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A refinement of an established method for computing confidence limits on system reliability.

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A REFINEMENT OF AN ESTABLISHED METHOD FOR
COMPUTING CONFIDENCE LIMITS ON
SYSTEM RELIABILITY

JAMES ANDREW GRESS

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A REFINEMENT OF AN ESTABLISHED

METHOD FOR COMPUTING CONFIDENCE

LIMITS ON SYSTEM RELIABILITY

by

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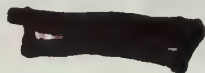
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ABSTRACT

A statement of a method [1] for obtaining an approx-
imate lower 100(1-α)% confidence limit on System Reliabil-
ity is made. A discussion of evaluating the accuracy of
this method on a digital computer is presented. Following
this the development of a continuity correction factor is
developed and the accuracies of this refinement for
several hypothetical systems are presented and discussed.

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CHAPTER I

PURPOSE OF THE STUDY

In past years our advancing technology has created
systems which are becoming more and more complex. In the
military this has been particularly true, and thus, has
produced a great need for a method of predicting the over-
all reliability of these systems. Many problems in the
area of reliability have not been solved. One of the
unsolved problems is the formulation of a method for compu-
ting a confidence interval on system reliability where the
reliability of the system is computed as the product of the
reliabilities of each of the components, and the sizes of
the test samples of the components are not all equal. If
the test samples are equal, there exist well-known methods
for computing a confidence interval. Because of the size
and complexity of present day systems it is necessary for
major contractors to have many sub-contractors. They may
be scattered througout the country and may produce a limited
supply of each component, of which very few may be used for
reliability testing. These factors plus the general practice
of testing before the system is assembled points out that
having the sample sizes all equal is a rare event.

An ad hoc procedure for computing an approximate con-
fidence interval has been developed, based upon Bernoulli
test data from unequal samples of the various components of
the system [1]. This technique will be called Method I
throughout this document.

An underlying assumption to this procedure is that the distribution of the unbiased estimator of the negative natural logarithm of system reliability may be approximated by a two-parameter Gamma probability distribution. Consequently, a discrete set of values are approximated by a continuous function. Realizing the inherent error in this approximation this Thesis has as its purpose

    (1) to develop a correction factor to reduce this error; and

    (2) to simulate the distribution of the lower confidence limit of various systems and examine the results achieved with and without this correction.

The fruitfulness of the investigation that follows is demonstrated by a discussion of the results. In later chapters, it will be explained that the α th percentile of the simulated distribution of the lower confidence limit, called $R_{s,L(\alpha)}$, and the true reliability of the system, R_s, should be the same; therefore, the difference will be used as a measure of the accuracy of Method I. Consider the following example: a system is made up of 15 components; 14 have a sample size of 20 to be tested; and each a failure probability of .995; one has a sample size of 150 and probability of failure .850. The true reliability, R_s, is .7924. The α th percentile point of the distribution of the lower $100(1-\alpha)\%$ confidence limit, without correction were computed

to be:

α	$R_{s,L(\alpha)}$	$\lvert R_{s,L(\alpha)} - R_s \rvert$
.05	.8170	.0246
.10	.8175	.0251
.20	.8000	.0076

Now by adding the correction factor the results became:

α	$R_{s,L(\alpha)}$	$\lvert R_{s,L(\alpha)} - R_s \rvert$
.05	.8113	.0190
.10	.8135	.0211
.20	.7969	.0045

The above example certainly did not have the best accuracy of the various systems chosen nor was it the worst. The following summary shows the average and variance of the error in all the cases studied.

Without correction:

α	Average Error	Variance of Error
.05	.0438	.00213
.10	.0305	.00076
.20	.0109	.00016

With correction:

α	Average Error	Variance of Error
.05	.0286	.00041
.10	.0168	.00015
.20	.0056	.00005

The next chapter plus the appendices contains a complete explanation of Method I and discusses some further approximations made in the method.

Chapter three discusses the digital computer's role in obtaining the distribution of the lower confidence limit of

the reliability by Method I. Also mentioned is the measure of accuracy used to evaluate the method's precision.

Chapter four outlines the reason for the correction factor and discusses the logic behind its development and application to this method.

In Chapter five the results of various computations are presented and discussed.

The last chapter summarizes the study and states the conclusions.

CHAPTER II

EXPLANATION OF METHOD I [1]

If a system consists of k components in logical series, the true system reliability, R_s, may be expressed as follows:

$$R_s = \prod_{i=1}^{k} p_i \qquad (1)$$

where p_i is the true reliability of the ith component. The component may be either continuous-operating or cycle type.

It is desired to find a lower $100(1-\alpha)\%$ confidence interval for R_s. That is, a statistic $R_{s,L(\alpha)}$ such that whatever the actual values of p_1, p_2,....p_k,

$$P[R_s \geq R_{s,L(\alpha)}] = 1-\alpha$$

The method proposed is to put n_i items of component i on test, i=1, 2, ..., k; under the environmental conditions defined in the mission and let each operate until failure or the mission time is reached, whichever occurs first. Letting

$$f_i = \text{number of components of type i}$$
$$\text{that didn't complete their mission.}$$

We define: $\hat{p}_i = 1-\hat{q}_i$ where $\hat{q}_i = \dfrac{f_i}{n_i}$ \qquad (2)

and $\qquad \hat{R}_s = \prod_{i=1}^{k} \hat{p}_i$

The random variable $=\ln\hat{R}_s$ is now used to obtain the lower confidence bound on R_s. This is accomplished by approximating the distribution of $-\ln\hat{R}_s$ by a two parameter gamma distribution and obtaining the confidence bound via the gamma distribution.

