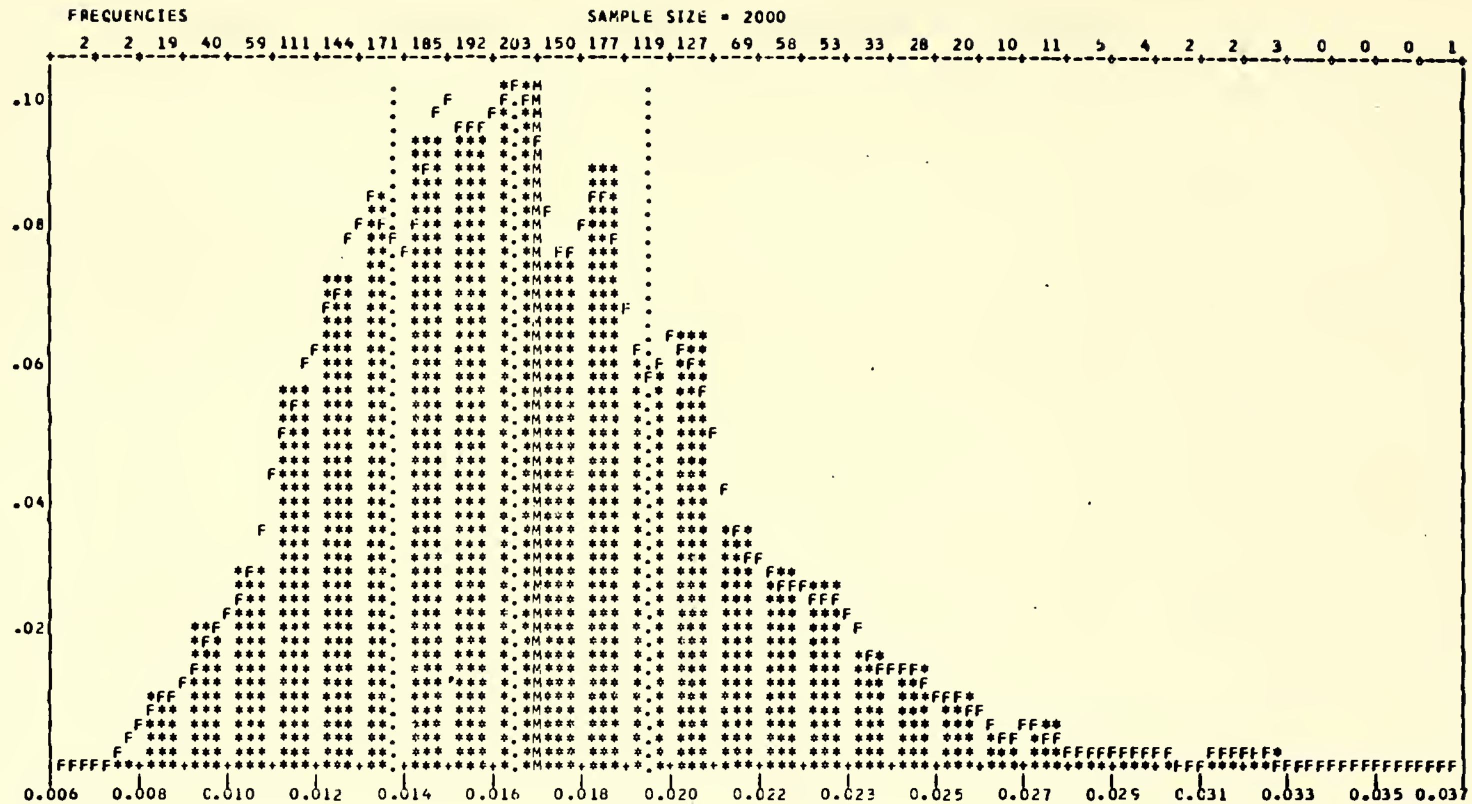
INTEGRAL SCUARE NCRM SAMPLE SIZE = 2000 TRIANGULAR WINOOW BANOWIOTH = RANGE / 10

ESTIMATED DENSITY FUNCTICN IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: *** = C.554E-03 UNITS
Y-SCALE: *** = 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

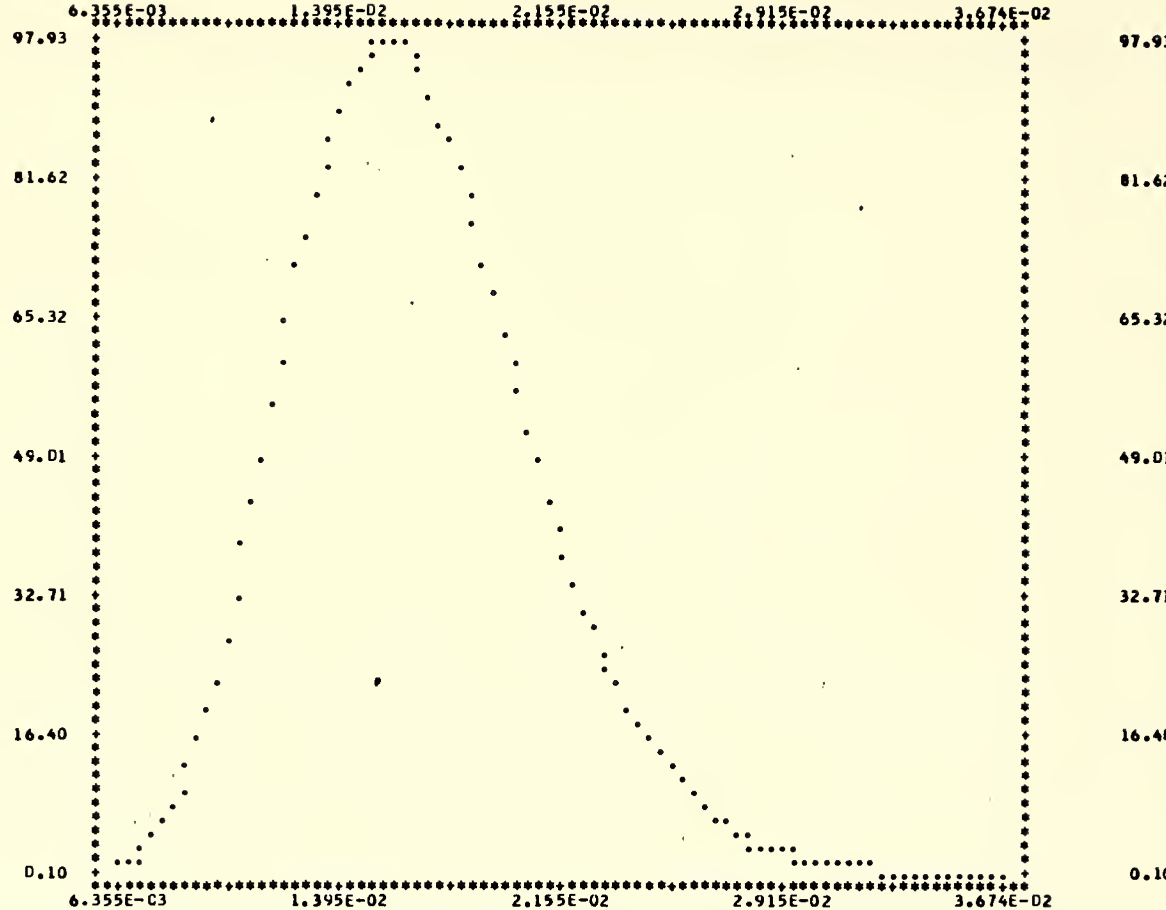


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.662469E-02	VARIANCE	1.633688E-05	M3	4.219656E-08	MINIMUM	6.355230E-03
MEDIAN	1.624629E-02	STD DEV	4.041892E-03	M4	9.751644E-10	.10 QUANTILE	1.178314E-02
TRIMEAN	1.630606E-02	COEF VAR	2.431259E-01	SKEWNESS	6.390331E-01	.25 QUANTILE (HINGE)	1.369802E-02
MIDMEAN	1.627684E-02	RANGE	3.038834E-02	KURTOSIS	6.537561E-01	.50 QUANTILE (MEDIAN)	1.624629E-02
GEOM MEAN	1.614773E-02	MIDSPREAD	5.335666E-03			.75 QUANTILE (HINGE)	1.903368E-02
HARM MEAN	1.567826E-02					.90 QUANTILE	2.196957E-02
						MAXIMUM	3.674357E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 1500

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)



X-SCALE: "*"= 0.380E-03 UNITS
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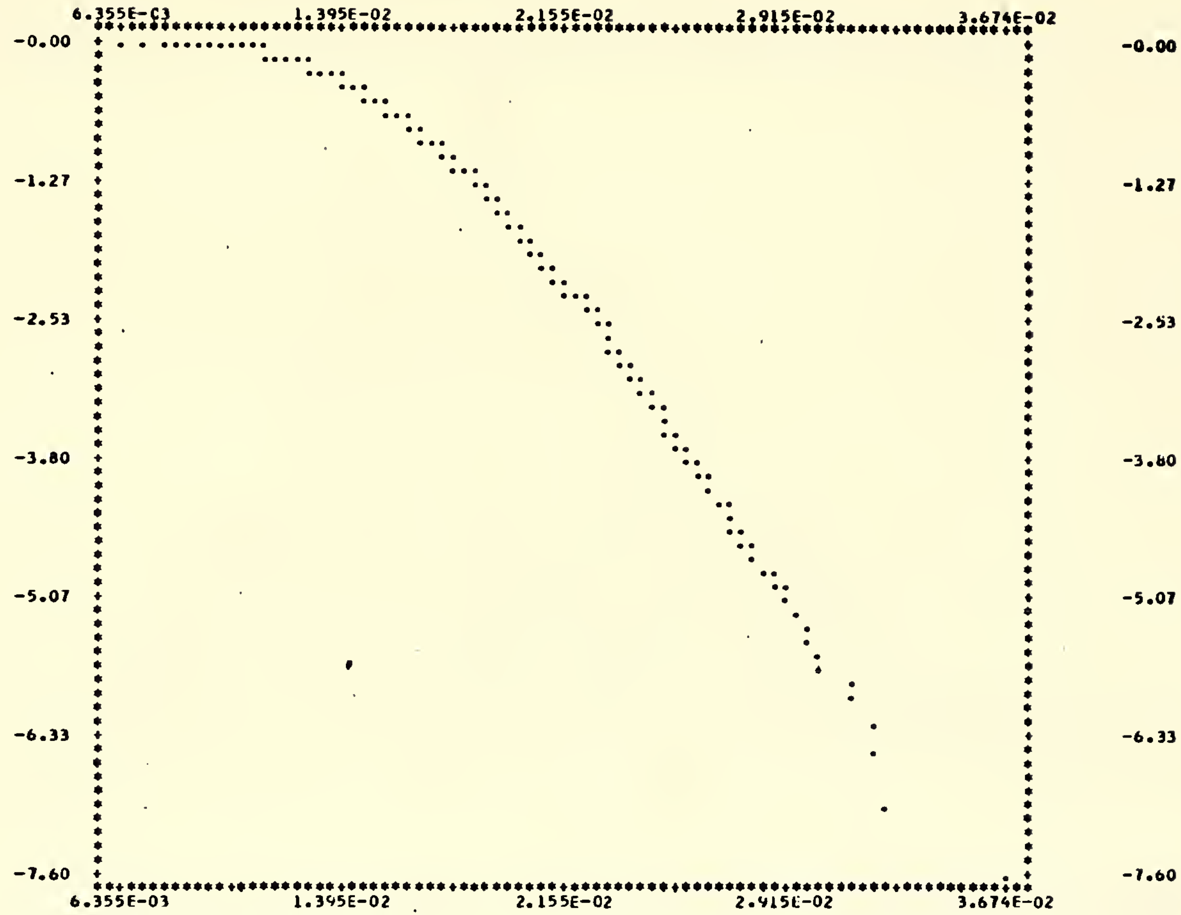
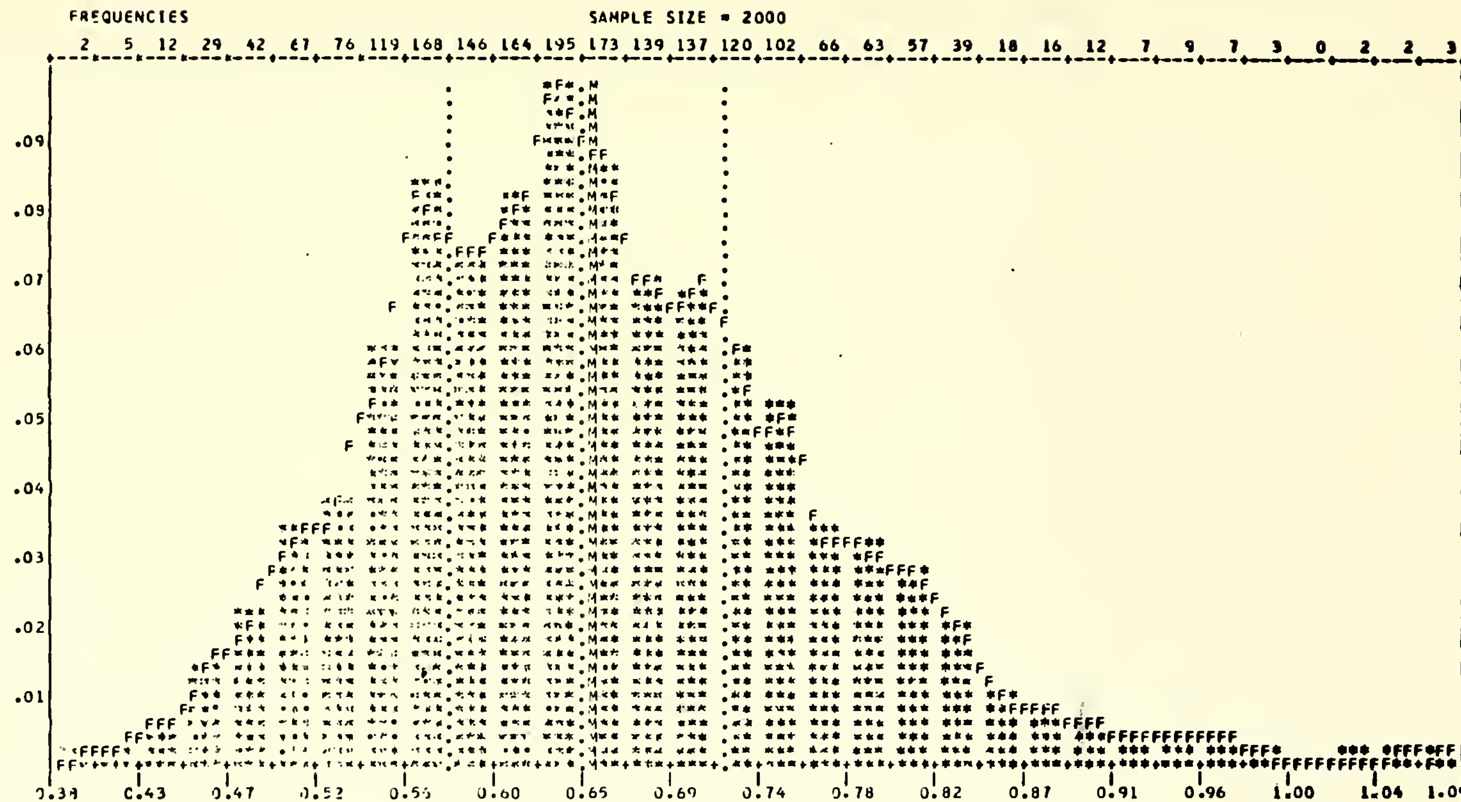
ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.380E-03 UNITS

Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

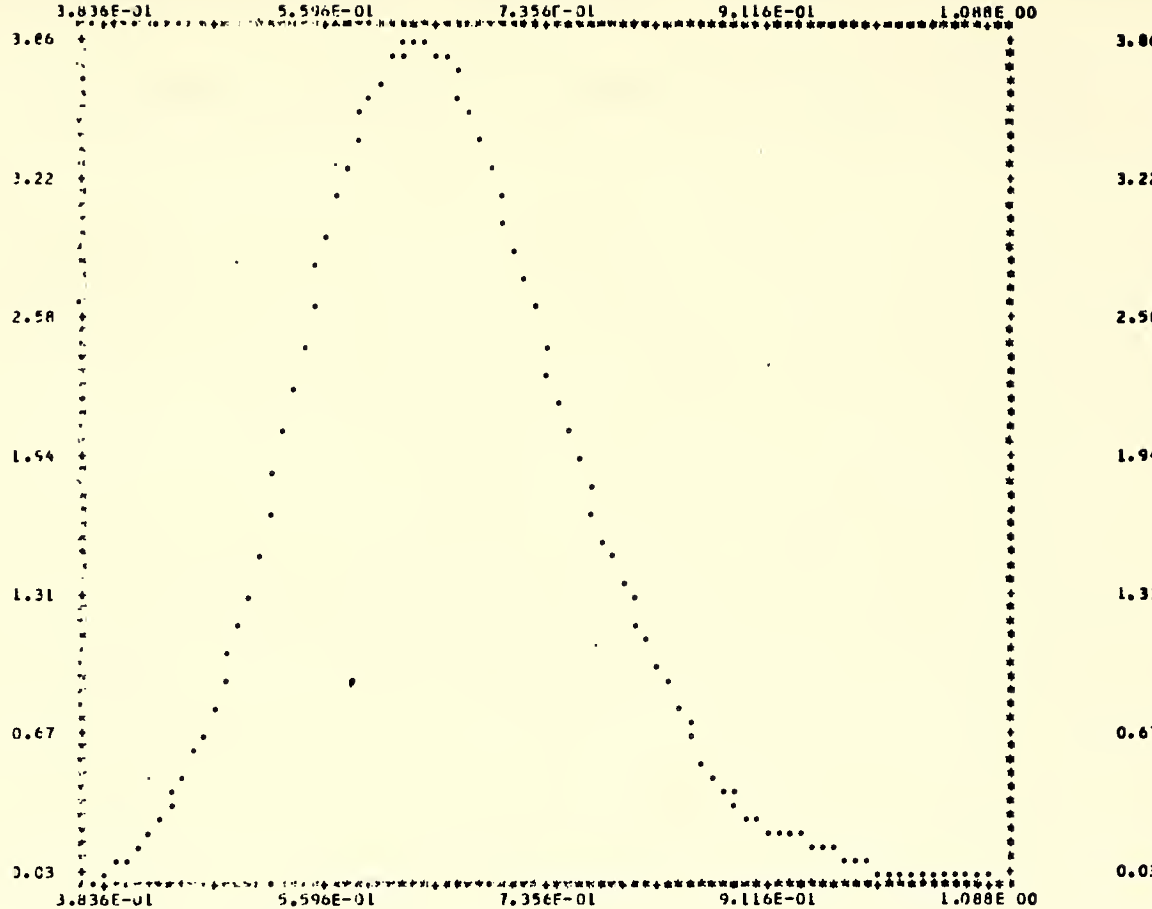


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	6.535984E-01	VARIANCE	1.033463E-02	M3	6.163854E-04	MINIMUM	3.836240E-01
MEDIAN	6.444693E-01	STD DEV	1.040895E-01	M4	4.209706E-04	.10 QUANTILE	5.301855E-01
TRIMEAN	6.464355E-01	COEF VAR	1.591586E-01	SKEWNESS	5.465521E-01	.25 QUANTILE (HINGE)	5.792165E-01
MIDMEAN	6.465046E-01	RANGE	7.039396E-01	KURTOSIS	5.861120E-01	.50 QUANTILE (MEDIAN)	6.444693E-01
GEOM MEAN	6.459207E-01	MIDSPREAD	1.383710E-01			.75 QUANTILE (HINGE)	7.175875E-01
HARM MEAN	6.379856E-01					.90 QUANTILE	7.926930E-01
						MAXIMUM	1.087564E 00

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 100

TRIANGULAR WINDOW. BANDWIDTH = 1/N



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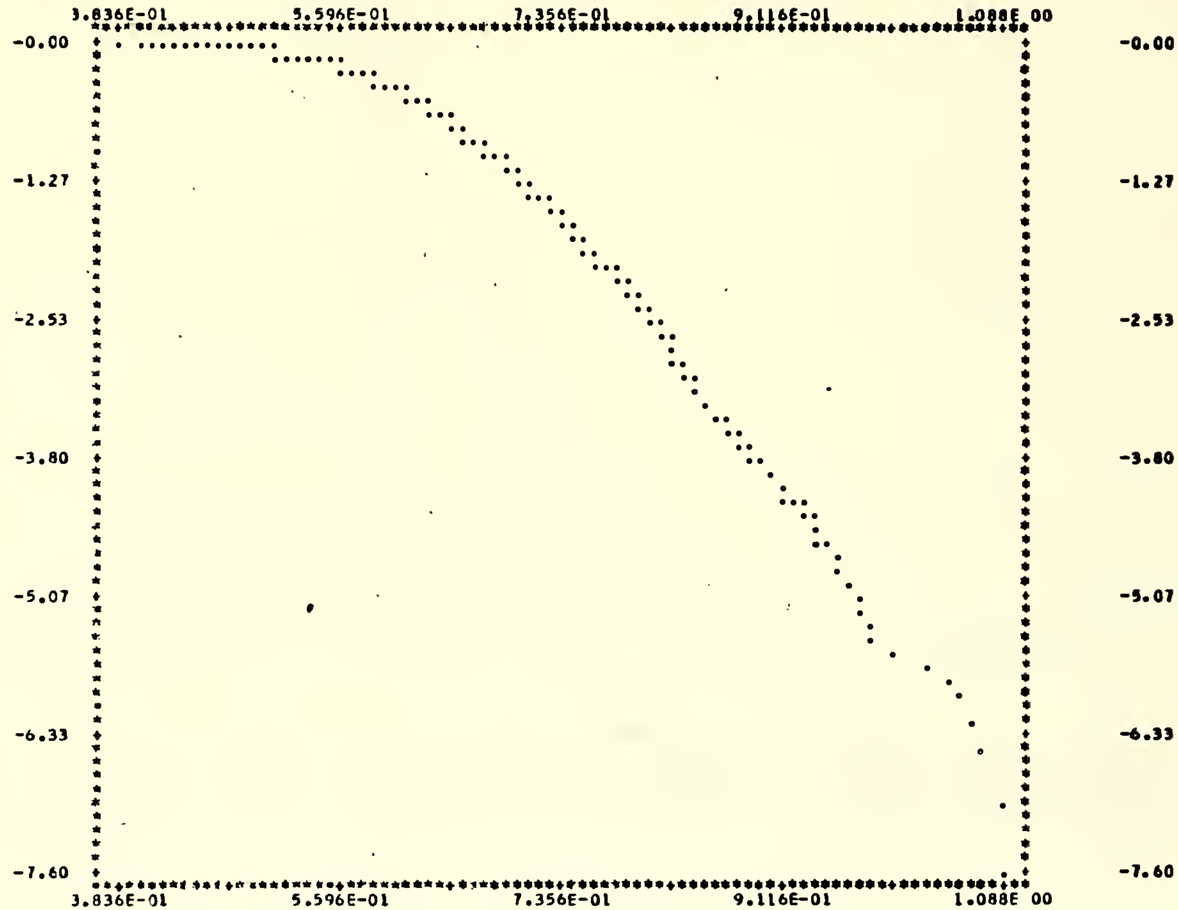
ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE

INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10

ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*". 0.880E-02 UNITS
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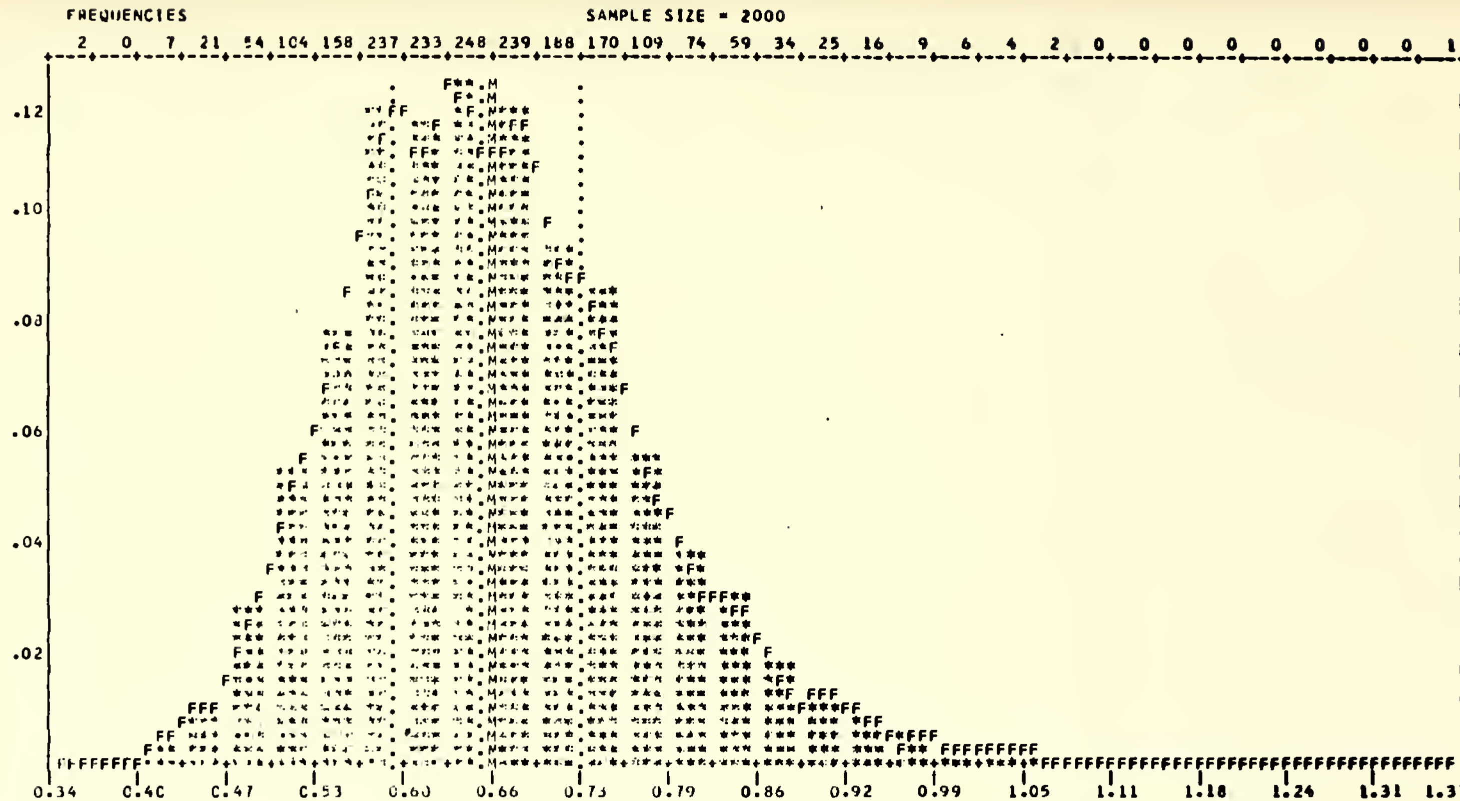
LN(1-I/N+1) VS X   EMPIRICAL LOG SURVIVAL FUNCTION

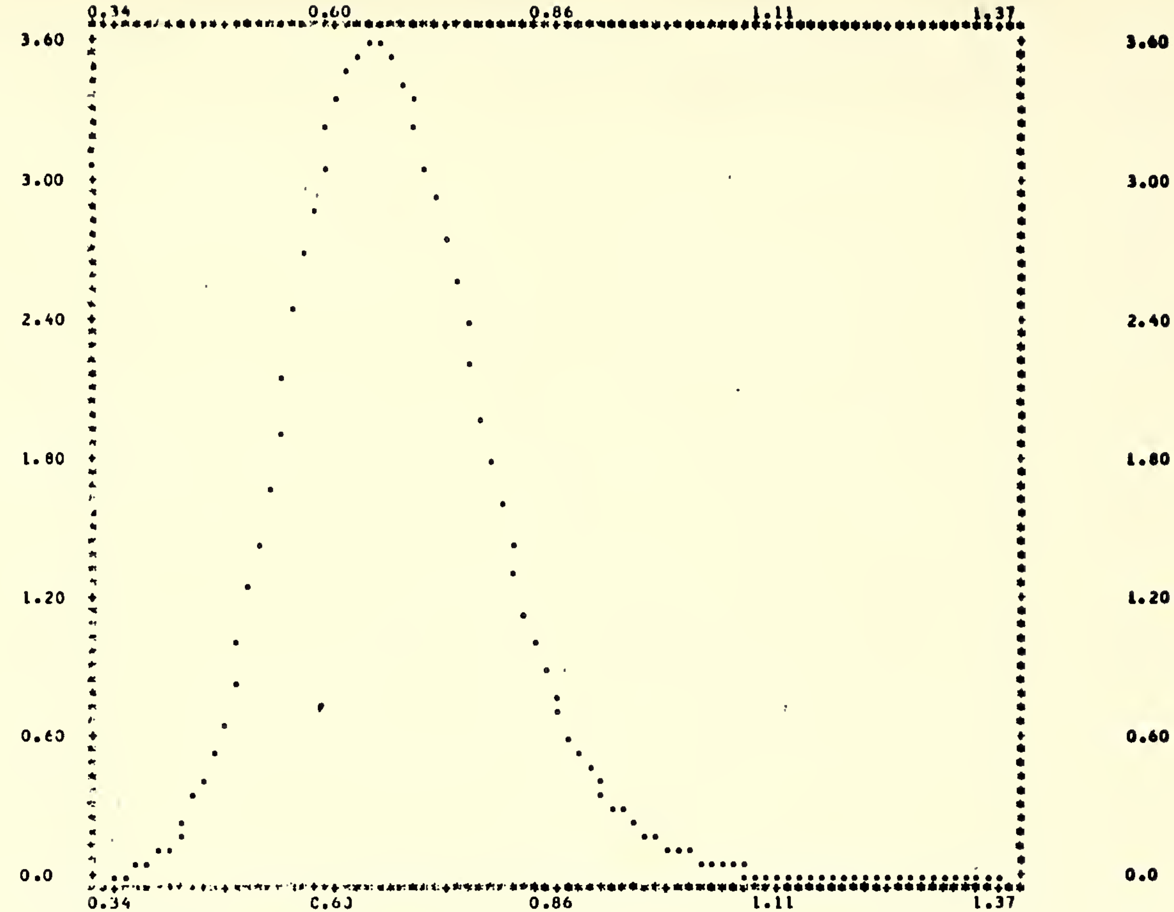


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	6.638315E-01	VARIANCE	1.145839E-02	M3	7.838549E-04	MINIMUM	3.401374E-01
MEDIAN	6.535550E-01	STD DEV	1.070439E-01	M4	5.552873E-04	.10 QUANTILE	5.374022E-01
TRIMEAN	6.559918E-01	COEF VAR	1.612515E-01	SKEWNESS	6.390730E-01	.25 QUANTILE (HINGE)	5.886273E-01
MIDMEAN	6.553867E-01	RANGE	1.032682E 00	KURTOSIS	1.229318E 00	.50 QUANTILE (MEDIAN)	6.535950E-01
GEOM MEAN	6.554613E-01	MIDSPREAD	1.395230E-01			.75 QUANTILE (HINGE)	7.281504E-01
HARM MEAN	6.472558E-01					.90 QUANTILE	8.010432E-01
						MAXIMUM	1.372820E 00

