Threshold effects in neutron elastic scattering.

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THRESHOLD EFFECTS IN NEUTRON ELASTIC SCATTERING

A DISSERTATION

SUBMITTED TO THE DEPARTMENT OF PHYSICS
AND THE COMMITTEE ON THE GRADUATE DIVISION
OF STANFORD UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By

John Thompson Wells

December 1962
I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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Approved for the University Committee on the Graduate Division:

Dean of the Graduate Division
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CHAPTER I

INTRODUCTION

We have attempted to demonstrate the Wigner cusp phenomenon in neutron elastic scattering and total cross sections near the thresholds for excitation of the first excited states of the dominant isotopes of Li, Fe, Zr, Ba, and Ce. The energy region investigated extended approximately 300 kev on either side of threshold.

Theoretical work (see, for example, Newton, Baz, Fonda, and Meyerhof) provides detailed predictions of possible cusp effects. The effects originate essentially in two fundamental physical principles: the conservation of incident flux, expressed in the theory as unitarity of the scattering matrix; and the uncertainty principle of Heisenberg, expressed in the theory as analyticity of the scattering matrix. The magnitude and shape of cusp anomalies can yield information useful in scattering analysis and in establishing spins and parities of excited states.

Cusp anomalies have been seen in proton elastic scattering near thresholds for (p,n) reactions. The theory shows that one essential requirement for a sharp cusp is that coulomb effects be absent from that reaction channel which provides the threshold. Any reaction with outgoing neutrons, such as (p,n) (d,n) (α,n) (n,n'), satisfies this requirement. 
The advantage of studying Wigner cusps in elastic neutron, rather than elastic proton scattering, for example, is the easier analysis caused by the absence of coulomb effects in the incoming channel. After estimating possible cusp magnitudes, we concluded that the effects of inelastic neutron scattering reactions on elastic neutron scattering might be quite noticeable for selected cases. We hoped thereby to find a method for spin assignment of excited states and to elucidate interaction mechanisms between nuclei and neutrons of few Mev energy.
CHAPTER II

THEORY OF WIGNER CUSPS

Previous articles on the theory of Wigner cusp phenomena have generally emphasized the simple spin cases (e.g., spinless particles); also, the results are not always given in a form readily compared with experiment. We would like to supplement the previous results by considering more general spin and angular momentum configurations. Our results will provide a convenient form for experimental comparisons. In addition, the form of the results for pure Wigner cusp effects will be particularly suitable for subsequent deduction of "energy-averaged" cusp expressions, as first derived in Ref. 6.

Pure Wigner Cusp Theory

We will compute the effect of reactions of the type \( X(n,n')X^* \) on neutron differential elastic and total cross sections near threshold energies. Blatt and Biedenharn\(^ {12} \) have derived necessary general expressions for cross sections. For completeness, we give in Appendix A a summary of the physical basis for the cross section formulas, starting with the wave functions and the scattering matrix, \( S \).

In applying these formulas we assume that elastic scattering with change of orbital angular momentum \( \ell \) may be neglected. Because of the parity rule requiring \( \Delta \ell = 0, 2, 4 \ldots \), elastic scattering with change of \( \ell \) turns out to be an unimportant phenomenon in most
In particular, in the present experiment this restriction is very mild, since, with one exception, the target nuclei were even-even. Hence, with $I = 0$, $i = \frac{1}{2}$, $s = \frac{1}{2}$ ($I$, $i$, and $s$ are target, projectile, and channel spin, respectively), there is no possibility of elastic scattering with change of $l$. The exceptional case is that of $\text{Li}^7$ ($I = 3/2$). In this case, $\Delta l \geq 2$ is limited by penetrability effects. For example, although incident $l = 0$ waves theoretically can be scattered elastically into $l = 2$ outgoing waves (with compound spin $J = 1$), it happens that in the energy region of interest $kR \approx 0.3$ and any process involving $l = 2$ is highly improbable. Consequently, for the present experiment, specializing the derivation to $\Delta l = 0$ is not a restriction on the validity of the results.

If one considers angular momenta situations in which there can be scattering involving change of channel spin $s$, but no change in channel energy, complexities arise in a derivation of cusp effects. To preserve clarity in the presentation, we will restrict our derivation to situations in which channel spin $s$ is separately conserved, in addition to the previous assumption that orbital angular momentum $l$ is conserved. It will be shown that this restriction does not affect the validity of the derivation as it applies to present experimental results. We will indicate the nature of the additional complexities in cusp theory, which are present without this restriction, at a suitable point in the derivation.

The following equations are numbered 3.16, 4.5, and 4.6 in Ref. 12. We have specialized them to elastic scattering with no change
of \( \alpha, l, s \) or \( i \). Accordingly, the differential elastic scattering cross section is given by

\[
\frac{d\sigma}{d\Omega_{\alpha;\alpha}} = \sum_{s=\leftarrow|l-1|} \left\{ \frac{(2s + 1)/[(2I + 1)(2i + 1)]}{((2I + 1)(2i + 1))} \right\} d\sigma_{\alpha s;\alpha s}
\]

(1)

where

\[
d\sigma_{\alpha s;\alpha s} = \left[ \frac{\alpha^2}{(2s + 1)} \right] \sum_{L=0}^{\infty} B_L(\alpha s;\alpha s) P_L(\cos \theta) d\Omega
\]

(2)

\[
B_L(\alpha s;\alpha s) = (1/4) \sum_{J_1} \sum_{J_2} \sum_{l_1} \sum_{l_2}
\]

\[
\times Z^2(l_1 J_1 l_2 J_2, s L)
\]

(3)

\[
\times \Re\left[ (1 - S_{\alpha l_1;\alpha s l_1})^* (1 - S_{\alpha l_2;\alpha s l_2}) \right]
\]

where \( \alpha \) refers to the type of incident particle and the state of the struck nucleus; \( \alpha \) is \( (2\pi)^{-1} \times \) the deBroglie reduced wavelength of the incident neutron/target nucleon system; \( s, I, \) and \( i \) are channel, target, and projectile spins, respectively; and \( l, J \) are orbital and total angular momentum, respectively. Subscripts 1 and 2 are dummy indices for use in summation. All sums are unrestricted, except that, in practice, only one such sum, for example, \( J_1 \), runs to infinity, because of the vanishing of \( Z \) coefficients. \( \Re \) stands for real part of the bracketed expression. \( S^J_{\alpha s l;\alpha s l} \) is the S-matrix element for elastic scattering with \( l \) and \( s \) conserved (see Appendix A).
\[ Z(l_1^1 J_1^1 l_2^1 J_2^1, s L) = i^{L-l_1^1+l_2^1} (2l_1^1 + 1)^{\frac{1}{2}} (2l_2^1 + 1)^{\frac{1}{2}} (2J_1^1 + 1)^{\frac{1}{2}} (2J_2^1 + 1)^{\frac{1}{2}} \times W(l_1^1 J_1^1 l_2^1 J_2^1, s L)(l_1^1 l_2^1 0 0 | l_1^1 l_2^1 L 0) \]

where \(W\) is the Racah coefficient. \(^{14}\) Tables of \(Z\) are available. \(^{15}\) The properties of \(W\) are discussed by Biedenharn, Blatt, and Rose. \(^{16}\)

The energy dependence of the elastic cross section is contained in \(S\). We will use the unitary and analytic properties of the S-matrix to obtain an expression for the elastic cross section in the vicinity of a reaction threshold.

Among the inelastic reactions which can proceed from a given incident partial wave, we consider only the one with lowest threshold energy. Note, however, that an excitation of a higher excited state is not excluded as long as its threshold has the lowest energy of those reactions which may proceed from a particular incident partial wave. Baz\(^3\) showed that the procedure for including higher inelastic thresholds of the same dominant partial wave is straight-forward in principle, although much more complicated. Since the present experiments were done in energy regions near first excited states, the case we consider is adequate, and the physical content of the results is not obscured by these complications.

Also, other true inelastic processes (e.g., neutron capture) occurring at lower energy than the threshold energy of interest are neglected.

In addition, we restrict the energy region in which the derivation is valid to that in which the reaction is dominated by a single
incident partial wave. In practice, such an energy region is usually sufficiently large to be of experimental interest. For example, an incident d wave contributes more than 90 percent of the cross section over an energy interval of 150 kev above threshold for inelastic neutron scattering leading to the 2\(^+\) first excited state of Ce\(^{140}\).\(^{17}\) One can explain this in terms of the optical model by noting that Ce has a large transmission coefficient for (low energy) outgoing s wave neutrons,\(^{18}\) and that sufficiently close to threshold, inelastic neutron scattering is always dominated by that incident partial wave which is associated (through angular momentum conservation) with an outgoing s wave.

Finally, to further simplify the presentation, but not as an essential limitation on the theory, we will only consider reactions dominated by outgoing s waves. The complete theory in this respect shows that reactions dominated near threshold by outgoing waves of orbital angular momentum \(l_o\), greater than zero, produce cusp effects in the \((l_o)^{th}\) derivative of the elastic cross section.\(^{11}\) Therefore, these cases have less experimental interest. With one exception, this restriction is not essential since we measured threshold effects of reactions with outgoing s waves. The exceptional case is the reaction Zr\(^{90}(n,n')\) Zr\(^{90}\)\(^*\) (1.75 Mev) which Tucker, et al. have found dominated by outgoing p waves already at about 30 kev above threshold.\(^{19}\) The necessary modifications to the theory in the case of Zr\(^{90}\) will be obvious.
Under the above assumptions the reaction [see Eq. (5) below] may be described by its partial reaction cross section

\[ \sigma^J_{r\alpha'\ell's'} = \pi \alpha' \cdot \frac{2}{J'} g^J_{s'}(1 - |S^J_{\alpha'\ell's'}; \alpha'\ell's'|^2) \]  

(4a)

where primes identify the particular channel quantities which couple to \( \ell' \), the dominant incident partial wave and \( g^J_{s'} \equiv (2J' + 1)/(2s' + 1) \) is a statistical factor obtained after coupling angular momenta, averaging over initial magnetic quantum numbers and summing over final magnetic quantum numbers (see Blatt and Weisskopf\textsuperscript{13} or Blatt and Biedenharn\textsuperscript{12}).

To simplify the notation, we now drop a number of subscripts. First \( \alpha' \) is superfluous since we always consider a neutron incident upon the ground state of a target nucleus. Second, we reduce the double subscript set on the S-matrix element to a single set; the meaning of \( S \), as given in Appendix A, should be kept in mind.

Equation (4a) becomes:

\[ \sigma^J_{\ell's'} = \pi \alpha' \cdot \frac{2}{J'} g^J_{s'}(1 - |S^J_{\ell's'}|^2) \]  

(4b)

Equation (4b) is essentially the statement that the scattering matrix \( S \) is unitary (see Appendix A). The corresponding expression for the total reaction cross section is:

\[ \sigma_r = \sigma_{r\ell'} = \sum_{s' = |I - i|}^{I+i} \frac{2s' + 1}{(2I + 1)(2I + 1)} \sum_{J' = |\ell' - s'|}^{\ell' + s'} \sigma^J_{r\ell's'} \]  

(5)

Inverting Eq. (4b), the S-matrix element may be written:

\[ S^J_{\ell's'} = (\exp 2i \cdot \delta^J_{\ell's'}) (1 - \sigma^J_{r\ell's'})^{\frac{1}{2}} \]  

(6)
Equation (6) defines \( \Delta \theta_j^{J', s'} \) as the (real) phase of the \( S \)-matrix element \( S_j^{J', s'} \). To simplify the notation, we have expressed \( \sigma_r^{J', s'} \) in units of \( \pi \hbar^2 g_s^{J'} \). This normalization will be used in the remaining discussion except in instances where clarity requires that the units be specifically restored.

Equation (6) is certainly valid above threshold. The usual procedure\(^2,3\) is to expand the complete phase (which may have real and imaginary parts) of \( S_j^{J', s'} \) in energy about threshold. One consequence of this procedure is that the fixed value of the real phase at threshold appears in the final expressions. On the other hand, Eq. (6) is quite general and there is no reason to assume that the form of Eq. (6) should not persist even away from threshold (above threshold). Consequently, we shall allow the real part of the phase of \( S \), i.e., \( \Delta \theta \), as well as \( \sigma_r \), to vary over the energy region of interest, if the physical situation requires that they do so. This leaves us free later to examine the consequences of energy averaging over a cusp region where the phase \( \Delta \theta \) and the cross section \( \sigma_r \) might be fluctuating rapidly because of the presence of many narrow resonances.

Using R-matrix theory, Meyerhof\(^7\) has shown that the \( S \)-matrix can always be written in a form equivalent to Eq. (6) if \( \ell_o = 0 \), independently whether the energy is above or below threshold. An analytic continuation in energy across threshold always exists, even in the presence of many narrow and possibly overlapping resonances which would produce rapid variation with energy in the phase and in the reaction cross section.
The prescription for continuing Eq. (6) below threshold, assuming outgoing s wave neutrons in the reaction, is given by Eq. (29a) in Ref. 7; essentially, one may let \( \sigma_r = |\sigma_r| \) above threshold go to \( i|\sigma_r| \) below threshold. It is interesting that in R-matrix theory this prescription actually results from a change in the effective logarithmic derivative of the reaction channel wave function from a pure imaginary number above threshold to a pure real number below threshold. This is related to the fact that outgoing wave function \( \exp(ik_o r)/r \) has to change to \( \exp(-|k_o|r)/r \) as the energy (E) is changed from above threshold \( (E_T) \) to below. \( k_o = \sqrt{2M(E - E_T)/\hbar^2} \), \( M \) = reduced mass of the outgoing neutron. This prescription becomes more complicated if one considers outgoing waves in the reaction channel with \( \ell_o \neq 0 \). The necessary modifications for extension of the present theory to these cases is also given in Ref. 7.