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To continue the development define:

$$S \equiv -\ln R_s = - \sum_{i=1}^{k} \ln (1-q_i) \qquad (3)$$

and, expanding the natural logarithm in an infinite series

$$S = - \sum_{i=1}^{k} \left[(-q_i) - \frac{1}{2}(-q_i)^2 - \frac{1}{3}(-q_i)^3 \ldots \right]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{\infty} \frac{q_i^{\ j}}{j} \qquad (4)$$

If each q_i is small, say, .15 or less, the above series can be approximated by the first two terms of the series, called T_i for ease of expression. That is,

$$S \doteq \sum_{i=1}^{k} \left[q_i + \frac{q_i^{\ 2}}{2} \right] \equiv \sum_{i=1}^{k} T_i \qquad (5)$$

In Appendix III [4], it is shown that the error due to the above truncation is quite small; in fact;

$$S \leq \sum_{i=1}^{k} T_i + \sum_{i=1}^{k} \frac{q_i^{\ 3}}{3(1-q_i)} \qquad (6)$$

In Appendix I [4], an unbiased estimator for T_i is developed and is called $\hat{T}_i$.

$$\hat{T}_i = a_i \hat{q}_i + b_i \frac{\hat{q}_i^{\ 2}}{2} \qquad (7)$$

where

$$a_i = \frac{2n_i - 3}{2(n_i - 1)} \qquad (8)$$

$$b_i = \frac{n_i}{n_i - 1} \qquad (9)$$

and as in (2)

$$\hat{q}_i = \frac{f_i}{n_i}$$

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An unbiased estimator is considered important here because the $\hat{T}_i$ are to be added to obtain:

$$\hat{S} = \sum_{i=1}^{k} \hat{T}_i \qquad (10)$$

Therefore, a positive or negative bias on all of the $\hat{T}_i$ would be undesirable. It also follows that $\hat{S}$ will be an unbiased estimator of S.

An approximate value for the variance of $\hat{S}$ can be shown to be (see Appendices II and IV of [4])

$$\text{Var}(\hat{S}) = \sum_{i=1}^{k} \text{Var}(T_i) \doteq \sum_{i=1}^{k} \frac{\hat{T}_i}{n_i} \qquad (11)$$

The distribution of $\hat{S}$ is now fitted with a two parameter gamma family. That is, it is assumed the probability distribution of $\hat{S}$ is given by the density function $f_{\hat{S}}$ (x,r,Θ) defined as:

$$f_S(x,r,\Theta) = \frac{1}{\Gamma(r)\Theta^r} x^{r-1} e^{-\frac{x}{\Theta}}, x>0, r>0, \Theta>0$$

$$= 0 \quad \text{elsewhere.}$$

Estimates of $\hat{r}$ and $\hat{\Theta}$ are found by the Method of Moments and the fact that $\frac{2\hat{S}}{\Theta}$ is a χ^2_{2r} random variable is used to obtain a confidence interval for $-\ln R_s$ and, thus, from this the confidence interval for R_s can be derived.

Since $\hat{S}$ is assumed to be gamma distributed:

$$E[\hat{S}] = \Theta r \qquad (12)$$

$$\text{Var}[\hat{S}] = \Theta^2 r \qquad (13)$$

But also from (10) that

$$E[\hat{S}] = \sum_{i=1}^{k} T_i$$

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and

$$\text{Var}[\hat{S}] \doteq \sum_{i=1}^{k} \frac{T_i}{n_i}$$

Thus, solving the above equations for r and Θ, the following expressions are obtained:

$$r = \left(\sum_{i=1}^{k} T_i \right)^2 \bigg/ \sum_{i=1}^{k} \frac{T_i}{n_i} \qquad (14)$$

$$\Theta = \sum_{i=1}^{k} \frac{T_i}{n_i} \bigg/ \sum_{i=1}^{k} T_i \qquad (15)$$

Thus, an estimator $\hat{r}$, for r is taken as

$$\hat{r} = \left(\sum_{i=1}^{k} \hat{T}_1 \right)^2 \bigg/ \sum_{i=1}^{k} \frac{\hat{T}_i}{n_i} \qquad (16)$$

Now to obtain the confidence interval the following procedure is used. Since $\frac{2\hat{S}}{\Theta}$ is a χ_{2r}^2 random variable and $\Theta r = -\ln R_s$, the following probability statements can be made:

$$1-\alpha = P\left[\frac{2S}{\Theta} \geq \chi_{1-\alpha,2r}^2 \right]$$

$$P\left[\Theta r \leq \hat{2}S \cdot \frac{r}{\chi_{1-\alpha,2r}^2} \right]$$

$$P\left[-\ln R_s \leq 2\hat{S} \cdot \frac{r}{\chi_{1-\alpha,2r}^2} \right]$$

$$P\left[-\ln R_s \leq 2\hat{S} \cdot \frac{r}{\chi_{1-\alpha,2r}^2} \right] \qquad (17)$$