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 200

TRIANGULAR WINDOW. BANDWIDTH = 1/N



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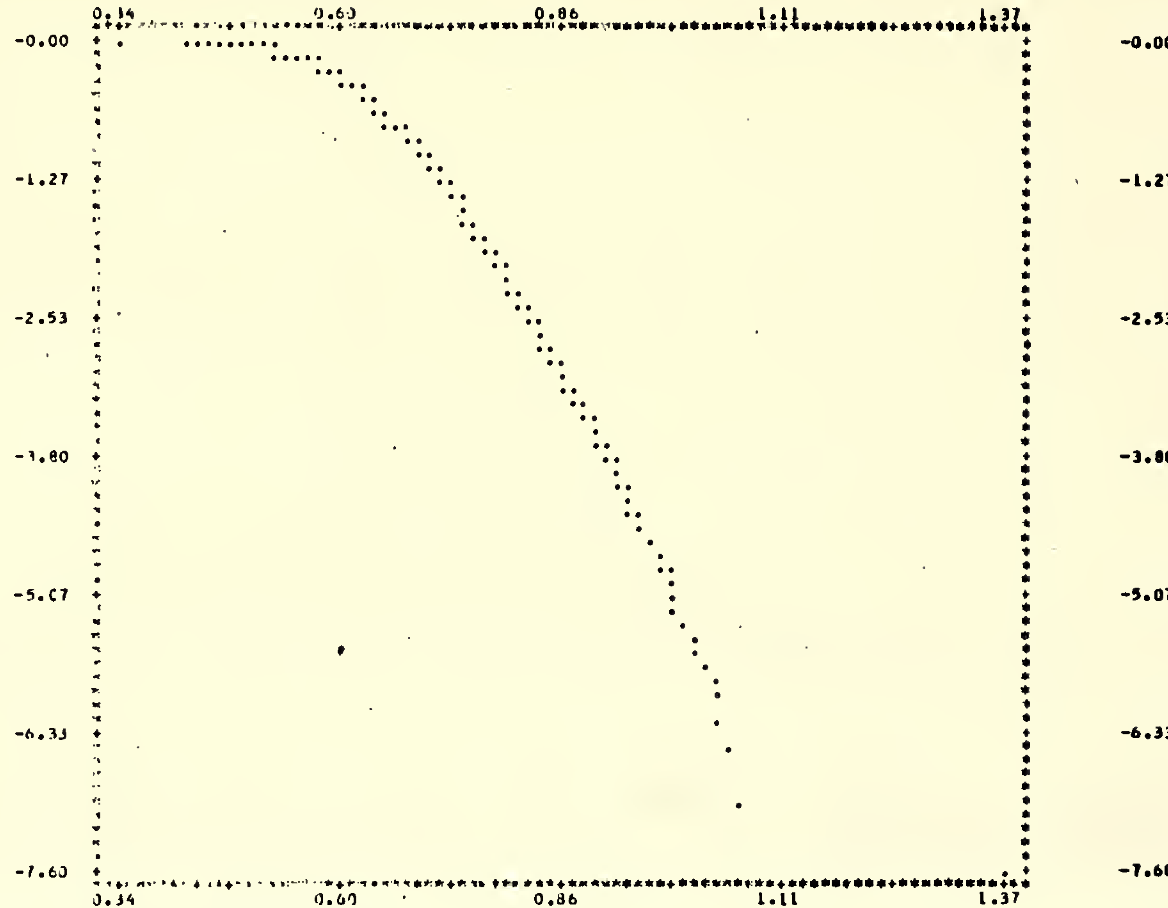
Y-SCALE: "*" = 0.600E-01 UNITS

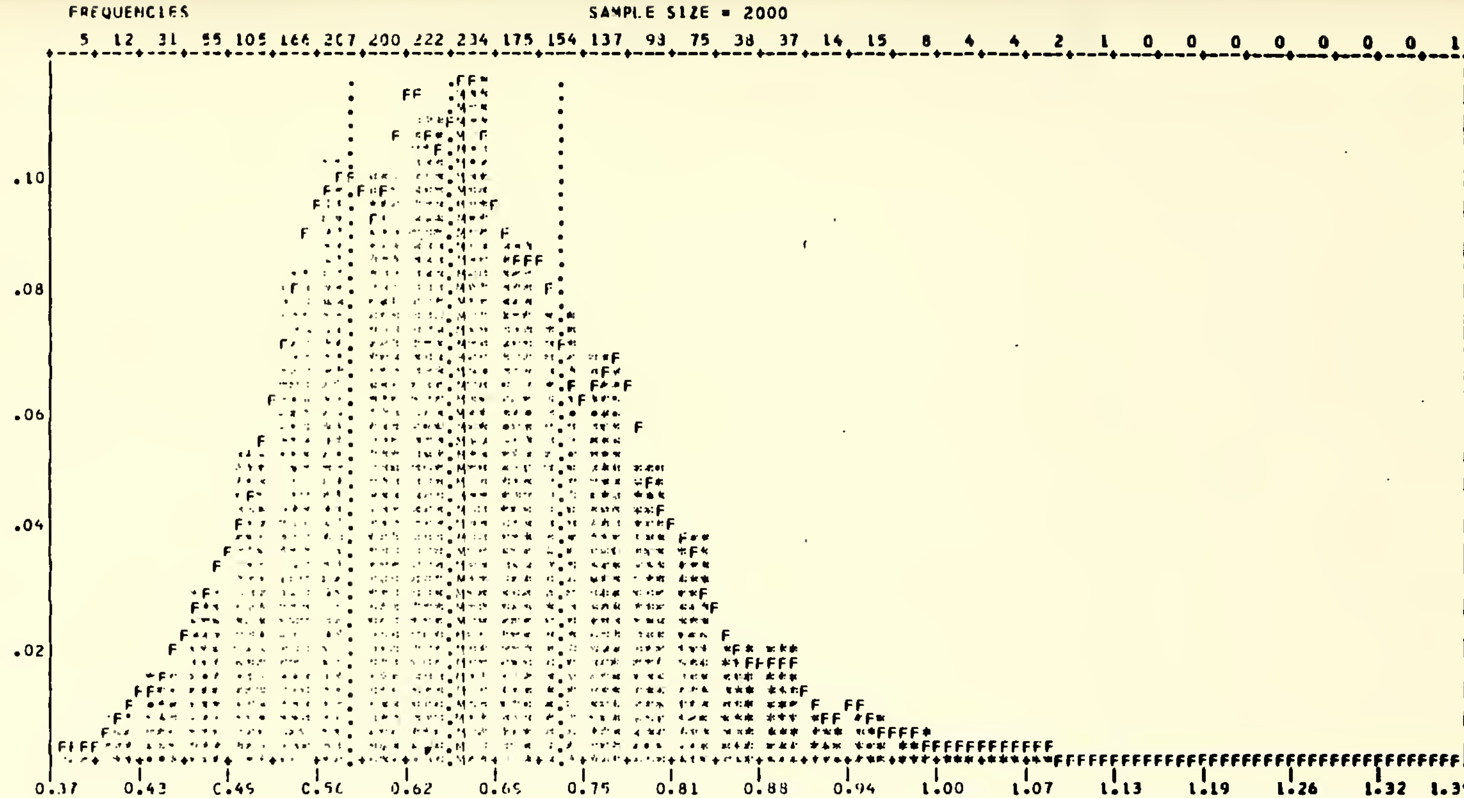
ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "." = 0.129E-01 UNITS

Y-SCALE: "." = 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

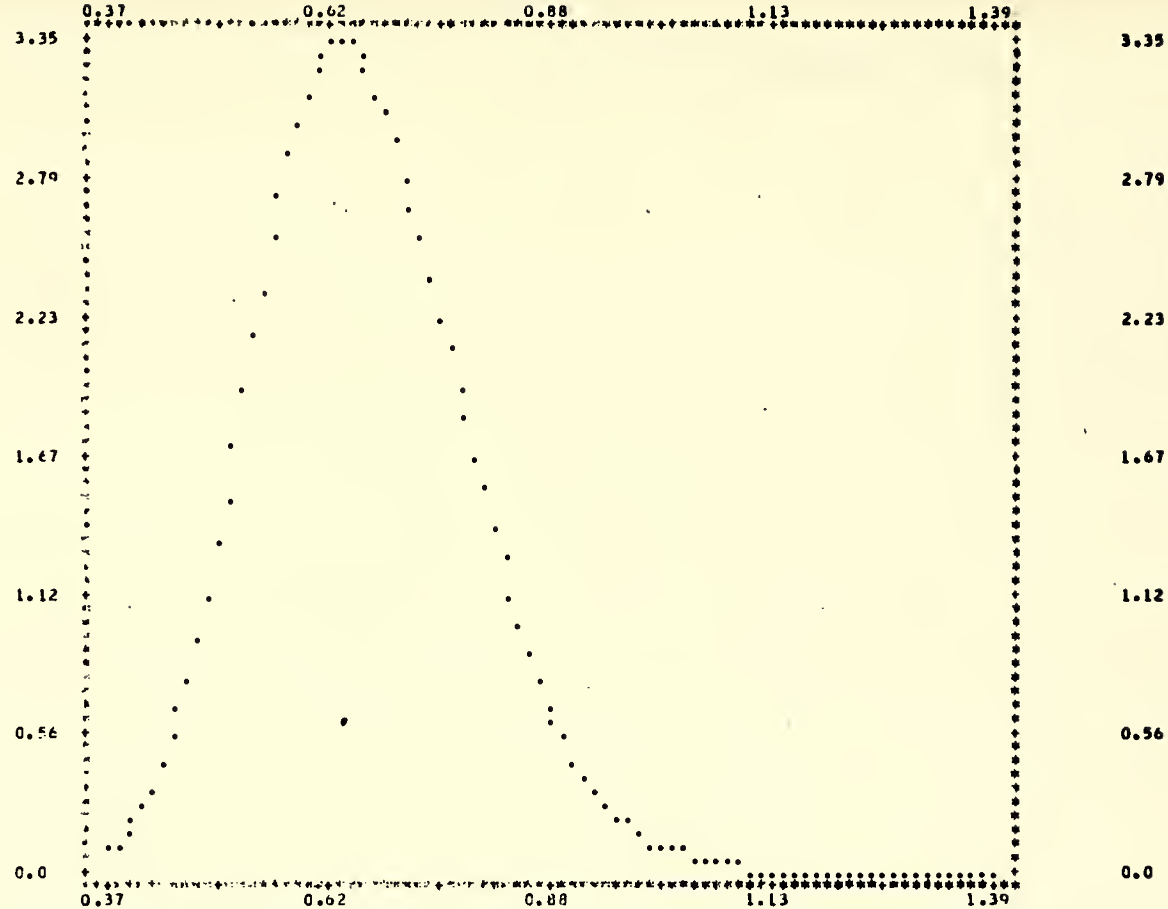


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	6.611907E-01	VARIANCE	1.326912E-02	M3	8.754442E-04	MINIMUM	3.669456E-01
MEDIAN	6.525854E-01	STD DEV	1.151910E-01	M4	6.908285E-04	.10 QUANTILE	5.241610E-01
TRIMEAN	6.538870E-01	COEF VAR	1.742184E-01	SKEWNESS	5.727507E-01	.25 QUANTILE (HINGE)	5.778513E-01
MIDMEAN	6.526204E-01	RANGE	1.018871E 00	KURTOSIS	9.236116E-01	.50 QUANTILE (MEDIAN)	6.525854E-01
GEOM MEAN	6.514053E-01	MIDSPREAD	1.538668E-01			.75 QUANTILE (HINGE)	7.317181E-01
HARM MEAN	6.417704E-01					.90 QUANTILE	8.125093E-01
						MAXIMUM	1.385818E 00

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 500

TRIANGULAR WINDOW. BANDWIDTH = 1/N



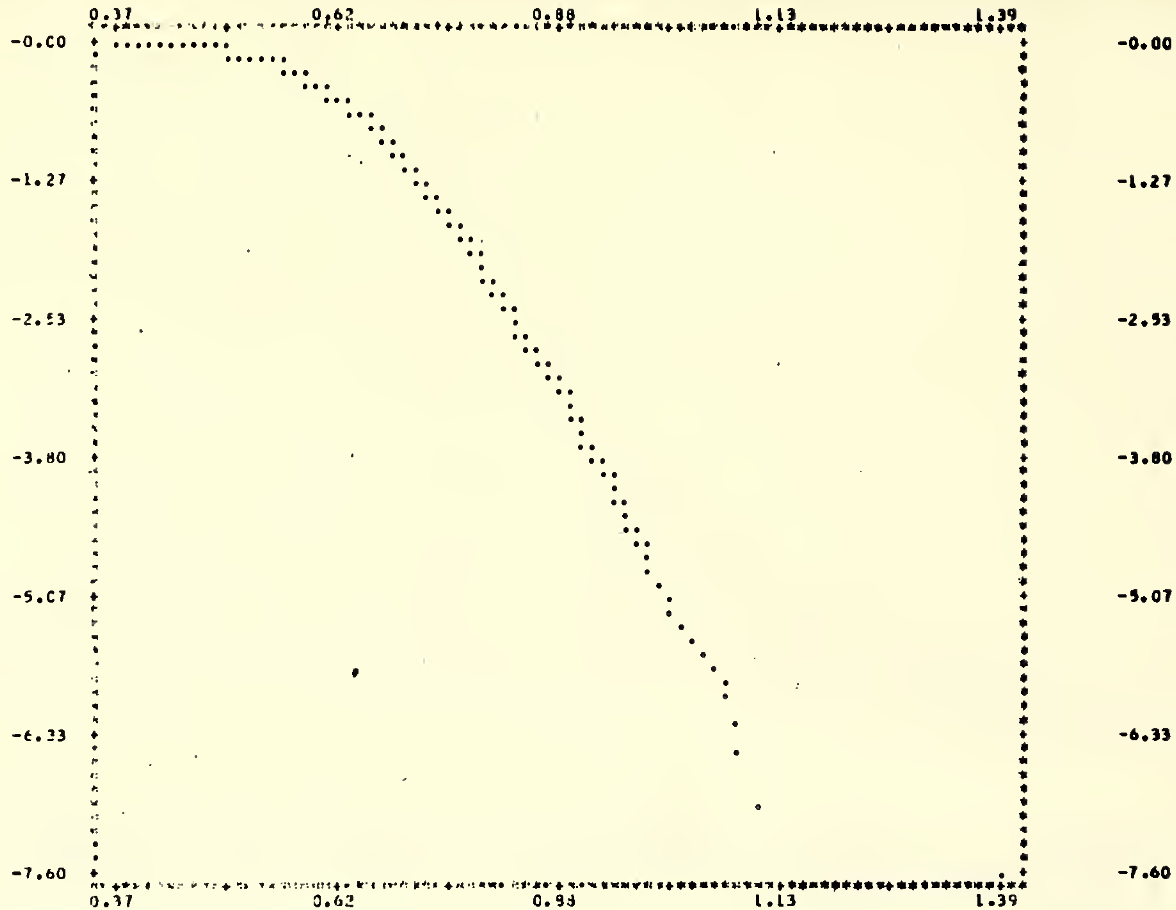
X-SCALE: "*"= 0.127E-01 UNITS

Y-SCALE: "*"= 0.558E-01 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE

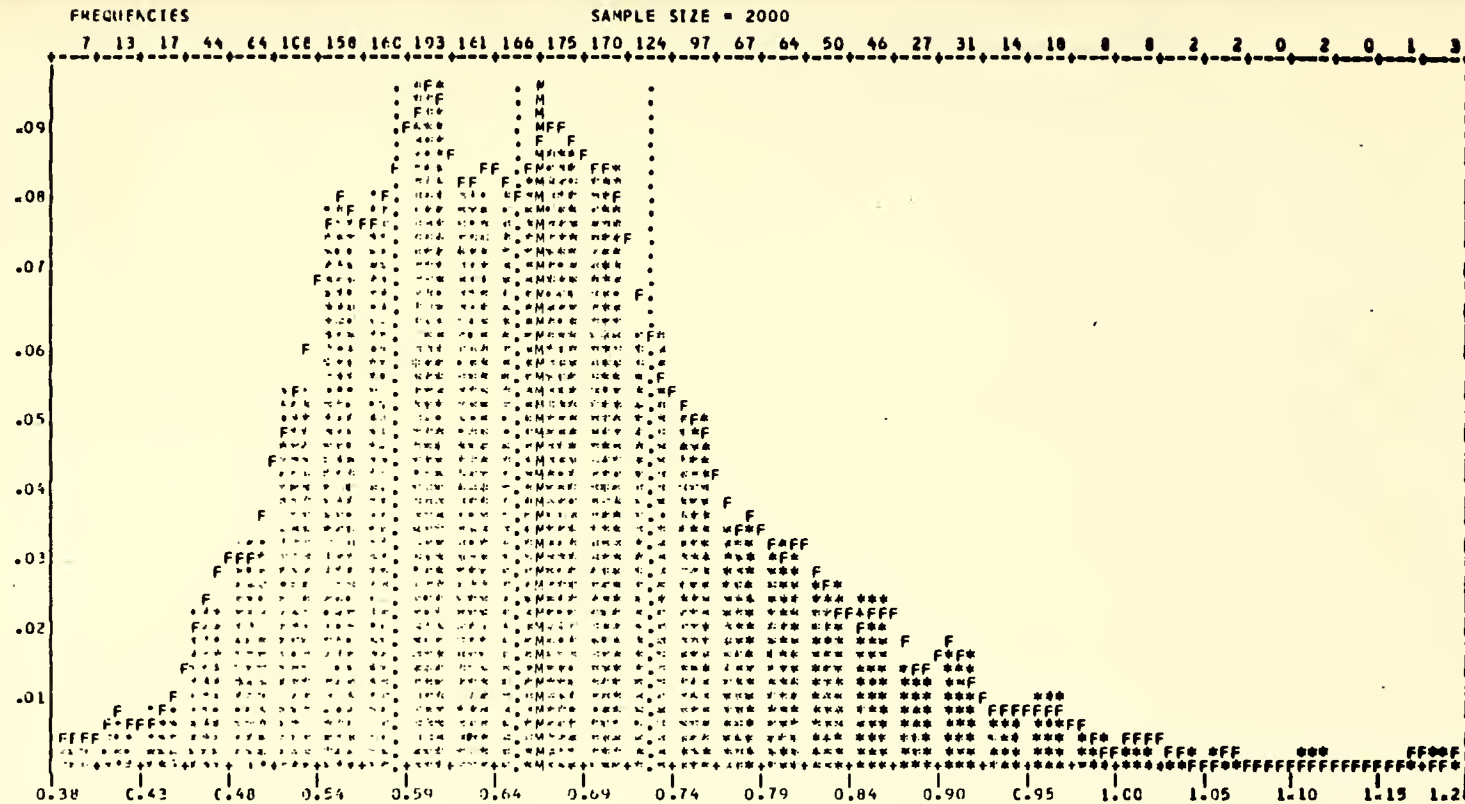
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000   TRIANGULAR WINDOW   BANDWIDTH = RANGE / 10

ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: = 0.127E-01 UNITS
Y-SCALE: = 0.127E 00 UNITS

LN(1-I/(N+1)) VS X EMPIRICAL LOG SURVIVAL FUNCTION

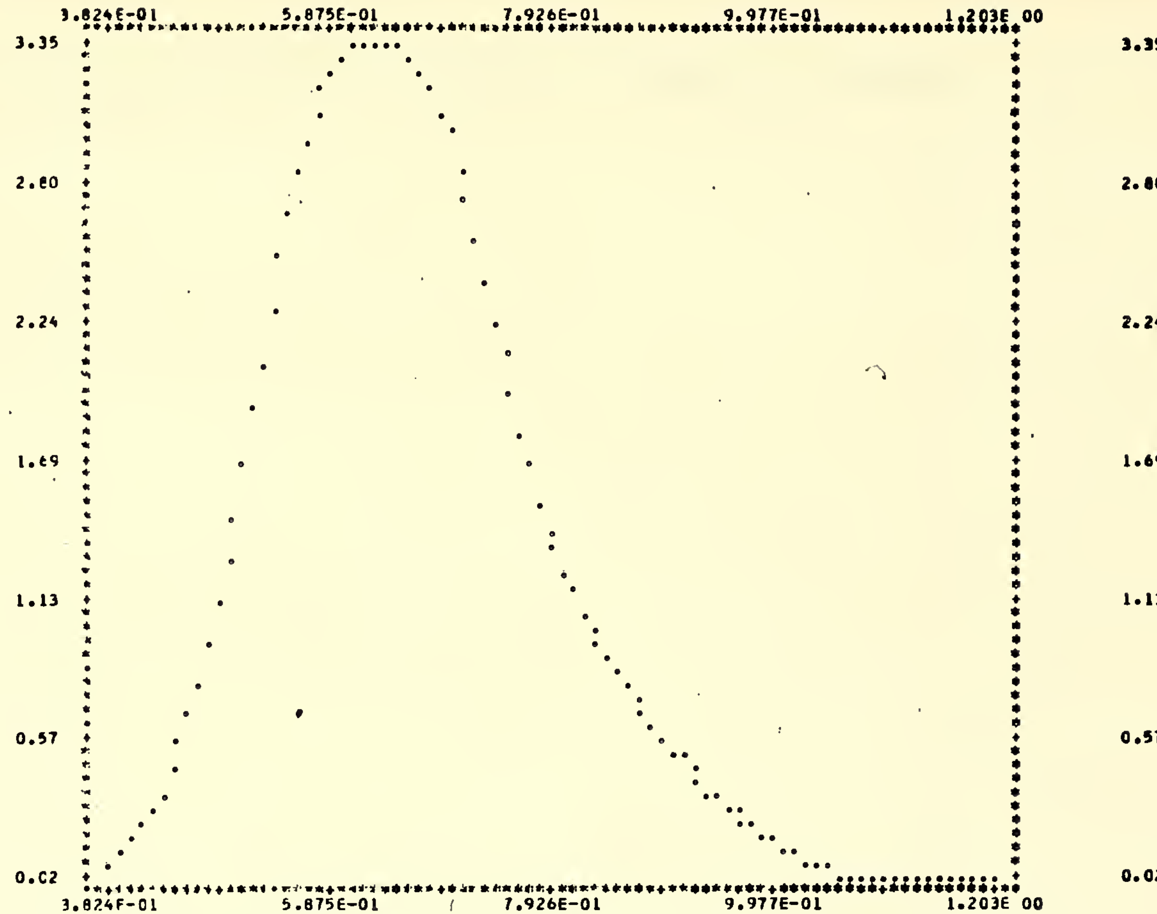


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	6.620630E-01	VARIANCE	1.420467E-02	M3	1.179938E-03	MINIMUM	3.823655E-01
MEDIAN	6.505351E-01	STD DEV	1.191834E-01	M4	7.649632E-04	.10 QUANTILE	5.245500E-01
TRIMEAN	6.510754E-01	COEF VAR	1.901815E-01	SKEWNESS	6.969673E-01	.25 QUANTILE (HINGE)	5.762172E-01
MIDMEAN	6.501917E-01	RANGE	8.204544E-01	KURTOSIS	7.912111E-01	.50 QUANTILE (MEDIAN)	6.505351E-01
GEOM MEAN	6.517717E-01	MIDSPREAD	1.508136E-01			.75 QUANTILE (HINGE)	7.270308E-01
HARM MEAN	6.414115E-01					.90 QUANTILE	8.218644E-01
						MAXIMUM	1.202820E 00

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 100

TRIANGULAR WINDOW, BANDWIDTH = 1/N



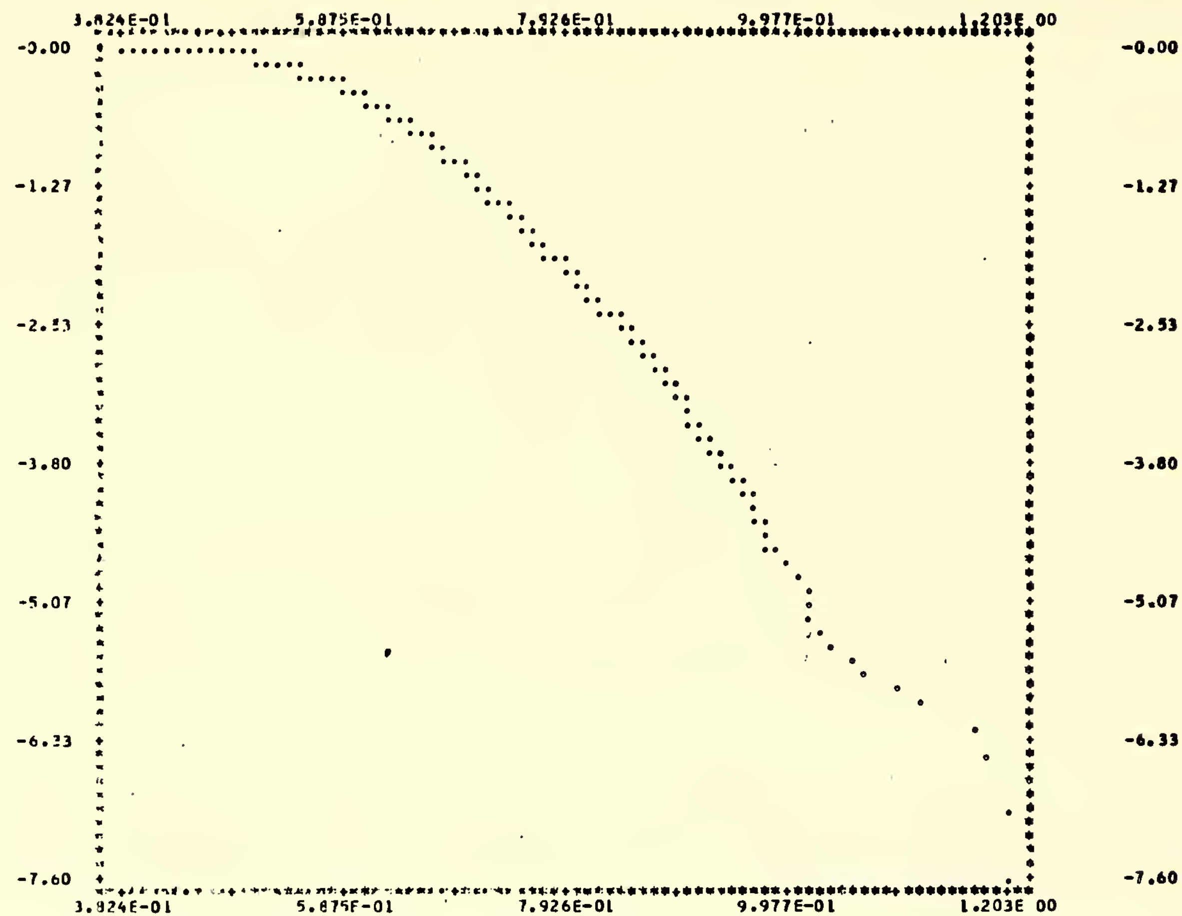
X-SCALE: "*"= 0.103E-01 UNITS

Y-SCALE: "*"= 0.556E-01 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE

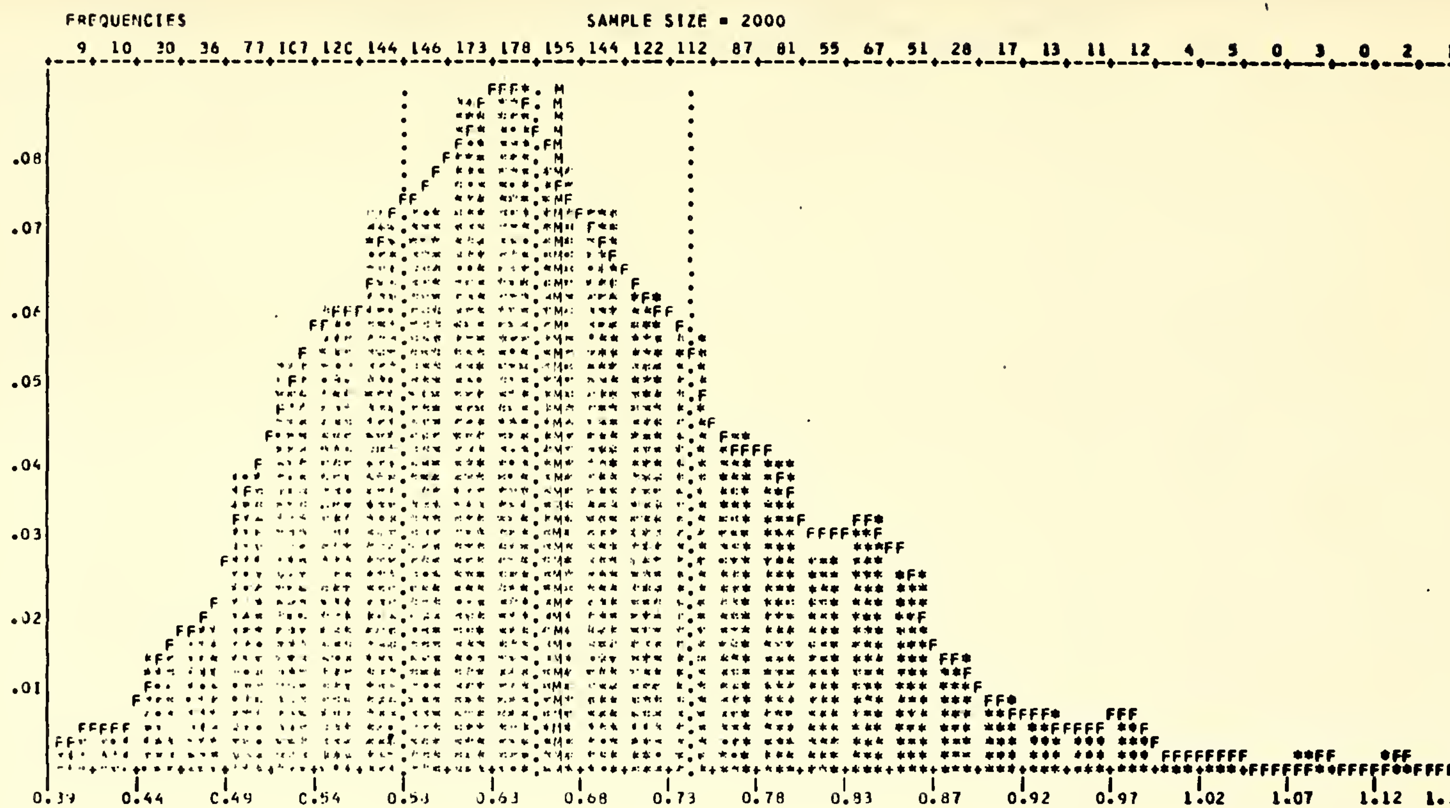
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10

ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.103E-01 UNITS
Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

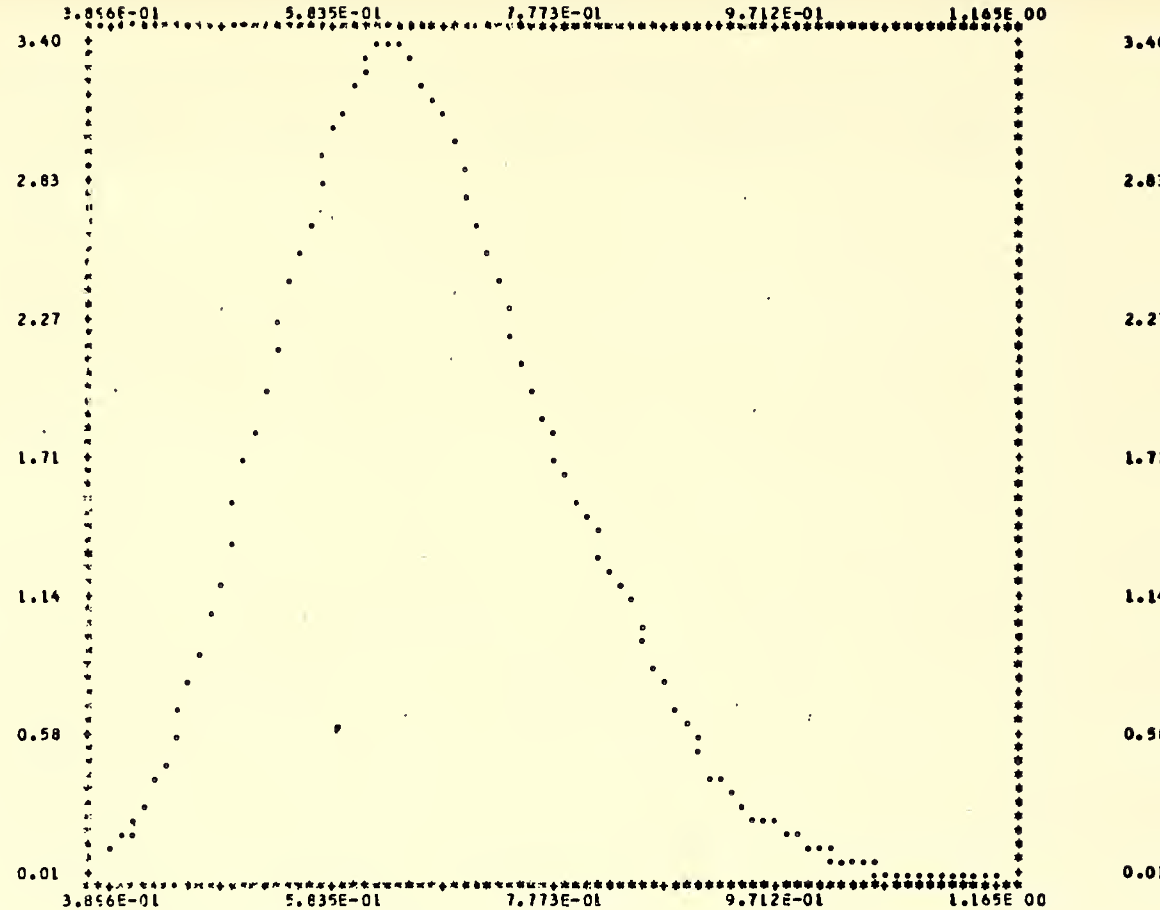


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	6.644731E-01	VARIANCE	1.421661E-02	M3	8.886582E-04	MINIMUM	3.896179E-01
MEDIAN	6.515089E-01	STD DEV	1.192334E-01	M4	6.607587E-04	.10 QUANTILE	5.196396E-01
TRIMEAN	6.551318E-01	COEF VAR	1.794406E-01	SKEWNESS	5.242528E-01	.25 QUANTILE (HINGE)	5.791879E-01
MIDMEAN	6.545200E-01	RANGE	7.754279E-01	KURTOSIS	2.692680E-01	.50 QUANTILE (MEDIAN)	6.515089E-01
GEOM MEAN	6.540395E-01	MIDSPREAD	1.583343E-01			.75 QUANTILE (HINGE)	7.375221E-01
HARM MEAN	6.436082E-01					.90 QUANTILE	8.303780E-01
						MAXIMUM	1.165046E 00

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

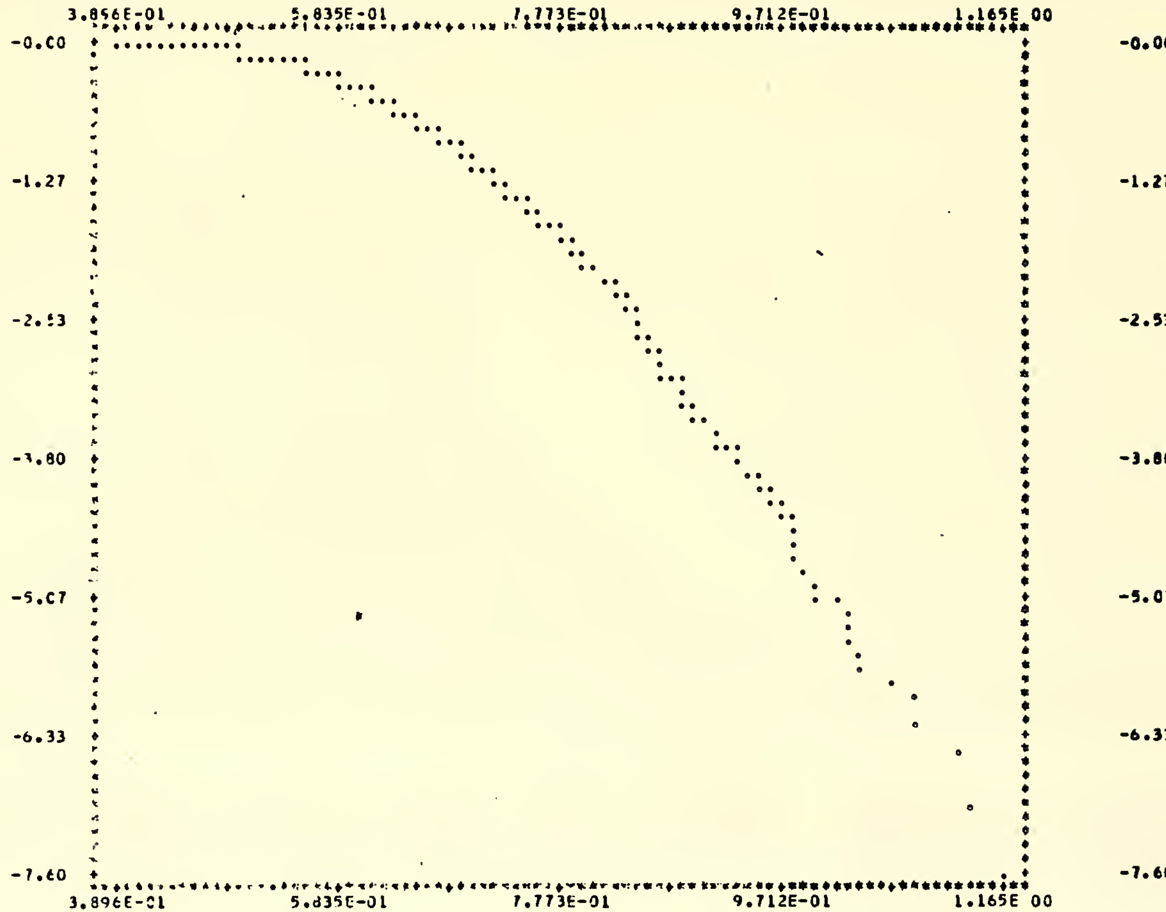
UNIFORM RANDOM VARIABLE SAMPLE SIZE N = 1500

TRIANGULAR WINDOW. BANDWIDTH = 1/N



X-SCALE: *** = 0.569E-02 UNITS
Y-SCALE: *** = 0.564E-01 UNITS

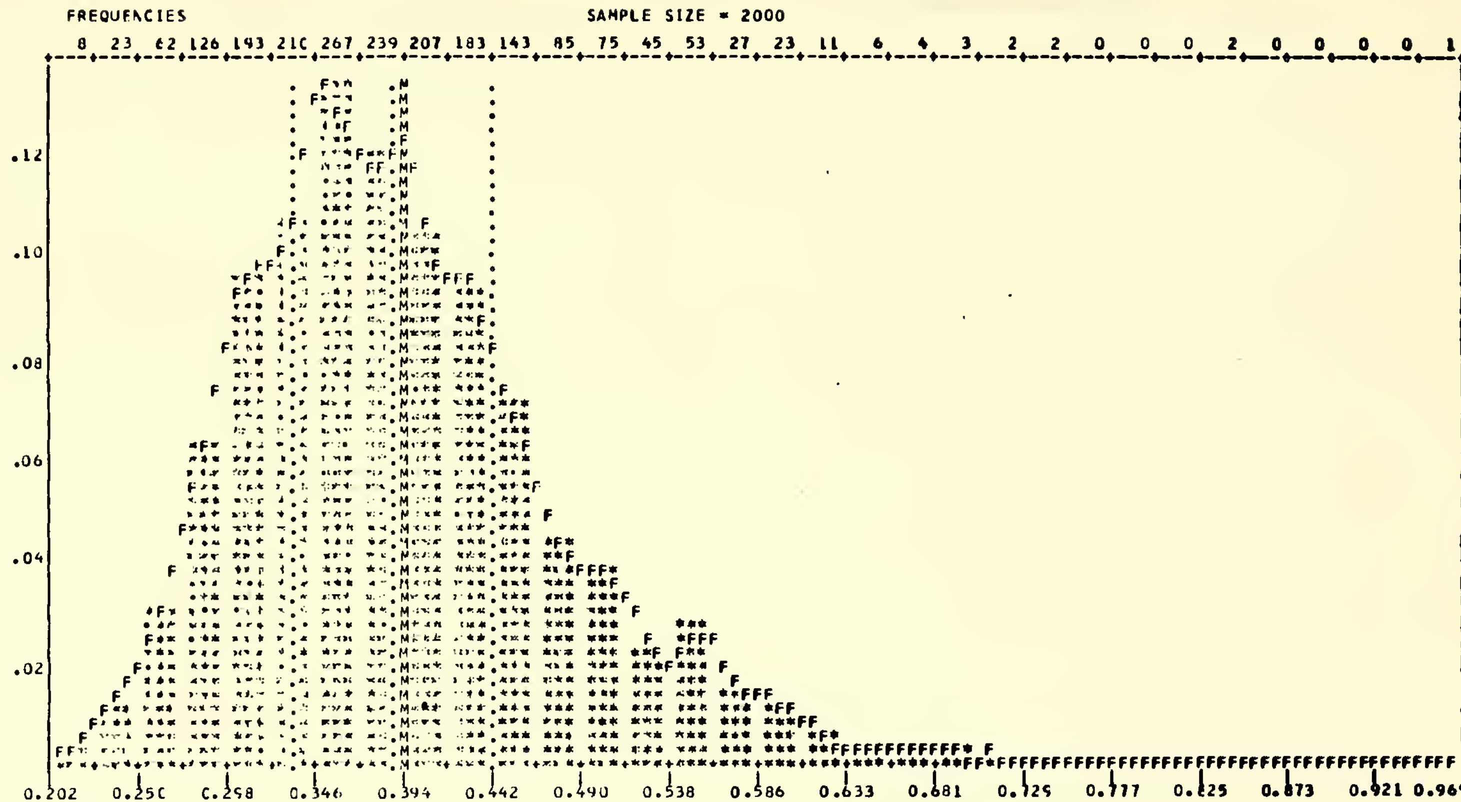
ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: *=* 1.969E-02 UNITS

Y-SCALE: *=* 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

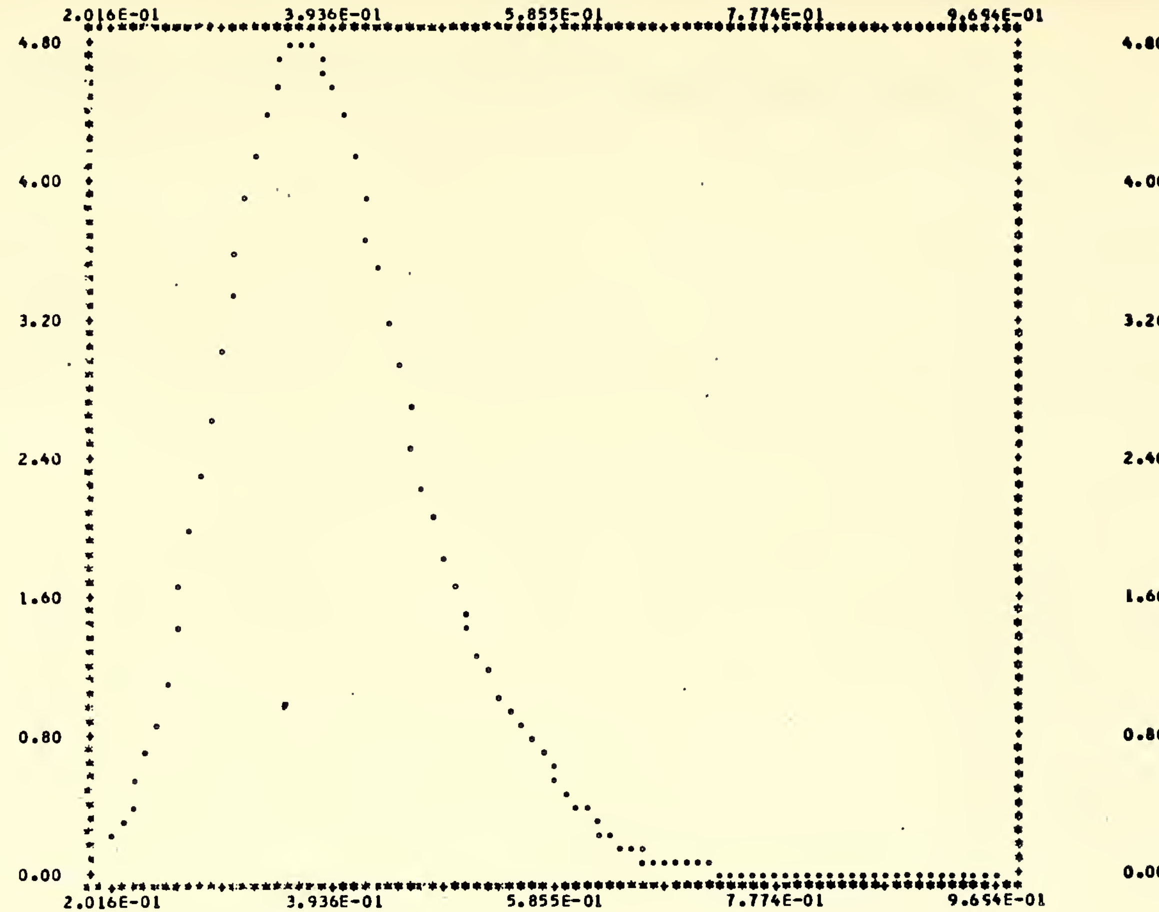


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	3.920965E-01	VARIANCE	7.312674E-03	M3	5.947638E-04	MINIMUM	2.016470E-01
MEDIAN	3.821697E-01	STD DEV	8.551419E-02	M4	2.694547E-04	.10 QUANTILE	2.539897E-01
TRIMEAN	3.837025E-01	COEFF VAR	2.180948E-01	SKEWNESS	9.511088E-01	.25 QUANTILE (HINGE)	3.324397E-01
MIDMEAN	3.823143E-01	RANGE	7.677135E-01	KURTOSIS	2.038873E 00	.50 QUANTILE (MEDIAN)	3.821697E-01
GEOM MEAN	3.833756E-01	MIDSPREAD	1.055921E-01			.75 QUANTILE (HINGE)	4.380317E-01
HARM MEAN	3.750882E-01					.90 QUANTILE	5.062481E-01
						MAXIMUM	9.693606E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 100

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)



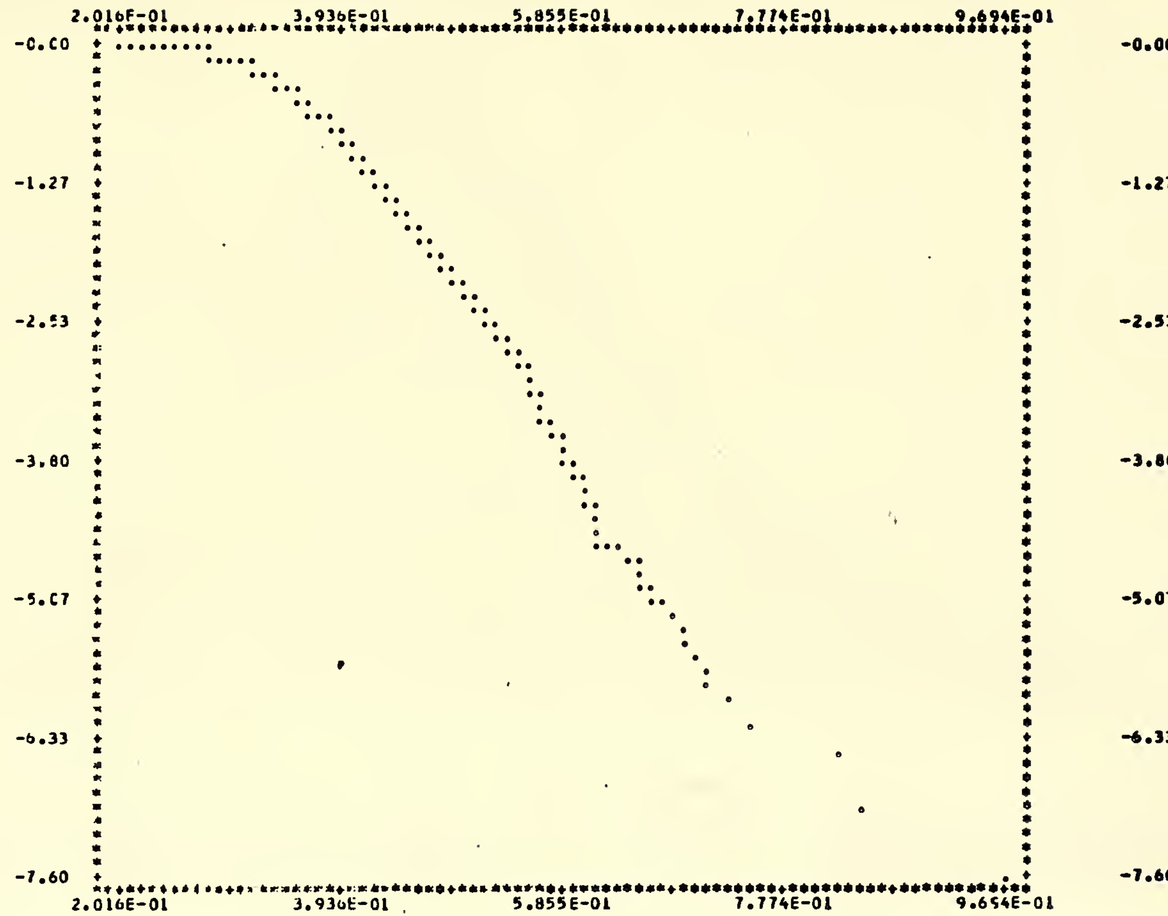
X-SCALE: "*"= C.960E-02 UNITS

Y-SCALE: "+"= C.800E-01 UNITS

ESTIMATED CENSITY FUNCTICN VS INTEGRAL SQUARE NORM RANDOM VARIABLE

INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10

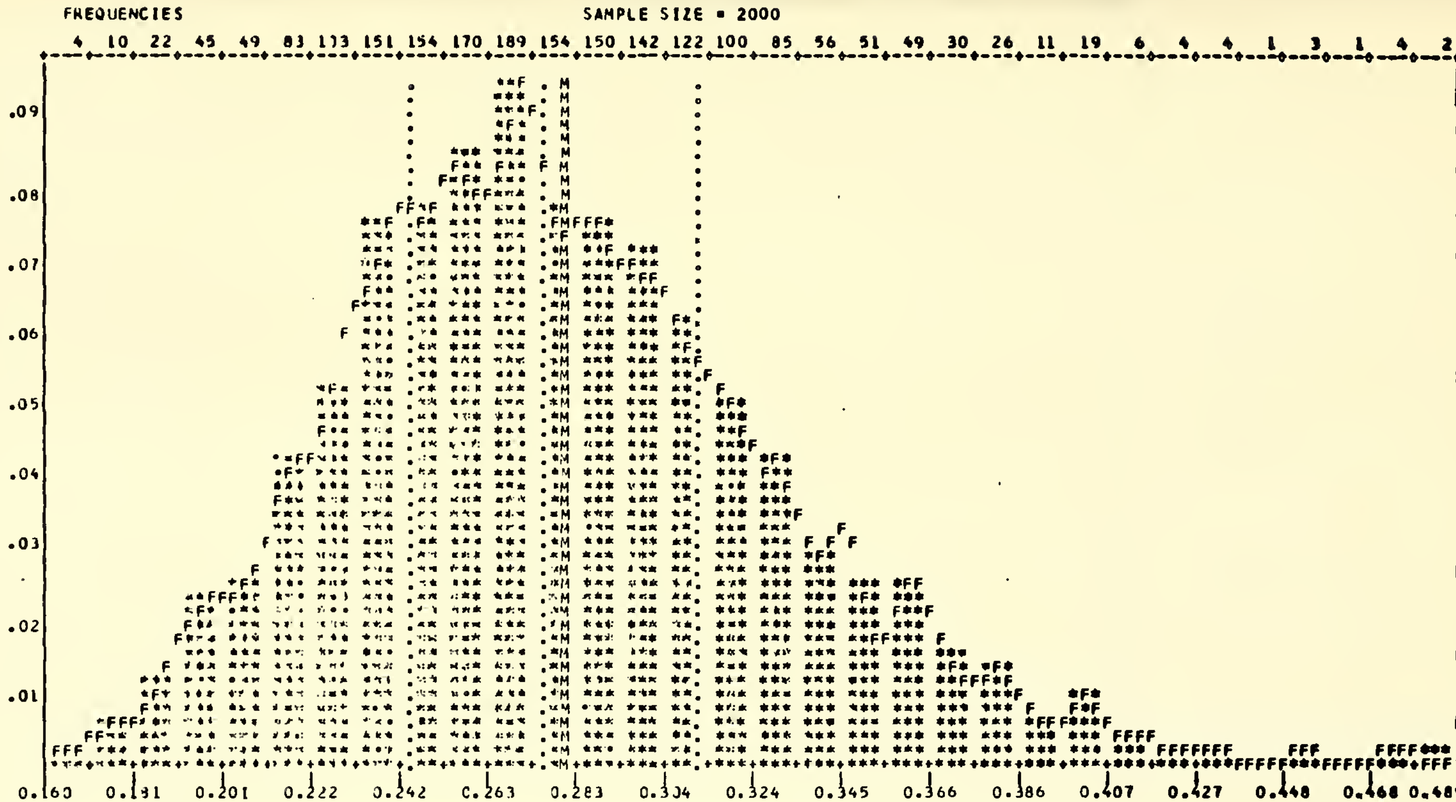
ESTIMATED CENSITY FUNCTICN IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.960E-02 UNITS

Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS . X EMPIRICAL LOG SURVIVAL FUNCTION



CENTRAL TENDENCY	
MEAN	2.800428E-01
MEDIAN	2.748137E-01
TRIMEAN	2.761929E-01
MIDMEAN	2.759345E-01
GEOM MEAN	2.757266E-01
HARM MEAN	2.715021E-01

SPREAD	
VARIANCE	2.490300E-03
STD DEV	4.990290E-02
COEF VAR	1.781974E-01
RANGE	3.286959E-01
MIDSPREAD	6.572706E-02

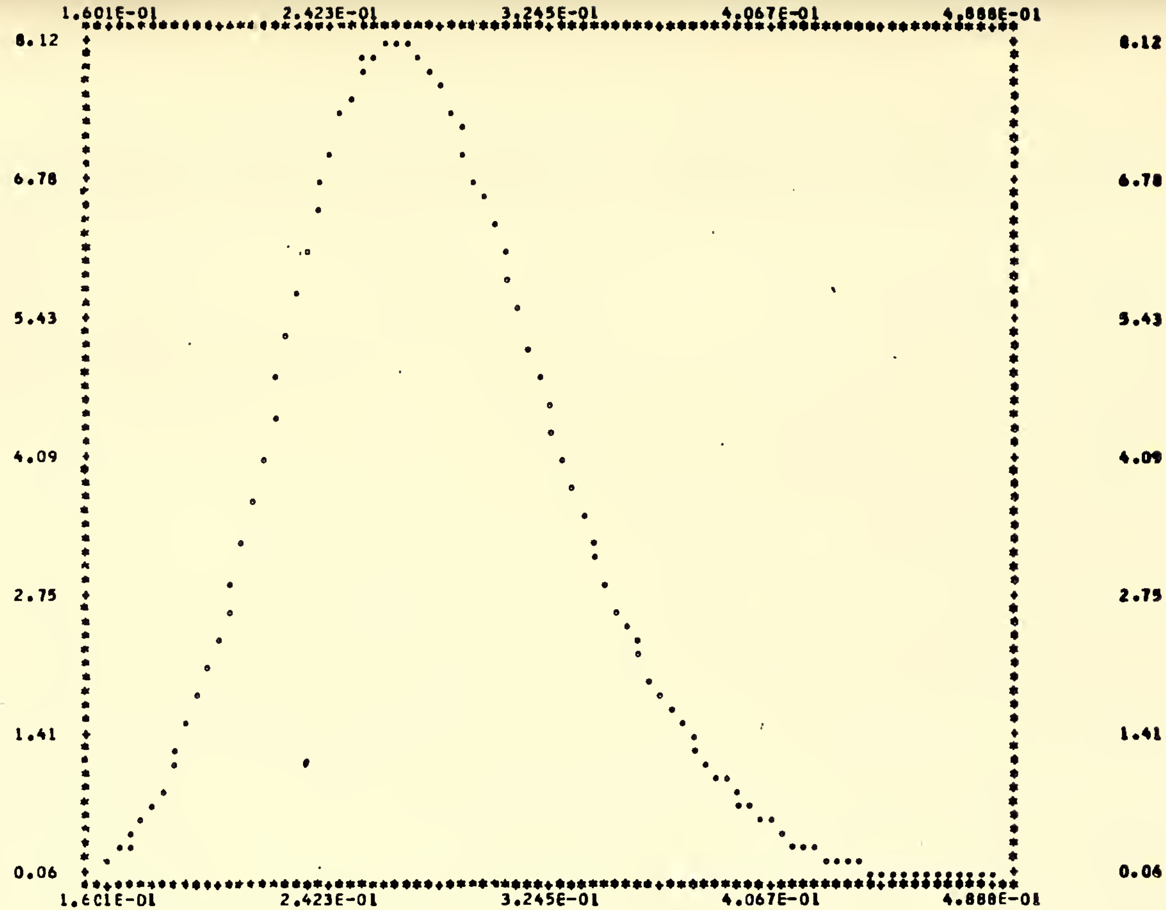
HIGHER CENTRAL MOMENTS	
M3	7.325548E-05
M4	2.227913E-05
SKEWNESS	5.894713E-01
KURTOSIS	5.924845E-01

DISTRIBUTION		
MINIMUM		1.601461E-01
.10 QUANTILE		2.203366E-01
.25 QUANTILE	(HINGE)	2.447091E-01
.50 QUANTILE	(MEDIAN)	2.748137E-01
.75 QUANTILE	(HINGE)	3.104361E-01
.90 QUANTILE		3.459940E-01
MAXIMUM		4.888420E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 200

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)