Using the fact that \( \sigma_r^{J'_{r'l'}} < < 1 \) near threshold, we can simplify Eq. (6) to

\[
S^{J'}_{l's'} \approx (\exp 2i S^{J'}_{l's'})(1 - \frac{1}{2} \sigma_r^{J'_{r'l'}})
\]

and, upon substituting the analyticity prescription into Eq. (7), we obtain

\[
S^{J'}_{l's'} = (\exp 2i S^{J'}_{l's'})(1 - \frac{1}{2} \sigma_r^{J'_{r'l'}} \left\{ \begin{array}{c} 1 \\ \text{if } E > E_T \\ \text{if } E < E_T \end{array} \right\})
\]

where, following Newton, \( \left\{ \begin{array}{c} A \\ B \end{array} \right\} \) means A if \( E_A > E_T \), and B if \( E_A < E_T \), where \( E_T \) is the reaction threshold energy. It should be
noted that \( \sigma_{rl's}' \), in Eq. (8) is strictly equal to \( \left| \sigma_{rl's}' (|E_\alpha - E_\alpha'|) \right| \); the accompanying \( \left\{ \frac{1}{i} \right\} \) contains the analytic properties of the expression.

We have separated \( S_{rl's}' \) into two terms. The first, \( \exp 2i S_{rl's}' \), can be interpreted as the "S-matrix element without reaction effects," if \( l_o = 0.7 \). When this term is substituted into Eqs. (1), (2), and (3), and the indicated sums are performed, the result is the "differential elastic cross section without reaction effects," which we designate by \( d\sigma^0,\alpha \). The second term, \( -(\exp 2i S_{rl's}') \frac{1}{2} \sigma_{rl's}' \left\{ \frac{1}{i} \right\} \), produces a cusp term after substitution into the same equations.

Accordingly, the Wigner cusp term \( \Delta d\sigma_{\alpha;\alpha} \) is defined by the set of equations

\[
d\sigma_{\alpha;\alpha} = d\sigma^0_{\alpha;\alpha} + \Delta d\sigma_{\alpha;\alpha} \tag{9}
\]

where

\[
\Delta d\sigma_{\alpha;\alpha} = \sum_{s' = |I-I|} \frac{(2s' + 1)}{(2I + 1)(2I + 1)} \Delta d\sigma_{\alpha s';\alpha s'} \tag{10}
\]

\[
\Delta d\sigma_{\alpha s';\alpha s'} = \left[ \frac{\mathcal{A}}{\alpha} (2s' + 1) \right] \sum_{L=0}^{\infty} \Delta B_L(\alpha s';\alpha s') P_L(\cos \theta) d\Omega \tag{11}
\]
\[ \Delta B_L(\alpha s'; \alpha s') = (1/4) \left\{ \sum_{J_1} \sum_{l_1} \sum_{J'} Z^2(l_1 J_1 l_1' J_1', s' L) \times \text{Re} \left[ (1 - \exp 2i \delta_{l_1 s'}^{J_1})^{*} (\exp 2i \delta_{l_1' s'}^{J_1'}) \frac{1}{2} \sigma_{r l' s'}^{J_1' i} \left\{ 1 \right\} \right] \right\} \]

\[ \times \sum_{J_1'} \sum_{l_2} \sum_{J_2} Z^2(l_1' J_1' l_2 J_2', s' L) \times \text{Re} \left[ (\exp 2i \delta_{l_1' s'}^{J_1'})^{*} \frac{1}{2} \sigma_{r l' s'}^{J_1' i} \left\{ 1 \right\} (1 - \exp 2i \delta_{l_2 s'}^{J_1'}) \right] \]

where \( \sigma_{r l' s'}^{J_1'} \) is still expressed in units of \( \pi \times 2 g_{s'} \) [see Eq. (6)].

The sums over \( l_1, J_1, l_2, J_2 \) in Eq. (3) are reduced because of the assumption that a single wave \( l' \) is dominant in the reaction.

Because \( Z(l_1 J_1 l_2 J_2', s L) = (-1)^L Z(l_2 J_2 l_1 J_1', s L) \) and \( \text{Re}(A^* B) = \text{Re}(A B^*) \), we may combine these sums, changing dummy indices and dropping the extraneous index \( \alpha \) for simplicity, to obtain

\[ \Delta B_L(s'; s') = (1/4) \sum_{J} \sum_{l} \sum_{J'} 2 Z^2(l J l' J', s' L) \times \text{Re} \left[ (1 - \exp 2i \delta_{l s'}^{J})^{*} (\exp 2i \delta_{l' s'}^{J'}) \frac{1}{2} \sigma_{r l' s'}^{J' i} \left\{ 1 \right\} \right] \]

\[ = (1/4) \sum_{J} \sum_{l} \sum_{J'} Z^2(l J l' J', s' L) \times 2 \sin \delta_{l s'}^{J}, \sigma_{r l' s'}^{J'}, \text{Re} \left\{ 1 \right\} \exp i \left( 2 \delta_{l' s'}^{J'}, - \delta_{l s'}^{J} \right) \]

Taking the \( \text{Re} \left[ \right] \) in the last equation yields
\[ \Delta B_L(s', s') = - \left( \frac{1}{4} \right) \sum J \sum \sum \frac{Z^2(J J' L L')}{J J'} \times \sigma^{J'}_{L' s'} 2 \sin \delta^J_{L s} \left\{ \begin{array}{c} \sin (2\delta^J_{L' s'} - \delta^J_{L s}) \\ \cos (2\delta^J_{L' s'} - \delta^J_{L s}) \end{array} \right\} \]  

\[ (12) \]

Within the limitations imposed during the derivation, Eqs. (9), (10), (11), and (12) describe expected Wigner cusp effects. The cusp effect is completely contained in the coefficients \( B_L \) in Eq. (2); that is, \( B_L = B^0_L + \Delta B_L \), where \( \Delta B_L \) is given by Eq. (12).

Since it is frequent practice (cf. Ref. 20) to analyze neutron scattering data in terms of the \( B_L \) coefficients, the utility of Eq. (12) for cusp interpretation is clear. Should a cusp effect be found in only one or two of several experimental \( B_L \) coefficients then the properties of the \( Z \) coefficients severely restrict the participating partial waves.

We now recall that the cusp predictions, Eqs. (9) to (12) are not generally valid if scattering is possible which conserves energy but not channel spin, because most detection systems are only energy-selective. In such cases, the threshold reaction is coupled (via unitarity of the S-matrix) to other off-diagonal (different initial and final state channel spin) elements, as well as a diagonal element ("pure" elastic scattering) of the S-matrix (see Eq. (4a)). An additional sum over final state channel spins is required in Eqs. (1) to (3). As a result additional cusp terms appear in Eq. (11). Unless polarization is also measured, comparisons of theory and experiment
are ambiguous in such cases. None of the present experimental results required this extension to cusp theory for comparison purposes.

The expected Wigner cusp in total cross sections \( \sigma_T \) is easily obtained if we recall that, by assumption, the interaction in the \((l')^{th}\) incident channel is purely elastic below threshold.

\[
\sigma_T \equiv \int (d\sigma/d\Omega)d\Omega + \sigma_{r l'} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
= \sigma^{\circ}_T + \left[ \int (\Delta d\sigma/d\Omega)d\Omega + \sigma_{r l'} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] \\
\equiv \sigma^{\circ}_T + \Delta \sigma_T
\]

(13)

where \( \sigma^{\circ}_T \) is the total cross section "without reaction effects."

Using the fact that \( \int P_L(\cos \theta)d\Omega = \delta^{K}_{L,0} \) (where \( \delta^{K}_{L,0} \) is a Kronecker delta) we obtain

\[
\Delta \sigma_T = \sum_{s' = |I-i|}^{I+i} \frac{(2s'+1)}{(2i+1)(2i+1)} \frac{\pi^2}{(2s'+1)} \Delta B_0(s';s') + \sigma_{r l'} \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

(14)

Substituting \( Z(l J l' J', s' 0) = \delta^{K}_{l l'} \delta^{K}_{J J'} (-1)^{-s'} (2J + 1)^{\frac{1}{2}} \), where \( \delta^{K}_{l l'} \) and \( \delta^{K}_{J J'} \) are Kronecker deltas, into the equation for \( \Delta B_L \) (with \( L = 0 \), Eq. (12), and putting this result into Eq. (14), one obtains

\[
\Delta \sigma_T = \sum_{s' = |I-i|}^{I+i} \frac{(2s'+1)}{(2i+1)(2i+1)} \sum_{J' = |l' - s'|}^{l' + s'} \left( \frac{\pi^2(2J'+1)}{(2s'+1)} \sigma_{r l' s'} \right) \\
\cdot \left\{ -2 \sin^2 \delta_{l' s'}^{J'} + \sigma_{r l'} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}
\]

(15)
where we have explicitly indicated the units of $\sigma_{rll's'}^{J'}$ (see Eq. (6)). Finally, recognizing that the summations in Eqs. (15) and (5) are identical, we may combine

$$\left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} \cdot \left\{ 2 \sin \frac{2\pi}{\delta_{l's'}}^{J'} \right\} = \left\{ \begin{array}{c} \cos 2\delta_{l's'}^{J'} \\ \sin 2\delta_{l's'}^{J'} \end{array} \right\}$$

and obtain the result

$$\Delta \sigma_T = + \frac{I+1}{\sum_{s'=[I-1]}^0} \left( \frac{(2s'+1)}{(2I+1)(2I+1)} \right) \Sigma_{J'=[l'-s']}^{l'+s'} \left( \frac{\pi^2(2J'+1)}{(2s'+1)} \sigma_{rll's'}^{J'} \right)$$

$$\cdot \left\{ \begin{array}{c} \cos 2\delta_{l's'}^{J'} \\ - \sin 2\delta_{l's'}^{J'} \end{array} \right\}$$

Equation (16), combined with Eq. (13), is the general expression for Wigner cusp effect in the total cross section, within the limitations imposed by the derivation.

**Wigner Cusps--Spinless Particles**

We now extract the special case of spinless particles from these expressions, for comparison with other work.

If all spins are zero, including that of the residual nuclei in the reaction, then Eq. (12) becomes

$$\Delta B_{L}(0;0) = -(1/4) \sum_{\ell} Z^2(\ell \ell 0 0, 0 L) 2 \sin \delta_{\ell} \sigma' \left\{ \begin{array}{c} \sin(2\delta_0 - \delta_{\ell'}) \\ \cos(2\delta_0 - \delta_{\ell'}) \end{array} \right\}$$
Since $Z(\ell \ell 0 0, 0 L) = (-1)^{L-\ell} (2\ell + 1)^{1/2} (\ell 0 0 0 | \ell 0 L 0)$

$$= \delta^K_{\ell L}, \text{ where } \delta^K_{\ell L} \text{ is a Kronecker delta,}$$

the sum in $\Delta B_L$ is reduced to one term, and using Eqs. (10) and (11) we obtain

$$\Delta \sigma = -\left(\frac{\pi^2}{4}\right) \sum_{\ell=0}^{\infty} \sigma^0_{r0} 2 \sin \delta_\ell \left\{ \frac{\sin(2\delta_0 - \delta_\ell)}{\cos(2\delta_0 - \delta_\ell)} \right\} P_\ell \, d\Omega$$

where $\delta_\ell = \delta^K_{\ell 0}$. This expression can be written as

$$\Delta \sigma \equiv 2 \Re \left[ \left(\frac{1}{2} \pi \right) \sum_{\ell=0}^{\infty} \left(1 - 2i \delta_\ell \right)P_\ell \right] \left(\frac{\sigma^0_{r0} \pi^2}{2\pi^2} \right) \frac{1}{2} \exp 2i \delta_0 \left\{ \frac{1}{i} \right\}$$

thus

$$\Delta \sigma = 2 \Re \left[ \left(1/\lambda \right) f^*(\theta) \frac{1}{2} \exp 2i \delta_0 \left\{ \frac{1}{i} \right\} \left(\sigma_r^{\exp} \right) \right]$$

(17)

where $f^*(\theta)$ is the complex conjugate of the scattered amplitude, with its explicit form evident by comparison with the preceding line, and where $\left(\sigma_r^{\exp} \right) \equiv (\pi \kappa^2 \sigma^0_{r0})$; i.e., the experimental reaction cross section.

This is just the cusp term obtained by Baz, for the case of spinless particles. Baz emphasized the explicit appearance of the amplitude, suggesting that there is additional theoretical significance in cusp data beyond cross section measurements (squared amplitudes) that are usually made. We would like to caution that most cases in practice involve non-zero spins; consequently scattered amplitudes, in general, are azimuth angle ($\varphi$) dependent. Comparison of theoretical
cusp effects with experiment may be undertaken if the theoretical expressions have been averaged and summed over magnetic quantum numbers. This is essentially what we have accomplished, and, in general, little vestige of complete amplitude forms is found in the resulting expressions.

On the other hand, it is useful to see from Eq. (17) that one can estimate expected Wigner cusp magnitudes from the expression

\[ |\Delta d\sigma| \approx \sigma_r (d\sigma)^{1/2}/\lambda \]

the remainder of the cusp expression being of order unity. The expected percentage cusp effect is inversely proportional to \((d\sigma)^{1/2}\) and exploratory experiments might well be undertaken at angles where the cross section is least.

The spinless particle result for total cross sections is a trivial reduction of Eq. (16) to

\[ \sigma_T = \sigma_T^0 + \sigma_r \exp \left\{ \frac{\cos 2\theta_0}{-\sin 2\theta_0} \right\} \]

This result is in agreement with those of others.\(^2,6\)

This concludes our discussion of pure Wigner cusp phenomena. Two other special cases, applicable to the present experiment, are deduced in Appendix D.
Theory of "Energy-Averaged" Wigner Cusps

We turn now to the problem of calculating the expected threshold effect, if the measured cross sections are effectively energy averages of the actual detailed cross sections, containing closely spaced resonance structure. That the resulting effects are related intimately to pure Wigner cusp phenomena was first discussed by Meyerhof.\(^6,7\) He showed that the energy average of Wigner cusp expressions leads to a predicted threshold effect which we shall call an "energy-averaged" Wigner cusp.

The physical situation we are considering is that the reaction and scattering cross sections are dominated by many closely spaced resonances. By closely spaced we mean a large number within the energy interval \(\Delta E\) over which the averaging is performed: either experimentally, because of finite incident beam resolution or because of overlap of the resonances; or theoretically, by choosing \(\Delta E > D\), where \(D\) is the average level spacing. No essential restrictions\(^7\) will be placed on the level width to spacing ratio, \(\Gamma/D\).