It has been noticed that this last statement has an important property; namely

$$\frac{2r}{\chi_{1-\alpha,2r}^2} \qquad (18)$$

is almost independent of r. As an example if $\alpha = .10$ for

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$r = 6$ the ratio is $\frac{2}{3.5}$ and for $r = 12$ the ratio is $\frac{2}{3.6}$.
It is observed that r can be varied by a factor of 2 and
still have nearly the same value. This effect is even
less for larger values of r. This suggests that the prob-
ability distribution of

$$\frac{[2\hat{r}]}{\chi^2_{1-\alpha,[2\hat{r}]}} \quad (19)$$

where $[2\hat{r}]$ denotes the smallest integer greater than or equal to $2\hat{r}$

has a very small variance. Therefore, (18) could be re-
placed by (19) in (17) and, thus, have the following state-
ment

$$1- \alpha \doteq P\left[-\ln R_s \leq \frac{\hat{S}[2\hat{r}]}{\chi^2_{1-\alpha,[2\hat{r}]}}\right] \quad (20)$$

(20) can be rewritten as

$$1-\alpha \doteq P\left\{R_s \geq \exp\left[\frac{-\hat{S}[2\hat{r}]}{\chi^2_{1-\alpha,[2\hat{r}]}}\right]\right\} \quad (21)$$

That is a $100(1-\alpha)\%$ lower confidence limit for R_s is

$$R_{s,L(\alpha)} = \exp\left[\frac{-\hat{S}[2\hat{r}]}{\chi^2_{1-\alpha,[2\hat{r}]}}\right] \quad (22)$$

CHAPTER III

SIMULATED DISTRIBUTION OF THE CONFIDENCE LIMIT

In order to evaluate the accuracy of the procedure, a digital computer is used to simulate the distribution of the S's. The following parameters are used as inputs to the computer program: k; n_i, $i = 1,\ldots, k$; and $p_i = 1-q_i$, $i = 1,\ldots, k$

In order to have an R_s to compare with the simulation the program first computes an R_s from

$$R_s = \prod_{i=1}^{k} p_i \quad .$$

The program next simulates values. A three digit random number is obtained using a uniform random number generator subroutine. If the random number generated is greater than p_i, a failure is "counted" by the computer; if the random number is less than or equal to p_i, no "count" is made. This random number generation is done n_i times for each p_i. Thus, the number of failures counted divided by the number of units of that component becomes an estimator for the unreliability of that component.

i.e., $q_i = \dfrac{f_i}{n_i}$

This process is repeated for all q_i; thus giving an estimator for each q_i. Once these estimators are simulated the following arithmetic operations are performed by the computer and stored for further use:

$$\hat{T}_i = a_i \hat{q}_i + \frac{b_i \hat{q}_i^{\,2}}{2} \tag{7}$$

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$$\hat{S} = \sum_{i=1}^{k} \hat{T}_i \qquad (10)$$

$$\hat{r} = (\sum \hat{T}_i)^2 \bigg/ \left(\sum \frac{\hat{T}_i}{n_i} \right) \qquad (23)$$

In order to complete the simulation of the distribution of $\hat{R}_{s,L(\alpha)}$, replications are made of the above procedure. This gives 500 values each of (10) and (23) which are used to compute 500 values of

$$-\ln\hat{R}_{s,L(\alpha)} = \hat{S}[2\hat{r}] \bigg/ \chi^2_{1-\alpha,[2\hat{r}]} \qquad (24)$$

for each value of α. These 500 points are then sorted, by size, by a separate subroutine. The computer now picks out the α th percentile of this distribution, call it A. Then $R_{s,L(\alpha)} = e^{-A}$ is computed and printed out for comparison with the true system reliability, R_s.

To develop a measure of accuracy for the simulation consider the underlying meaning of the probability statement

$$P[R_s \geq R_{s,L(\alpha)}] = 1 - \alpha \qquad (25)$$

If the probability density function of $R_{s,L(\alpha)}$ is plotted on a coordinate system, and R_s is taken to be a point on the abscissa, then $100(1-\alpha)\%$ of the area under the curve lies above and $\alpha\%$ lies below R_s. Therefore, R_s should be equal to the α th percentile of the distribution of $R_{s,L(\alpha)}$ which we have called exp (-A).

Thus, $|R_s - \exp(-A)| \qquad (26)$ is a measure of the accuracy of the procedure under investigation.

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CHAPTER IV

DEVELOPMENT OF CONTINUITY CORRECTION FACTOR

In the previous discussion $\hat{S}$ was assumed to be a gamma distributed random variable. Since a digital computer simulation is used to generate the values of $\hat{S}$ it is realized that the actual distribution is discrete. Therefore, the program is estimating a gamma distribution. As in the case of estimating a binomial with a normal distribution a more accurate result can be had if the discrete values are slightly shifted.

The following is the development of a correction factor (commonly called a continuity correction factor). Consider the following:

$$\hat{S} = \sum_{i-1}^{k} T_i$$

$$\hat{T}_i = a_i \cdot \frac{f_i}{n_i} + \frac{b_i}{2}\left(\frac{f_i}{n_i}\right)^2 \tag{27}$$

where $a_i = \dfrac{2n_i-3}{2(n_i-1)}$ and $b_i = \dfrac{n_i}{n_i-1}$.