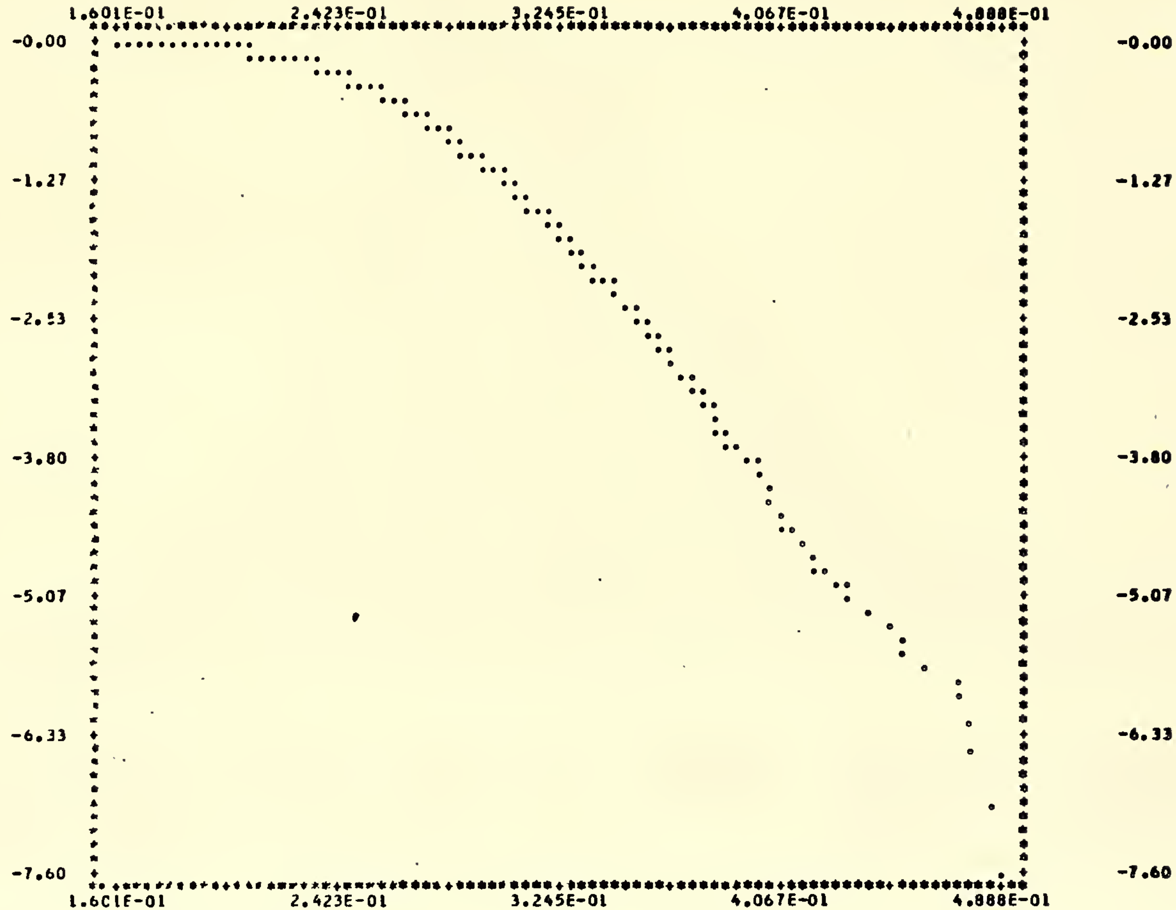
X-SCALE: "*"= 0.411E-02 UNITS

Y-SCALE: "+"= 0.134E 00 UNITS

ESTIMATED DENSITY FUNCTICN VS INTEGRAL SQUARE NCRM RANDOM VARIABLE

INTFGRAL SCUARE NCRM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10

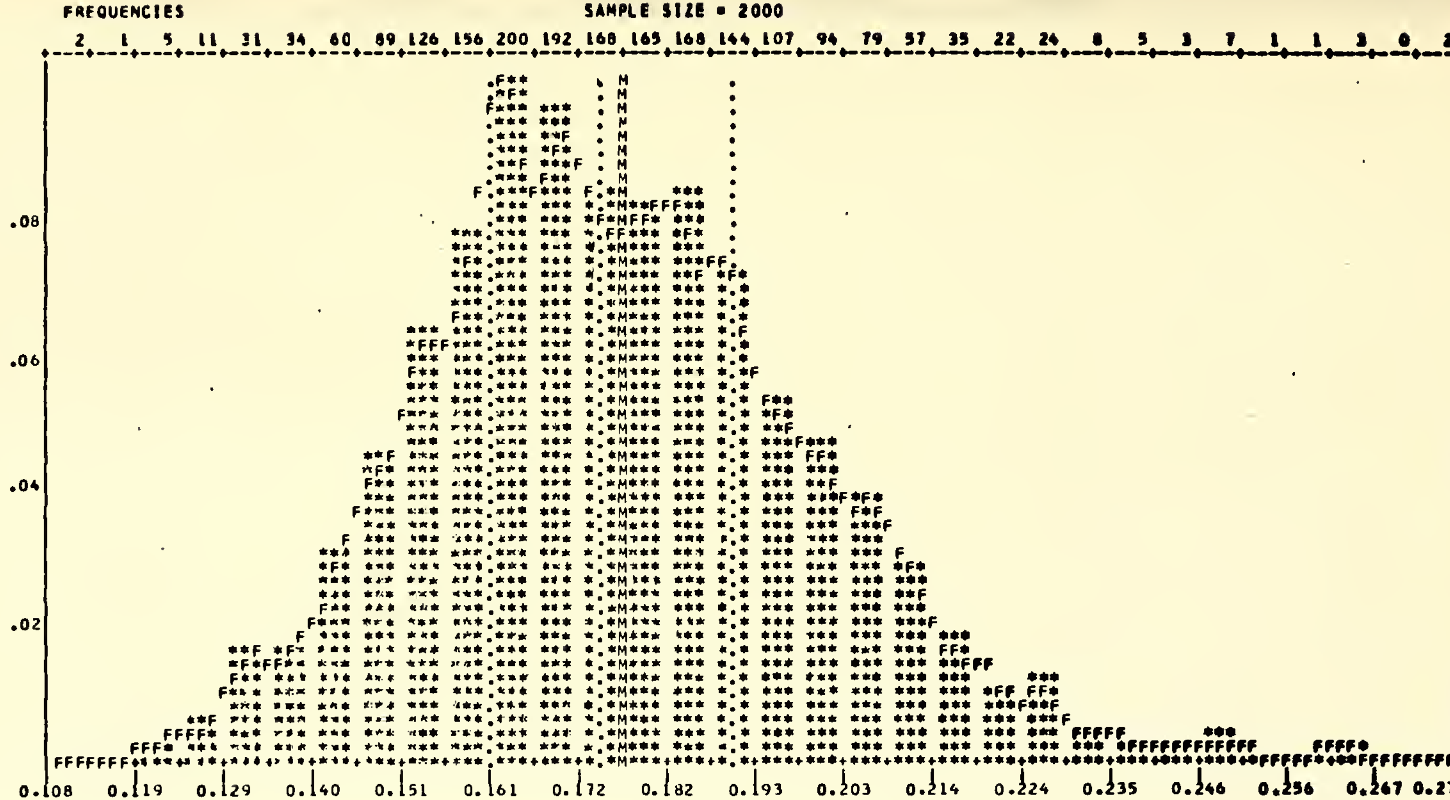
FSTIMATED CENSITY FUNCTICN IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.411E-02 UNITS

Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

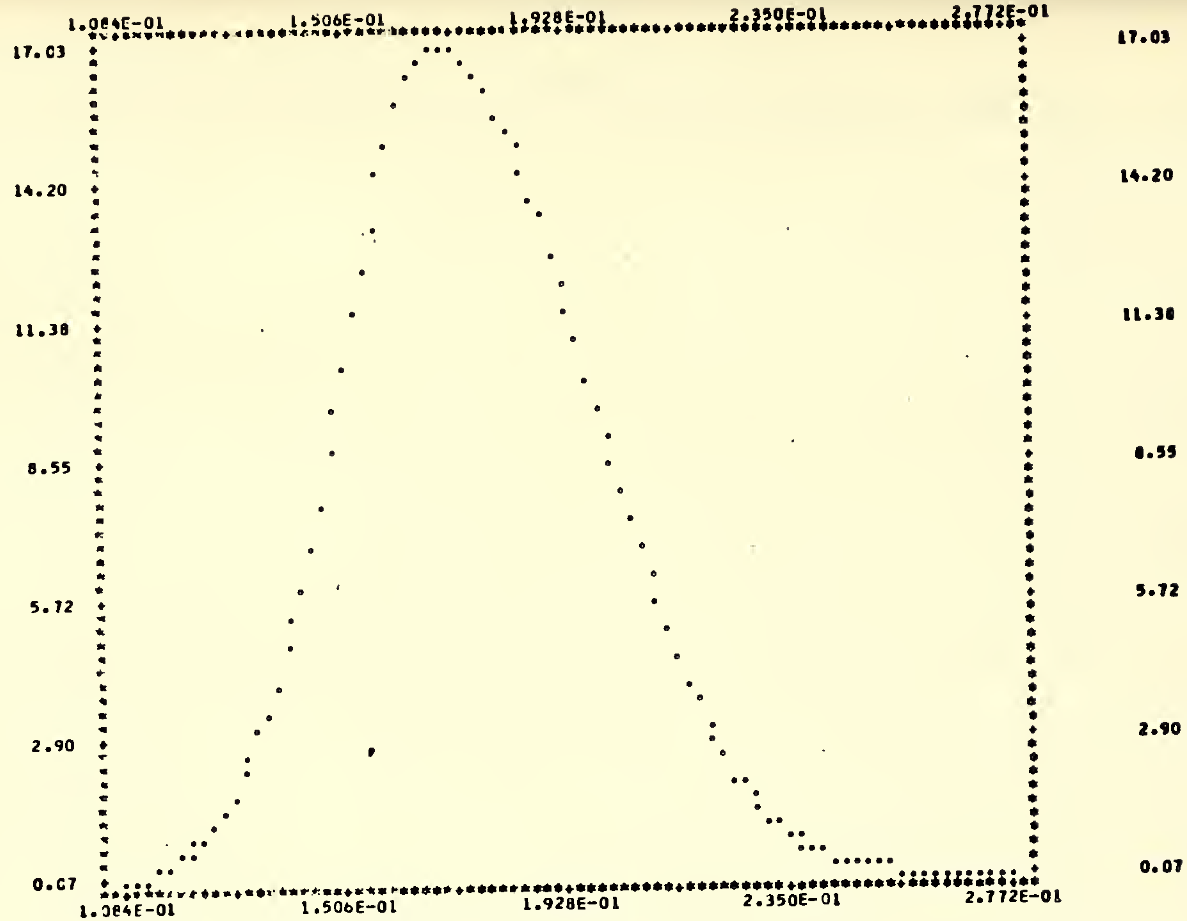


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.763979E-01	VARIANCE	5.243721E-04	M3	5.343630E-06	MINIMUM	1.083923E-01
MEDIAN	1.746314E-01	STD DEV	2.289917E-02	M4	9.649602E-07	.10 QUANTILE	1.487721E-01
TRIMEAN	1.751256E-01	COEF VAR	1.298154E-01	SKEWNESS	4.450174E-01	.25 QUANTILE (HINGE)	1.607113E-01
MIDMEAN	1.745445E-01	RANGE	1.688312E-01	KURTOSIS	5.093794E-01	.50 QUANTILE (MEDIAN)	1.746314E-01
GEOM MEAN	1.749359E-01	MIDSPREAD	2.981722E-02			.75 QUANTILE (HINGE)	1.905285E-01
HARM MEAN	1.734888E-01					.90 QUANTILE	2.066078E-01
						MAXIMUM	2.772235E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 500

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)



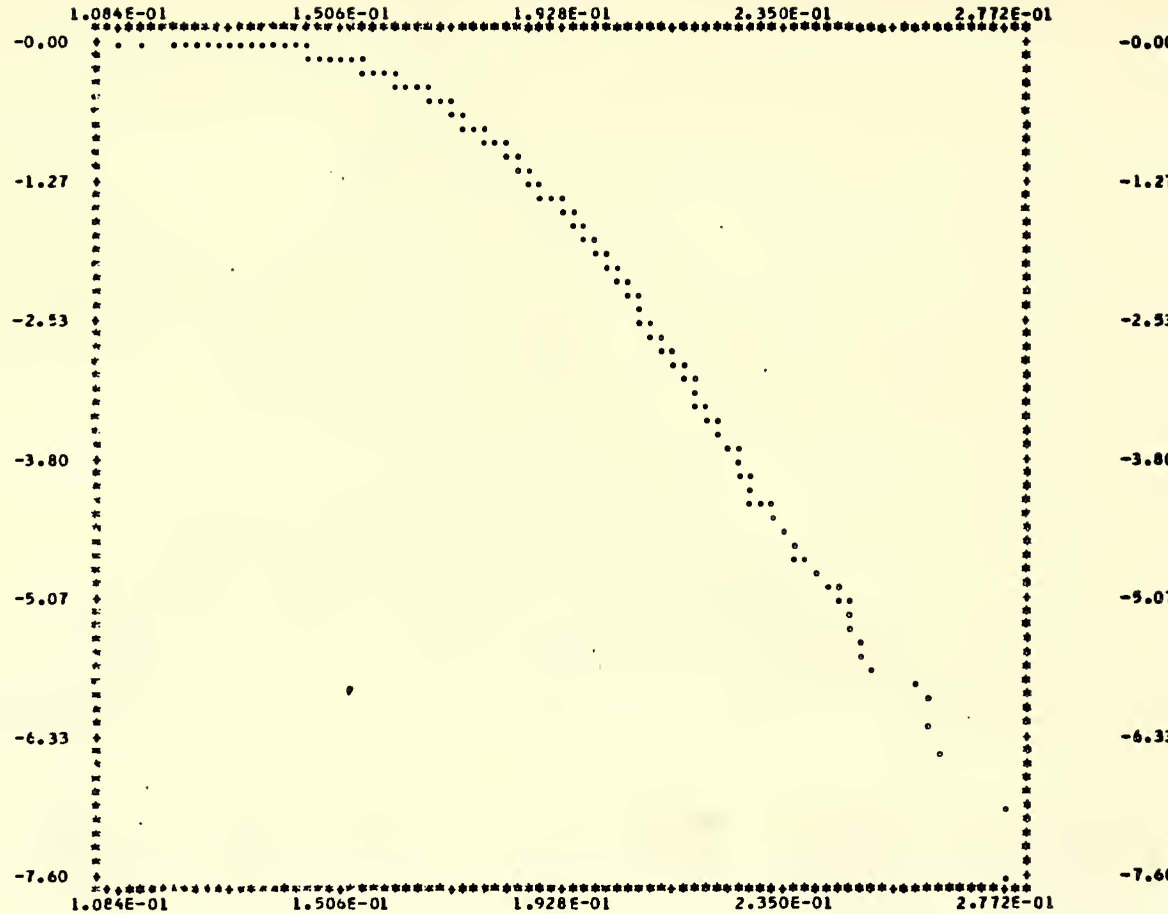
X-SCALE: *"* 0.211E-02 UNITS

Y-SCALE: *"* 0.283C 00 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE

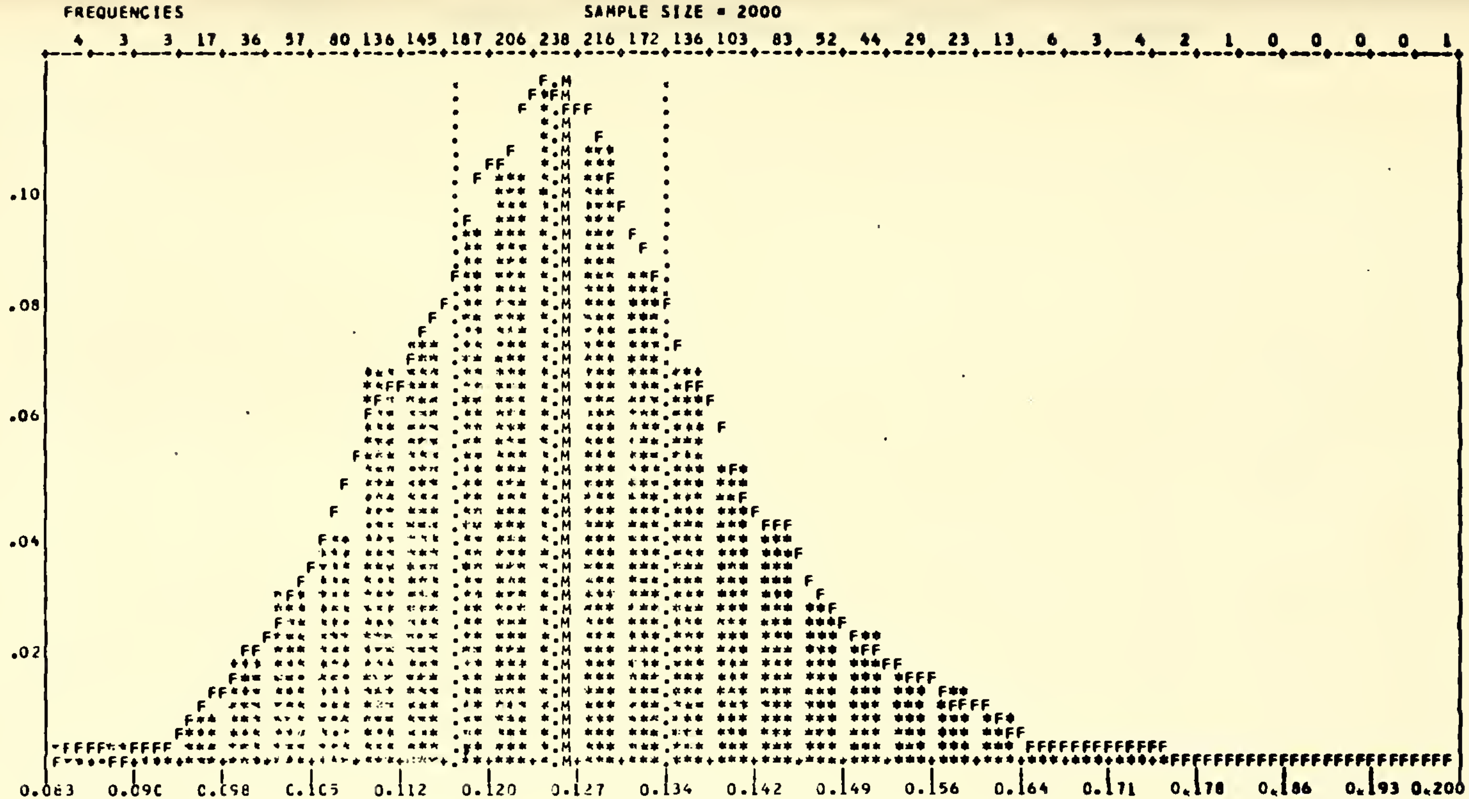
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10

ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.211E-02 UNITS
Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

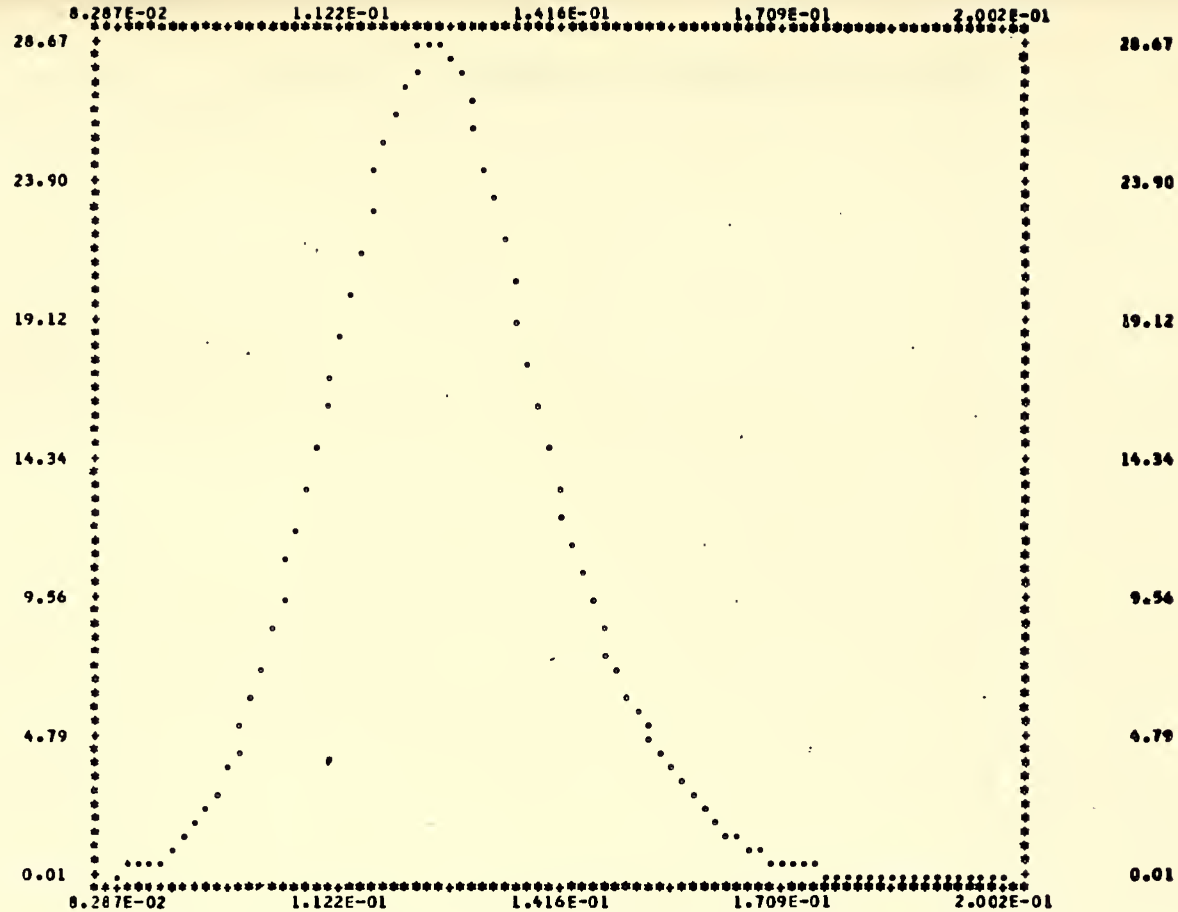


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.258343E-01	VARIANCE	1.999531E-04	M3	1.085470E-06	MINIMUM	8.286673E-02
MEDIAN	1.251185E-01	STD DEV	1.414048E-02	M4	1.427055E-07	.10 QUANTILE	1.085095E-01
TRIMEAN	1.252340E-01	COEF VAR	1.123737E-01	SKEWNESS	3.839066E-01	.25 QUANTILE (HINGE)	1.164791E-01
MIDMEAN	1.251930E-01	RANGE	1.173736E-01	KURTOSIS	5.693092E-01	.50 QUANTILE (MEDIAN)	1.251185E-01
GEOM MEAN	1.250451E-01	MIDSPREAD	1.774102E-02			.75 QUANTILE (HINGE)	1.342201E-01
HARM MEAN	1.242683E-01					.90 QUANTILE	1.442502E-01
						MAXIMUM	2.002404E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

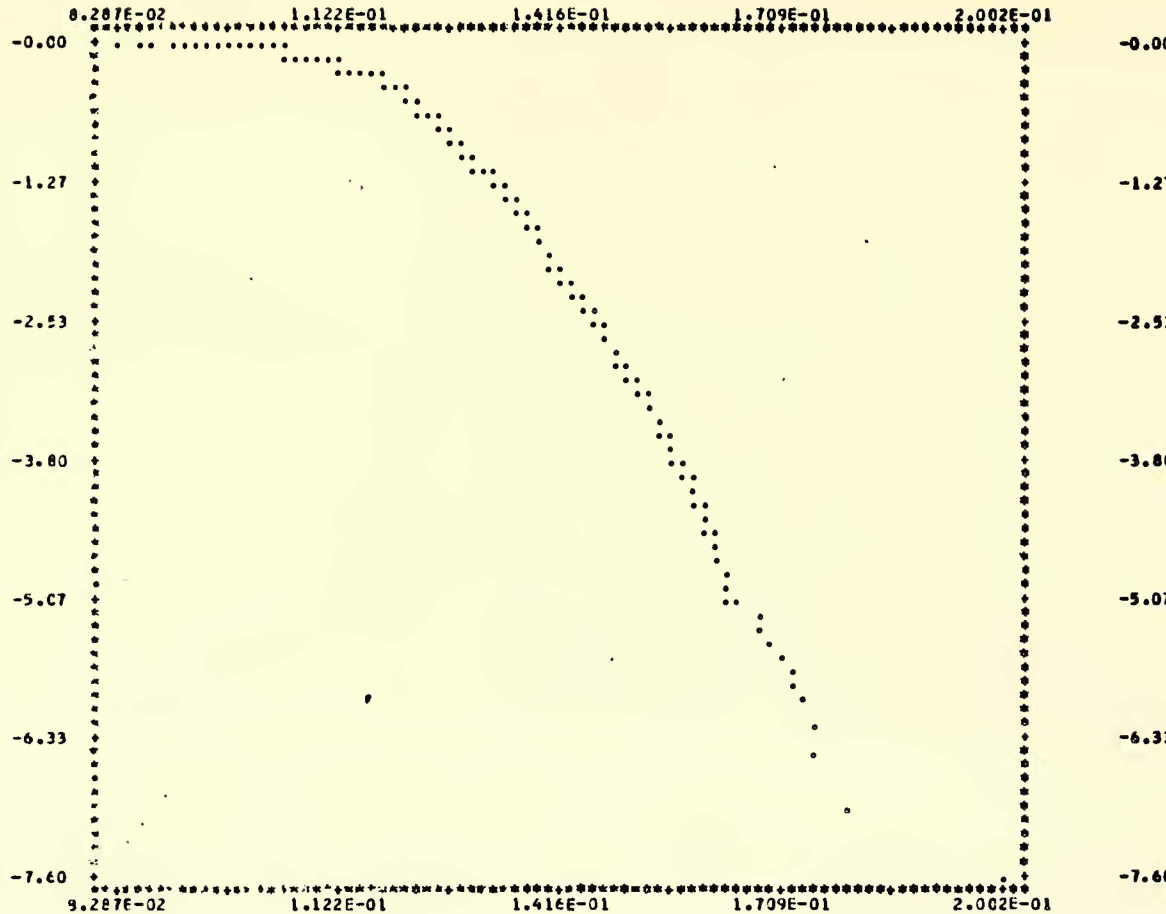
CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 1000

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)



X-SCALE: "*"= 0.147E-02 UNITS
Y-SCALE: "*"= 0.478E 00 UNITS

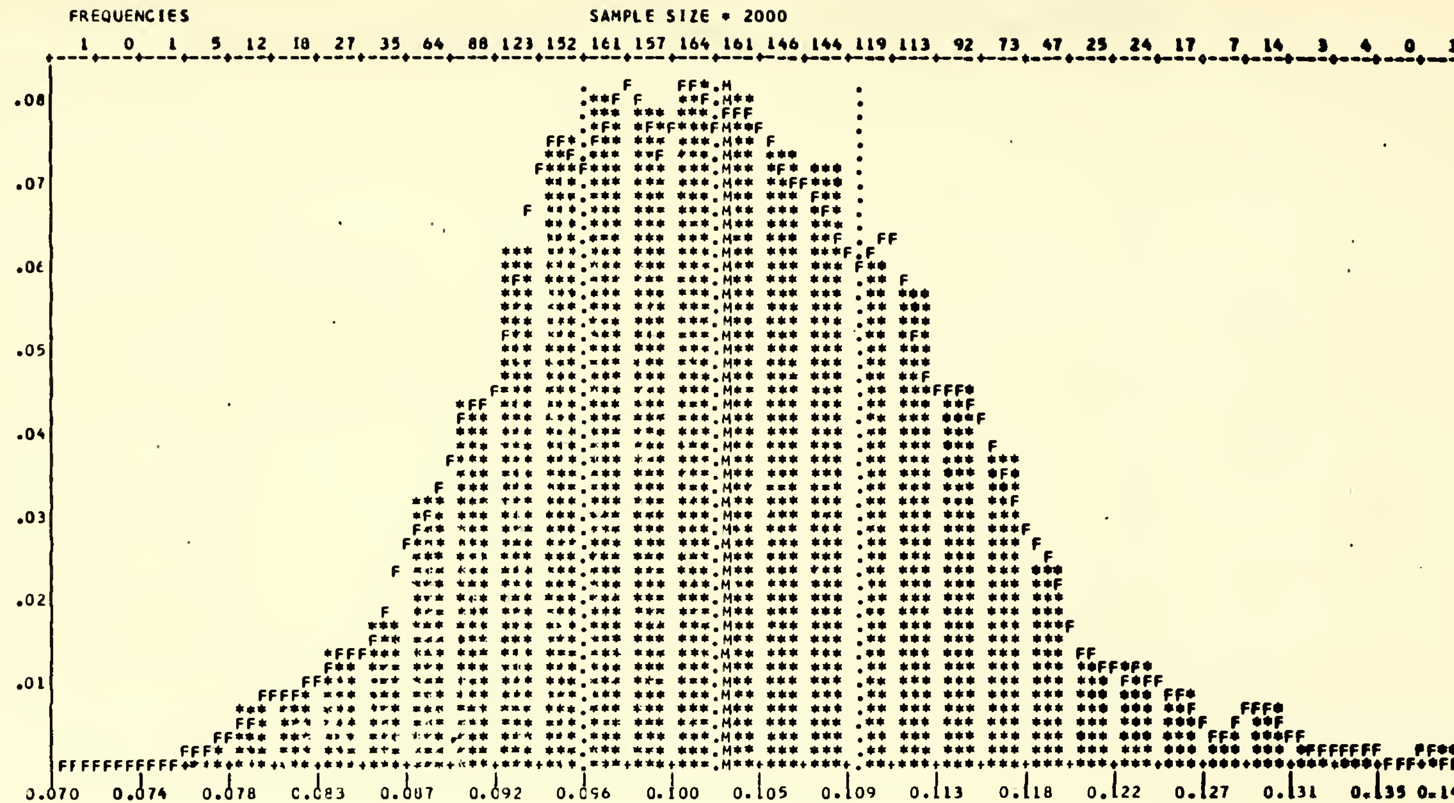
ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.147E-02 UNITS

Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

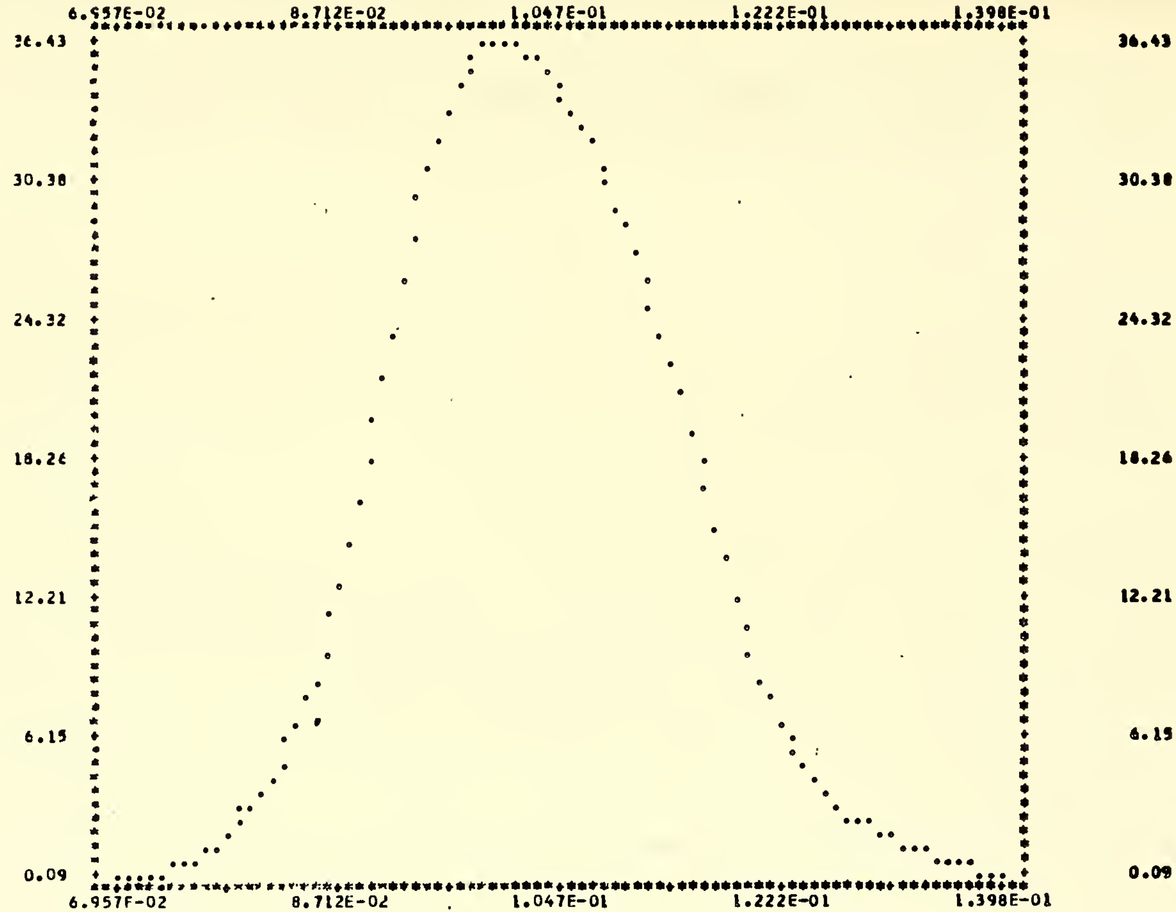


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.028634E-01	VARIANCE	1.049400E-04	M3	2.663205E-07	MINIMUM	6.956708E-02
MEDIAN	1.023059E-01	STD DEV	1.024403E-02	M4	3.324176E-08	.10 QUANTILE	9.018344E-02
TRIMEAN	1.025003E-01	COEF VAR	9.958857E-02	SKEWNESS	2.477381E-01	.25 QUANTILE (HINGE)	9.549809E-02
MIDMEAN	1.024408E-01	RANGE	7.019967E-02	KURTOSIS	1.857281E-02	.50 QUANTILE (MEDIAN)	1.023099E-01
GEOM MEAN	1.023560E-01	MIDSPREAD	1.438510E-02			.75 QUANTILE (HINGE)	1.098832E-01
HARM MEAN	1.018493E-01					.90 QUANTILE	1.160563E-01
						MAXIMUM	1.397668E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