Noting that the energy dependence of differential elastic cusp effects is contained in \(\Delta B_L(s';s')\) (see Eq. (12)) we can isolate the energy dependent quantities:

\[
\langle \sigma_{r,l's'}^{J'} \, 2 \sin \delta_{ls}^J \left\{ \frac{\sin(2\delta_{l's'}^{J'}) - \delta_{l's'}^J}{\cos(2\delta_{l's'}^{J'}) - \delta_{l's'}^J} \right\} \rangle
\]

Brackets, \(\langle \rangle\), imply that an energy average is to be taken over an interval containing many resonances. The phases \(\delta_{l's'}^J\) and...
\( \delta_{\ell s}' \) are rapidly varying with energy over the averaging interval and we will assume they are uncorrelated for \( J \neq J' \) and/or \( \ell \neq \ell' \). This is the well known statistical assumption for the compound nucleus. Intuition might suggest that uncorrelated phases would lead immediately to breakup of expression (18) into a simple form, for example, the product of the averages of the factors. It turns out that such a breakup is by no means obvious, because of the presence of \( \sigma_{r\ell s}' \) within the averaging brackets.

Using R-matrix theory and applying the random sign assumption to certain value quantities \( \gamma_{\lambda} \) which will be discussed, Meyerhof showed that

\[
\left\langle \sigma_{r\ell s}' \left\{ \frac{2 \sin^2 \delta_{\ell s}'}{\sin 2\delta_{\ell s}'} \right\} \right\rangle \approx \left\{ \begin{array}{c} \left\langle \sigma_{r\ell s}' \right\rangle \\ 0 \end{array} \right\} \tag{19}
\]

and

\[
\left\langle \sigma_{r\ell s}' \left\{ -\cos 2\delta_{\ell s}' \right\} \right\rangle \approx \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \tag{20}
\]

The value quantities \( \gamma_{\lambda} \) mentioned above were introduced by Wigner and Eisenbud to parametrize the terms of the R-matrix, each of which has a resonance denominator. Thus, these quantities correspond to resonances and could be indexed in our notation by \( \gamma_{\lambda,\ell s}' \), where \( \lambda \) specifies a particular resonance of the set associated with angular momenta \( J, \ell \) and spin \( s \). The random sign assumption leading to Eqs. (19) and (20) is made for value quantities of different index.
J, \( l \), or \( s \), but is not justified for relating value quantities of the same \( J, l \) and \( s \), but different \( \lambda \); i.e., different resonances initiated by the same partial wave may have correlated phases. In particular, the reaction cross section appearing in the averaging brackets in Eqs. (19) and (20) proceeds through resonances correlated in phase to the remaining expression in the bracket, and it is surprising that the simple results, Eqs. (19) and (20) were obtained.\(^7\)

In Appendix C, we show how Eqs. (19) and (20) can be applied to expression (18) to give the result

\[
\langle \sigma_{rl's'}^{J'} \rangle \approx \left\{ \delta^K_{\ell \ell'}, \delta^K_J \right\} \delta_{l's'}^{J'} \delta_{l's'}^{J'}
\]

where \( \delta^K \) is a Kronecker delta.

Part of the proof of Eqs. (19) and (20), which led to Eq. (21), depends upon a cancellation of path integrals used in the energy averaging process, which will be complete only if normally slowly varying quantities remain slowly varying throughout the averaging interval. For the case of outgoing \( s \) waves in the reaction channel, one of these quantities (the logarithmic derivative of \( R \)-matrix theory) does not vary slowly right at reaction threshold. In essence, this difficulty is just the result of rediscovering pure Wigner cusp phenomena as one formally approaches very close to threshold in the averaging process.
This means that the derivation of Eqs. (19) and (20) is not strictly valid arbitrarily close to threshold. The actual experimental result would depend on the exact energy distribution of the beam. As long as the energy distribution of the beam does not overlap the threshold, the results, i.e., Eqs. (19) and (20) and consequently Eq. (21), will be applicable.

After justifying Eq. (21) we have completed the derivation of "energy-averaged" cusp expressions. Recalling Eq. (12), which contains expression (18), and using Eq. (21), we obtain

$$\langle \Delta B_L(s';s') \rangle = - (1/4) \sum_{J'} Z^2(c_{J'} J' \ell' s', s' L) \left\{ \begin{array}{c} \sigma_{rJ's'} \\ 0 \end{array} \right\} (22)$$

the sums over $\ell$ and $J$ have vanished because of $S^K_{J\ell}$, and $S^K_{JJ'}$ in Eq. (21). The "energy-averaged" differential cross section cusp term is obtained by substituting Eq. (22) in Eqs. (9-11).

Substitution of relation (20) in Eq. (16) gives the expected result for the total cross section,

$$\langle \Delta q_T \rangle = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right]$$

(23)

That is, there is no "energy-averaged" cusp in the total cross section. This is consistent with the basic optical model assumption, used in relating the model to experiment, that the energy-averaged total cross section, being linearly dependent upon the scattering matrix elements, is smooth in energy even across reaction thresholds. We will come back to this point when summarizing the theory.
"Energy-Averaged" Cusps--Spinless Particles

As an example, for comparisons to previous work, consider the special case of spinless bombarding and target particles

\[ I = i = s' = 0 \ ; \ \ell' = J' \]

Equation (22) becomes

\[ \langle \Delta B_L(0;0) \rangle = - \frac{1}{4} Z^2(\ell' \ell' \ell' \ell', 0 L) \left\{ \sigma^{\ell' \ell' 0}_{rL'0} \right\} \]

Using \( Z^2(\ell' \ell' \ell' \ell', 0 L) = (2\ell' + 1)^2 (\ell' \ell' 0 0 | \ell' \ell' L 0)^2 \), indicating explicitly the units of \( \langle \sigma^{\ell'}_{rL'0} \rangle \), and by substituting Eq. (22a) into Eqs. (9-11), we obtain

\[ \langle \Delta d\sigma_{\alpha;\alpha} \rangle = - (2\ell' + 1) (\pi \times \frac{2(2\ell' + 1)}{4\pi}) \sum_{\ell} (\ell' \ell' 0 0 | \ell' \ell' L 0)^2 \]

\[ \times P_L d\Omega \]

\[ \equiv - (2\ell' + 1)(\langle \sigma^{\ell'}_{rL'} \rangle_{\text{exp}} / 4\pi) (P_{\ell'})^2 \left\{ \frac{1}{0} \right\} , \]

where \( \langle \sigma^{\ell'}_{rL'} \rangle_{\text{exp}} \) means units have been restored to \( \langle \sigma^{\ell'}_{rL'} \rangle \) and experimental values may be substituted for this quantity. Equation (24) is the same as the formula, Eq. (16) of Ref. 6.

"Energy-Averaged" Cusps--Spin \( \frac{1}{2} \) Incident Particles

We give the results for a second special case of Eq. (22) of particular interest in comparison with the present experimental data, where the incident particle is a neutron, the target nucleus is even-even, and incident d waves dominate the reaction.
\[ I = 0, \, i = \frac{1}{2}, \, J' = l' \pm \frac{1}{2}, \quad l' = 2, \quad \text{and} \quad s' = \frac{1}{2} \]

\[
\langle \Delta \sigma(l' = 2) \rangle = - \left( \langle \sigma_{r l'} = 2 \rangle_{\exp} / 4\pi \right) \sum_{J' = l' - \frac{1}{2}}^{l' + \frac{1}{2}} R_{J'}
\times \sum_{L} \left[ Z^2(2J', 2J', \frac{1}{2}, L) / (2J' + 1) \right] P_L \begin{pmatrix} 1 & \hat{0} \end{pmatrix} d\Omega 
\tag{25}
\]

where \( R_{J'} \equiv \langle \sigma_{r l'} / 2 \rangle_{\exp} / \langle \sigma_{r l'} \rangle_{\exp} \)

Although not immediately obvious, Eq. (25) is identical to Eq. (16') in Ref. 6. To compare, one must identify \( R_{l'} \equiv (l' + 1) / (l' + 1 + l' - r) \), \( R_{l'} = l' - \frac{1}{2} = l' - \frac{1}{2} / (l' + 1 + l' - r) \), where \( \bar{r} \) is defined and used in Ref. 6.

Since Eq. (25) is important to present experimental comparisons, it will be useful to give the terms in detail. It turns out that the cusp prediction, Eq. (25), is quite insensitive to the ratio \( R_{J'} \); recall that \( R_{J'} \) was defined as the fraction of the reaction probability which originates from compound resonances of a particular \( J' \), for example \( l' + \frac{1}{2} \). We make the reasonable choice \( R_{J'} = \frac{1}{2} (2J' + 1) / (2l' + 1) \) which is the same as the fraction of the total number of incident states of the \( (l') \)th partial wave which have total angular momentum \( J' \). With this choice of \( R_{J'} \), Eq. (25) becomes

\[
\langle \Delta \sigma(l' = 2) \rangle = - \left( \langle \sigma_{r l'} = 2 \rangle_{\exp} / 4\pi \right) \begin{pmatrix} 1 & \hat{0} \end{pmatrix}
\times (2l' + 1)^{-1} \sum_{J', L} \frac{1}{2} Z^2(2J', 2J', \frac{1}{2}, L) P_L (2l' + 1) d\Omega
\]

\[
= - \left( \langle \sigma_{r l'} = 2 \rangle_{\exp} / 4\pi \right) \begin{pmatrix} 1 & \hat{0} \end{pmatrix}
\times [1 + 38/35 P_2(\cos \theta) + 18/35 P_4(\cos \theta)] d\Omega 
\tag{26}
\]
Equation (26) actually applies to any scattering experiments on medium and high atomic numbered even-even targets at energies where random phase concepts are useful, and near the threshold of a reaction dominated by incident \( \ell \) waves. The outgoing particle in the reaction channel must be a neutron, as mentioned previously, in order to obtain a sharp cusp.

In conclusion, we remark that the expressions for "energy-averaged" Wigner cusps depend only upon statistical assumptions, and not upon a particular model, for example, the optical model; rather, these expressions simply give the change in cross sections expected at reaction thresholds, due to unitarity of the scattering matrix. Meyerhof showed, however, that these statistical assumptions are consistent with the optical model assumption that the scattering matrix element consists of a smooth plus fluctuating part.\(^{22}\)

Also, we note that "energy-averaged" Wigner cusp expressions predict no below-threshold effect. Consequently, in regions of mass number and energy where "energy-averaged" theory applies, it is not necessary to specify the dominant outgoing wave in the reaction channel, in contrast to pure Wigner cusp theory (see the discussion just before Eq. (7)). In particular, outgoing \( p \) wave dominance of the reaction in the \( \mathrm{Zr}^{90} \) experiment, discussed earlier, causes no difficulty in analysis. Determination of the dominant incident partial wave in the reaction and Eq. (26) suffice.
Neutron Differential Elastic Cross Section Measurements

Neutrons were produced by the $T(p,n)$ reaction, using tritium loaded zirconium and titanium targets for the proton beam from a High Voltage Engineering Company 3-Mev Van de Graaff generator stabilized to about 1 kev. Various tritium targets, described in Table I, of from 10 to 70 kev thickness at 1.02 Mev proton energy (as determined by the rise method at threshold) were used. An eccentric circular motion was imposed upon the tritium target. Consequently, the proton beam struck a ring-shaped area on the target, which minimized cooling difficulties and also effectively averaged over nonuniform tritium deposition in the target, thereby producing a time integrated neutron flux very nearly proportional to accumulated proton charge.

The neutron beam was collimated by a rectangular conical opening in LiCO$_3$-loaded paraffin source shielding of characteristic thickness 18 in. The geometry and shielding of the experiment is shown in Fig. 1. The collimation angle was such that the energy spread in the neutron beam due to $T(p,n)$ kinematics was approximately 4 kev. A standard long counter placed at $0^\circ$ with respect to the beam (see Fig. 1) was used to monitor the neutron yield.
Fig. 1. Experimental Arrangement--Plan View: 1. Tritium loaded target; 2. Air-water spray coolant; 3. Eccentric wobble for additional cooling effect; 4. Paraffin shielding (loaded with LiCO$_3$); 5. Paraffin (LiCO$_3$ loaded) rectangular conical collimator; 6. Scatterer in "In Beam" position; 7. Scatterer in "Out" position; 8. Anthracene or stilbene crystal (a right circular cylinder, 2" dia x 1" long); 9. 6810A RCA photomultiplier tube; 10. Preamplifier (linear and nonlinear gamma suppression outputs); 11. Long Counter (BF$_3$) in monitor position.
The neutron beam was allowed to strike the scatterer, which was made slightly smaller than the aperture of the neutron beam, at about 10 in. from the collimator end. The lithium scatterer was a right circular cylinder oriented vertically; the remaining scatterers used were in the shape of a slab, oriented with their normals parallel to the floor and at 40° to the incident beam direction. The scatterers used are described in detail in Table II. Scatterer thicknesses were between 1/5 and 5/8 neutron mean free paths at the energies of experimental interest. These rather thick scatterers were necessary to obtain sufficient counting rates. Scattered neutrons were detected at two angles by organic scintillation crystals (2 in. dia x 1 in. long anthracene and 2 in. dia x 3/4 in. long stilbene) placed one on each of the transmission and reflection sides of the scatterer at distances varying from 6 to 10 in. and with the crystals oriented as shown in Fig. 1. The scatterer to crystal distance varied with detection angle and was set at the maximum consistent with a neutron background/effect ratio less than two. A saturated space-charge pulse-shape discrimination output of the detection preamplifiers was used to suppress gamma-ray background. Typical neutron pulse spectra, given in Fig. 2, illustrate that we also used an integral discriminator to bias out inelastic neutrons.