The smallest difference between two values of S will be the smallest change in one of the T_i terms. For a T_i the smallest change is when f_i is increased by one for that respective n_i that is the max (n_i)

$$i=1,2, \ldots.k$$

Let j be that i $\ni$

$$n_j = \max (n_i)$$

$$i = 1, 2, \ldots k$$

and in $\hat{T}_j$ increase f_j by one to obtain $\hat{T}_j{}^*$; that is

$$\hat{T}_j{}^* = a_j \frac{(f_j+1)}{n_j} + \frac{b_j}{2}\left(\frac{f_j+1}{n_j}\right)^2 \qquad (28)$$

Define the correction factor as

$$C_1 = \frac{1}{2} (\hat{T}_j{}^* - \hat{T}_j) \qquad (29)$$

where

$$\hat{T}_j = \frac{2n_j-3}{2(n_j-1)} \cdot \frac{f_j}{n_j} + \frac{n_j}{2(n_j-1)} \cdot \frac{f_j{}^2}{n_j{}^2}$$

$$= \frac{(2n_j-3) f_j+f_j{}^2}{2n_j (n_j-1)} \qquad (30)$$

$$Tj^* = \frac{(2n_j-3)}{2(n_j-1)} \cdot \frac{(f_j+1)}{n_j} + \frac{n_j}{2(n_j-1)} \cdot \frac{(f_j+1)^2}{n_j{}^2}$$

$$= \frac{f_j{}^2+(2n_j-1) f_j+2(n_j-1)}{2n_j (n_j-1)} . \qquad (31)$$

Then

$$C_1 = \frac{1}{2} \cdot \frac{[f_j{}^2+(2n_j-1)f_n+2(n_j-1)] - [(2n_j-3)f_n+f_n{}^2]}{2n_j(n_j-1)}$$

$$= \frac{2n_j+2f_j-2}{4n_j(n_j-1)}$$

$$= \frac{n_j+f_n-1}{2n_j{}^2(n_j-1)} . \qquad (32)$$

The corrected $\hat{S}$ is now defined to be

$$\hat{S}^* = \hat{S}+C_1 \qquad (33)$$

It can be seen from the above calculations that $\hat{S}*$ will have a value even if all the T_i's turn out to be zero. Keeping this in mind and looking at the equation for $\hat{r}$ in equation (23)

$$\hat{r} = \left(\sum_{i=1}^{k} \hat{T}_i \right)^2 \Bigg/ \sum_{i=1}^{k} \frac{\hat{T}_i}{n_i} = (\hat{S}*)^2 \Bigg/ \sum_{i=1}^{k} \frac{\hat{T}_i}{n_i}$$

it is observed that in this case $\hat{r}$ becomes infinite when the denominator is zero. Obviously this doesn't make sense; therefore, a correction factor is added to the denominator. Because this correction factor has little intuitive basis other than keeping $\hat{r}$ from becoming infinite, it was picked quite arbitrarily. The following correction factor was chosen:

$$C_2 = \frac{\hat{T}_j^* - \hat{T}_j}{n_j} \tag{34}$$

and added to $\sum_{i=1}^{k} \frac{\hat{T}_i}{n_i}$ giving a new $\hat{r}$ ie. $\hat{r}*$. This factor proved to be adequate and didn't appear to bias the final results; at least, not in the region of accuracy desired in the model.

The two corrected values are now combined in the probability statement of equation (21)

$$1-\alpha = P \left[R_s \geq \exp \left(-\hat{S}^* [2\hat{r}^*] \Big/ \chi^2_{1-\alpha, [2\hat{r}^*]} \right) \right] \tag{35}$$

giving the associated lower confidence limit

$$R_{s,L(\alpha)} = \exp \left(-\hat{S}^* [2\hat{r}^*] \Big/ \chi^2_{1-\alpha, [2\hat{r}^*]} \right) \tag{36}$$

which was simulated in the computer.

CHAPTER V

RESULTS OF SIMULATIONS

There were thirteen different combinations of param-
eters used, representing as many hypothetical systems.
Table I lists the input parameters and the results. Col-
umn (1) is self-explanatory. Column (2) is the number of
components in the system. Colums (3) gives the size of
the sample tested for each respective component. Column
(4) is the probability of success for each component of
the system. Column (5) is the system reliability as com-
puted directly from the input parameters (2), (3) and (4).
Column (6) lists the three values of α for which a lower
confidence limit was computed. Column (7) is the exponen-
tial power of the α th percentile of the distribution.

The first four cases in Table I are examples of systems
with different component sample sizes. In Cases 5 through
13, the sample size is constant in each case. These last
cases show the effect that system size, sample size and
probabilities have on the accuracy of the method. The
effects noted will be discussed after Table II is presented.

A further check was made on the method by computing
Case 10 four times, each time starting the random number
generator at a different point. The results of this test
showed there wasn't any difference in the first four decimal
places of the confidence limit estimate. From this, it is
concluded the 500 replications of each case is enough to
smooth out any fluctuations in the generated numbers.

TABLE I

RESULTS OF COMPUTER SIMULATION WITHOUT
CONTINUITY CORRECTION FACTOR

(1) CASE	(2) k	(3) n_i	(4) P_i	(5) R_s	(6) α	(7) exp(-A)
1	13	150 19 90 5 75 125 100 63 125 125 18 59 28	.995 .900 .985 .980 .979 .995 .988 .970 .982 .995 .980 .968 .967	.7233	.05 .10 .20	.7185 .7270 .7292
2	15	250 30 40 20 120 75 15 100 130 90 65 60 70 60 130	.990 i=1,---,15	.8601	.05 .10 .20	.8610 .8626 .8606
3	15	20 150 20 20 20 20 20 20 150 20 150 20 150 20 150	.995 i=1,---,14 .850 i=15	.7920	.05 .10 .20	.7270 .7480 .7891
4	15	20 i=1,-,14 150 i=15	.995 i=1,---,14 .850 i=15	.7924	.05 .10 .20	.8170 .8175 .8000
5	5	50	.99 i=1,---, 5	.9510	.05 .10 .20	1.0000 .8958 .9142
6	10	50	.99 i=1,---,10	.9044	.05 .10 .20	.8359 .8604 .8894
7	15	50	.99 i=1,---,15	.8601	.05 .10 .20	.7985 .8325 .8506