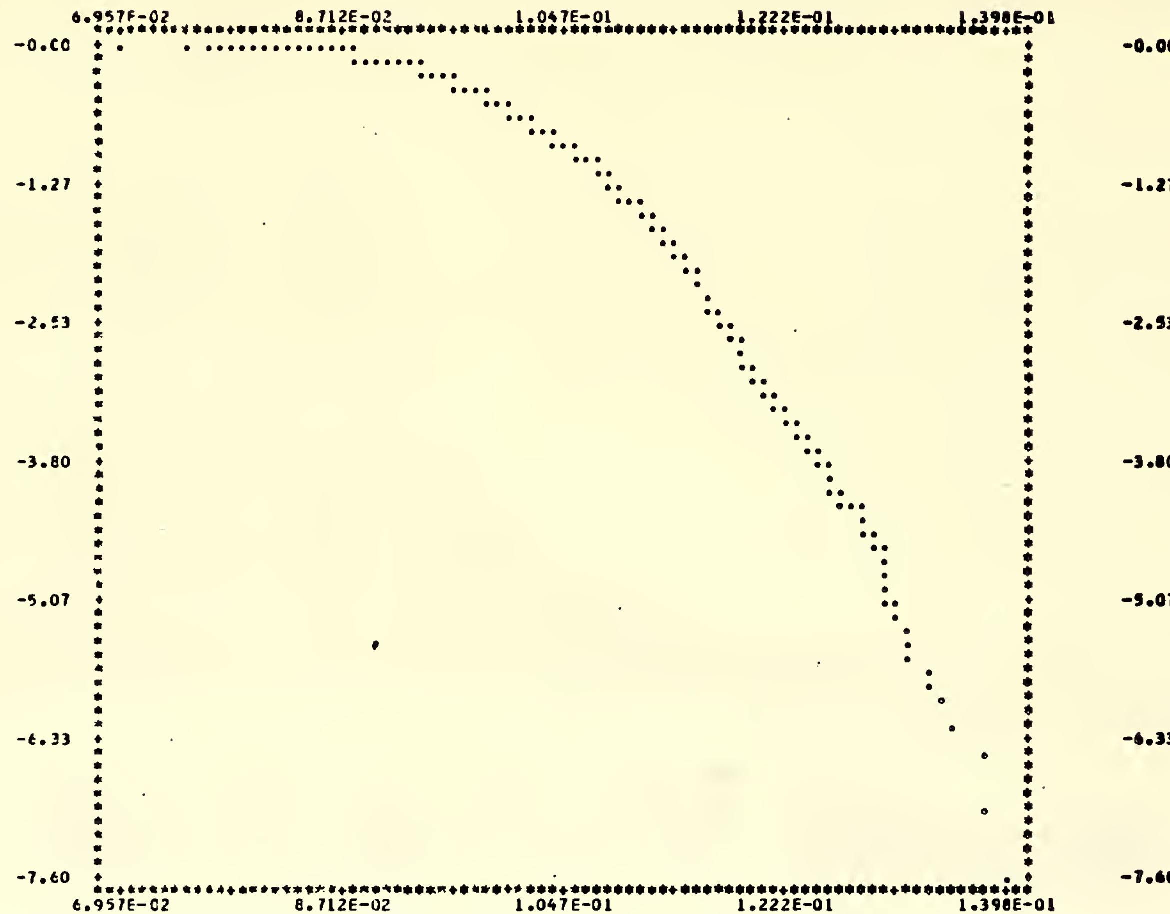
CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 1500

TRIANGULAR WINDOW. BANDWIDTH = 1/SQRT(N)



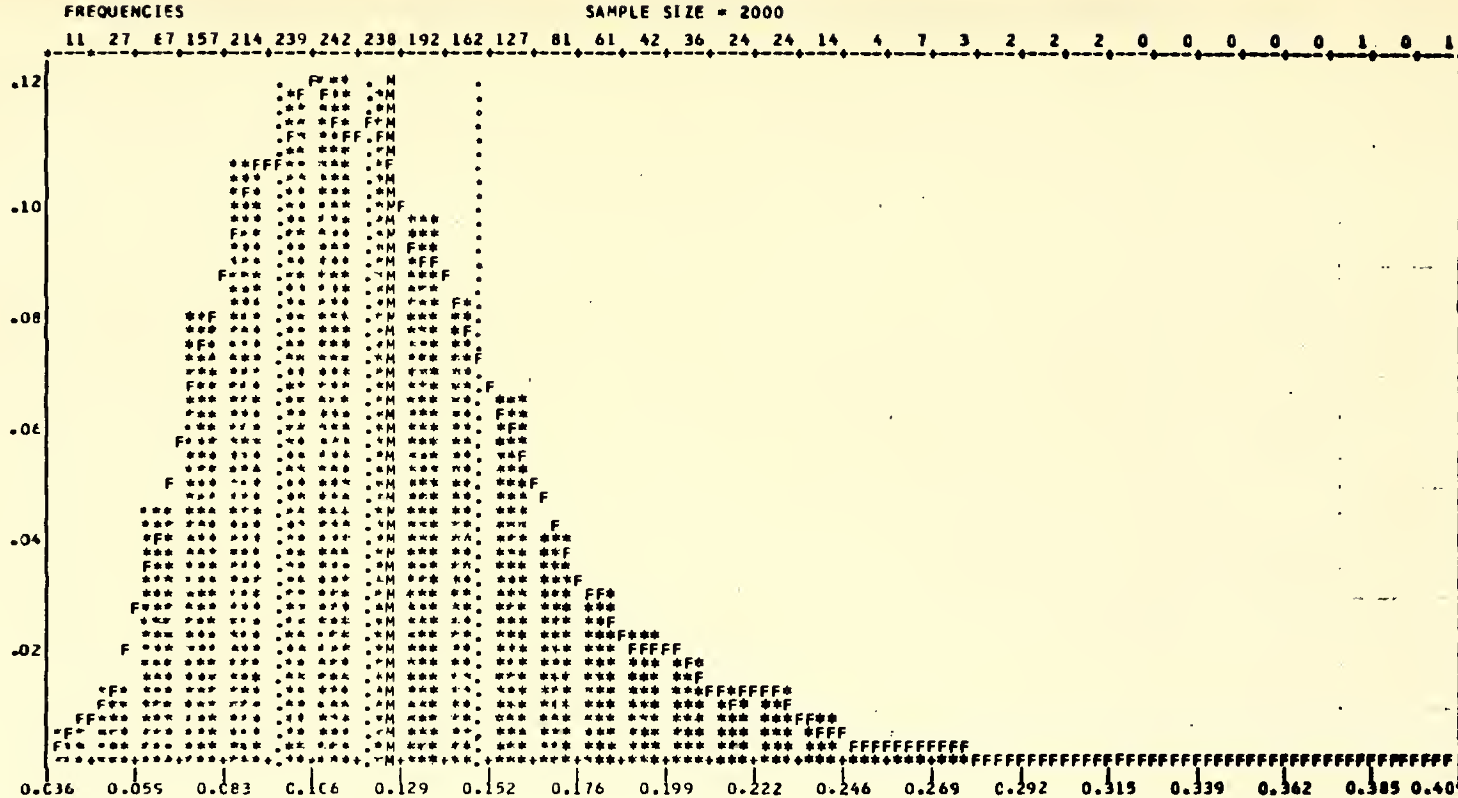
X-SCALE: "=" = 0.877E-03 UNITS
Y-SCALE: "*" = 0.606E 00 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: *=* 0.877E-03 UNITS
Y-SCALE: *=* 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

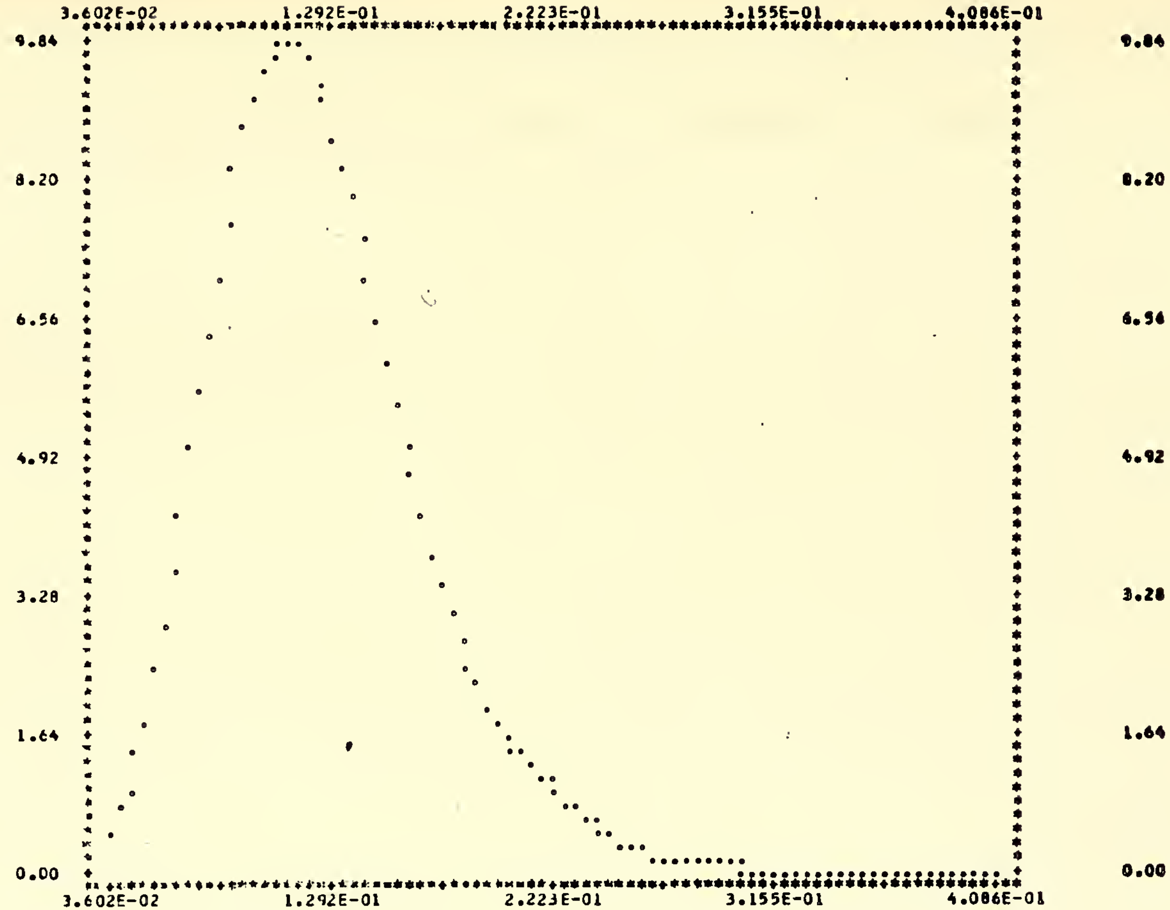


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.247246E-01	VARIANCE	1.804262E-03	M3	8.461093E-05	MINIMUM	3.601930E-02
MEDIAN	1.187804E-01	STD DEV	4.247661E-02	M4	1.787450E-05	.10 QUANTILE	7.716137E-02
TRIMEAN	1.198417E-01	COEF VAR	3.405631E-01	SKEWNESS	1.104020E 00	.25 QUANTILE (HINGE)	9.434485E-02
MIDMEAN	1.192868E-01	RANGE	3.726265E-01	KURTOSIS	2.490786E 00	.50 QUANTILE (MEDIAN)	1.187804E-01
GEOM MEAN	1.180715E-01	MIDSPREAD	5.311632E-02			.75 QUANTILE (HINGE)	1.474612E-01
HARM MEAN	1.117418E-01					.90 QUANTILE	1.797985E-01
						MAXIMUM	4.086459E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 100

TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)



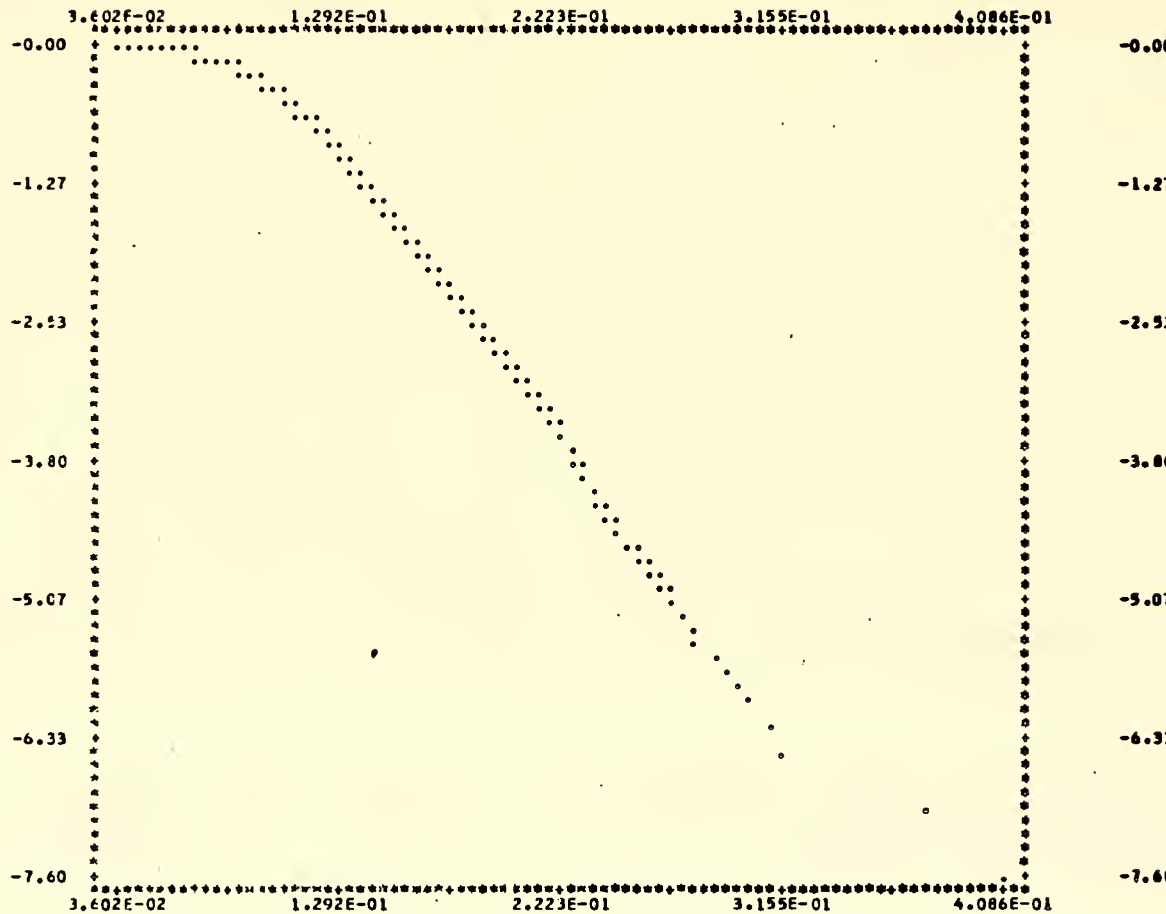
X-SCALE: ***= 0.466E-02 UNITS

Y-SCALE: ***= 0.164E 00 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE

INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10

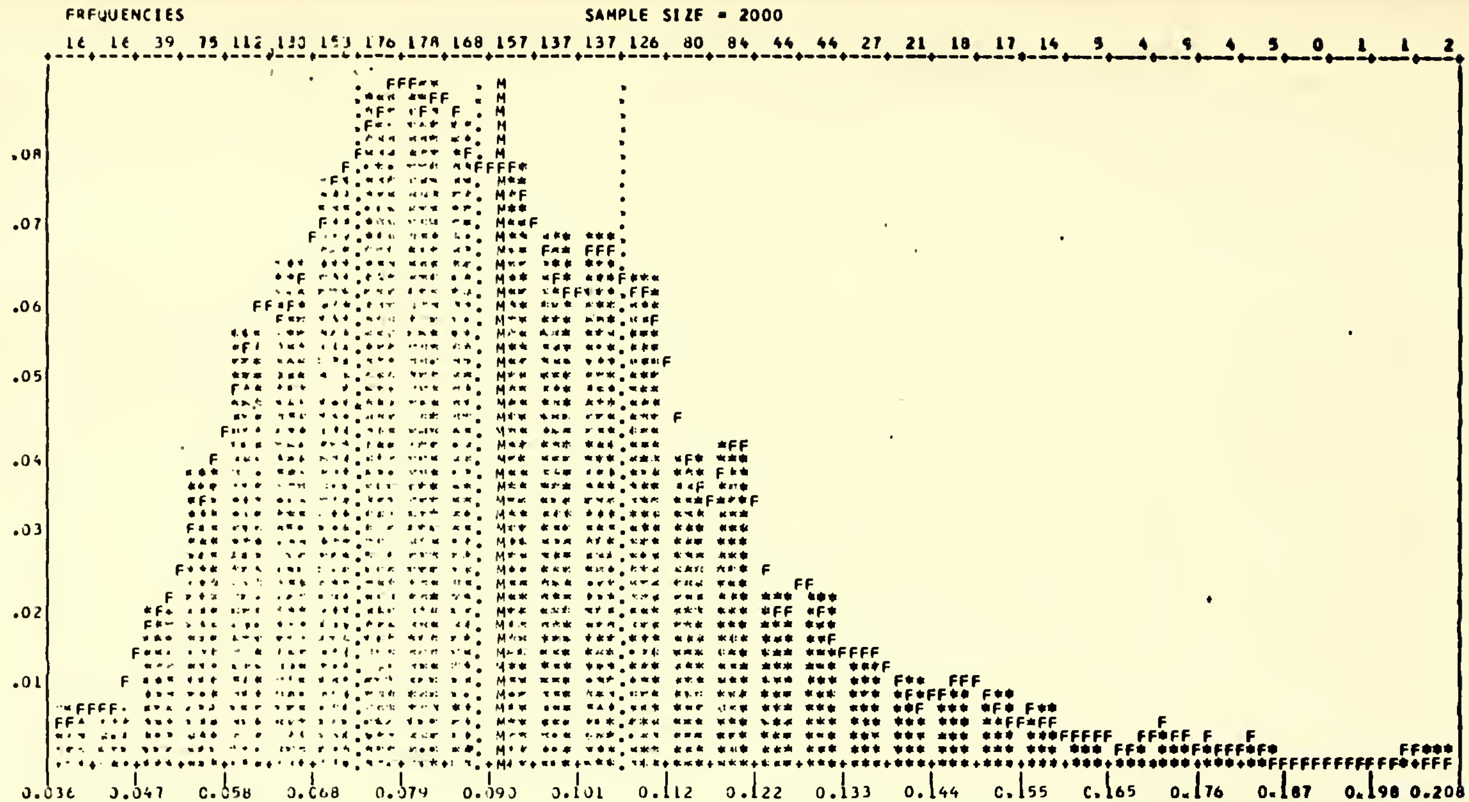
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "+"= 0.466E-02 UNITS

Y-SCALE: "+"= 0.127E 00 UNITS

LN(I-I/N+I) VS X   EMPIRICAL LOG SURVIVAL FUNCTION

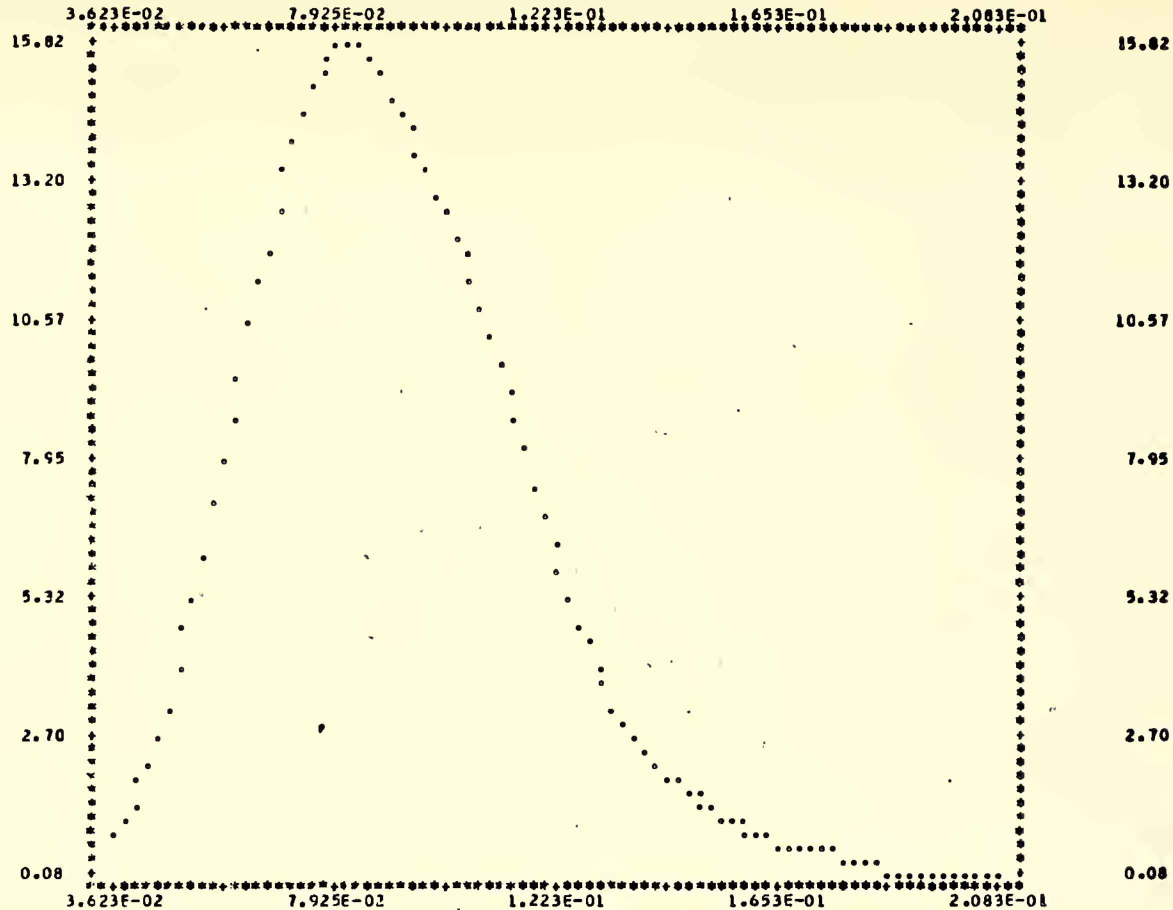


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	9.105271E-02	VARIANCE	6.739579E-04	M3	1.354842E-05	MINIMUM	3.623084E-02
MEDIAN	8.773708E-02	STD DEV	2.596070E-02	M4	1.798639E-06	.10 QUANTILE	6.072954E-02
TRIMEAN	8.858252E-02	COEFF VAR	2.851172E-01	SKEWNESS	7.743542E-01	.25 QUANTILE (HINGE)	7.252294E-02
MIDMEAN	8.847809E-02	RANGE	1.720661E-01	KURTOSIS	9.598465E-01	.50 QUANTILE (MEDIAN)	8.773708E-02
GEOM MEAN	8.750981E-02	MIDSPREAD	3.381014E-02			.75 QUANTILE (HINGE)	1.063331E-01
HARM MEAN	8.405179E-02					.90 QUANTILE	1.236336E-01
						MAXIMUM	2.082970E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 200

TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)



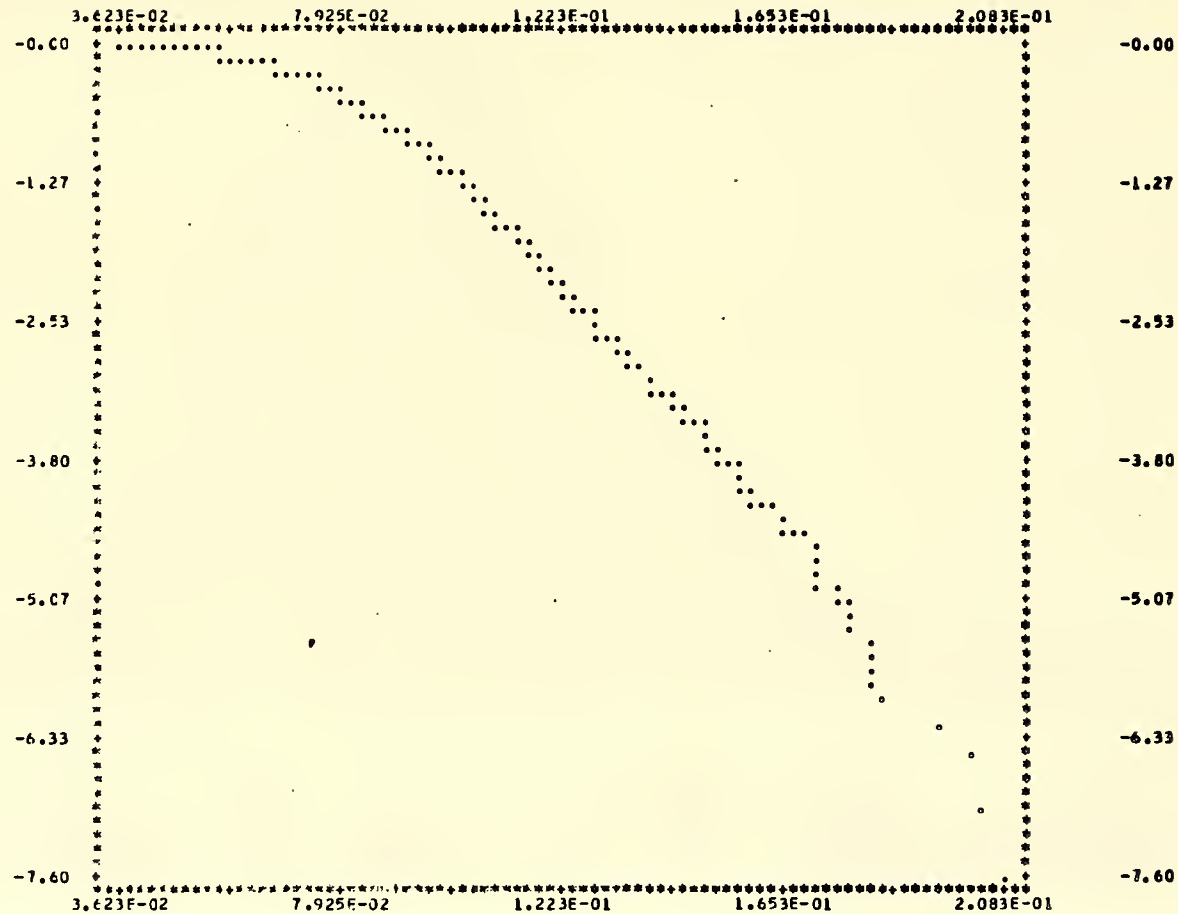
X-SCALE: "*"= 0.215E-02 UNITS

Y-SCALE: "*"= 0.262E 00 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE

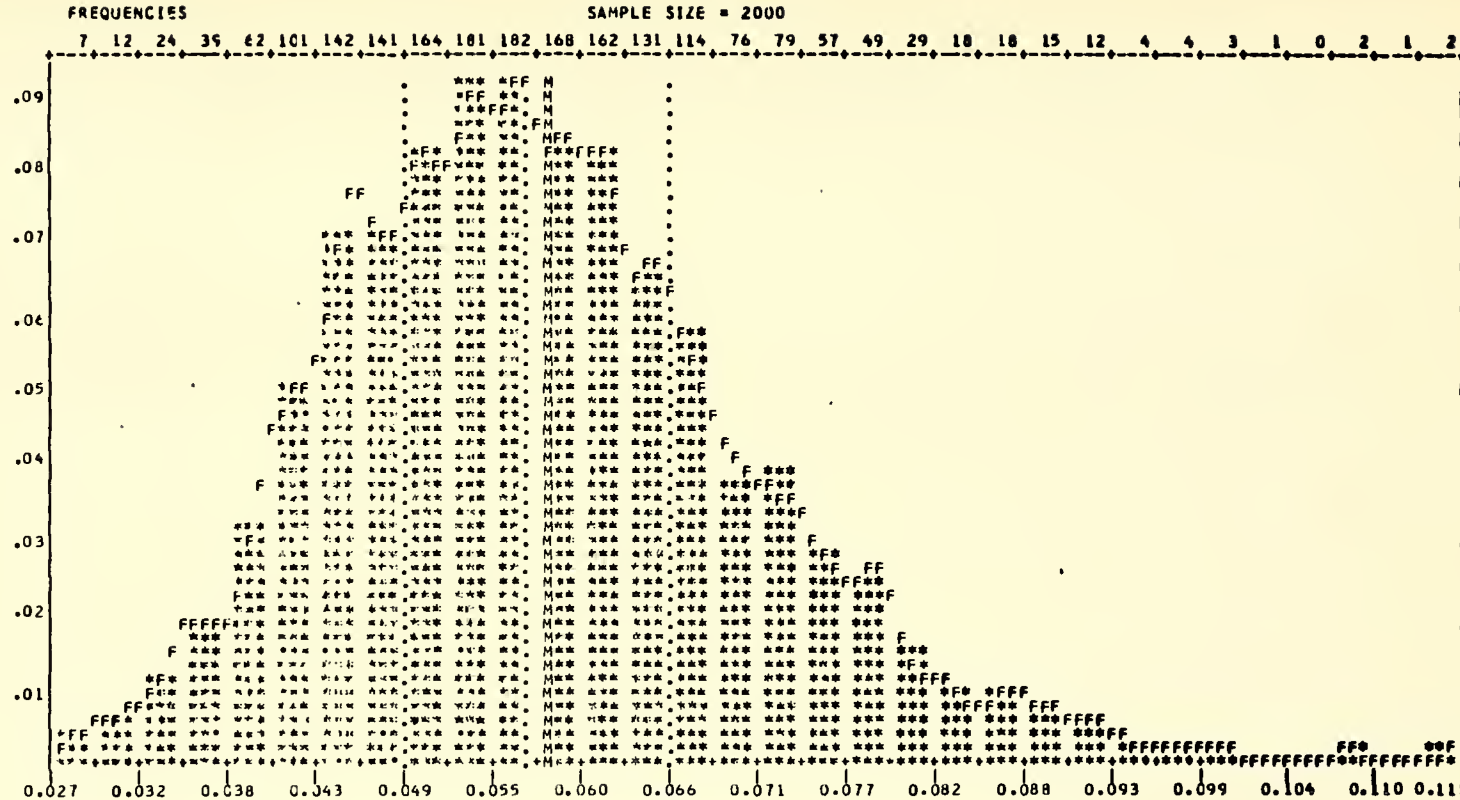
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10

ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "+"= 0.215E-02 UNITS
Y-SCALE: "+"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

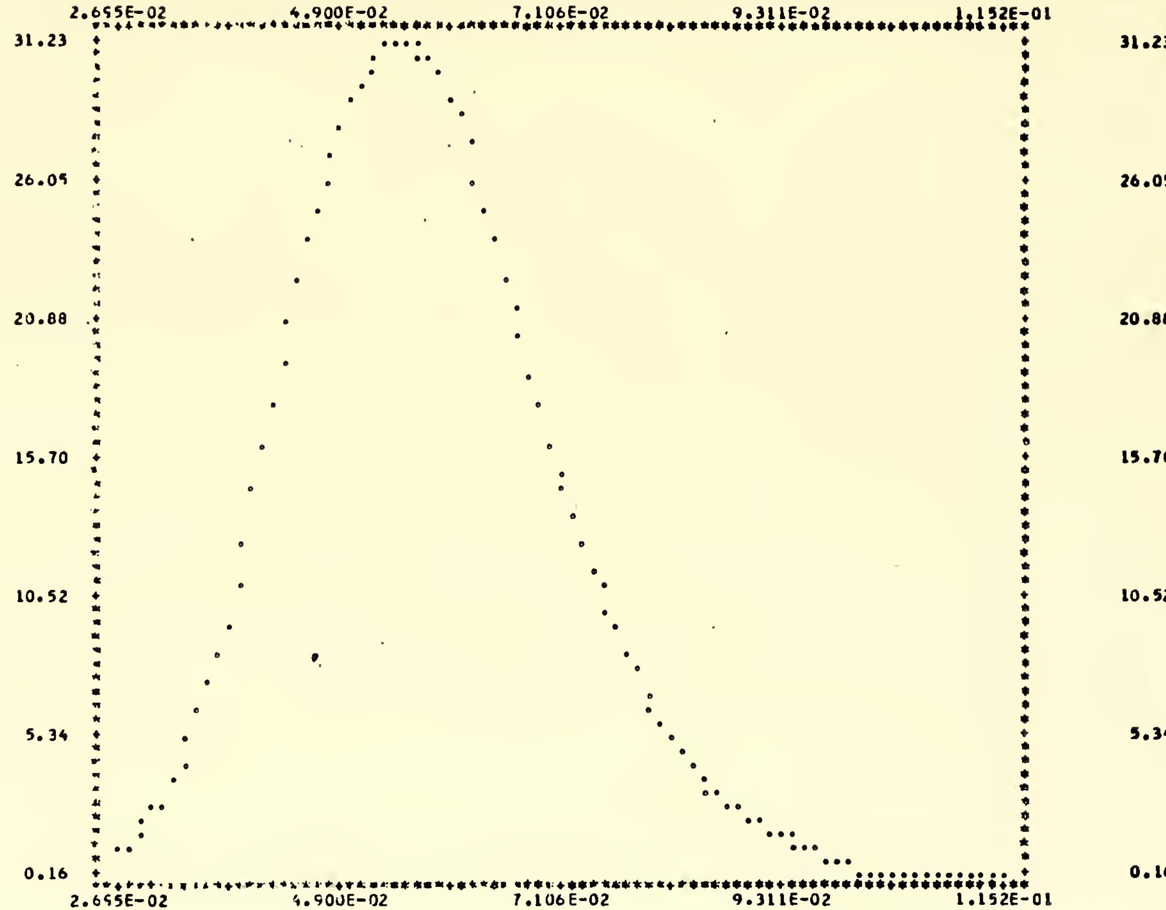


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	5.762135E-02	VARIANCE	1.634320E-04	M3	1.244043E-06	MINIMUM	2.695018E-02
MEDIAN	5.649266E-02	STD DEV	1.278405E-02	M4	9.818217E-08	.10 QUANTILE	4.231457E-02
TRIMEAN	5.664164E-02	COEF VAR	2.218631E-01	SKEWNESS	5.954288E-01	.25 QUANTILE (HINGE)	4.845238E-02
MIDMEAN	5.662918E-02	RANGE	8.821827E-02	KURTOSIS	6.758566E-01	.50 QUANTILE (MEDIAN)	5.649266E-02
GEOM MEAN	5.623742E-02	MIDSPREAD	1.667655E-02			.75 QUANTILE (HINGE)	6.512892E-02
HARM MEAN	5.486799E-02					.90 QUANTILE	7.440829E-02
						MAXIMUM	1.151685E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

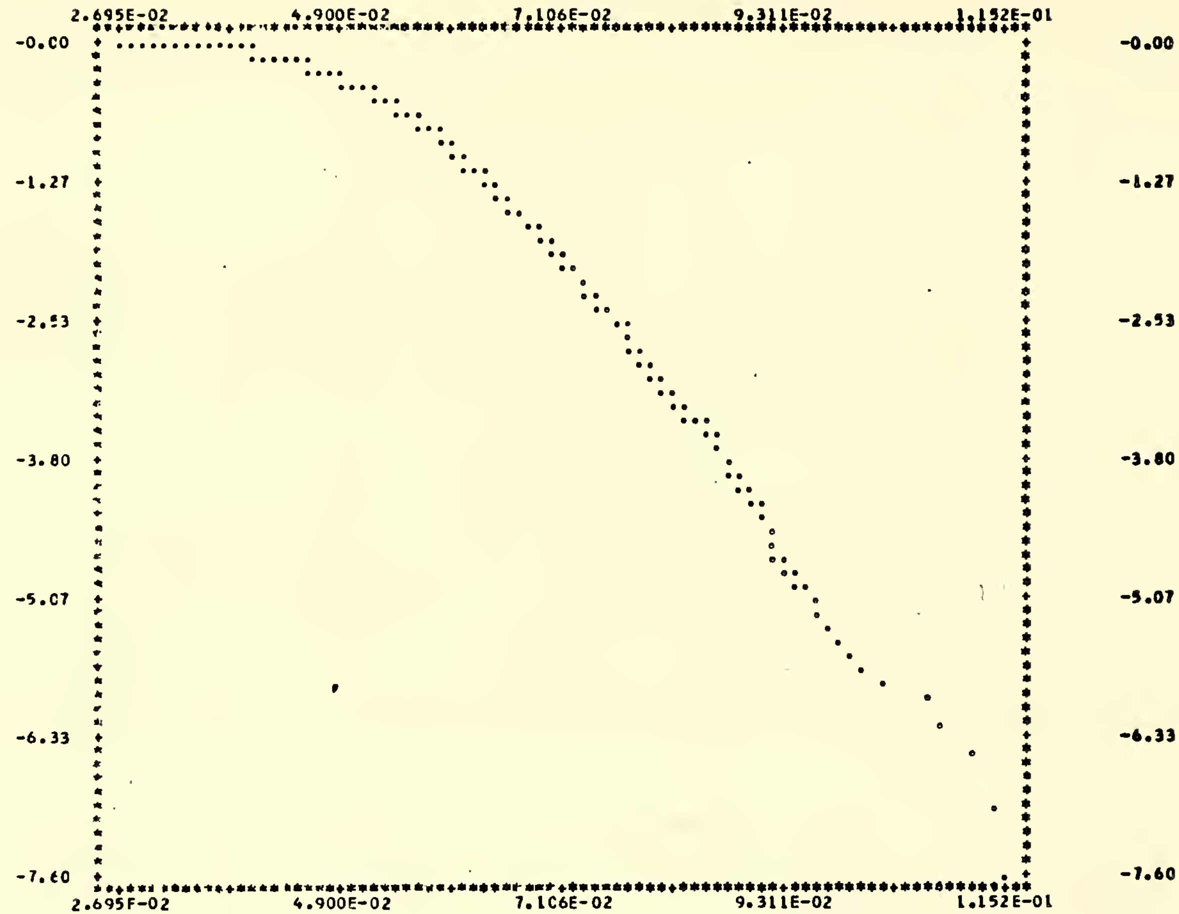
CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 500

TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)



X-SCALE: "*"= 0.110E-02 UNITS
Y-SCALE: "*"= 0.518E 00 UNITS

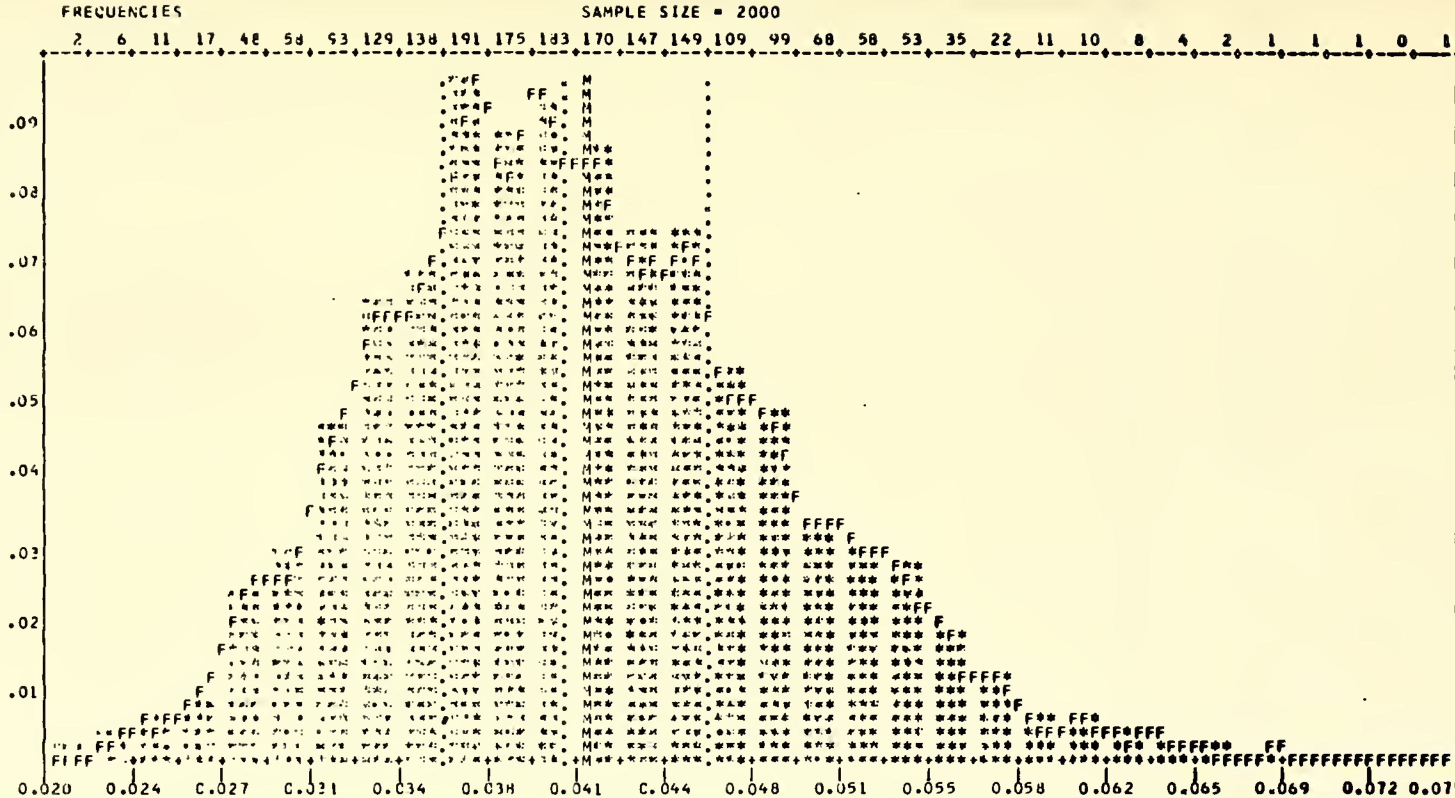
ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "+"= 0.110E-02 UNITS

Y-SCALE: "+"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

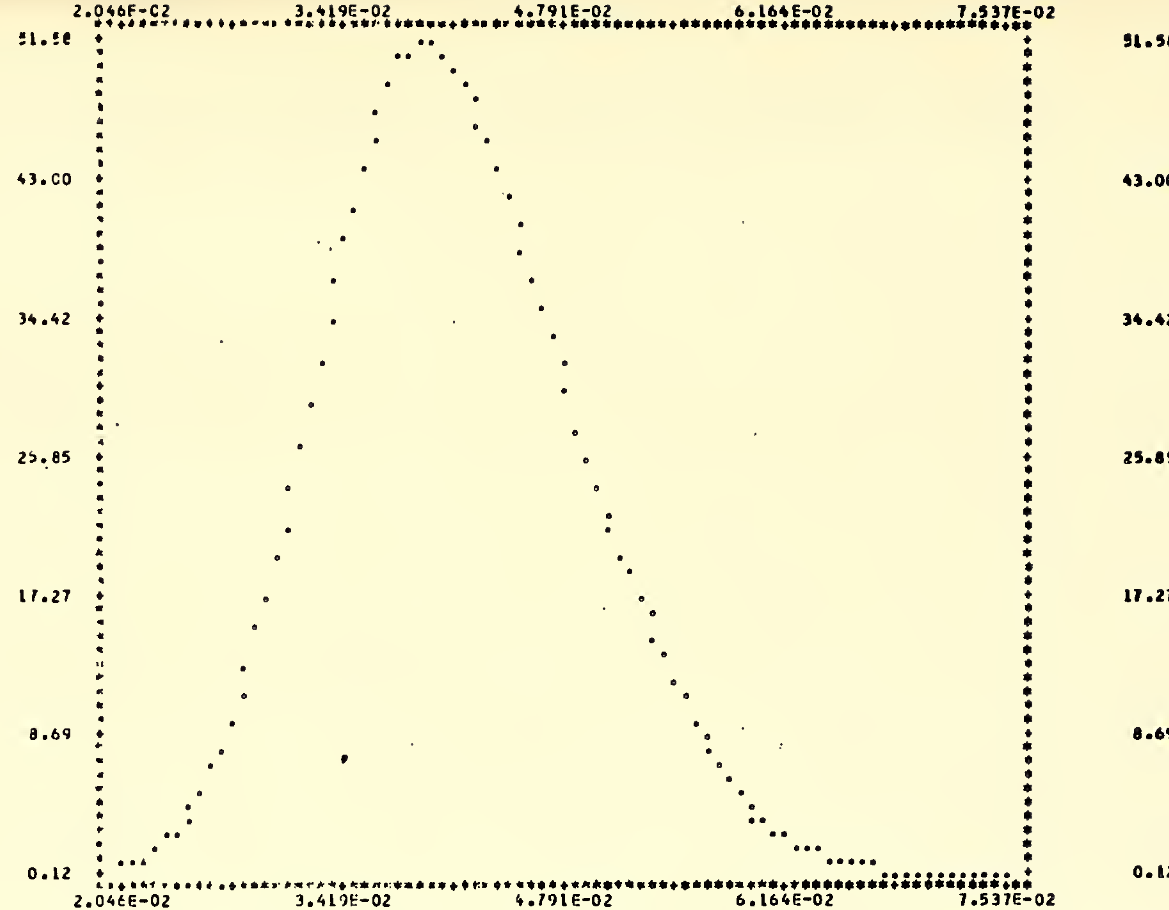


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	4.115786E-02	VARIANCE	5.824008E-05	M3	1.850365E-07	MINIMUM	2.045888E-02
MEDIAN	4.053637E-02	STD DEV	7.631518E-03	M4	1.090378E-08	.10 QUANTILE	3.190529E-02
TRIMEAN	4.074845E-02	COEF VAR	1.852406E-01	SKEWNESS	4.163175E-01	.25 QUANTILE (HINGE)	3.589308E-02
MIDMEAN	4.070301E-02	RANGE	5.490798E-02	KURTOSIS	2.146444E-01	.50 QUANTILE (MEDIAN)	4.053637E-02
GEOM MEAN	4.045803E-02	MIDSPREAD	1.013494E-02			.75 QUANTILE (HINGE)	4.602802E-02
HARM MEAN	3.979941E-02					.90 QUANTILE	5.154393E-02
						MAXIMUM	7.536685E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 1000

TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)



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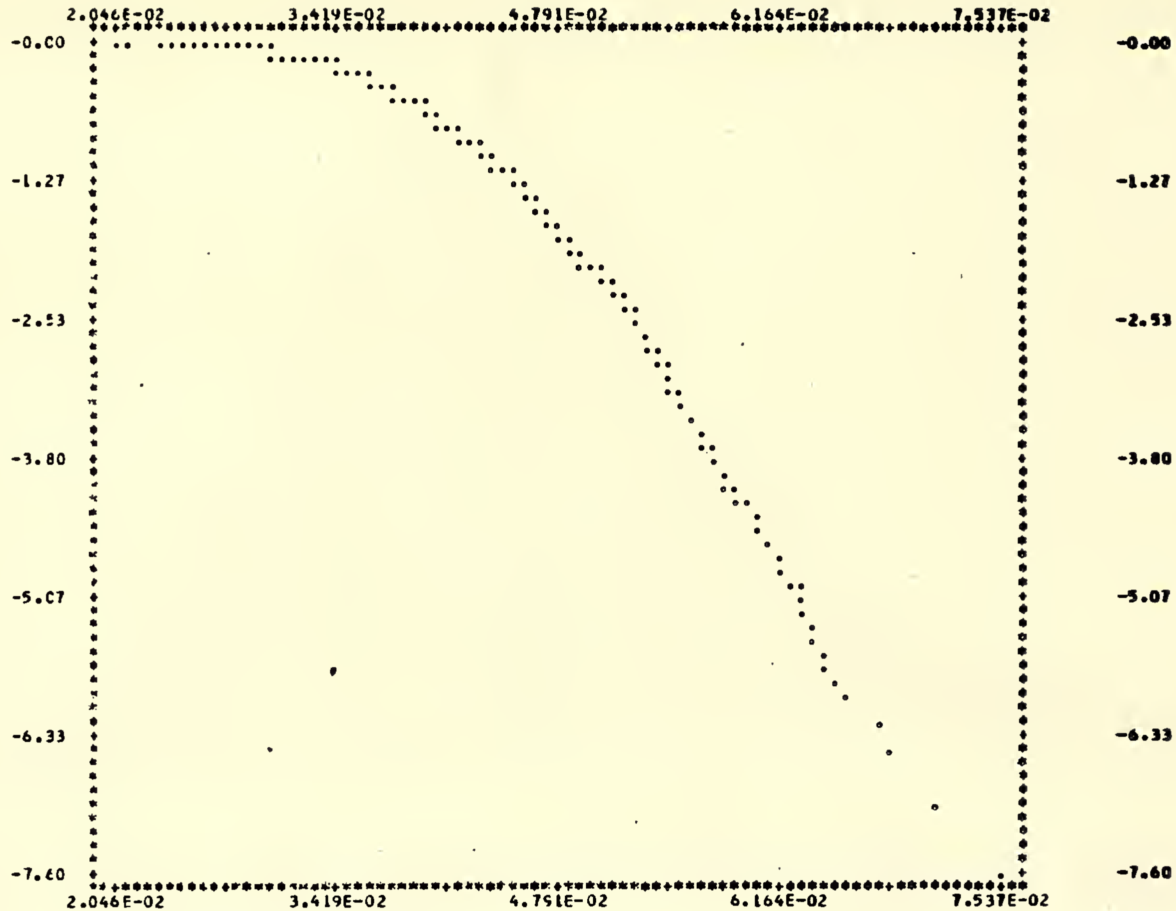
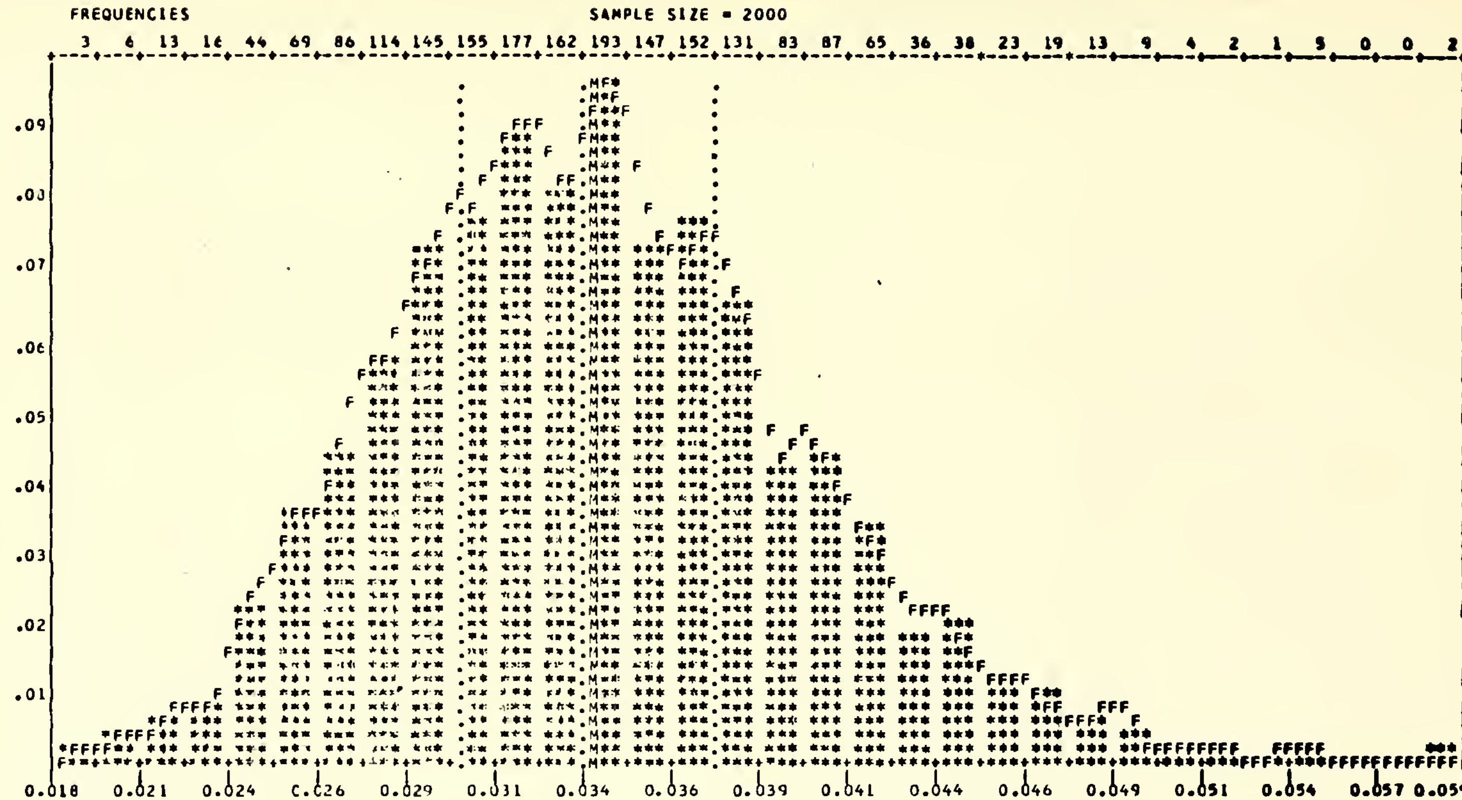
Y-SCALE: "*"= 0.858E 00 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "+"= 0.686E-03 UNITS
Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X   EMPIRICAL LOG SURVIVAL FUNCTION

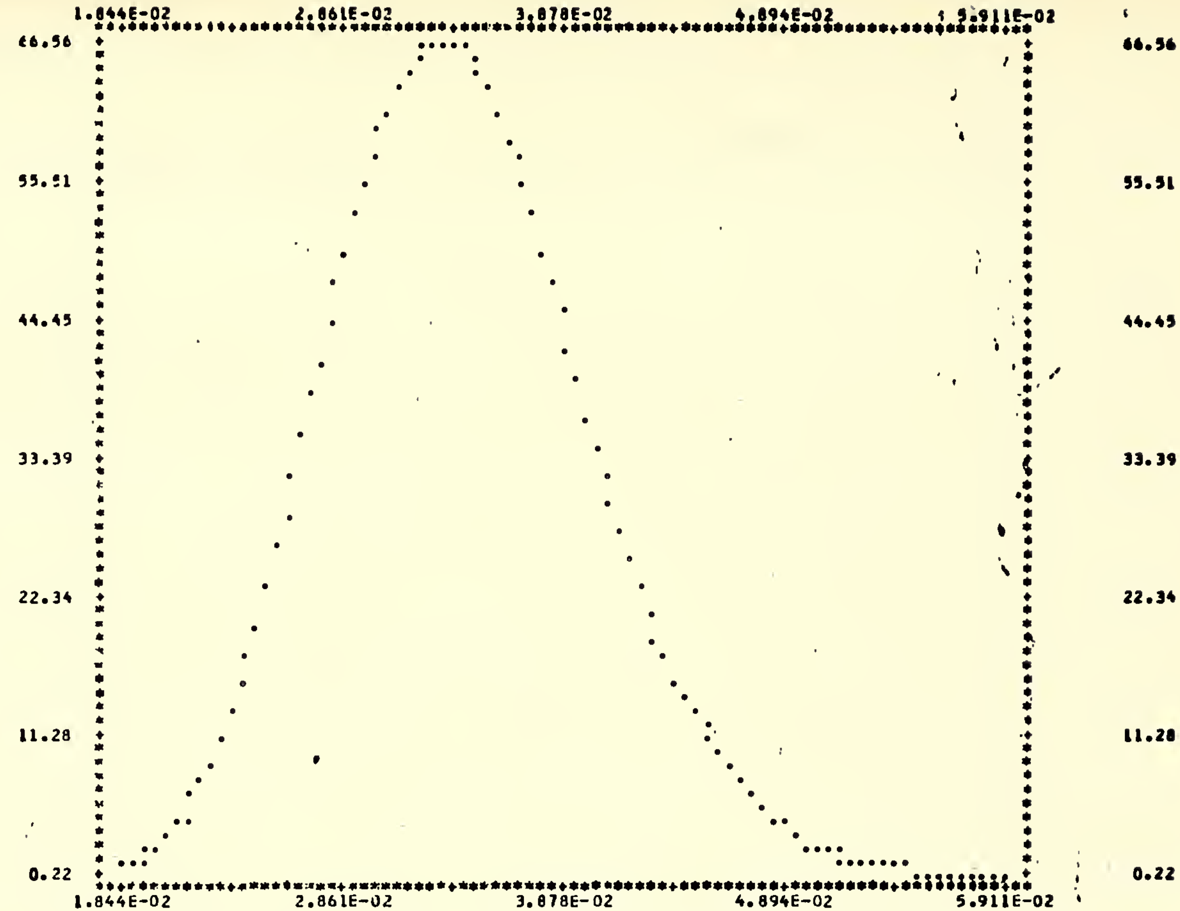


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	3.404245E-02	VARIANCE	3.410765E-05	M3	8.600512E-08	MINIMUM	1.844407E-02
MEDIAN	3.375116E-02	STD DEV	5.840175E-03	M4	3.904848E-09	.10 QUANTILE	2.685221E-02
TRIMEAN	3.375556E-02	COEF VAR	1.715556E-01	SKEWNESS	4.317644E-01	.25 QUANTILE (HINGE)	2.990774E-02
MIDMEAN	3.372147E-02	RANGE	4.066641E-02	KURTOSIS	3.566065E-01	.50 QUANTILE (MEDIAN)	3.375116E-02
GEOM MEAN	3.354767E-02	MIDSPREAD	7.704485E-03			.75 QUANTILE (HINGE)	3.761223E-02
HARM MEAN	3.305505E-02					.90 QUANTILE	4.161298E-02
						MAXIMUM	5.911048E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 1500

TRIANGULAR WINDOW. BANDWIDTH = 3/SQRT(N)