To facilitate neutron background subtraction, the scatterers were suspended on thin wires from an overhead crane so that the scatterer could be alternately removed and replaced in the beam at each neutron energy (see Fig. 1).
### TABLE I

Description of Tritium-Loaded Targets

<table>
<thead>
<tr>
<th>Index</th>
<th>Made by</th>
<th>Yield (\frac{\text{neutrons/sterad}}{\text{microcoulomb}})</th>
<th>Thickness ((\text{at } T(p,n) \text{ threshold } 1.019 \text{ Mev}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>HAR</td>
<td>(1.5 \times 10^6)</td>
<td>22</td>
</tr>
<tr>
<td>2</td>
<td>HAR</td>
<td>(0.37 \times 10^6)</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>HAR</td>
<td>(4.8 \times 10^6)</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>OR</td>
<td>(0.33 \times 10^6)</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>OR</td>
<td>(1.1 \times 10^6)</td>
<td>11</td>
</tr>
</tbody>
</table>

HAR = Harwell, England (Tritium-loaded titanium)  
OR = Oak Ridge Natl. Lab., Tenn. (Tritium-loaded zirconium)

### TABLE II

Description of Scatterers

<table>
<thead>
<tr>
<th>Scatterers Used in (d\sigma) Measurements</th>
<th>Material</th>
<th>Dimensions (inches)</th>
<th>Chemical Purity (%)</th>
<th>Angle between slab and beam (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cerium a.</td>
<td>4.0 x 3.0 x 0.63</td>
<td>99.8</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Cerium b.</td>
<td>3.25 x 2.75 x 0.19</td>
<td>99.8</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Barium</td>
<td>4.0 x 3.0 x 1.19</td>
<td>97.5</td>
<td>45</td>
<td></td>
</tr>
<tr>
<td>Zirconium</td>
<td>4.0 x 1.5 x 0.56</td>
<td>98</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Iron</td>
<td>4.0 x 2.5 x 0.52</td>
<td>98</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Lithium</td>
<td>3.05 dia x 4.4 high</td>
<td>~ 100</td>
<td>Rt. cir. cylinder</td>
<td></td>
</tr>
</tbody>
</table>

Samples Used in \(\sigma_T\) (Transmission) Measurements

- Cerium: \(1.0 \times 1.25 \times 0.935\) long  
  - The "long" dimension was parallel to the beam axis.
- Barium: \(1.0 \text{ dia.} \times 1.87\) long
- Iron: \(1.0 \times 1.13 \times 0.97\) long
Fig. 2. Pulse spectra from detection equipment at neutron energy of 0.5 Mev. a. Gamma-Ray-Suppression Spectra (nonlinear pulses):
1. With the detector (2" dia x 3/4" long stilbene crystal) in the neutron beam and with the pulse height analyzer (PHA) recording all preamplifier nonlinear output pulses. 2. With the proton beam blocked by a beam stopper from hitting the tritium target, with a Na$^{22}$ source taped on the crystal, and with the PHA recording all preamplifier nonlinear output pulses. 3. With the proton beam on the tritium target, with the detector crystal in the neutron beam, and with a Na$^{22}$ source on the crystal. Two integral discriminator biases were set, one on each of the linear and nonlinear spectra, and the PHA had been adjusted to accept nonlinear pulses which were associated with a time coincidence of discriminated linear and nonlinear pulses. b. Linear Spectra (linear preamplifier output pulses associated with a time coincidence of discriminated linear and nonlinear pulses): A. With the detector crystal in a typical experimental position (90° lab) and the scatterer in the experimental position in the neutron beam. B. With the detector in the same experimental position as in A, and with the scatterer retracted to its "Out" position (see Fig. 1). C. (A - B) = elastically scattered neutron pulse spectrum.
\( \gamma \) SUPPRESSION SPECTRA

INTEGRAL DISCRIMINATOR

\[
\begin{align*}
\text{RELATIVE UNITS} & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
\text{PULSE HEIGHT RELATIVE UNITS} & \quad 0 \quad 50 \quad 100 \quad 150 \quad 200
\end{align*}
\]

LINEAR SPECTRA

INTEGRAL DISCRIMINATOR

\[
\begin{align*}
\text{NUMBER OF PULSES INCIDENT NEUTRON} & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
\text{INTEGRAL DISCRIMINATOR} & \quad 0 \quad 50 \quad 100 \quad 150 \quad 200
\end{align*}
\]
Neutron Total Cross Section Measurements

The same neutron beam, source shielding, collimation, and gamma suppression described above were used in measuring total cross sections.

Neutrons were detected in this case by a stilbene crystal with dimensions 1/2 in. dia x 1/2 in. long mounted at 0° to the incident beam. Transmission samples were right parallelepipeds whose area normal to the beam (≈ 1 sq. in.) was sufficient to shield the crystal from direct neutrons. (Neutrons were produced over a proton beam spot area of about 1/4 in. x 1/4 in.) The samples were 10 in. from the collimation (28 in. from the neutron source) and 20 in. from the crystal. Sample thicknesses were chosen to allow approximately 60 percent neutron transmission at energies of interest in the experiment. The sample and a 9 in. long 1 in. dia. tungsten shield were mounted on thin supporting rods above the ends of two arms of a rotatable horizontal cross. A third arm had only a thin supporting rod attached. Thus it was possible to count transmitted, background, and direct neutrons in sequence at each neutron energy by rotating the cross.

Apparatus

It remains to describe the electronics systems necessary to reduce crystal scintillation events to (elastically scattered) neutron counts. A detailed discussion of our gamma suppression circuit is contained in Appendix D. The block diagram, Fig. 3, will be useful in following the present discussion.
Fig. 3. Block diagram of electronics, showing the function of each major component.
SYSTEM II DUPLICATES I FOR A SECOND ANGLE

LONG COUNTER PREAMPLIFIER

LINEAR AMPLIFIER

SCALER COUNTER

3µs DELAY

256 CHANNEL PULSE HEIGHT ANALYZER

GATE

SYSTEM I PREAMPLIFIER

LINEAR OUTPUT

γ SUPPRESSSION OUTPUT

CATHODE FOLLOWER BANK

LINEAR AMPLIFIER

INTEGRAL DISCRIMINATOR

1.35µs DELAY

SLOW COINCIDENCE UNIT

CATHODE FOLLOWER

SCALER COUNTER

EXPERIMENTAL STATION

CONTROL AND DATA ROOM
For each of two detection systems an RCA 6810A photomultiplier and preamplifier (Fig. 22) produced linear and nonlinear (gamma suppression) outputs in response to a crystal scintillation. These two outputs fed into a double delay line and RC coupled linear amplifier, respectively.

Each of the amplifier outputs drove an adjustable bias circuit (integral discriminator) which generated a standard square 20 volt output for each input pulse that exceeded the bias level. After time synchronization was established by the use of a suitable delay cable, these pulses were allowed to enter a standard coincidence unit. A coincidence output pulse was obtained for each simultaneous occurrence of input pulses. Thus, with properly adjusted bias levels on each of the linear and nonlinear pulses, a coincidence output pulse signalled the detection of an elastically scattered (linear pulse above linear bias) neutron (nonlinear pulse above nonlinear bias). These pulses were the experimental counts.

A 256-channel pulse-height analyzer (PHA) was a valuable tool in setting up the gamma suppression circuit, the amplifier gains, and the discriminator biases. We also used the PHA as a monitor on the overall electronics performance during a run; these monitor spectra had a specific role in data analysis for some of the experiments (to be discussed). Typical pulse spectra are shown in Fig. 2. The coincidence output pulses also served to gate the PHA when spectra such as those shown in Fig. 2 were taken.

A standard Hansen and McKibben long counter was employed to
monitor the neutron yield. Direct neutrons were moderated by a paraffin cylinder and some were captured by boron\textsuperscript{10} in a one-in. dia. BF\textsubscript{3} detection tube. The subsequently emitted alpha particle initiated a pulse which was amplified and counted as a neutron event. The efficiency of the counter was approximately $2 \times 10^{-3}$ and was fairly constant for a range of neutron energies from 0.5 Mev to 3 Mev. A 1/8 in. cadmium and 3 in. paraffin shield on the sides and back of the counter reduced floor scattered neutrons to less than 15 percent of direct neutrons. Although absolute neutron yields of our tritium targets were measured using the long counter (see Table I), these yields were not required in the data analysis.

Procedure

**Differential Elastic Cross Section**

Since the experimental interest centered on effects near the energies of first excited states, initial adjustments of the apparatus were made at these energies. For each experiment the nonlinear preamplifier controls (see Appendix D) were adjusted for optimum gamma-ray suppression, while the crystal was located in the scatterer "in beam" position, later called the "calibration position." Care was taken to insure that the crystal plane was parallel to the incident beam axis. This aspect corresponded to the aspect of the crystal toward scattered neutrons when it was in experimental position (see Fig. 1). After these adjustments were completed, the neutron yield and overall efficiency of the crystal/electronics system were calibrated by counting
direct target neutrons at several neutron energies above and below threshold. This was done for each of two crystal detector systems.

After the crystals were moved to the experimental position (Fig. 1), the same energy range was covered in energy increments of approximately one-half incident neutron-energy spread. At each energy, counts were recorded both with the scatterer in and with the scatterer out of the beam, thus providing a basis for subtraction of neutron background. On a succeeding run down in energy the same increments were used, but the points interleaved those from the run up; thus, we checked data reproducibility.

All counts were normalized to a fixed amount of accumulated proton charge. When using 30-50 microamperes proton current, we found that the necessary counting time was typically 10 minutes for each energy point.

Long counter monitor neutron counts were also recorded at each point. Monitor neutron counts with the scatterer out of the beam served both as a check on the condition of the tritium target, which was subjected at times to as much as 70 watts of heat flow, and as the basis for a small correction of the data analysis for target yield variation.

**Total Cross Sections**

As stated above, the gamma suppression circuit was adjusted at a neutron energy corresponding to an excited state of interest. After carefully aligning the sample and shield between the proton beam spot
and crystal detector, we took a sequence of counts at each energy point. By rotating the supporting horizontal cross on which sample and shield were mounted we were able to place first the sample, then the shield, and last the duplicate sample support (for a count of direct neutrons) into the beam. After each such set of counts the neutron beam energy was increased by approximately one-half incident beam-energy spread until a desired energy range was covered below and above threshold. A subsequent run down in energy, with interleaving points, verified reproducibility.
CHAPTER IV

DATA REDUCTION AND ANALYSIS

Data Normalization

The basic normalization used in the experiment was to a fixed accumulated proton charge, which implies an assumption that the resulting time-integrated neutron flux was also a fixed quantity, except for the smooth change in the T(p,n) reaction cross section with energy, and except for a ±1 percent uncertainty introduced by our proton current integrator. As mentioned before, the eccentric circular motion of the tritium target minimized any imperfection in this assumption, by averaging over nonuniformities of tritium deposition. However, small variations in integrated neutron flux from the true smooth shape of the T(p,n) reaction cross section might have been present when the neutron energy was changed, because of changes in proton beam intensity distribution which occur when one sets the accelerator to a new energy. If corrections were not made this could have produced scatter in the data, as plotted versus energy, in excess of statistical fluctuations. For this reason, monitor neutron counts, which had very small statistical errors, were used to renormalize the differential cross section effect counts to a smooth curve drawn through monitor neutron data; this curve was assumed to have the true smooth shape of the neutron yield versus neutron energy. The normalization was then a percentage
correction to each data point depending upon the percent deviation of
the monitor counts from the smooth curve. This correction was usually
less than one percent and never more than three percent. It was gen-
erally impossible to distinguish by eye the difference between corrected
and uncorrected data, but in seeking cusp effects, we wished to take
all precautions necessary to establish the relative magnitude of the
data points plotted versus neutron energy.

Data Averaging

The original data for experiments on zirconium, cerium and
barium showed reproducible fluctuations in addition to a well defined
drop in magnitude over a broad region above reaction thresholds. It
turned out that energy averaging the original data did not mask the
essential structure or features. Figure 4 illustrates this point,
where we have used differential elastic cross section data from a
cerium experiment. Therefore, we were able to improve the statistical
errors without significant loss of structural detail and also to estab-
lish a standard energy averaging interval (ΔE = 50 kev) for presenting
measurements at many angles.

Insofar as we were concerned with pure Wigner cusp effects, we
could not use data averaging procedures in the experiments on lithium
and iron. Here, the magnitude of the structure in the cross sections
turned out to exceed that expected of cusp effects; in the experiments
on iron, the structure was particularly marked. A successful search
for a cusp effect would depend upon distinguishing this basic structure
Fig. 4. Neutron differential elastic (90°) cross section of natural cerium for three widths of energy averaging interval.
(a) Original data; \( \Delta E = 7 \) kev; statistical errors \( \pm 5\% \). (b) Same data averaged over \( \Delta E = 25 \) kev; statistical errors \( 2.5\% \). (c) Result of a larger averaging interval, \( \Delta E = 50 \) kev; statistical errors approximately \( 1\% \). The averaging in (c) was selected as the standard for cusp analysis of cerium. The loss of structural detail upon successive averaging is apparent.
in the cross sections from a true cusp effect. We illustrate with Fig. 5, which gives iron total cross section measurements, and which shows that use of 25 kev incident neutron energy resolution washed out significant structure elucidated by experiments with 7 kev spread in incident neutron energy.

In addition to washing out structure, a poor-resolution incident beam also has the well known effect of distorting sharp features in cross sections. We observed this in differential cross section measurements; a comparison of Fig. 11 and Fig. 14 shows a feature shift of about +10 kev as the result of using 30-kev compared to 7-kev energy resolution. This amount of feature shift would make a cusp analysis impossible.

In reporting some of the lithium differential cross sections we have energy averaged the data for clarity in presentation (see Fig. 10). It is clear from the foregoing discussion that in so doing we conceded that there was no cusp effect in the data and that remaining structure must be attributed strictly to the elastic scattering cross section of the lithium nucleus.

Data Analysis

Differential Cross Sections

We give our calculation of the measured differential elastic cross sections using (1) the basic data, which consisted of the counts $C = C_i \text{(scatterer in)} - C_o \text{(scatterer out)}$, (2) geometrical factors,
Fig. 5. The total neutron cross section of natural iron, using neutron beams of two energy spreads: (a) $\Delta E = 25$ kev; (b) $\Delta E = 7$ kev. The size of the symbol indicates the size of the statistical errors.
and (3) an overall efficiency measured in a calibration run. In order to make a detailed comparison of gross structure, cusp effects in cross sections, or angular distributions with theory, or with other experimental work, additional corrections for multiple scattering and finite scatterer effects had to be applied. These corrections are discussed in Appendix E and tabulated in Table III.

Measured cross sections will be given in the center of mass (CM) frame and will be expressed as functions of mean polar CM angle \( \theta_0 \) (CM)) and mean CM incident neutron energy \( \bar{E} \) (CM)). Our reason for presenting CM data is that theoretical formulas for cusp phenomena and threshold energies are more conveniently expressed in CM terms. The quantities \( d\sigma \) (CM), \( \theta \) (CM), and \( E \) (CM) were computed from the corresponding quantities expressed in the laboratory frame, using well known transformation formulas.\(^{25}\) In the present experiment the differences between lab (laboratory) and CM quantities were negligible except in the cases of iron and lithium.