TABLE I (Continued)

RESULTS OF COMPUTER SIMULATION WITHOUT
CONTINUITY CORRECTION FACTOR

(1) CASE	(2) k	(3) n_i	(4) p_i	(5) R_s	(6) α	(7) $\exp(-A)$
8	30	20	.99 i=1,---,30	.7397	.05 .10 .20	.5767 .6647 .7059
9	5	100	.95 i=1,---, 5	.7738	.05 .10 .20	.7636 .7652 .7696
10	15	20	.995 i=1,---,14 .85 i=15	.7924	.05 .10 .20	.6862 .7048 .7666
11	15	40	.995 i=1,---,14 .85 i=15	.7924	.05 .10 .20	.7683 .7825 .7935
12	15	50	.995 i=1,---,14 .85 i=15	.7924	.05 .10 .20	.7704 .7829 .7925
13	15	100	.995 i=1,---,14 .85 i=15	.7924	.05 .10 .20	.7780 .7885 .7918

Table II lists the results of the cases with and without the correction factor added. Columns (1), (2) and (3) were described before. Column (4) is the exponential power of the α th percentile of the distribution with the correction added. Column (5) shows the value of the measure of accuracy used in evaluating Method I with correction added. Column (6) is the same as (5) but without the correction added.

From Table II, it is quite apparent, on the whole, that the error was reduced by the addition of the correction factor. It also appears the correction has less effect on cases where the sample sizes (n_i) vary [Cases: 1, 2, 3, 4], but still, obviously, worth the effort. Cases 5, 6 and 7 seemed to indicate the number of components (k) in the system has no effect on accuracy. Cases 10, 11, 12 and 13 appear to point out that as the sample size (n_i) increases the accuracy versus probabilities [also increases] This concludes the discussion of the most observable facts that can be viewed from the tables.

? [   ] two words added by library. not by the author.

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TABLE II

COMPARISON OF METHOD I WITH (*) AND

WITHOUT CONTINUITY CORRECTION FACTOR

(1) CASE	(2) R_s	(3) α	(4) $\exp(-A*)$	(5) $\exp(-A*)-R_s$	(6) $\exp(-A)-R_s$
1	.7233	.05 .10 .20	.7184 .7270 .7291	.0049 .0037 .0058	.0048 .0037 .0038
2	.8601	.05 .10 .20	.8610 .8626 .8606	.0010 .0025 .0006	.0010 .0026 .0006
3	.7924	.05 .10 .20	.7219 .7459 .7873	.0705 .0465 .0051	.0654 .0444 .0033
4	.7924	.05 .10 .20	.8113 .8135 .7969	.0190 .0211 .0045	.0246 .0251 .0076
5	.9510	.05 .10 .20	.9961 .9269 .9522	.0039 .0241 .0012	.0490 .0552 .0368
6	.9044	.05 .10 .20	.8628 .8953 .8998	.0415 .0090 .0045	.0685 .0440 .0150
7	.8601	.05 .10 .20	.8384 .8480 .8567	.0216 .0120 .0033	.0616 .0276 .0095
8	.7397	.05 .10 .20	.6830 .7070 .7220	.0567 .0327 .0177	.1630 .0750 .0338
9	.7738	.05 .10 .20	.7635 .7651 .7696	.0103 .0086 .0042	.0102 .0085 .0042
10	.7924	.05 .10 .20	.7495 .7591 .7684	.0429 .0330 .0240	.1062 .0876 .0258

TABLE II (Continued)

COMPARISON OF METHOD I WITH (*) AND

WITHOUT CONTINUITY CORRECTION FACTOR

(1) CASE	(2) R_s	(3) α	(4) exp $(-A*)$	(5) exp $(-A*)-R_s$	(6) exp $(A-A)-R_s$
11	.7924	.05 .10 .20	.7714 .7820 .7931	.0210 .0104 .0007	.0241 .0099 .0011
12	.7924	.05 .10 .20	.7700 .7826 .7923	.0224 .0098 .0001	.0220 .0095 .0001
13	.7924	.05 .10 .20	.7779 .7884 .7917	.0145 .0040 .0007	.0144 .0039 .0006

CHAPTER VI

SUMMARY AND CONCLUSIONS

In the preceding four chapters a procedure [1] has been explained which will approximate a lower $100(1-\alpha)\%$ confidence interval on system reliability. The simulation of this method on a computer has been explained and the technique used to measure the accuracy of the method was stated. A continuity correction factor has been developed which enables the discrete computations of a digital computer to more closely approximate a continuous distribution.

The simulation was run for thirteen cases with and without the continuity correction factor. These were not an adequate amount of cases on which to base any concrete conclusions. It is felt these cases are sufficient to show the merit of the correction factor. The following summary as stated in Chapter I demonstrates the validity of the procedure.