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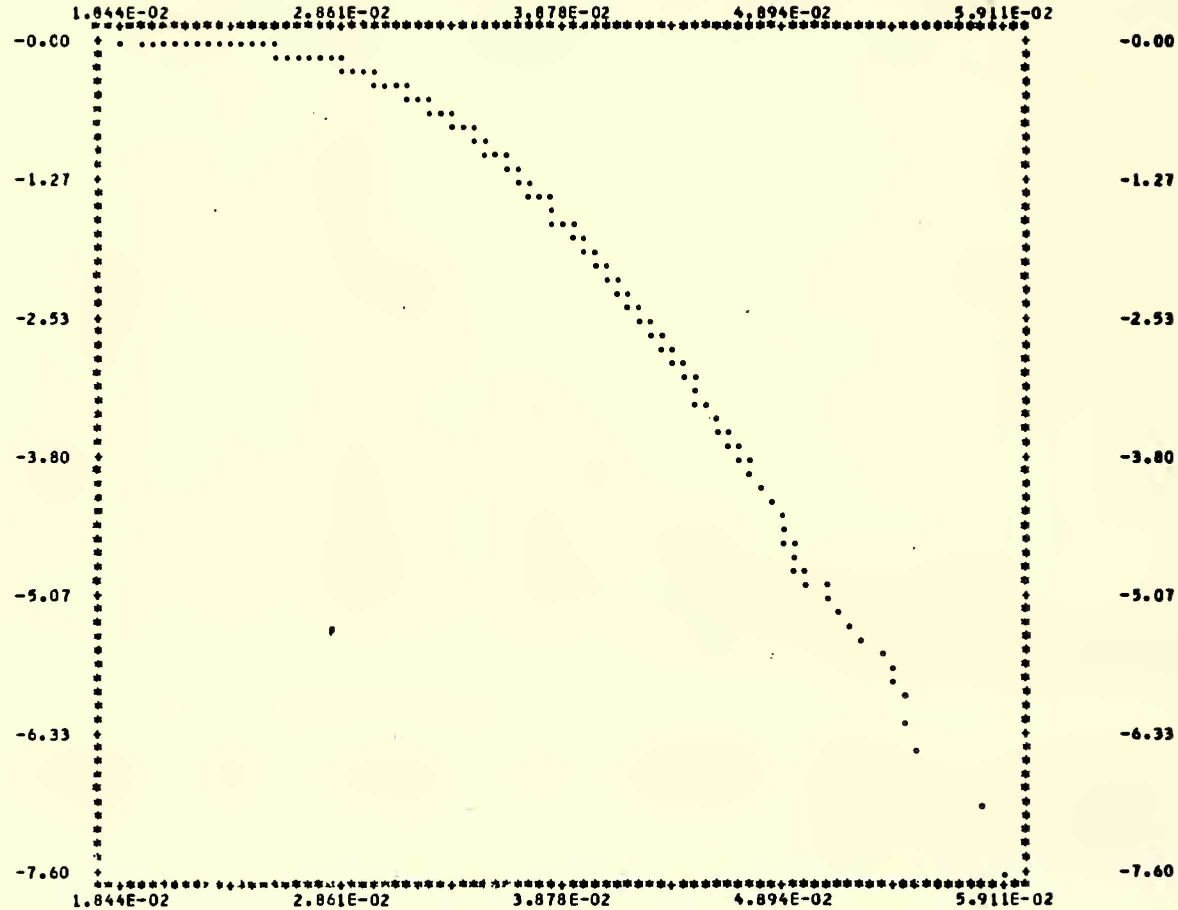
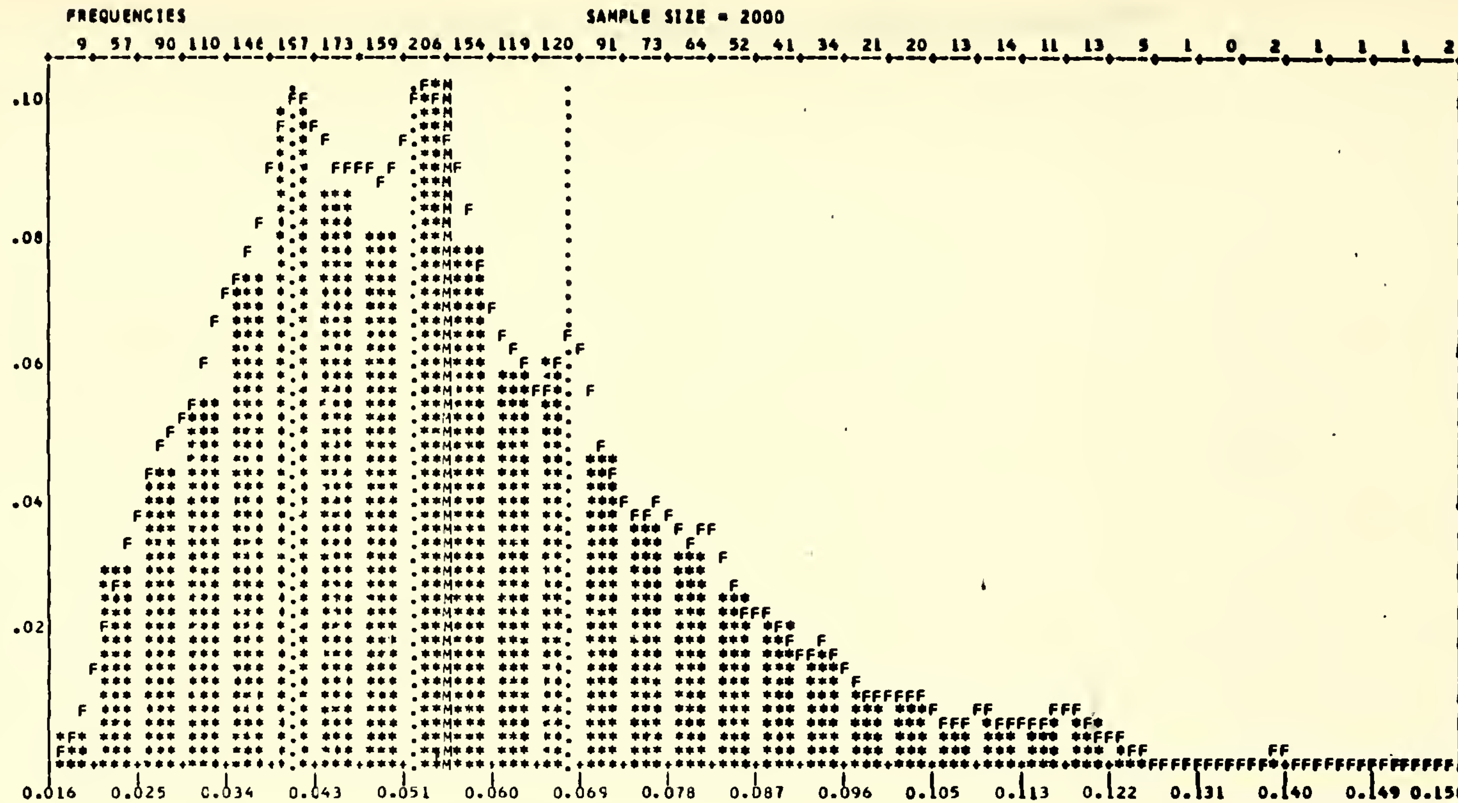
Y-SCALE: "*"= 0.111E 01 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.508E-03 UNITS
Y-SCALE: "+"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

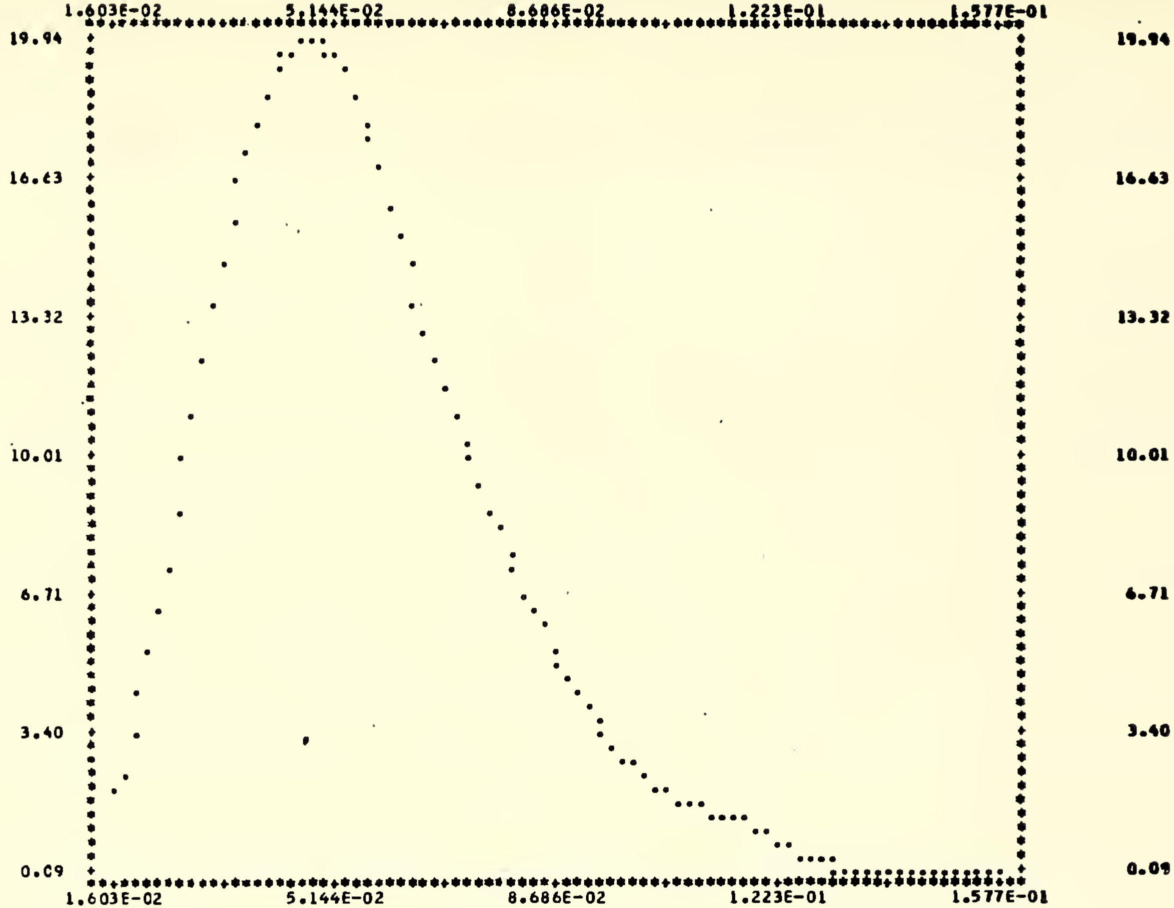


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	5.583763E-02	VARIANCE	4.645335E-04	M3	9.521510E-06	MINIMUM	1.602586E-02
MEDIAN	5.256751E-02	STD DEV	2.155304E-02	M4	8.931187E-07	.10 QUANTILE	3.143855E-02
TRIMEAN	5.332820E-02	COEF VAR	3.859948E-01	SKEWNESS	9.509995E-01	.25 QUANTILE (HINGE)	4.019031E-02
MIDMEAN	5.280042E-02	RANGE	1.416712E-01	KURTOSIS	1.138807E 00	.50 QUANTILE (MEDIAN)	5.256751E-02
GEOM MEAN	5.196208E-02	MIDSPREAD	2.779719E-02			.75 QUANTILE (HINGE)	6.798750E-02
HARM MEAN	4.828836E-02					.90 QUANTILE	8.459371E-02
						MAXIMUM	1.576971E-01

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 100

TRIANGULAR WINDOW. BANDWIDTH = 20/SQRT(N)



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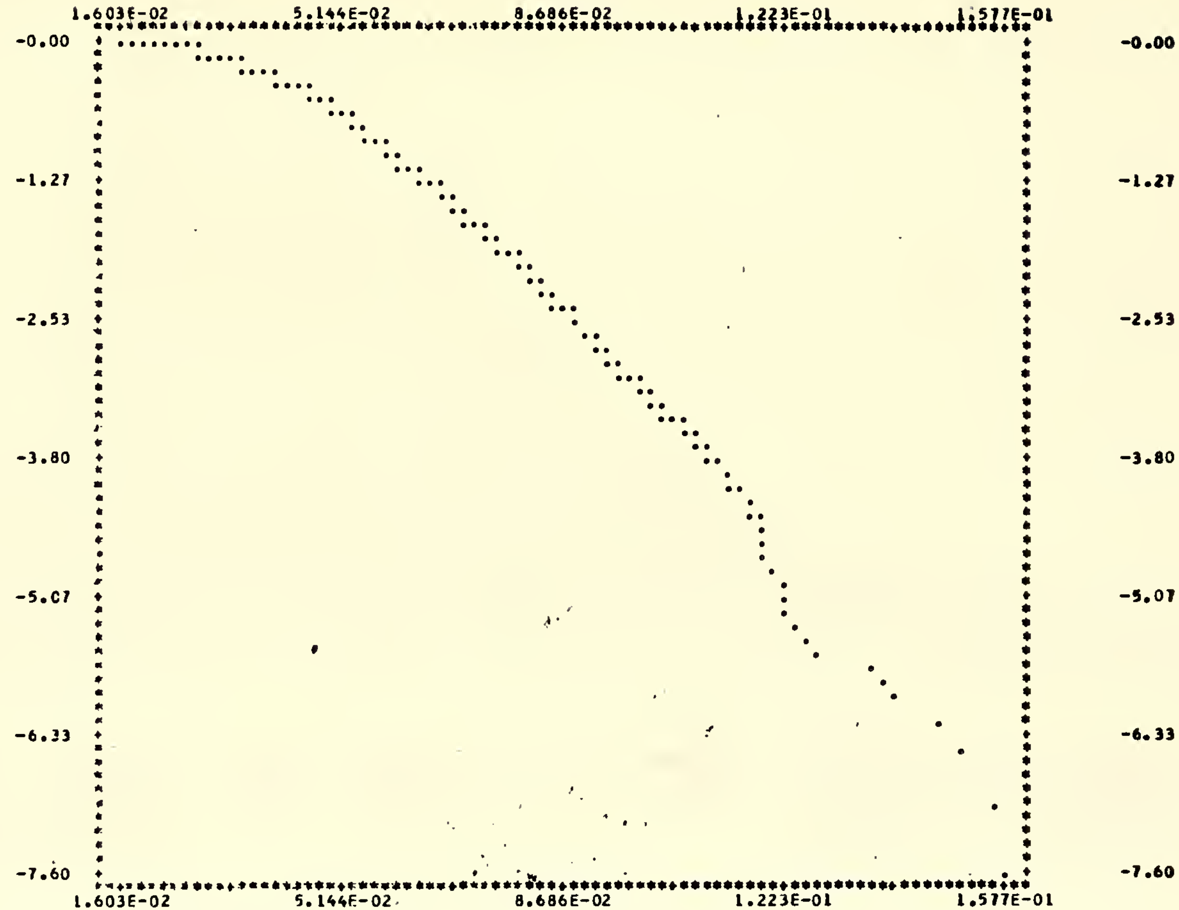
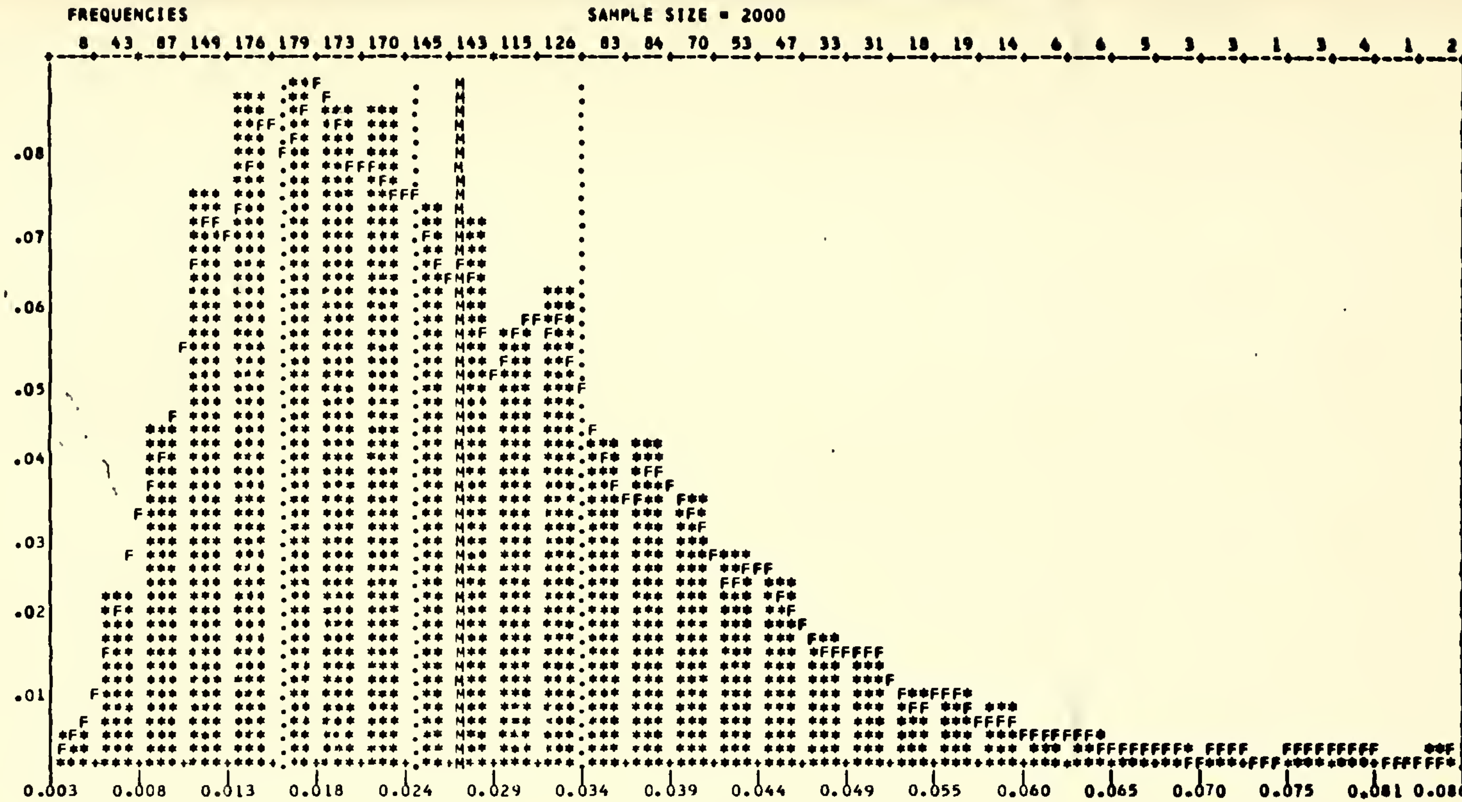
Y-SCALE: "*"= 0.331E 00 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.177E-02 UNITS
Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X   EMPIRICAL LOG SURVIVAL FUNCTION

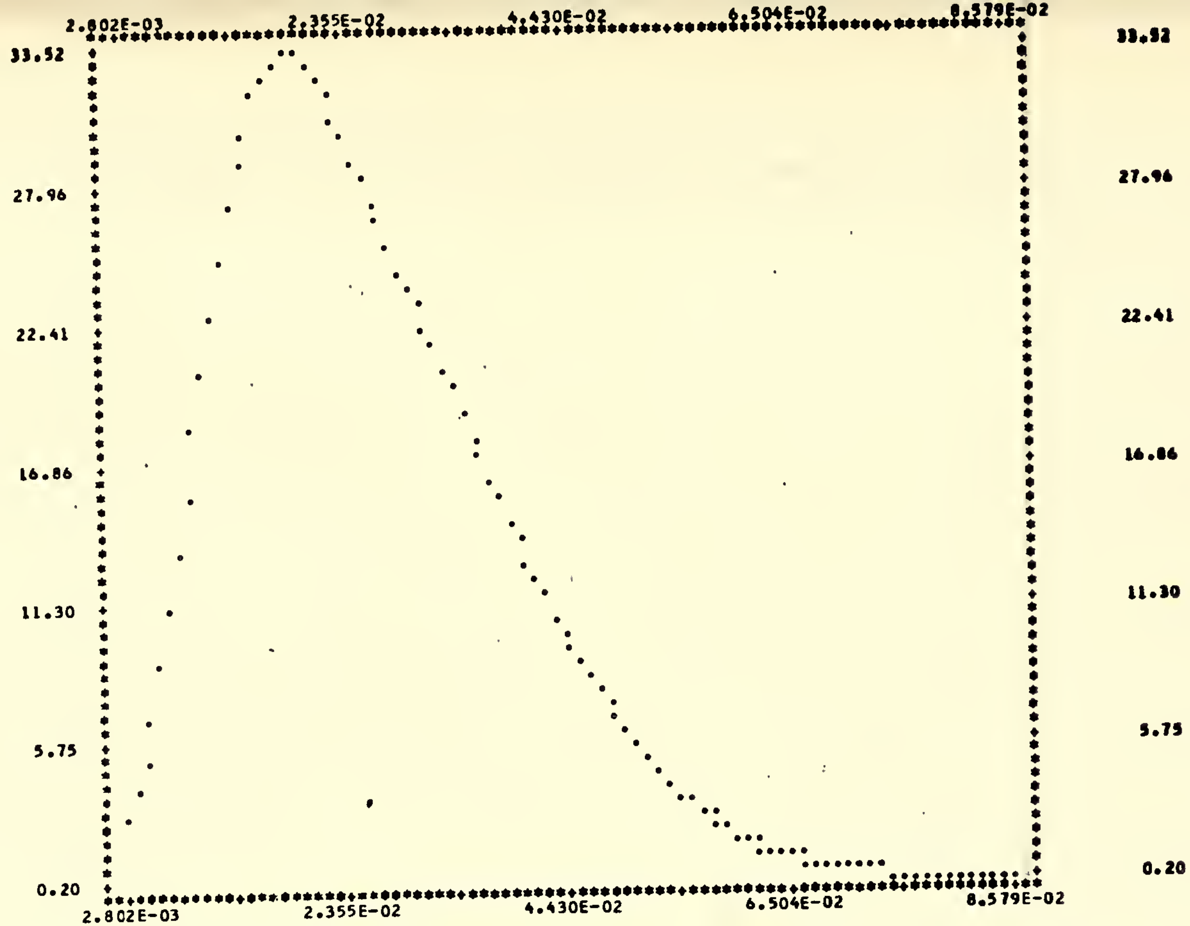


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	2.623237E-02	VARIANCE	1.708773E-04	M3	2.220775E-06	MINIMUM	2.802486E-03
MEDIAN	2.378632E-02	STD DEV	1.307201E-02	M4	1.225009E-07	.10 QUANTILE	1.166754E-02
TRIMEAN	2.436662E-02	COEF VAR	4.983159E-01	SKEWNESS	9.942096E-01	.25 QUANTILE (HINGE)	1.626213E-02
MIDMEAN	2.421955E-02	RANGE	8.298862E-02	KURTOSIS	1.195372E 00	.50 QUANTILE (MEDIAN)	2.378632E-02
GEOM MEAN	2.314303E-02	MIDSPREAD	1.736961E-02			.75 QUANTILE (HINGE)	3.363174E-02
HARM MEAN	2.013422E-02					.90 QUANTILE	4.411876E-02
						MAXIMUM	8.579111E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

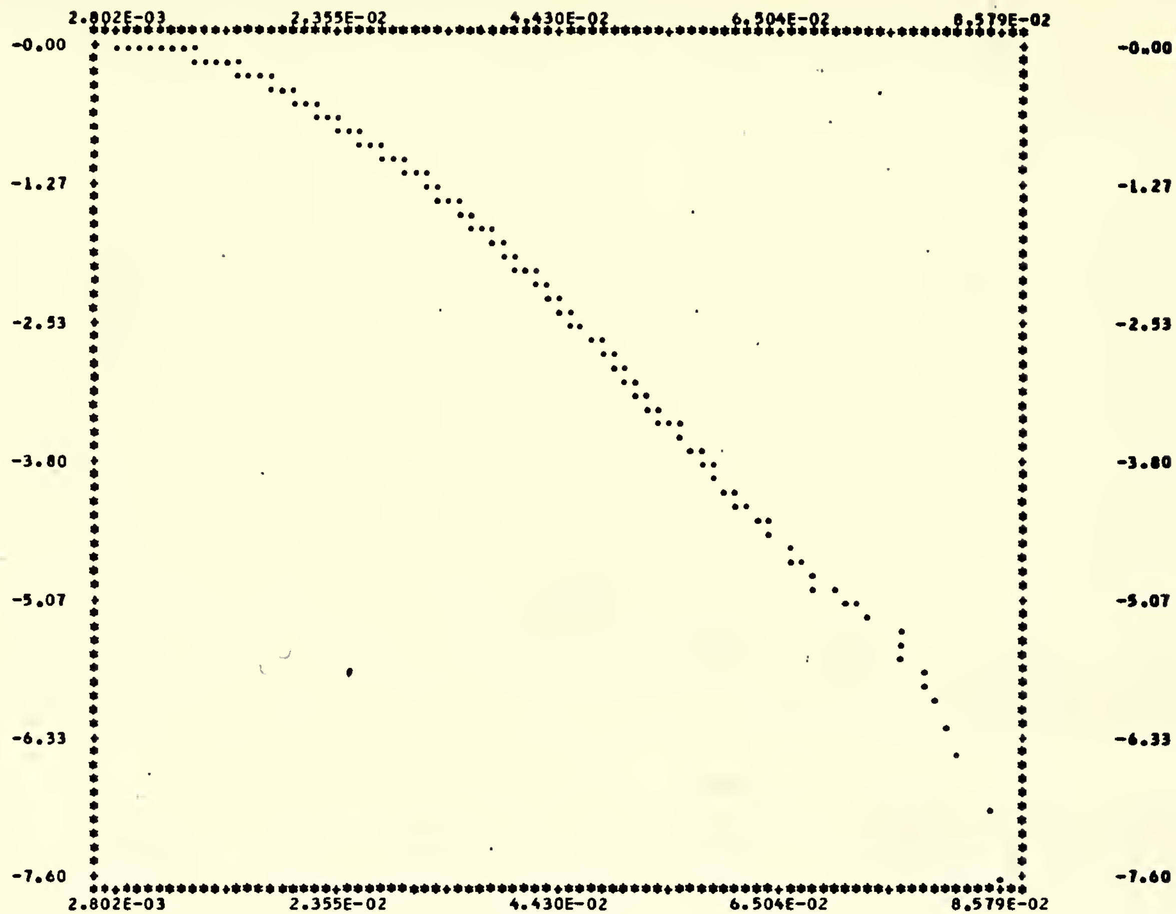
CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 200

TRIANGULAR WINDOW. BANDWIDTH = 20/SQRT(N)



X-SCALE: "*"= 0.104E-02 UNITS
Y-SCALE: "*"= 0.555F 00 UNITS

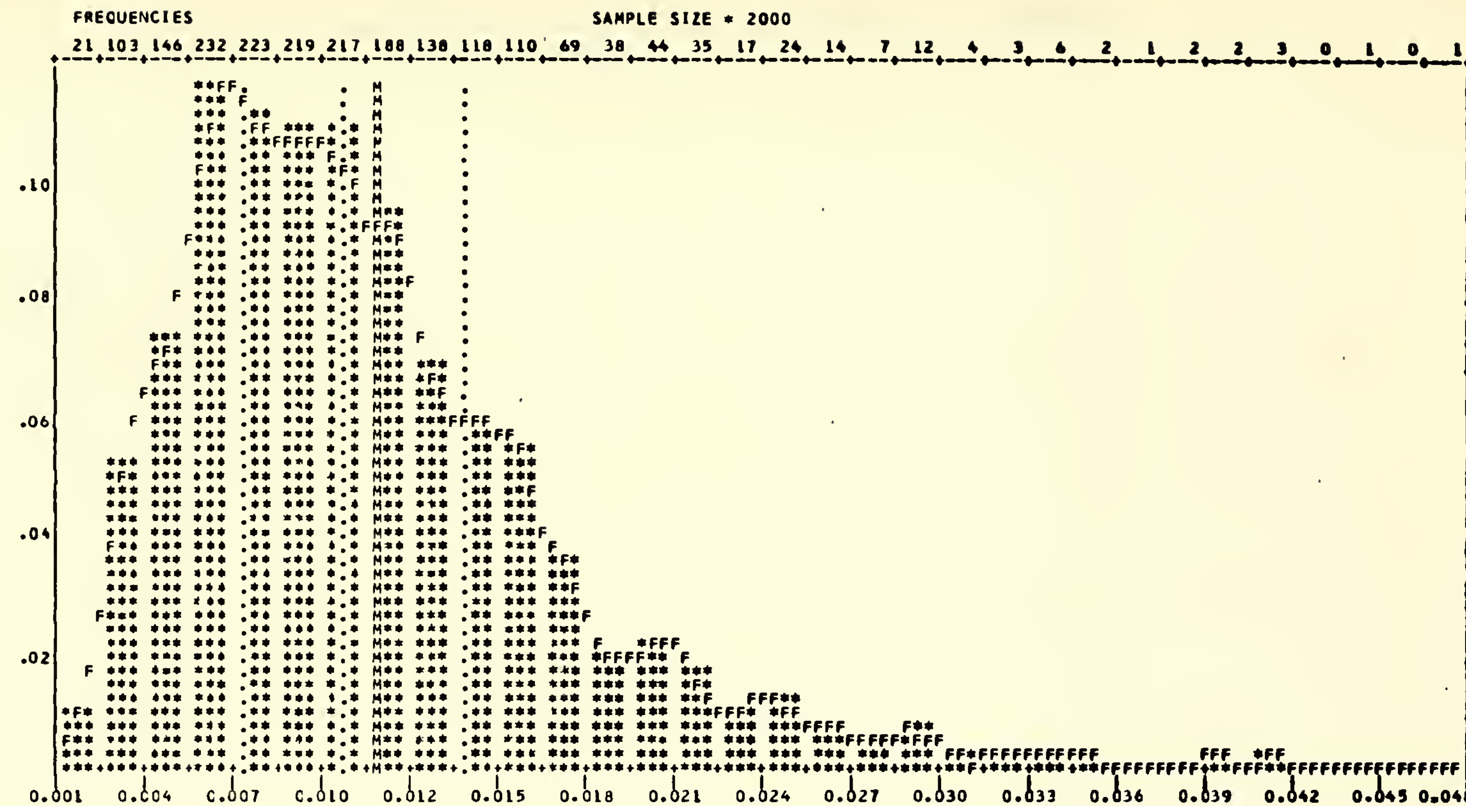
ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.104E-02 UNITS

Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X   EMPIRICAL LOG SURVIVAL FUNCTION

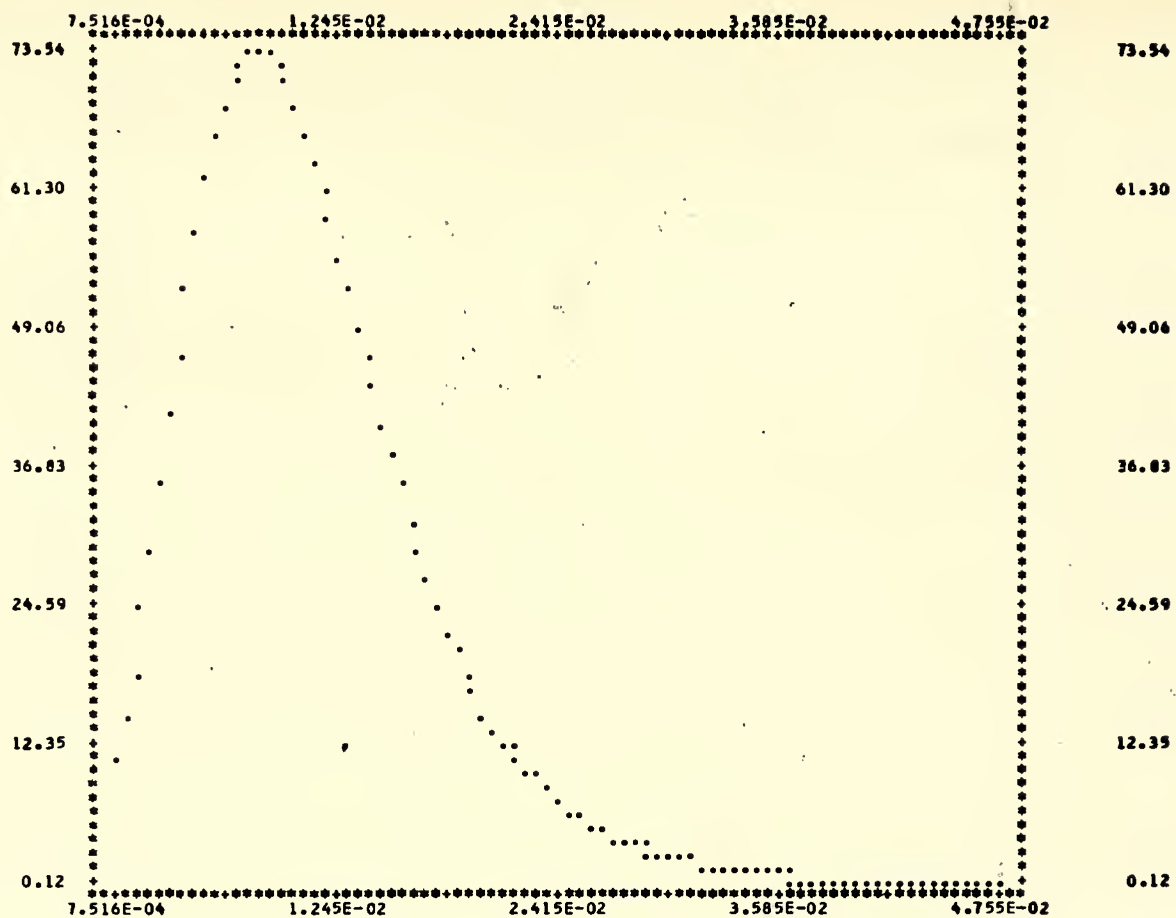


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION		
MEAN	1.104308E-02	VARIANCE	3.803970E-05	M3	3.256197E-07	MINIMUM		7.516474E-04
MEDIAN	9.863541E-03	STD DEV	6.167632E-03	M4	8.821925E-09	.10 QUANTILE		4.510917E-03
TRIMEAN	1.010973E-02	COEF VAR	5.585063E-01	SKEWNESS	1.387891E 00	.25 QUANTILE	(HINGE)	6.590415E-03
MIDMEAN	1.000951E-02	RANGE	4.679744E-02	KURTOSIS	3.096622E 00	.50 QUANTILE	(MEDIAN)	9.863541E-03
GEOM MEAN	9.487070E-03	MIDSPREAD	7.531032E-03			.75 QUANTILE	(HINGE)	1.412145E-02
HARM MEAN	7.965840E-03					.90 QUANTILE		1.887139E-02
						MAXIMUM		4.754909E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 500