\( \bar{E} \) (lab) was determined by subtracting one-half the incident neutron energy spread from the maximum lab energy of the neutron beam. Incident neutron energy spread was computed, assuming statistical independence of the two contributions, from (1) the geometrical spread, fixed by the neutron collimation angle and \( T(p,n) \) kinematics to be 4 kev, at 1.5-Mev neutron energy; and (2) the proton energy spread caused by the thickness of the metal target, containing tritium. The latter spread was computed from the target thickness at \( T(p,n) \) threshold\(^{23}\) \([1.019-\text{Mev proton energy (lab)}]\) as determined from the slope of
the rise at threshold and conversion factors relating proton stopping power of the target at threshold to the stopping power at the energy of interest.\textsuperscript{25} Neutron energy spread is directly related to proton energy spread by $T(p,n)$ kinematics.\textsuperscript{26}

**Maximum** incident neutron lab energy present in the incident beam was determined from maximum proton lab energy by the same kinematical relations, where the proton energy was calibrated in two ways. First, by using the accurately known $T(p,n)$ threshold energy we effectively calibrated the Van de Graaf machine analyzing magnet at 1.019 Mev. The magnet field was measured with a proton magnetic resonance probe, between the pole faces, to obtain with high precision an RF frequency which could be associated with 1.019 Mev proton energy. We used the intersection of the straight line extrapolation of the lead slope of the $T(p,n)$ yield versus frequency curve and the frequency axis as indicating the true threshold. Since proton resonance frequency and the magnetic field ($B$) are linearly related ($f = \bar{\mu} \cdot B/h$, where $f$ is frequency, $\bar{\mu}$ is the proton magnetic moment, and $h$ is Planck's constant), and the analyzing magnet is a momentum selector, then maximum proton lab energy is $E_{p}^{\text{max}}(\text{lab}) = 1.019 f_{TH}^{2} f_{TH}^{2}$ Mev where $f_{TH}^{2}$ is the previously described threshold proton resonance frequency.

Second, because $C^{12}$ has a sharp resonance at 2.076-Mev lab neutron energy, a long counter of the type used in this experiment has an anomaly in its efficiency versus energy curve at 2.076 Mev due to the increased cross section of $C^{12}$ in the paraffin moderator.\textsuperscript{23} Thus, a dip occurs in monitor neutron counts at this neutron energy. The
proton resonance frequency at this dip established a calibration point for the mean neutron energy which served as a check on calibration by the first method. The two calibration points conveniently straddled the proton energy region involved in the present experiments.

Initially, the "effect counts" \( C \) were normalized as described in the first section of this chapter, giving normalized effect counts, \( C_N \).

For a given scattering sample and incident neutron lab energy \( E \) the relation between \( C_N \) and the differential cross section is:

\[
C_N = Y f_x[(E - E_R), R_x] f_D(E - E_R) \Omega_x \Omega_s \\
\times [1 - \exp(- n \sigma_T d)] \frac{d\sigma(\theta, E)}{\sigma_T} \tag{27}
\]

where \( Y \) is the neutron yield of the target per unit solid angle; \( f_x[(E - E_R), R_x] \) is the intrinsic crystal efficiency at the lab energy of the scattered neutron and at the crystal-to-scatterer distance, \( R_x \); \( E_R \) is the lab recoil energy of the residual nucleus; \( f_D(E - E_R) \) is a discriminator correction to be discussed; \( \Omega_x \) and \( \Omega_s \) are the solid angles subtended by the crystal to the scatterer center and by the scatterer to the proton beam spot, respectively; \( nd \) is the number of scattering nuclei per unit area in the direction of the incident beam; \( \sigma_T \) is the total cross section of the scatterer material; and \( d\sigma(\theta, E) \) is the differential elastic cross section per unit solid angle. Since \( d\sigma(\theta, E) \) has been assumed constant [otherwise Eq. (27) would contain integrals over solid angle of the finite detector and energy spread of
the beam] over the angular region described by $\Omega_x$ and the energy region included by the energy spread of the incident neutrons, the symbols $\theta$ and $E$ refer to mean polar angle and mean neutron beam energy respectively [i.e., $\theta_{\text{lab}}$ and $E_{\text{lab}}$].

The corresponding formula for "calibration counts" is

$$C_c = Y \Omega_c f_x(E,R_c) f_D(E), \quad (28)$$

where $\Omega_c$ is the solid angle subtended by the crystal to the proton beam spot and $R_c$ is the associated scatterer to tritium target distance.

Taking the ratio Eq. (27)/Eq. (28) yields

$$d\sigma(\theta,E) = \frac{f_x(E,R_c) f_D(E)}{f_x[(E - E_R), R_x] f_D(E - E_R)} \frac{\Omega_c}{\Omega_x \Omega_s}$$

$$\times \sigma_T [1 - \exp(- n \sigma_T d)]^{-1} K(C_N/C_c) \quad (29)$$

where $K$ is a constant normalizing the effect and calibration counts to the same accumulated proton charge.

The factors $f_D$ are necessary corrections which arise because the discriminator bias on detected linear pulses was fixed. As a result, the counted fraction of the total detected elastic neutron spectrum is a function of energy. This is shown in detail by Fig. 6 and is explained further in the figure caption. In the detection of scattered neutrons, the factor $f_D$ becomes $f_D(E - E_R)$ because the lab energy of the scattered neutrons rather than that of the incident neutrons determines the discriminator correction. In all except the
Fig. 6. Linear pulse spectra for elastically scattered neutrons at three energies, showing the energy dependence of detection efficiency due to fixed integral discrimination bias. The spectra were obtained with the detector at 125° (lab) to a lithium scatterer and are the result of subtracting the spectrum with scatterer out of the neutron beam from the spectrum with scatterer in the beam. This figure only illustrates an experimental point in the data analysis. Actual runs were made with the linear bias set at a fixed voltage, corresponding to approximately 1/2 maximum pulse height for a neutron energy equal to the mean energy of a complete run. In this example, the bias would be set at about pulse height 70, but the run could not extend as low as 0.28-Mev neutron energy.
\( \bar{E}_n(\text{CM}) = 0.65 \text{ Mev} \)

\( \bar{E}_n(\text{CM}) = 0.48 \text{ Mev} \)

\( \bar{E}_n(\text{CM}) = 0.28 \text{ Mev} \)
experiment on Li and Fe $f_D(E - E_{R_x}) \approx f_D(E)$ because the lab recoil energy of the residual nucleus was negligible. Thus the ratio $f_D(E)/f_D(E - E_{R_x})$ in Eq. (29) was $\approx 1$. For the lithium and iron experiments we determined this ratio by comparison of pulse height analyzer spectra of linear pulses taken during "calibration" and "effect" runs.

We simplified the ratio $f_x(E, R_x)/f_x[(E - E_{R_x}), R_x]$, which is a ratio of intrinsic crystal efficiencies, in a reasonable way, by assuming when necessary (in lithium and iron experiments) that the energy dependence of $f_x$ was separable from the distance dependence, $R_x$. Recalling the calibration equation, Eq. (28), we see that

$$\frac{C_c}{[Y(E) \Omega_c f_D(E)]} = f_x(E, R_x)$$

(30)

Thus, if we divide the plot of $C_c$ versus $E$ by the incident flux, $Y(E)$, obtained from monitor neutron counts, and by $f_D(E)$, obtained from linear pulse spectra, we see that the energy dependence of $f_x$ is determined. The same energy dependence, by assumption, applied to $f_x(E - E_{R_x}, R_x)$, hence by translation of the curve $f_x$ along the energy axis by an amount $E_{R_x}$ (calculated from kinematical relations\textsuperscript{25}) we obtained the ratio $f(E - E_{R_x}, R_x)/f_x(E, R_x)$ insofar as the (separated) energy dependence is concerned, i.e.,

$$f_x(E, R_x)/f_x[(E - E_{R_x}), R_x] \approx \epsilon_x(E) f_x(R_x)/f_x(R_x),$$

(31)

where $\epsilon_x(E)$ is the just determined energy dependence of the $f_x$ ratio.

It will be useful to write

$$\Omega_c/\Omega_x \Omega_s = \epsilon_1(R_x^2/A_s),$$

(32)
where \( A_s \) is the scatterer area projected perpendicularly to the beam axis and \( \epsilon_1 \) is a small correction for the degree to which the three solid angles are not exactly expressible as projected area/distance squared.

Substituting (31) and (32) in Eq. (29) we obtain

\[
\begin{align*}
\sigma(\theta, E) &= \epsilon_1(E) \frac{f_x(R_c)}{f_x(R_x)} \epsilon_1(R_x^2/A_s) \frac{f_D(E)}{f_D(E - E_R)} \\
&\times \sigma_T(1 - \exp(- n \sigma_T d))^{-1} K C_N/C_c
\end{align*}
\]

The quantity \( \epsilon_1 f_x(R_c)/f_x(R_x) = \epsilon_2(R_x) \) represents the inverse of the effective crystal efficiency as a function of distance, \( R_x \), normalized to its value at \( R_x = R_c \). If we make the reasonable assumption that this quantity is very close to unity for \( R_x \) equal to and greater than \( R_c \) (\( R_c \) was 28 in. in our experiment) then neutron counts versus distance can be compared to inverse square law behavior (normalized at large distances) to obtain \( \epsilon_2(R_x) \). This measurement was made using a PoBe neutron source at several distances \( R_x \) from the crystal, to obtain relative counts in unit time for each position. The results are shown in Fig. 7. Large corrections would be necessary for distances less than 5 in. In the present experiments the factor \( \epsilon_2(R_x) \) was typically 1.10 to 1.20 (\( R_x \) varied from 5.5 to 9 in. depending upon the polar angle \( \theta \) of the cross section measurement).

In view of the foregoing reductions we may write the most useful form of the expression for cross section:
Fig. 7. The deviation from \((\text{distance})^{-2}\) efficiency response of a 2" dia \times 1" long stilbene crystal, oriented with a diameter parallel to the distance coordinate. Curve A: \((\text{Distance})^{-2}\) fitted to the data with largest abscissa. Curve B: Smooth curve through the experimental points. Curve C: Calculated curve using graphical analysis and efficiency corrections from Marion and Fowler.\(^{23}\)
$$d\sigma_{\text{CM}}(E_{\text{CM}}') = (R_x^2/A_s) \epsilon_2(R_x) \epsilon_3 \epsilon_4(E)[f_D(E)/f_D(E - E_R)]$$

$$\times \sigma_T(1 - \exp(-n \sigma_T d))^{-1} \ln[(C_D - C_B)/(C_T - C_B)]$$

(34)

where we have included a factor $\epsilon_3$ denoting the conversion of the cross section to the CM frame. For all except the experiments on iron and lithium, the factor $\epsilon_4(E) f_D(E)/f_D(E - E_R)$ was unity, which simplified data analysis.

**Total Cross Sections**

The data analysis for total cross sections is described by

$$\sigma_T(E_{\text{CM}}) = \epsilon_3(ND)^{-1} \ln[(C_D - C_B)/(C_T - C_B)]$$

(35)

where $\sigma_T(E_{\text{CM}})$ is the total cross section as a function of mean center of mass neutron energy, $\epsilon_3$ is (as before) a symbol of the conversion of the energy dependence from lab to CM frame, ND is the number of nuclei per unit area in the sample, $\ln$ means natural logarithm, $C_D$ is the number of direct neutron counts, $C_B$ is the number of background counts, $C_T$ is the number of transmitted counts, and where $C_D, C_B$, and $C_T$ are all functions of neutron energy and are all normalized to a fixed accumulated proton charge. No monitor neutron normalization was applied because only count ratios are involved with the machine energy at one setting.
Error Analysis

Differential Elastic Cross Sections

Contributions to the uncertainty in the absolute magnitude of the differential elastic cross sections were present in each factor of Eq. (34), reproduced below:

\[
d\sigma_{CM}(\theta_{CM}, E_N(CM)) = (\frac{R_N^2}{A_s}) \epsilon_3(R_x) \epsilon_4(E) (\frac{f_D(E)}{f_D(E - E')}) \times \sigma_T(1 - \exp(-n \sigma d))^{-1} K \frac{C_N}{C_c}
\]

(34)

Statistical errors entered the results through the factor \(C_N/C_c\). Those in \(C_c\) were negligible. Those which were in \(C_N\) are separately described in the figure captions for each set of experimental data. In general, the statistical errors were rather small (from 1 to 3 percent); hence, they had little effect on the precision of the absolute differential cross section measurements.

Experimental uncertainties in the factors of Eq. (34) are discussed below:

1. Factor \(C_N/C_c\). There were two experimental uncertainties in \(C_c\). The detector was located at the scatterer "in beam" position during the calibration run. Its distance from the neutron source was about 28 in. The precision with which the crystal could be placed at the exact scatterer position was \(\pm 1/8\) in. Hence, there was an \(\pm 1\) percent uncertainty from this source. Also, we include in \(C_c\) an error associated with the precision to which the crystal was oriented...
with respect to the incident neutrons. The aspect of the crystal relative to the scattered neutrons, when the crystal was in detection position (see Fig. 1), should have been the same, because deviations produce differences in effective crystal efficiency. We assign an additional uncertainty of ±2 percent to $C_N$, as a measure of the precision with which the crystal aspects were adjusted.

2. Factor $(R_x^2/A_s)$. The experimental uncertainty in $R_x$, the crystal to scatterer distance, was ±1/8 in. for mean $R_x$ about 7 in.; thus, $R_x$ had an extreme uncertainty of ±2.5 percent. The dimensions and aspect (with respect to the beam axis) of the scatterers were not precisely known, which led to a ±1 percent uncertainty in $A_s$. A total ±6 percent uncertainty was assigned factor $R_x^2/A_s$.

3. Factor $\sigma_T(1 - \exp(-n \sigma_T d))^{-1}$. Uncertainties in the value of $\sigma_T$ enter this factor only in second order terms in $n \sigma_T d$. The precision to which the factor $nd$ was known determined the experimental error. A ±1 percent uncertainty was assigned this factor.