Without Correction:

α	Average Error	Variance of Error
.05	.0438	.00213
.10	.0305	.00076
.20	.0109	.00016

With Correction:

α	Average Error	Variance of Error
.05	.0286	.00041
.10	.0168	.00015
.20	.0056	.00005

From Tables I and II in Chapter V, it can be seen the average error for those cases where sample size (n_i)

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is varied would be less than shown above. It must then be concluded that Method I with the correction factor will be an effective method for approximating the lower $100(1 - \alpha)$% confidence limit for system reliability.

It should be pointed out that another study; obviously, could be done in this same area. By varying the input parameters (k, n_i, p_i) and determining their effects, one could possibly establish limits on their variation to keep within a desired degree of accuracy.

In closing, I would like to express my appreciation to Dr. W. Max Woods of the Operations Analysis Department for pointing out this problem and serving as a very willing advisor. I also wish to extend my thanks to Dr. Rex H. Shudde for supplying the basic computer program for Method I which required slight alteration to meet my needs.

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DERIVATION OF $\hat{T}_i$, THE UNBIASED ESTIMATOR OF T_i

In order that $\hat{T}_i$ be an unbiased estimator of T_i, it is necessary that

$$T_i = E[\hat{T}_i]$$

Rewriting the above expression, using (5) and (7)

$$q_i + \frac{q_i^2}{2} = E\left[\frac{a_i f_i}{n_i} + \frac{b_i f_i^2}{2}\right]$$

and, substituting (2)

$$q_i + \frac{q_i^2}{2} = E\left[\frac{a_i f_i}{n_i} + \frac{b_i f_i^2}{2n_i^2}\right]$$

$$= \frac{a_i}{n_i} E[f_i] + \frac{b_i}{2n_i^2} E[f_i^2]$$

$$= \frac{a_i n_i q_i}{n_i} + \frac{b_i}{2n_i^2} (n_i^2 q_i^2 - n_i q_i^2 + n_i q_i)$$

$$= a_i q_i + \frac{b_i q_i^2}{2} (1 - \frac{1}{n_i}) + b_i q_i$$

$$= (a_i + \frac{b_i}{2n_i}) q_i + (b_i - \frac{b_i}{n_i}) \frac{q_i^2}{2}$$

Then equating coefficients of q_i and $\frac{q_i^2}{2}$

$$a_i + \frac{b_i}{2n_i} = 1$$

$$b_i - \frac{b_i}{n_i} = 1$$

and solving simultaneously

$$a_i + \frac{b_i}{2n_i} = 1$$

$$\frac{1}{2}(b_i) - \frac{1}{2}\frac{(b_i)}{n_i} = \frac{1}{2}(1)$$

$\Rightarrow$
$$a_i + \frac{b_i}{2} = \frac{3}{2}$$

$\Rightarrow$
$$a_i = \frac{1}{2}(3 - b_i)$$

Substituting back

$$\frac{1}{2}(3 - b_i) + \frac{b_i}{2n_i} = 1$$

$$3 - b_i + \frac{b_i}{n_i} = 2$$

$\Rightarrow$
$$b_i = \frac{n_i}{n_i - 1}$$

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and again substituting

$$a_i = \frac{1}{2} \left(3 - \left(\frac{n_i}{n_i - 1} \right) \right)$$

$$= \frac{2n_i - 3}{2(n_i - 1)}$$

Thus,

$$\hat{T}_i = \frac{2n_i - 3}{2(n_i - 1)} \hat{q}_i + \frac{n_i}{n_i - 1} \frac{\hat{q}_i^2}{2}$$

is an unbiased estimator of T_i.

APPENDIX II

COMPUTATION OF VARIANCE OF $\hat{S}$

$$\text{Var } (\hat{S}) = \text{Var}\left(\sum_{i=1}^{k} \hat{T}_i\right) = \sum_{i=1}^{k} \text{Var } (\hat{T}_i)$$

$$\text{Var } (\hat{T}_i) = E[\hat{T}_i^2] - E^2 [\hat{T}_i]$$

$$= E[\hat{T}_i^2] - T_i^2$$

$$E[\hat{T}_i^2] = E[(a_i \hat{q}_i + \frac{b_i \hat{q}_i^2}{2})^2]$$

$$= E[a_i^2 \hat{q}_i^2 + a_i b_i \hat{q}_i^3 + \frac{b_i^2 \hat{q}_i^4}{4}]$$

$$= \frac{a_i^2}{n_i^2} E[f_i^2] + \frac{a_i b_i}{n_i^3} E[f_i^3] + \frac{b_i^2}{4n_i^4} E[f_i^4]$$

From [3]

$$E[f_i^2] = n_i^2 q_i^2 - n_i q_i^2 + n_i q_i$$

$$E[f_i^3] = E[(f_i - E[f_i])^3] + 3E[f_i]E[f_i^2] - 2E^3[f_i]$$

$$= n_i^3 q_i^3 - 3n_i^2 q_i^3 + 2n_i q_i^3 + 3n_i^2 q_i^2 - 3n_i q_i^2 + n_i q_i$$

$$E[f_i^4] = E[(f_i - E[f_i])^4] + 4E[f_i]E[f_i^3] - 6E^2[f_i]E[f_i^2] +$$

$$3E^4[f_i]$$

$$= n_i^4 q_i^4 - 6n_i^3 q_i^3 + 11n_i^2 q_i^4 - 18n_i^2 q_i^3 + 7n_i^2 q_i^2 -$$