TRIANGULAR WINDOW. BANDWIDTH = 20/SQRT(N)



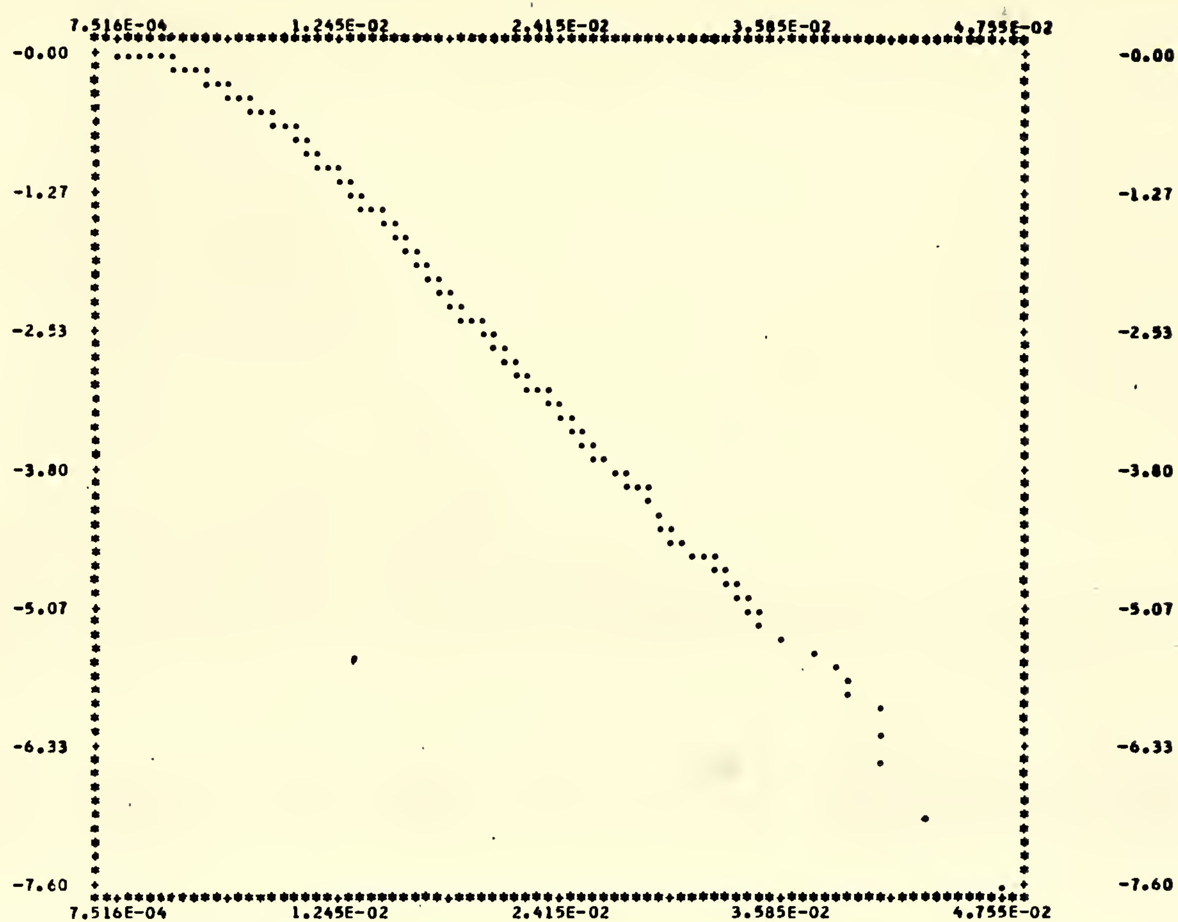
X-SCALE: "*"= 0.585E-03 UNITS

Y-SCALE: "*"= 0.122E 01 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE

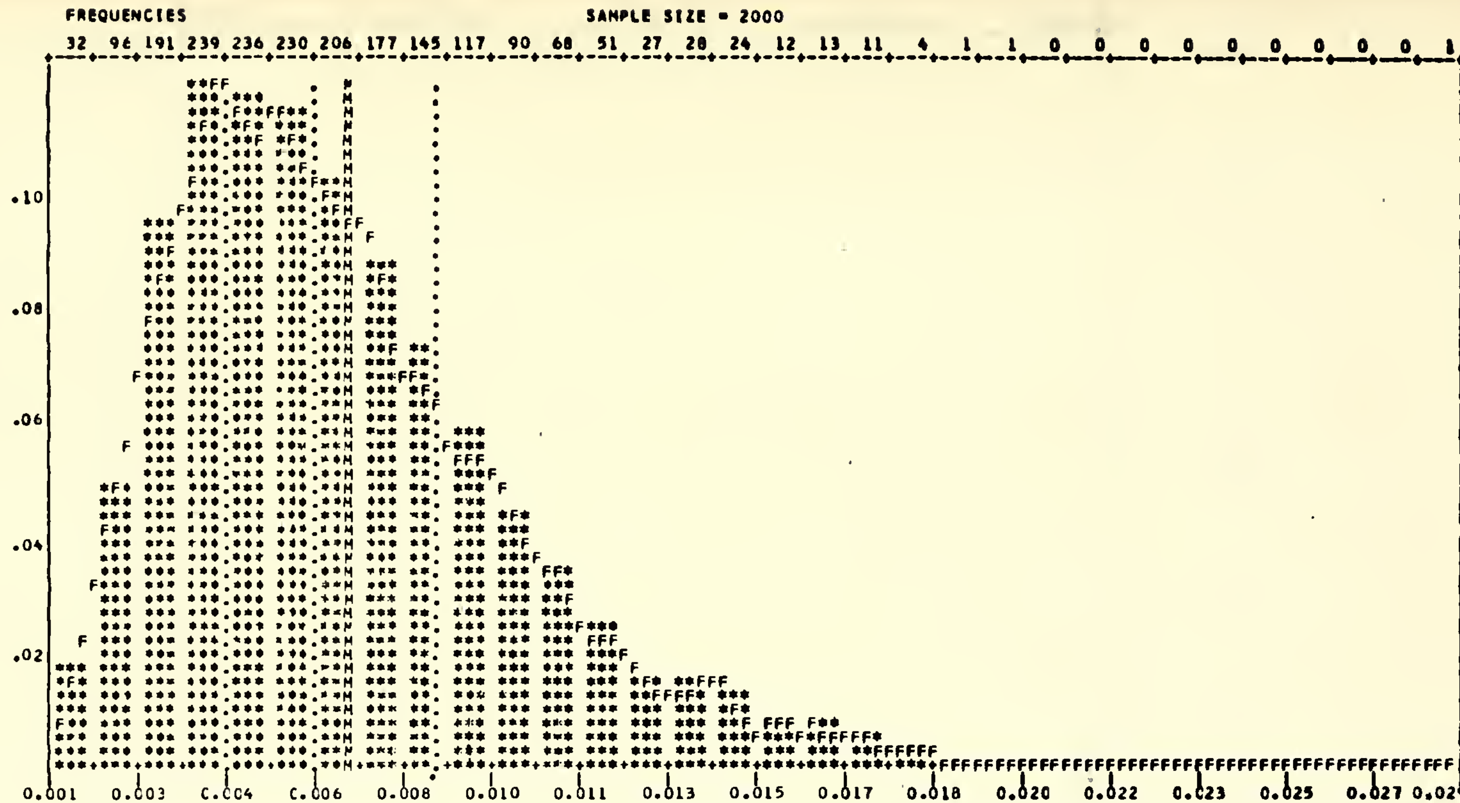
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10

ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.585E-03 UNITS
Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

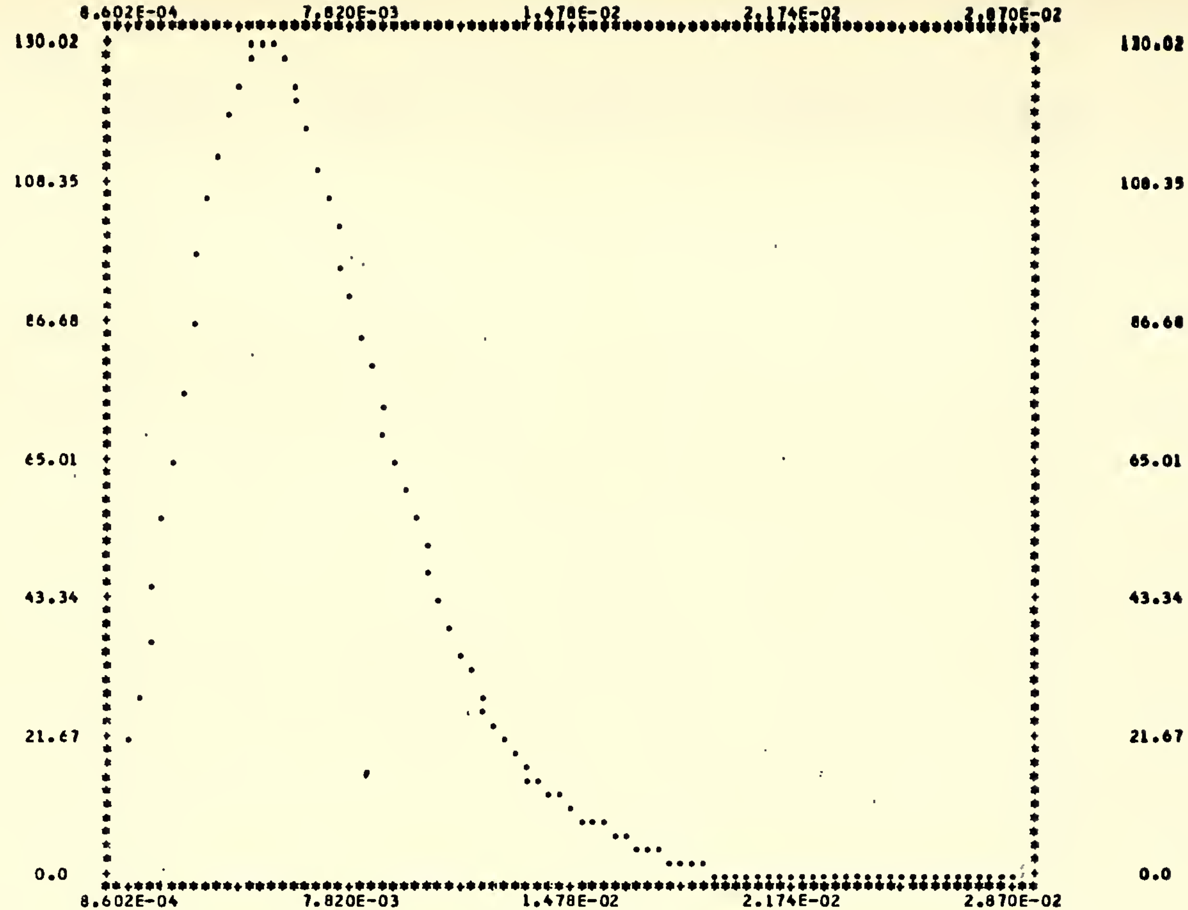


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	6.554537E-03	VARIANCE	1.049055E-05	M3	3.509636E-08	MINIMUM	8.602038E-04
MEDIAN	5.582481E-03	STD DEV	3.238912E-03	M4	5.101948E-10	.10 QUANTILE	2.959801E-03
TRIMEAN	6.118555E-03	COEF VAR	4.941482E-01	SKEWNESS	1.032914E 00	.25 QUANTILE (HINGE)	4.169609E-03
MIDMEAN	6.053288E-03	RANGE	2.784064E-02	KURTOSIS	1.635954E 00	.50 QUANTILE (MEDIAN)	5.982481E-03
GEOM MEAN	5.794682E-03	MIDSPREAD	4.170049E-03			.75 QUANTILE (HINGE)	8.339658E-03
HARM MEAN	5.046826E-03					.90 QUANTILE	1.090107E-02
						MAXIMUM	2.870085E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

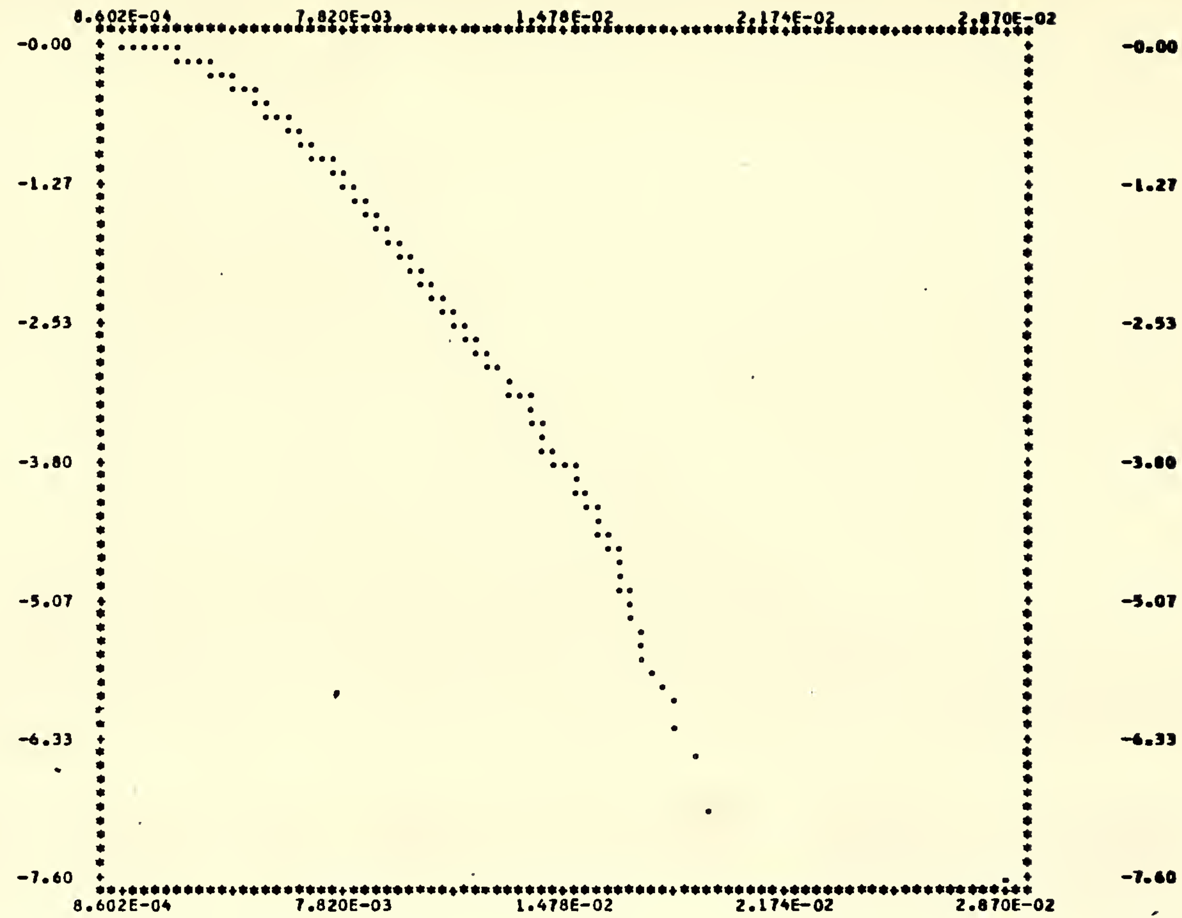
CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 1000

TRIANGULAR WINDOW. BANDWIDTH = 20/SQRT(N)



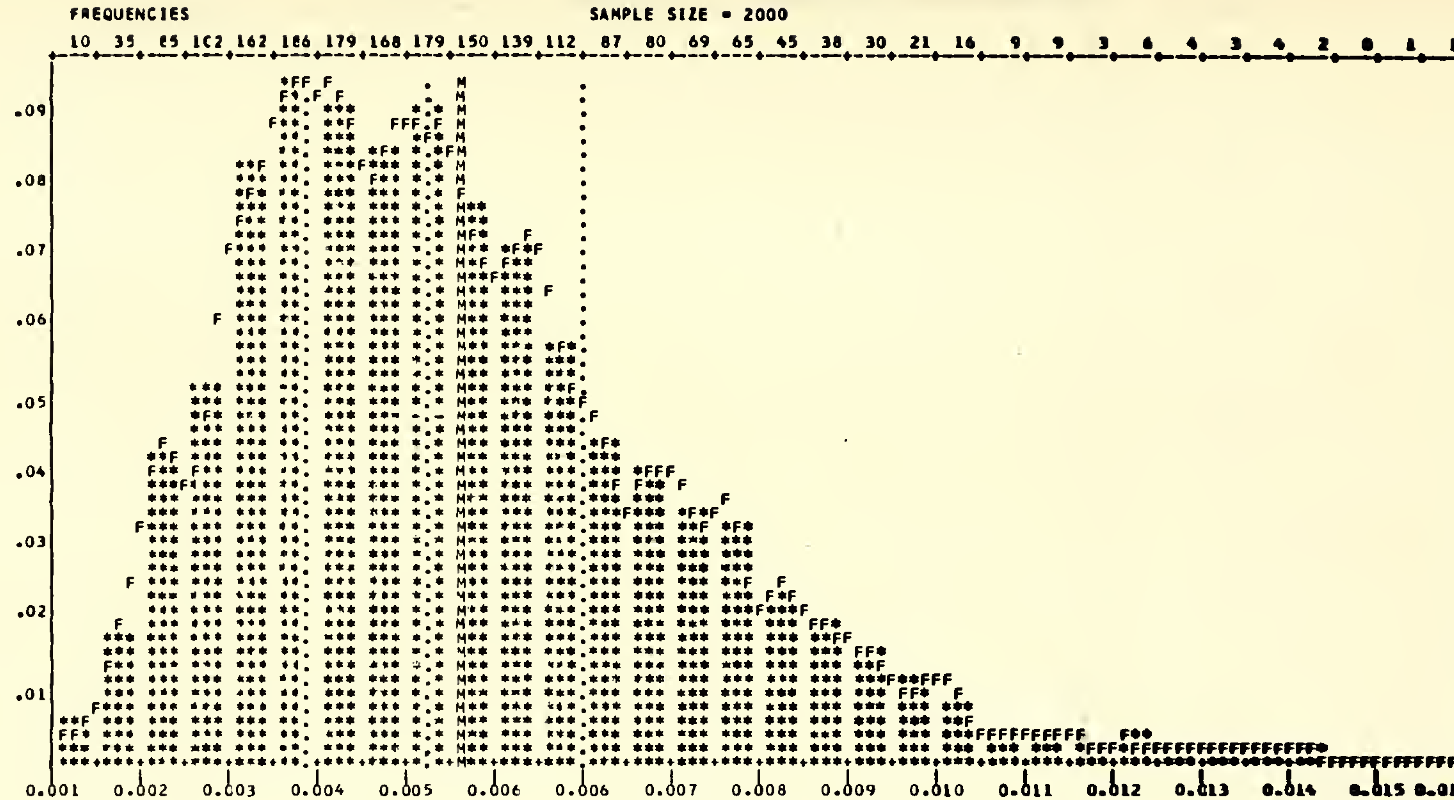
X-SCALE: "*"= 0.348E-03 UNITS
Y-SCALE: "*"= 0.217E 01 UNITS

ESTIMATED DENSITY FUNCTION VS INTEGRAL SQUARE NORM RANDOM VARIABLE
INTEGRAL SQUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINDOW BANDWIDTH = RANGE / 10
ESTIMATED DENSITY FUNCTION IS EVALUATED AT 100 EQUALLY SPACED POINTS



X-SCALE: "*"= 0.348E-03 UNITS
Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X   EMPIRICAL LOG SURVIVAL FUNCTION

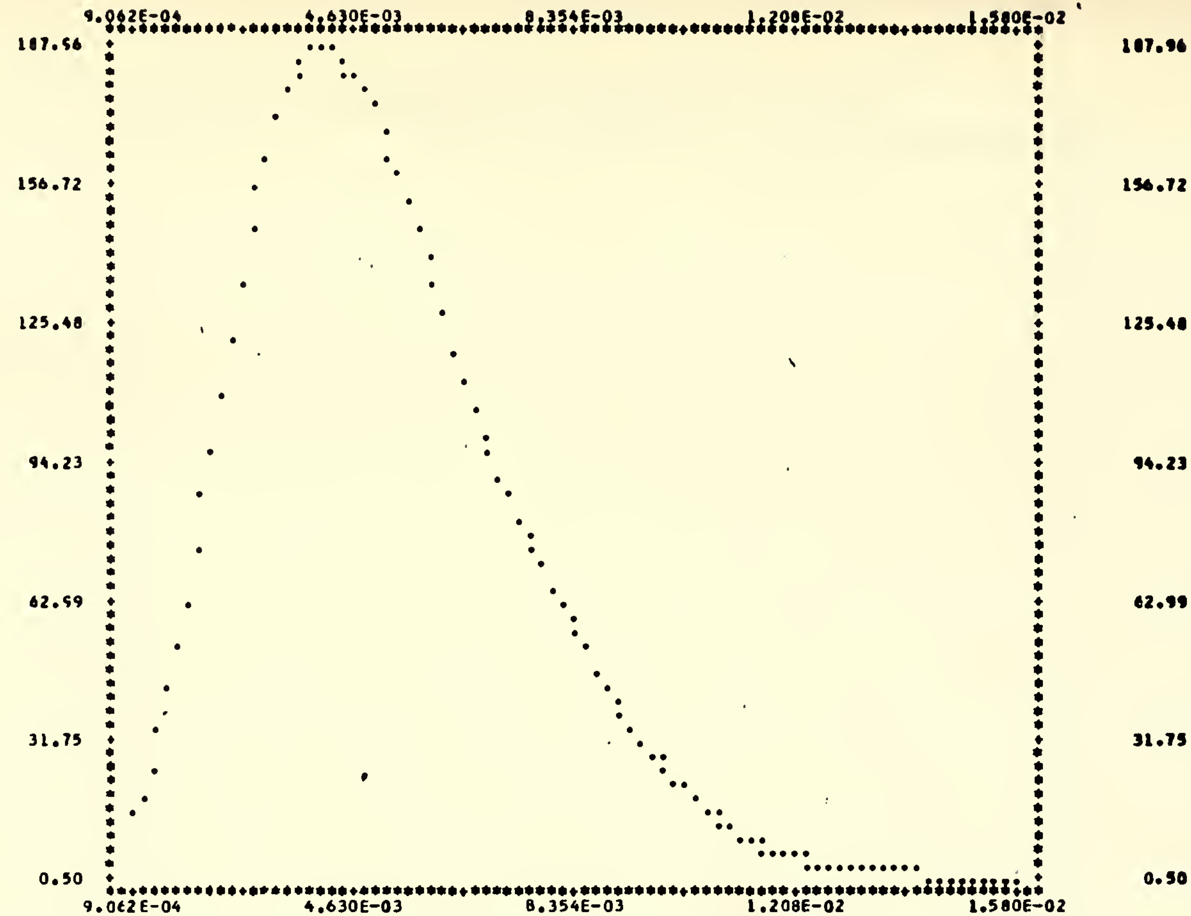


CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	5.186867E-03	VARIANCE	5.136757E-06	M3	1.065400E-08	MINIMUM	9.061934E-04
MEDIAN	4.822433E-03	STD DEV	2.266441E-03	M4	1.060338E-10	.10 QUANTILE	2.665251E-03
TRIMEAN	4.900172E-03	COEF VAR	4.369577E-01	SKEWNESS	9.151226E-01	.25 QUANTILE (HINGE)	3.502941E-03
MIDMEAN	4.861325E-03	RANGE	1.489536E-02	KURTOSIS	1.018522E 00	.50 QUANTILE (MEDIAN)	4.822433E-03
GEOM MEAN	4.714068E-03	MIDSPREAD	2.949947E-03			.75 QUANTILE (HINGE)	6.452888E-03
HARM MEAN	4.249580E-03					.90 QUANTILE	8.202024E-03
						MAXIMUM	1.580156E-02

INTEGRAL SQUARE NORM SAMPLE SIZE M = 2000

CAUCHY RANDOM VARIABLE SAMPLE SIZE N = 1500

TRIANGULAR WINDOW. BANDWIDTH = 20/SQRT(N)



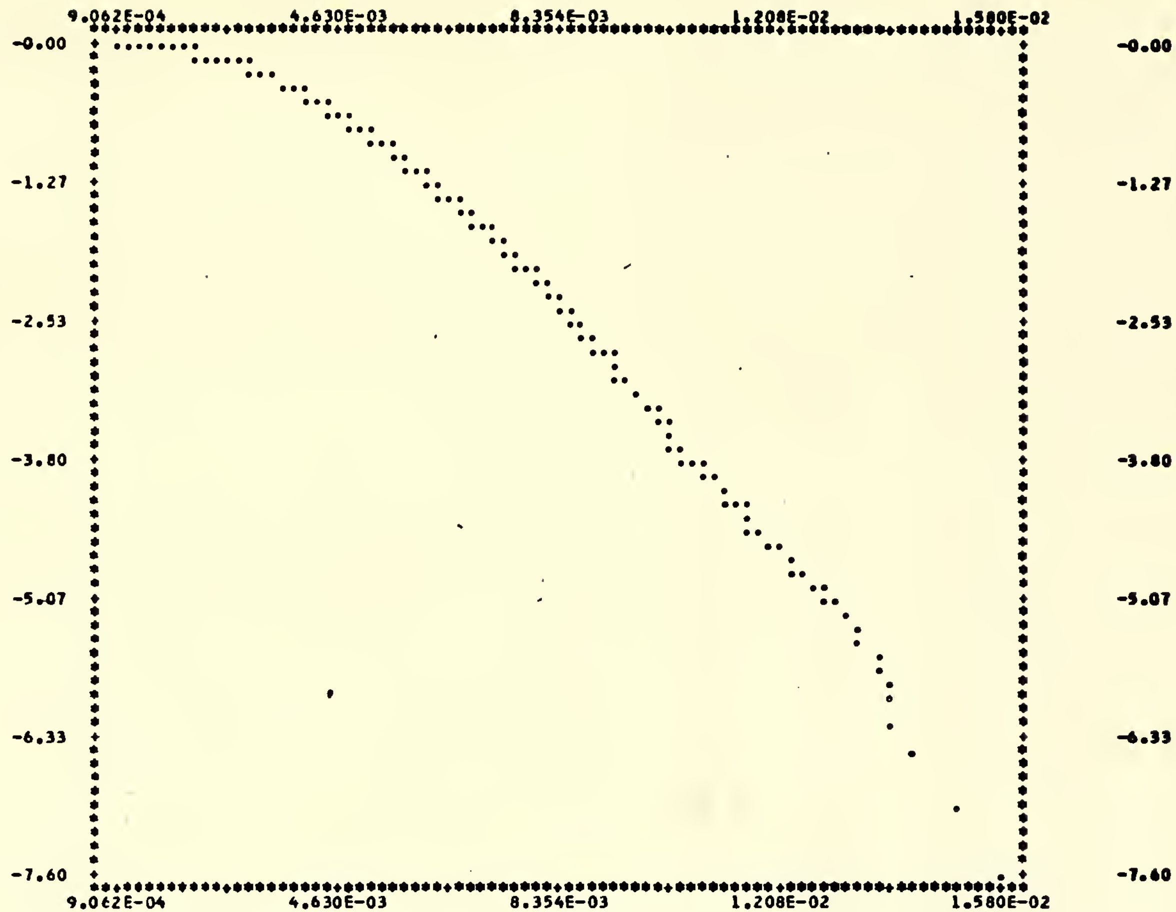
X-SCALE: "*"= 0.186E-03 UNITS

Y-SCALE: "*"= 0.312E 01 UNITS

ESTIMATEO OENSITY FUNCTION VS INTEGRAL SQUARE NCRM RANOOM VARIABLE

INTEGRAL SCUARE NORM SAMPLE SIZE = 2000 TRIANGULAR WINOOW BANOWIOTH = RANGE / 10

ESTIMATEO CENSITY FUNCTICN IS EVALUATED AT 100 EQUALLY SPACEO POINTS



X-SCALE: "*"= 0.186E-03 UNITS
Y-SCALE: "*"= 0.127E 00 UNITS

LN(1-I/N+1) VS X EMPIRICAL LOG SURVIVAL FUNCTION

LIST OF REFERENCES

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2. Bickel, P. J. and Rosenblatt, M., (1973). On some global measures of the deviations of density function estimates. The Annals of Mathematical Statistics, I, 1071-1095.

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5. Wegman, E. J., (1972). Non-parametric probability density estimation: I. A summary of available methods. Technometrics, 14, No. 3.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The distribution of a measure of distance between a probability density function and its estimate is examined through Monte Carlo methods. The estimate of the density function is that proposed by Rosenblatt using sums of weight functions centered at the observed values of the random variables. The weight function used in all cases was triangular, but both uniform and Cauchy densities were tried for different sample sizes and bandwidths. The simulated distributions appear in all cases to be close to Gamma distributions, but it

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has not been possible to relate the parameters to the population distribution characteristics or to the window shape and bandwidth.

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