4. Factor K. Although the current integrator used in determining the normalization to fixed proton charge is rated at ±1 percent reproducibility, the mean normalization for several counting periods of the same accumulated charge was quite precise. So also was the mean normalization for several counting periods of a different fixed proton charge (i.e., the counting period for calibration points was much less than that for runs involving detection of scattered neutrons). The uncertainty in $K$ was at most ±0.5 percent.
5. Factor $\epsilon_2(R_x)$. This factor was obtained by comparing experimental points with a theoretical (distance)$^{-2}$ behavior for effective efficiency versus distance (see Fig. 7). Figure 7 shows that the data did not precisely fix the normalization of the curve at large distances. The figure also shows some discrepancy between experimental and calculated points (discussed elsewhere in this report). Considering $R_x = 7$ in. typical, we estimated ±3 percent uncertainty in $\epsilon_2(R_x)$.

6. Factor $\epsilon_3$. This factor is symbolic, representing conversions involved in quoting results in terms of mean CM energies, CM angles, and CM intensities, where CM is center of mass. Quantities $M_1/M_2$ and $M_1/(M_1 + M_2)$, which are precise in the present usage, dominate these conversion formulas, the uncertainty in lab angles and energies playing a minor role. An extreme uncertainty ±0.5 percent was assigned factor $\epsilon_3$.

7. Factor $\epsilon_4(E) f_D(E)/f_D(E - E_R)$. In all except the lithium and iron experiments, this factor is unity to good precision. In lithium and iron cross sections it is a source of additional uncertainty. These uncertainties, which are potentially large for a single data point, are reduced if one observes a smooth trend in the factor when it is determined for points at many energies. We estimated a ±2 percent uncertainty for iron cross sections and ±5 percent uncertainty for lithium cross sections from this factor.

Some of the experimental errors, given in 1 through 7, were correlated; for example, $R_x^2$ is anti-correlated to $\epsilon_2(R_x)$. For the purposes of this experiment, detailed examination of such correlations
was not necessary and the procedure adopted was to assume independence of the errors; thus, the sum of the above percent uncertainties applied.

We concluded that exclusive of statistical errors and those errors associated with multiple scattering corrections (see Appendix E), the absolute differential cross sections of zirconium, barium, and cerium were known to within \( \pm 14 \) percent; of iron to within \( \pm 16 \) percent; and of lithium to within \( \pm 19 \) percent.

**Total Cross Sections**

Measurements of total cross sections were accomplished in a neutron background of less than 5 percent of direct neutron counts which minimized second order background corrections to the data. The samples had small dimensions perpendicular to the incident beam direction, being just sufficient to completely shield the detector from direct neutrons. The sample-to-detector and sample-to-source distances were large compared to the detection crystal dimensions. Neutron transmission through the samples was approximately 60 percent of the incident flux. All these are excellent experimental conditions for measurement of total cross sections.\(^ {24} \) A crude calculation indicated that in-scattered neutron corrections to the cross section were at most 2 percent. The main constraint on the amount of in-scattering was the small solid angle the detector crystal presented to in-scattered neutrons. An uncertainty of \( \pm 3 \) percent was assigned to account for second order background scattering effects and sample in-scattering corrections which we did not calculate in detail. Thus, we are assuming in effect
that the uncertainty in our rough calculation is larger than the correction. A ±1 percent extreme error was estimated for the measurement of ND, the number of scattering nuclei per unit area.

Hence, exclusive of statistical errors which are given separately for each experiment, the total uncertainty in our absolute total cross section measurements was ±4 percent.
CHAPTER V

EXPERIMENTAL RESULTS AND COMPARISON WITH THEORY

Energy and Angle Dependence of the Cross Sections

Although the basic purpose of the experiment was to search for threshold effects in neutron cross sections, data such as angular distributions and differential elastic and total cross sections were obtained which are interesting in their own right, exclusive of possible cusp effects. These data will be discussed briefly, with emphasis on illustrations to clarify the discussions.

The measured angular distribution of 1.6-Mev neutrons scattered elastically from natural cerium is presented in Fig. 8. The data points are cross plotted from measurements as a function of energy at separate angles. As shown in the figure, an optical model prediction of Moldauer,\textsuperscript{17} modified for multiple scattering and finite scatterer effects (see Appendix E) agrees very well in shape and, except at back angles, in magnitude, with this data.

Figure 9 shows the measured angular distribution of 600 kev (CM) neutrons elastically scattered from natural lithium. The data of this experiment compares favorably with that from a compilation of other experiments by Howerton,\textsuperscript{27} which is also shown. This substantiates our estimate that multiple scattering corrections were small for the lithium experiments (see Appendix E). The shape of the distribution
Fig. 8. Angular distribution of neutrons elastically scattered from natural cerium at $E_n = 1.6$ Mev. Curve 1: A Moldauer optical model prediction. Curve 2: Curve 1 modified with a multiple scattering correction. Curve 3: Curve 2 modified with calculated experimental finite scatterer effects. Triangles are experimental points from the present experiment.
Fig. 9. Angular distribution of neutrons elastically scattered from natural lithium at $E_n = 0.6$ Mev. Open circles are data from the present experiment. This data is not corrected for multiple scattering. Triangles are data from Howerton's compilation. 27
$\frac{d\sigma}{E_N(CM)} = 0.6\,\text{meV}$
is interesting, being peaked backwards, presumably as a result of interference between the strong $p$ wave resonance\textsuperscript{28} at 0.25 Mev and an $s$ wave background. This backward peaking means that one should look at forward angles for cusp effects, because the theoretical percentage effect is inversely proportional to $(d\sigma)^{1/2}$ (see Chapter II).

In Fig. 10 we show some of the differential cross section data, versus energy, from which the angular distribution, Fig. 9, was cross-plotted. At the lower energies the upper edge of the $p$ wave resonance\textsuperscript{28} at 0.25 Mev is in evidence. The wavy structure in all the cross sections is probably not an experimental effect; it is quite surprising, because for a light nucleus such as lithium one might expect rather smoothly varying cross sections at these energies, except for isolated resonances. The inelastic cross section to the first excited state of $^7\text{Li}$, as measured by Freeman, Lane, and Rose\textsuperscript{29} shows similar structure.

The differential elastic cross section of iron, versus energy, was measured for four angles using a rather thick neutron target (#3, Table I) in the vicinity of 0.847 Mev. The results are shown in Fig. 11. It was later determined that a much thinner neutron target was needed to resolve the structure and to search for cusps (see Fig. 14; target #5, Table I was used).

Energy and Angle Dependence of Threshold Features

We discuss here the experimental results as they pertain to Wigner and "energy-averaged" Wigner cusps. Frequent reference will be made, explicitly and implicitly, to the predictions of cusp theory, Chapter II.
Fig. 10. Neutron differential elastic cross sections (CM) of natural lithium versus center of mass (CM) energy for five angles (CM). The data are not corrected for multiple scattering. The original experimental results have been averaged such that statistical errors in these points are approximately $\pm 1\%$.
Fig. 11. Neutron differential elastic cross sections of natural iron versus energy for four angles. The data is not corrected for multiple scattering. Incident neutron energy spread is approximately 25 kev. Statistical errors are indicated by the size of the symbol.
Lithium (92.6% Li$^7$)

The cusp results for lithium were essentially negative. The theory of Appendix B, Case I, applies. We saw in the preceding section that the largest percentage cusp effect in $d\sigma$ is expected at forward angles. Figure 12(b) shows that an experimental effect was indeed present at 34° (CM) in the vicinity of 0.478 Mev, the first excited state of Li$^7$. However, using the reaction cross section magnitude of Freeman, Lane, and Rose, and the applicable theory, we find that the observed effect was at least a factor 5 too large to be explained as a Wigner cusp; an effect of approximately 2 percent is predicted 50 kev from threshold. Figure 12(a) compares our measurement of lithium total cross section with that from Howerton's compilation. Although the two sets of data agree, no Wigner cusp is in evidence in either. We had hoped to see an effect in the total cross section, because the results of Lane, et al., (see Fig. 13) appear to show a break near 0.5 Mev in the coefficient $B_0$ of the cross section expansion (see Chapter II). The energy scale change in Fig. 13 at about 0.5 Mev may bias this interpretation of their data.

Iron (91.7% Fe$^{56}$)

With iron distinct features appear in all cross sections at the energy of interest ($2^+$ first excited state Fe$^{56}$, 0.847 Mev). This is exemplified in Fig. 14 where a resonance feature appears, apparently at an identical energy (just above 0.847 Mev) in the total cross section, in the $145^\circ$ differential elastic cross section and in the reaction
Fig. 12. Neutron cross sections of natural lithium versus energy. (a) Total neutron cross section; triangles are data from Howerton's compilation$^2$ and solid circles are the energy-averaged ($\Delta E = 20$ kev) data from the present experiment. (b) $34^\circ$ CM differential elastic cross section of the present experiment.
Fig. 13. Differential cross section for scattering of neutrons by Li$^7$. Reproduced from the work of A. Langsdorf, Jr., R. O. Lane, and J. E. Monahan.
Fig. 2. The differential cross section for scattering of neutrons by Li$^7$. 
Fig. 14. (a) Neutron differential elastic (145°) cross section of natural iron versus energy. (b) Total neutron cross section of natural iron. (c) Reaction cross section Fe$^{56}$ (n, n', γ (.847 Mev)) Fe$^{56}$. Statistical errors are the size of the symbols in (a) and (b), but are ±10% in (c). The smooth curves show the result of removing a calculated Wigner cusp effect under one possible assumption on the phase of the scattering amplitude.
cross section $\text{Fe}^{56}(n, n')\text{Fe}^{56*}(0.847 \text{ Mev})$. The latter cross section was measured by Tucker, et al., using the same tritium target.\textsuperscript{19} Another resonance feature which seems to be a "mirror image" of the aforementioned resonance across threshold is evident in these data, and in all the differential cross sections at other angles investigated [see Fig. 15 (six angles)]. Application of the most general theory is hopeless because of the large number of unknown phases. At least six phases contribute to the cusp, even with a restriction on participating incident partial waves to those with orbital angular momentum $l$ less than 3; see Appendix B. A reasonable and tractable special case of the theory, which might apply to the iron data is discussed and explicitly evaluated in Appendix B, Case II. Unfortunately, the special assumptions involved allow cusp analysis only over a narrow energy region, entirely above threshold. The results of this analysis are superimposed on the data in Fig. 15 for differential cross sections (6 angles) and in Fig. 16 for the total cross section. The results are neither startling nor conclusive, although by removal of the cusp effect the peak energies and widths of the above threshold resonance in the differential elastic, total and inelastic cross sections can be brought into exact coincidence.

**Cerium-Barium-Zirconium**

We consider now the results obtained for targets of higher atomic number in the region where the optical model is expected to apply because compound resonances are closely spaced—cerium, barium,
null
Fig. 15. Neutron differential elastic cross sections of natural iron versus energy for six angles. The data is not corrected for multiple scattering. (a) gives the data over the full energy range of the run. The lined box shows the section of the data which is expanded into (b). In (b) statistical errors are indicated by the symbol size. Open triangles show the result of removing a calculated Wigner cusp effect, under one possible assumption on the phase of the scattering amplitude.
(a) $d\sigma(125^\circ)_{Fe}$

(b) Mean CM Neutron Energy (MeV) vs. Barns/Sterad for $^{2+}Fe^{56}$
Fig. 16. Total cross section of natural iron. The size of statistical errors is indicated by the size of the symbol. Triangles show the result of removing a calculated Wigner cusp effect under one possible assumption on the phase of the scattering amplitude.
\( \sigma_T \) Iron

\( 2^+ \text{Fe}^{56} \)

Barns

Mean CM Neutron Energy Mev
and zirconium. The theory of "energy-averaged" Wigner cusps is then also expected to apply (see Chapter II).

Composite data for cerium (88.5% Ce$^{140}$), given in Fig. 17, is representative of the cusp analyses we present for all three elements. A special feature of this figure, however, is that we present the original data, to show the degree of scatter on interleaving runs (most of which is not statistical, but experimentally reproducible). In the remaining figures presented for this group, the data was averaged over an energy interval suitable to reduce statistical errors to about ±1 percent (refer to Chapter IV). This procedure standardized data presentation for a large number of angles.

This and other figures in this group also show cross section trends predicted by the optical model. These were obtained from the work of Campbell, et al. 30 The optical model shapes were interpolated from very few points given in the reference and should not be considered a precise computation; hence, they were normalized to the experimental data at threshold energy. The magnitudes of the predicted cross sections 30 were consistent with the estimated experimental errors (Chapter IV) and the corrections given in Appendix E.

A cusp effect is quite apparent in Fig. 17(b) by comparison of the optical model trends with the experimental cross section. To compare the magnitude of the cusp effect with theory one could subtract the experimental cross section from the optical model prediction and plot the difference as a function of neutron energy. But, as mentioned before, the optical model trends are somewhat uncertain and, furthermore,
Fig. 17. The experimental inputs to a cerium "energy-averaged" Wigner cusp analysis. (a) Energy-averaged ($\Delta E = 40$ kev) experimental inelastic cross section Ce$^{140}(n, n' \gamma (1.597$ Mev)) Ce$^{140}$. (b) Original data for elastic 95° scattering of neutrons from natural cerium. The two symbols indicate the reproducibility of the data on successive interleaving energy runs. The size of the symbols indicates the size of the statistical errors. (c) Same data as (b) corrected, point by point, for a calculated "energy-averaged" Wigner cusp. The smooth curves in (b) and (c) are optical model predictions of Campbell, et al., normalized at 1.6 Mev.
the experimental cross section has natural fluctuations, which will be discussed below. Hence, it was thought more appropriate to remove from the experimental cross sections the theoretical cusp effect and to show (a) that in the resultant curve there would be no more cusp effect—at least within the natural fluctuations of the cross section, and (b) that the resultant curve would follow the optical model trend, which theoretically represents the cross section without threshold effects.