$$6n_i q_i^4 + 12n_i q_i^3 - 7n_i q_i^2 + n_i q_i$$

Therefore

$$E[\hat{T}_i^2] = \frac{(2n_i - 3)^2}{2^2 (n_i - 1)^2 n_i^2} \{n_i^2 q_i^2 - n_i q_i^2 + n_i q_i\} +$$

$$\frac{(2n_i - 3) n_i}{2(n_i - 1)^2 n_i^3} \{n_i^3 q_i^3 - 3n_i^2 q_i^3 + 3n_i^2 q_i^3 +$$

$$3n_i^2 q_i^2 - 3n_i q_i^2 + n_i q_i\} +$$

$$\frac{n_i}{4(n_i - 1)^2 n_i^4} \{n_i^4 q_i^4 - 6n_i^3 q_i^4 + 6n_i^3 q_i^3 + 11n_i^2 q_i^4 -$$

$$18n_i^2 q_i^3 + 7n_i^2 q_i^2 - 6n_i q_i^4 + 12n_i q_i^3 - 7n_i q_i^2 +$$

$$n_i q_i\}$$

$$E[\hat{T}_i^2] = \frac{(4n_i^2 - 12n_i + 9)(n_i q_i^2 - q_i^2 + q_i) + (4n_i - 6)}{4(n_i - 1)^2 n_i}$$

$$\frac{(n_i^2 q_i^3 - 3n_i q_i^3 + 2q_i^3 + 3n_i q_i^2 - 3q_i^2 + q_i)}{4(n_i - 1)^2 n_i} +$$

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$$\frac{(n_i^3 q_i^4 - 6n_i^2 q_i^4 + 6n_i^2 q_i^3 + 11n_i q_i^4 - 18n_i q_i^3 +}{4(n_i - 1)^2 n_i}$$

$$\frac{7n_i q_i^2 - 6q_i^4 + 12q_i^3 - 7q_i^2 + q_i\ )}{4(n_i - 1)^2 n_i}$$

$$= \frac{n_i^3 (q_i^4 + 4q_i^3 + 4q_i^2) + n_i^2\ (-6q_i^4 - 12q_i^3 -}{4n_i (n_i - 1)^2}$$

$$\frac{4q_i^2 + 4q_i) + n_i (11q_i^4 + 8q_i^3 - 2q_i^2 - 8q_i) +}{4n_i (n_i - 1)^2}$$

$$\frac{(-6q_i^4 + 2q_i^2 + 4q_i)}{4n_i (n_i - 1)^2}$$

Let

$$A = q_i^4 + 4q_i^3 + 4q_i^2$$

$$B = -6q_i^4 - 12q_i^3 - 4q_i^2 + 4q_i$$

$$C = 11q_i^4 + 8q_i^3 - 2q_i^2 - 8q_i$$

$$D = -6q_i^4 + 2q_i^2 + 4q_i$$

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Then

$$E[\hat{T}_i^2] = \frac{An_i^3 + Bn_i^2 + Cn_i + D}{4n_i^3 - 8n_i^2 + 4n_i}$$

$$= \frac{A}{4} + \frac{B + 2A}{4n_i} + \frac{(C + 2B + 3A)n_i + (D - B - 2A)}{4n_i(n_i - 1)^2}$$

$$= T_i^2 + \frac{-q_i^4 - q_i^3 + q_i^2 + q_i}{n_i} + \frac{(2q_i^4 - 4q_i^3 + 2q_i^2)n_i}{4n_i(n_i - 1)^2}$$

$$\frac{-(2q_i^4 - 4q_i^3 + 2q_i^2)}{4n_i(n_i - 1)^2}$$

$$= T_i^2 + \frac{q_i}{n_i} + \frac{q_i^2}{2n_i} + \frac{q_i^2}{2n_i} - \frac{(q_i^3 + q_i^4)}{n_i} + \frac{q_i^2(q_i - 1)^2}{2n_i(n_i - 1)}$$

$$E[\hat{T}_i^2] - T_i^2 = \mathrm{Var}(\hat{T}_i) = \frac{T_i}{n_i} + \frac{q_i^2}{n_i}\left[\frac{1}{2} - (q_i + q_i^2) + \right.$$

$$\left. \frac{(q_i - 1)^2}{2(n_i - 1)} \right]$$

which, when simplified, gives

$$\mathrm{Var}(\hat{T}_i) = \frac{T_i}{n_i} + \frac{q_i^2}{n_i}\left[\frac{3q_i^2 - n_i(2q_i^2 + 2q_i - 1)}{2(n_i - 1)} \right]$$

$$= \frac{T_i}{n_i} + \frac{q_i^2}{2(n_i - 1)}\left[1 + \frac{3q_i^2}{n_i} - 2q_i^2 - 2q_i \right]$$

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which we approximate by

$$Var(\hat{T}_i) \doteq \frac{T_i}{n_i}$$

Thus

$$Var(\hat{S}) \doteq \sum_{i=1}^{k} \frac{T_i}{n_i}$$

and the truncation error is

$$\sum_{i=1}^{k} \left[\frac{q_i^2}{2(n_i - 1)} \left(1 + \frac{3q_i^2}{n_i} - 2q_i^2 - 2q_i \right) \right]$$

This error is further discussed in Appendix IV.