The procedure for removing a theoretical cusp from the measured cross section was to multiply the reaction cross section (i.e., inelastic neutron scattering cross section) by the isotopic percentage of the isotope of interest, by the appropriate multiple scattering and finite scatterer correction (the reciprocal of Table III (Appendix E) values), and by the theoretical angle dependent factor (Chapter II). This negative quantity was subtracted from the experimental data [see, for example, Eq. (26)].

\[
\text{Cerium (88.5\% Ce}^{140})
\]

We now consider the specific case of cerium, for which Eq. (26) applies. Tucker, et al., measured the reaction cross section [Fig. 17(a)] Ce\(140\)(n, n\(^\prime\))Ce\(140\*)(1.60 Mev).\(^{19}\) As previously mentioned, incident d waves dominate the reaction to at least 150 kev above threshold, hence \(\ell' = 2\) in Eq. (26). The results of removing the theoretical cusp effect are shown in Fig. 17(c). There is little doubt that the theoretical "energy-averaged" Wigner cusp, with \(\ell' = 2\) is a satisfactory description of the change at threshold. Note Eq. (26) predicts
that if incident $s$ waves were imagined to dominate this reaction, rather than the actual $d$ waves, the resulting expected cusp, at polar angle 90°, would be a factor 20/13 or about 50 percent larger. The plot corresponding to Fig. 17(c), corrected for an expected ($l' = 0$) cusp, would show an upward break at threshold and would follow neither the below threshold data trend nor the optical model trend. Thus we can conclude the reaction is not dominated by incident $s$ waves. This type of information can be useful in assigning excited state spins, for example.

The differential elastic cross section of cerium was measured at ten angles, versus energy. These results are shown in Fig. 18. The points obtained after removing a calculated cusp are also given for each angle. In all cases, the adjusted data so obtained follow the trend of the below threshold data and the fluctuations in these is the same above and below threshold. In general, the optical model curves agree nicely with the modified cross sections; that is, they follow what the theory defines as "cross section without reaction." There is no reason to expect this agreement to be perfect, since the optical model was constructed to explain gross structures only, over a wide range of atomic number and energy. We should recall the assumption, made for Eq. (26), that the fraction (called $R_{J'}$ in Chapter II) of the reaction probability for proceeding from compound nucleus states of a particular $J'$ is the same as the fraction of the dominant incident partial wave states (all with the same orbital angular momentum $l'$) which have $J = J'$. Other assumptions for $R_{J'}$ lead to cusp
Fig. 18. The neutron differential elastic cross section of natural cerium versus energy for ten angles. The data is not corrected for multiple scattering. Statistical errors are ±1%. The points are an energy average (ΔE = 50 kev) of the original experimental data. For each angle the left side is the measured cross section. The right side is the cross section with a calculated "energy-averaged" Wigner cusp removed. The calculation is based on the excitation of the 1.6 Mev state in Ce$^{140}$. No cusp contributions from the $0^+$ (1.90 Mev) or $4^+$ (2.08 Mev) states are included. The smooth curves are optical model predictions of Campbell, et al., normalized at 1.6 Mev. 30
magnitude predictions differing from those used here by as much as 30 percent, at certain angles. Although not the case in the present work, it is possible that a more refined experiment could be used to measure $R_J$; such a measurement could assist in the understanding of neutron nucleus interaction mechanisms, particularly if it turned out that a particular reaction preferred one value of compound nucleus $J'$ to another, even though resonances were closely spaced and possibly overlapping.

**Barium \((71.7\% \text{ Ba}^{138})\)**

The measured cross sections of barium and the results of the appropriate cusp analysis are compiled in Fig. 19. The reaction cross section $\text{Ba}^{138} (n, n')\text{Ba}^{138*} (1.427 \text{ Mev})$ was measured by Tucker, et al.\(^{19}\) Again the spin of the first excited 1.427 Mev level is $2^+$ and the ground state $0^+$. That $s$ wave neutrons dominate the reaction very far from threshold is not so clear in this case, since the reaction shape is distorted by resonance structure. However, optical model calculations show that barium is near a maximum in the $s$ wave strength function.\(^{18}\) As in cerium then, the reaction is considered to proceed by the $l' = 2$ incident partial wave. The results of theoretical comparison are satisfactory. The cusp effect is smaller than in cerium because of the smaller isotopic percentage and the smaller reaction cross section near threshold. In Fig. 19(c) we see that it would be difficult to distinguish a cusp at all were it not for a comparison with the optical model trend which happens to be rising.
Fig. 19. Neutron differential elastic (90° and 125°) cross sections of natural barium versus energy. (a) Energy-averaged (ΔE = 20 kev) inelastic cross section for Ba^{138}(n, n' γ (1.427 Mev)) Ba^{138}. Statistical errors are ±5%. The left sides of (b) and (c) are an energy average (ΔE = 2, kev) of the original experimental data. This data is not corrected for multiple scattering. Statistical errors are ±1%. The right sides of (b) and (c) show the results of removing calculated "energy-averaged" Wigner cusps. The smooth curves are optical model predictions of Campbell, et al., normalized at 1.427 Mev.


Zirconium (51.5% Zr\textsuperscript{90})

The measured cross sections of zirconium and the results of cusp analysis are compiled in Fig. 20. The inelastic reaction Zr\textsuperscript{90}(n, n')Zr\textsuperscript{90*}(1.75 Mev) was measured by Tucker, et al.\textsuperscript{19} The spin of the 1.75-Mev first excited state is 0\textsuperscript{+} and the ground state 0\textsuperscript{+}. Tucker also showed that p wave outgoing neutrons dominate the reaction not far above threshold. Hence we assume that the reaction proceeds from incident p waves, i.e., \( l' = 1 \). For Zr\textsuperscript{90} the low isotopic percentage and small reaction cross section near threshold combine to produce a small calculated cusp effect. At 60° polar angle the predicted cusp effect is too small to be distinguished in the data. Note also that at 60° the optical model trend does not agree well with the measured cross section slope. At 130° the cusp is distinguishable and the theoretical cusp fit is satisfactory.

Total Cross Sections

The theoretical "energy-averaged" Wigner cusps which were used to explain the preceding experimental results (Ce, Ba, Zr) are based on the same assumptions that underlie the optical model.\textsuperscript{22} These assumptions also lead to predictions that the energy average of the total cross sections should be smooth across thresholds [see Eq. (23)]. Our measurements of the total cross sections of cerium and barium are given in Fig. 21. The predicted cusp effect for the total cross section of zirconium—even if the energy-averaged theory would not apply—would be so small that we did not investigate this case. For completeness,
Fig. 20. Neutron differential elastic (60° and 130°) cross sections of natural zirconium versus energy. (a) is a 40 kev average of the reaction cross section $\text{Zr}^{90}(n, n') \text{Zr}^{90}(1.75 \text{ Mev})$. Statistical errors are ±5%. The left sides of (b) and (c) are an energy average ($\Delta E = 50 \text{ kev}$) of the measured cross sections. This data is not corrected for multiple scattering. Statistical errors are ±1%. The right sides of (b) and (c) show the results of removing calculated "energy-averaged" Wigner cusps. The smooth curves are optical model predictions of Campbell, et al., normalized at 1.75 Mev.
\[ \sigma_r \]
\[ Zr^{90}(n,n')Zr^{90}(1.752) \]

![Graphs showing neutron cross-sections and angular distributions for Zr.](image)
Fig. 21. Neutron total cross sections for natural barium, (a); cerium, (b); and zirconium, (c). Statistical errors are indicated by the size of the symbols. The data on barium and cerium were measured in the present experiment. The data on zirconium is taken from Howerton's compilation.\textsuperscript{27} In (b) circles are the total cross section of cerium; triangles are integrated elastic cross sections, corrected for multiple scattering; and squares are the sum of the integrated elastic and of the reaction cross section (corrected for isotopic percentage) $\text{Ce}^{140}(n, n' \gamma(1.6 \text{ Mev})){\text{Ce}}^{140}$.
the zirconium total cross section data from Howerton is also presented in Fig. 21, showing the inadequacy of widely spaced energy points for the present study. The data for barium and cerium were taken with an energy spread of about 30 kev, yet they are certainly not smooth functions of energy, displaying marked structure. The mean values of these fluctuations do not seem to drop markedly as threshold is crossed, suggesting agreement with the theory. If, however, the theory of energy-averaged cusps does not apply, but one assumes instead a pure Wigner cusp phenomenon, the cusp predicted is at most a 5 percent effect which would be masked by the fluctuations. Hence, the total cross section data cannot be considered a verification of either assumption.

That the fluctuations are not experimental is supported by the differential cross section data for cerium at forward angles (Fig. 18) which shows the same structure. Note that the structure does not contain recognizable resonance shapes. It is possible that the considerations of Ericson on statistical fluctuations in cross sections might apply here. He shows that if a compound system is characterized by many overlapping resonances, each of width $\Gamma$, and with relative random phase of the value quantities $\gamma_{\lambda}$ of resonance theory, cross sections would have a fluctuating structure of average width, $\Gamma$.

In the case of cerium only, sufficient differential elastic data was taken to permit integration over all solid angle. The total elastic cross sections so obtained for several energies, after correction for multiple scattering, are plotted in Fig. 21(b). The drop at threshold (1.6 Mev) is evident, as, in accord with any theory of
threshold effects, flux conservation alone requires the total elastic cross section to drop above threshold for a reaction. In the case of Ce$^{140}$, the experimental differential cross sections all drop, so the total elastic cross section does also. It is reassuring that, when the inelastic scattering cross section is corrected for isotopic percentage and added to the integrated elastic cross section, the resultant cross section (i.e., the total cross section) is smooth across threshold. The agreement in magnitude between the latter curve and the directly measured total cross section is considered somewhat fortuitous.
CHAPTER VI

CONCLUSIONS

The experiments have shown that under suitable conditions the threshold effects expected in neutron elastic scattering can be observed. From an experimental point of view particular attention must be paid to the choice of energy resolution consistent with observing a cusp effect in reasonable time.

Explicit theoretical expressions have been obtained for expected Wigner and "energy-averaged" Wigner cusp effects under general angular momentum conditions, within some reasonable restrictions imposed in the derivation. The effects were found to be contained in the coefficients, $B_L$, of the Blatt-Biedenharn formulation of the general scattering problem. Since many survey experiments involving neutron scattering have been analyzed in terms of the $B_L$ coefficients, we make the obvious suggestion that such results be examined near excited state energies for possible tests of cusp theory.

The expected Wigner cusp in lithium cross sections was expected to be an approximately 2 percent effect, 50 kev from threshold. Our experiment was not sufficiently refined to pick out an effect of this size.

Resonance structure in the measured cross sections of iron made a Wigner cusp interpretation ambiguous. Features occurred at the threshold energy, but their analysis in terms of cusp theory was inconclusive.
The measured differential elastic cross sections of Zr, Ba, and Ce dropped at inelastic thresholds. Analysis of the drops in terms of "energy-averaged" Wigner cusp theory resulted in good agreement between theory and experiment. This indicates that the usual energy-averaging assumptions underlying the optical model are indeed applicable to these nuclei. Also, the cusp theory shows\(^7\) that there cannot be a large amount of direct interaction between incident neutrons and these nuclei at the neutron energies used in these experiments (\(\approx 2\) Mev), or else below threshold effects would also be seen.
APPENDIX A

THE PHYSICAL BASIS FOR THE S-MATRIX AND ITS USE
IN CROSS SECTION FORMULAS

We will review the principles of scattering analysis and the
definition of the S-matrix as presented by Blatt and Biedenharn. They pointed out that general expressions for cross sections in terms
of the S-matrix are rarely given in standard texts.

The general formulas are applicable to any collision process
in which two particles collide and two particles emerge. However, in
this appendix, and after a certain stage in the development, only the
special case of elastic scattering with no \( l \) change will be con-
sidered.

The channel index \( \alpha \) defines the type of incoming particle and
the state of the struck nucleus. The channel spin, \( s \), is the total
spin angular momentum of the channel, obtained by vector addition of
the intrinsic spin, \( i \), of the incident particle and the spin, \( I \),
of the struck nucleus. The orbital angular momentum (in the center of
mass system), \( l \), is combined with \( s \) to form the total angular
momentum, \( J \). Only in this particular appendix, primes will be used
to refer to outgoing or final state quantities. \( J \) is conserved during
the collision, hence the values of \( l' \) and \( s' \) must be consistent
with \( J' = J \). Similarly, the parity \( \pi \), of the initial system, is
conserved, and \( \pi = \pi' \).
The probability amplitude $S^J_{\alpha's'\ell';\alpha s \ell}$ for a collision with total angular momentum $J$, from channel $\alpha s \ell$ to channel $\alpha's'\ell'$ is referred to as an element of the scattering matrix. The scattering matrix, or $S$-matrix, is defined by considering asymptotic forms of the collision wave functions, as follows.

Consider one definite value of $J$ and $J_z = M$. At any particular energy, $E$, the channel wave number, $k_{\alpha'}$, and relative speed, $v_{\alpha'}$, in each channel $\alpha$ are given by the energetics of the reaction. Only those channels which are "open" ($k_{\alpha}$ real) are considered. Let $\varphi_{\alpha s}$ be the product wave function of the target nucleus and incident particle. The spin and angle dependence of a wave function of total angular momentum $J$, $J_z = M$, orbital angular momentum $\ell$, and channel spin $s$ is given by

$$\psi^M_{J s} = \sum_{m_{\ell}=-l}^{l} \sum_{m_s=-s}^{s} (lsm_{\ell} m_s jm) Y_{l m} (\theta, \phi) \chi_{s,m_s}$$  \hspace{1cm} \text{(A1)}$$

where $\chi_{s,m_s}$ is the incident spin wave function and $Y_{l m}$ is the well known spherical harmonic of angular momentum $\ell$ and projection $m$.

Using these definitions, the most general wave function in channel $\alpha s$ with total angular momentum quantum numbers $J, M$, consists of the superposition of incoming and outgoing spherical waves. Asymptotically (at sufficiently large distances) we can write (for neutral particles)

$$\psi_{\alpha s}(JM) = \frac{1}{r_a(v_{\alpha})^{1/2}} \sum_{jm} \chi_{s,m} \left< a_{JM} \exp[-i(k_{\alpha} r_{\alpha} - \frac{1}{2} \ell \pi)] - B_{\alpha s \ell} \exp[i(k_{\alpha} r_{\alpha} - \frac{1}{2} \ell \pi)] \right>$$  \hspace{1cm} \text{(A2)}$$
Substitution of Eq. (A2) into the wave equation yields a unique connection between the outgoing wave coefficients, $B$, with the ingoing wave coefficients, $A$. Thus

$$B_{\alpha's'\ell'} = \sum_{\alpha s} \sum_{\ell} S_{\alpha's'\ell'};_{\alpha s\ell} A_{\alpha s\ell}$$

(A3)

Eq. (A3) defines the scattering matrix, $S$. With these definitions it is evident that $S$ is unitary (conserving the wave function normalization in time), symmetric (reciprocity theorem), and independent of $M$.