APPENDIX III

ANALYSIS OF THE ERROR DUE TO TRUNCATION

In the transitions from (4) to (5) an infinite
series was truncated to its first two terms. Examine
the inner summation of (4)

$$\sum_{j=1}^{\infty} \frac{q_i^{\,j}}{j} = q_i + \frac{q_i^{\,2}}{2} + \sum_{j=3}^{\infty} \frac{q_i^{\,j}}{j} \qquad (37)$$

$$= T_i + R_2$$

where R_2 is the remainder of the series after two terms
have been written out. If we test the nth and $(n + 1)^{st}$
terms of this series by the ratio test,

$$\frac{q_i^{\,n+1} \,/\, n + 1}{q_i^{\,n} \,/\, n} = \frac{n q_i}{n + 1}$$

we see that this expression is always less than $q_i \leq 1$,
which implies the series is convergent. Then by Theorem
24, p. 328 of [2]

$$\lim_{n \to \infty} \left| \frac{n q_i}{n+1} \right| = q_i$$

which implies that

$$R_n \leq q_i^{\,n+1} \,/\, (n + 1)(1 - r), \ r \geq q_i$$

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Since $r \geq q_i$, let us choose $r = q_i \leq 1$ so as to make R_n as small as possible

$$R_n \leq q_i^{n+1} / (n+1)(1-q_i)$$

and evaluating at $n = 2$

$$R_2 \leq q_i^3 / 3(1-q_i) \qquad (38)$$

Since $q_i << 1$ in most cases of interest, choose $q_i = .15$, for example, which is about the largest value it will ever take on, and we see that $R_2 \leq 1.4 \times 10^{-3}$. Substituting equation (38) into (37),

$$\sum_{j=1}^{\infty} \frac{q_i^j}{j} \leq T_i + q_i^3 / 3(1-q_i) \qquad (39)$$

and substituting (39) into (4)

$$S \leq \sum_{i=1}^{k} (T_i + q_i^3 / 3(1-q_i))$$

which upon distributing the summation sign becomes equation (6). For $q_i = .05$, still quite a large value for q_i to take on, and $k = 15$

$$S \leq \sum_{i=1}^{k} T_i + 6 \times 10^{-5},$$

so the approximation

$$S \doteq \sum_{i=1}^{k} T_i$$

is quite good.

INVESTIGATION OF ERROR DUE TO APPROXIMATION IN Var $(\hat{S})$

In Appendix II the Variance of the random variable $\hat{S}$ is computed, and it is seen that the value used in (11) before differs from the true variance by a factor

$$\sum_{i=1}^{k} \left[\frac{q_i^2}{2(n_i-1)} \left(1+ \frac{3q_i^2}{n_i} - 2q_i^2 - 2q_i \right) \right] \tag{40}$$

Upon embarking upon an investigation of the size of this term it is quickly seen that it is a task which, due to the amount of work involved, is beyond the scope of this thesis. A few general comments on this term are in order, however.

Being a function of both the q_i and n_i, equation (40) may, for certain combinations of these parameters, actually be less than zero and reduce the variance of $\hat{S}$. In order to determine when this occurs, it would be necessary to use a digital computer to compute and plot for each set of n_i, i=1,..., k, a curve over the range of $0 \leq q_i \leq 1$; and, similarly, to compute and plot for each set of q_i, i=1,...,k, a curve over the range of interest of n_i, $2 \leq n_i \leq N$, where N might be 150.

If the summand of (40) is plotted as a function of a single q_i, with the n_i held fixed, the curve is seen to be bell-shaped over the range $0 \leq q \leq 1$, but within the realistic range that q_i may be expected to take on $0 \leq q_i \leq .15$, (40) is a non-decreasing function, and strictly increasing unless a point of inflection occurs.

If the summand of (40) is plotted as a function of a single n_i, where n_i is of course interger-valued, it is seen to be a monotone-decreasing step-function, asymptotic to the n_i axis.

As a crude bound, on the size of the error, consider the summand of (40)

$$\frac{q_i^2}{2(n_i-1)} \left(1 + \frac{3q_i^2}{n_i} - 2q_i^2 - 2q_i \right)$$

$$< \frac{q_i^2}{2n_i} \left(1 + \frac{3q_i^2}{n_i} - 2q_i^2 - 2q_i \right)$$

$$< \frac{q_i^2}{2n_i} \left(1 + \frac{3q_i^2}{n_i} \right)$$

$$< \frac{1}{2n_i} \left(1 + \frac{3}{n_i} \right)$$

which implies that

$$\sum_{i=1}^{k} \left[\frac{q_i^2}{2(n_i-1)} \left(1 + \frac{3q_i^2}{n_i} - 2q_i - 2q_i \right) \right] < \sum_{i=1}^{k} \frac{1}{2n_i} \left(1 + \frac{3}{n_i} \right)$$

which is obviously not a least upper bound, but an example is informative.

Let n = 20 for all i, and k = 15

$$\sum_{i=1}^{k} \frac{1}{2n_i} \left(1 + \frac{3}{n_i} \right) \doteq .4$$

which admittedly is of an undesirable magnitude for an omit-
ted term, but (40) is undoubtedly much less than .4.

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13. ABSTRACT

A statement of a method [1] for obtaining an approximate
lower $100(1-\alpha)\%$ confidence limit on System Reliability
is made. A discussion of evaluating the accuracy of this
method on a digital computer is presented. Following
this the development of a continuity correction factor
is developed and the accuracies of this refinement for
several hypothetical systems are presented and discussed.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
System Reliability						
Confidence Limits						
Computer Simulation of Probability Distribution						
Unequal Component Sample Sizes						
Continuity Correction Factor						