It remains to decompose the incident plane wave into the functional forms we are using. The incident wave is $\varphi_{\alpha s} \exp(ikz)X_{s,m_s}$. First, one writes

$$\exp(ikz)X_{s,m_s} = (4\pi)^{\frac{1}{2}} \sum_{l=0}^{\infty} i^l (2l + 1)^{\frac{1}{2}} \left( \frac{\pi}{2hr} \right)^{\frac{1}{2}} J_{l+\frac{1}{2}}(kr) Y_{l,0}(\theta)X_{s,m_s}$$

where

$$Y_{l,0}(\theta)X_{s,m_s} = \sum_{J=|l-s|}^{l+s} \sum_{M=-J}^{J} (\ell s m_s | \ell s JM) Y_{J, M}$$

Using the asymptotic form of $J_{l+\frac{1}{2}}$ for $kr > l$ the following expression is obtained for the plane wave in channel $\alpha, s$ with spin direction given by $m_s$:

$$\exp(ikz X_{s,m_s} \varphi_{\alpha s} \approx \frac{i(\pi)^{\frac{1}{2}}}{k \alpha} \varphi_{\alpha s} \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \sum_{l=|J-s|}^{J+s}$$

$$\times (\ell s m_s | \ell s JM) i^l (2l + 1)^{\frac{1}{2}} Y_{J, M}$$

$$\times \left\{ \exp[-i(k_r \alpha - \frac{1}{2} l \pi)] - \exp[i(k_r \alpha - \frac{1}{2} l \pi)] \right\}$$

(A4)
null
Comparing Eq. \((A^4)\) with the standard form, Eq. \((A2)\), we see that the amplitudes of the ingoing spherical waves are given by

\[
A_{JM}^{\alpha \beta \ell} = i \chi_{\alpha} (\pi \nu_{\alpha})^{1/2} (\ell s \alpha m_s \mid \ell s JM) i^{l/2} (2l + 1)^{1/2} \tag{A5}
\]

The amplitudes of the outgoing spherical waves are determined from Eq. \((A3)\). As in the main text, we now specialize to elastic scattering with no change of \(l\) or \(s\). They are

\[
B_{JM}^{\alpha \beta \ell} = i \chi_{\alpha} (\pi \nu_{\alpha})^{1/2} \sum_{\ell = |J - s|} \times (\ell s \alpha m_s \mid \ell s JM) i^{l/2} (2l + 1)^{1/2} S_{JM}^{\alpha \beta \ell} \tag{A6}
\]

In order to investigate cross sections we only need that part of the plane wave solution, Eq. \((A4)\), which is due to the scattering interaction; i.e., we want to obtain \(\psi_{\text{scattered}}\) from \(\psi = \psi_{\text{incident}} + \psi_{\text{scattered}}\). Therefore, we substitute Eqs. \((A5)\) and \((A6)\) into Eq. \((A2)\) and subtract the incident wave, Eq. \((A4)\); the result is

\[
\psi_{\text{scattered}} = i \chi_{\alpha} (\pi \nu_{\alpha})^{1/2} \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \sum_{\ell = |J - s|} \times \times (\ell s \alpha m_s \mid \ell s JM) i^{l/2} (2l + 1)^{1/2} \times \exp[i(kr_{\alpha} - \frac{1}{2} l \pi)] \times [1 - S_{JM}^{\alpha \beta \ell}] \chi_{JM}^{M} \tag{A7}
\]

If the spin angle function \(\chi_{JM}^{M}\) is decomposed using Eq. \((A1)\), then Eq. \((A7)\) can be used to define the scattered amplitude, \(q_{JM}^{\alpha \beta \ell}  \chi_{JM}^{M}\)

\[
\psi_{\text{sc}}(\alpha \beta m_s \mid \ell s m_s) = i \chi_{\alpha} \frac{\exp i k r_{\alpha}}{r} \varphi_{\alpha s} \times \sum_{m_s = -s}^{s} q_{JM}^{\alpha \beta \ell}  \chi_{JM}^{M} (\theta \varphi) \chi_{s, m_s} \tag{A8}
\]
where \( q \) is explicitly

\[
q_{\alpha m_s'; \alpha m_s}(\theta, \varphi) = \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \sum_{l=|J-s|}^{l} \sum_{\mu=-l}^{l} \frac{1}{2} \mu \frac{l}{2} \left( \frac{1}{2} l s_m \right) \left( l s_m' \right) (l s_m' \mid l s JM) 
\times (1 - S_{\alpha l; \alpha l'} Y_{l \mu}(\theta, \varphi)) 
\]

The differential cross section for the scattering \( \alpha m_s \rightarrow \alpha m_s' \) is given by

\[
d\sigma_{\alpha m_s'; \alpha m_s} = 4\alpha^2 \left| q_{\alpha m_s'; \alpha m_s}(\theta, \varphi) \right|^2 d\Omega \quad \text{(A10)}
\]

For unpolarized incident beams, we must average Eq. (A10) over initial spin direction \( m_s \) and sum over final spin direction \( m_s' \):

\[
d\sigma_{\alpha s; \alpha s} = (2s + 1)^{-1} \sum_{m_s=-s}^{s} \sum_{m_s'=-s}^{s} d\sigma_{\alpha m_s'; \alpha m_s} \quad \text{(A11)}
\]

This expression is greatly simplified by sum rules for Clebsch-Gordan coefficients, and it is from this calculation, as performed by Blatt and Biedenharn, that one obtains the useful final cross section expressions, involving \( Z \) coefficients.

Finally, to obtain the differential elastic cross section, one must average over the possible values of incident channel spin, \( s \).

\[
d\sigma_{\alpha; \alpha} = \sum_{s=|I-1|}^{I+1} \frac{2s + 1}{(2I + 1)(2I + 1)} d\sigma_{\alpha s; \alpha s} \quad \text{(A12)}
\]

Equations (A12) and (A11), as explicitly worked out in Ref. 12, are just Eqs. (1), (2) and (3) of the main text.
APPENDIX B

SPECIAL CASES OF INTEREST TO THE PRESENT EXPERIMENT

Case I. Cusp effects in neutron scattering from Li near the threshold for excitation of the first excited (0.478 Mev) 1/2\(^-\) state of Li\(^7\) (ground state spin and parity, 3/2\(^-\)).

The large resonance in the elastic cross section at about 0.25-Mev neutron energy has been analyzed and found to correspond to a 3\(^+\) state in Li\(^8\).\(^{28}\) The reaction cross section Li\(^7\)(n, n')Li\(^7\*)(0.478 Mev) has a broad resonance at 1.35-Mev neutron energy, probably associated with a compound 1\(^+\) state.\(^{29}\) Also, the inelastic scattering is dominated near threshold by outgoing s waves, with compound angular momentum J\(^\pi\) = 1\(^-\).\(^{29}\) Finally, any process involving incident partial waves with \(l > 1\) is unlikely at 0.5 Mev because kR \approx 0.3. We conclude that in the vicinity of 0.48-Mev incident (center of mass) neutron energy only s and p waves participate significantly in elastic scattering. Also, incident s waves dominate the inelastic scattering. The latter conclusion results from the known spins and parities of the ground and first excited state of Li\(^7\), and the assumption that no process with \(l > 1\) is likely. With s waves dominant in the outgoing reaction channel the only possible incident channel leading to the reaction is characterized by \(J' = 1, s' = 1,\) and \(l' = 0\). Using these values, and \(I = 3/2, i = 1/2,\) in Eq. (12), one obtains

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\[
\Delta B_L(1;1) = -\frac{1}{4} \sum_{J \ell} \mathcal{Z}^2(J 0 1, 1 L) \\
\times 2 \sin \delta_{1 \ell} \sigma_{R01} \left\{ \begin{array}{c}
\sin(\Theta_{01}^{1} - \Theta_{11}^{J}) \\
\cos(\Theta_{01}^{1} - \Theta_{11}^{J})
\end{array} \right\}
\]  

(B1)

Restricting ourselves to \( \ell < 2 \), as indicated above, we need these \( \mathcal{Z} \) coefficients

\[
\mathcal{Z}^2(0101,10) = 3, \quad \mathcal{Z}^2(1101,11) = 3, \quad \mathcal{Z}^2(1201,11) = 5
\]

and all other \( \mathcal{Z}^2(\ell J 0 1, 1 L) \) values are zero. Therefore,

\[
\Delta B_0(1;1) = -\frac{3}{4} \sigma_{R01} \left\{ \begin{array}{c}
2 \sin^2 \delta_{01}^{1} \\
\sin 2\Theta_{01}^{1}
\end{array} \right\}
\]

\[
\Delta B_1(1;1) = -(3/2) \sigma_{R01} \left[ \sin \delta_{11}^{1} \left\{ \begin{array}{c}
\sin(2\Theta_{01}^{1} - \Theta_{11}^{1}) \\
\cos(2\Theta_{01}^{1} - \Theta_{11}^{1})
\end{array} \right\} \\
+ \frac{5}{3} \sin \delta_{11}^{2} \left\{ \begin{array}{c}
\sin(2\Theta_{01}^{1} - \Theta_{11}^{2}) \\
\cos(2\Theta_{01}^{1} - \Theta_{11}^{2})
\end{array} \right\} \right]
\]  

(B2)

Equations (B2) might be compared to data such as that of Langsdorf, Lane and Monahan, reproduced herein as Fig. 13. Unfortunately, the energy scale in this figure changes in the region of interest (0.5 Mev lab). We present the figure primarily to illustrate the utility of cusp expressions in the form of Eqs. (B2). The cusp prediction for the differential elastic cross section is obtained by substituting Eq. (B2) in Eqs. (9-11) to obtain
Δσ = -(3/8) \left( \sigma_{r\ell=0} \right)_{\text{exp}} \exp \left\{ \begin{array}{c} 2 \sin^2 \frac{\delta}{10} \\ \sin \frac{2\theta}{10} \end{array} \right\}

+ 2 P_\ell (\cos \theta) \left[ \begin{array}{c} (5/3) \sin \frac{\theta}{10} \\ \cos \frac{2\theta}{10} \end{array} \right] \right] dΩ

(B3)

Using Eq. (16), we obtain the total cross section cusp term,

Δσ_T = (3/8) \left( \sigma_{r\ell=0} \right)_{\text{exp}} \left\{ \begin{array}{c} \cos \frac{2\theta}{10} \\ \sin \frac{2\theta}{10} \end{array} \right\}

(B4)

where \( \left( \sigma_{r\ell=0} \right)_{\text{exp}} \) in Eqs. (B3) and (B4) is the experimental reaction cross section near threshold.

For the case of \( ^7\text{Li} \) we recall the restriction imposed on the derivation leading to Eq. (12), that the amount of scattering with change of \( s \), but no change in energy or orbital angular momentum \( \ell \), be small. There are no good reasons for believing this to be true in the present case. Change of \( \vec{s} = \vec{r} + \vec{i} \) from \( |s| = 2 \) to \( |s_o| = 1 \) (\( s_o \) means outgoing "elastic" channel spin), or vice versa, in the \( ^7\text{Li} \) case is essentially a spin flip followed by a recoupling of \( s \) and \( \ell (\ell \geq 1) \), and this is just the type of interaction involved in the reaction leading to the first excited state. We have seen, however, that in fact \( \ell \) waves (\( \ell' = 0 \)) dominate the \( ^7\text{Li} \) reaction; thus "spin flip" elastic scattering (which requires \( \ell \geq 1 \)) does not couple to the reaction. Hence, Eqs. (B3) and (B4) apply. Using the reaction cross section
value (100 kev above threshold) of Freeman, Lane, and Rose,\textsuperscript{29} and the present experimental values of $d\sigma$ (34° CM), we can estimate the expected percent effect to be

$$\Delta d\sigma/d\sigma^0 = (3/32 \pi) \sigma_T/d\sigma^0 \approx (3/32 \pi) \cdot 05/05 \approx 3 \text{ percent effect}$$

($\sim$ 100 kev from threshold)

Correspondingly, for the expected cusp in $\sigma_T$ we obtain

$$\Delta \sigma_T/\sigma_T^0 = (3/8)(0.05/1.1) \approx 2 \text{ percent effect}$$

(100 kev from threshold)

Case II. Cusp effects in neutron scattering from iron near the threshold for excitation of the first excited state ($2^+$, 0.847 Mev) of Fe\textsuperscript{56} (ground state spin and parity $0^+$).

Equations (9) to (12) for pure Wigner cusp effects apply to this case because the resonance structure in the cross sections is apparently resolved. We must compare Eq. (9), the expression for change in cross section, with experiment. There are too many possible phases involved in a general treatment of Eq. (9) to make cusp interpretation meaningful; therefore, we sought a tractable and reasonable simplification.

The experimental data, see Fig. 14, display a resonance at an identical energy in the reaction cross section, in the total cross section, and at six angles in the differential elastic cross section. On this evidence, we assumed that in the vicinity of this resonance the
scattering and reaction proceed through the same compound state of total momentum, \( J' \), and are dominated by one incident partial wave, \( \ell' \).

With this assumption, and substituting \( I = 0, i = 1/2, s' = 1/2 \) in Eq. (12) we obtain

\[
\Delta B_L(\alpha \frac{1}{2}; \alpha \frac{1}{2}) = -\frac{1}{4} \sum_{\ell' \frac{1}{2}} \left\{ \frac{2 \sin^2 \delta_{\ell' \frac{1}{2}}}{\sin 2 \delta_{\ell' \frac{1}{2}}} \right\} \sigma^{J'}_{r\ell' \frac{1}{2}} \tag{B5}
\]

where \( J' \) must be either \( \ell' + \frac{1}{2} \) or \( \ell' - \frac{1}{2} \), depending on the nature of the compound resonance. By substituting Eq. (B5) in Eqs. (9) to (11), one obtains

\[
\Delta d\sigma_{\alpha;\alpha} = \frac{(\sigma^{J'}_{r\ell' \frac{1}{2}})_\text{exp}}{4\pi} \left\{ \frac{2 \sin^2 \delta_{\ell' \frac{1}{2}}}{\sin 2 \delta_{\ell' \frac{1}{2}}} \right\} \times \sum_{L=0}^{\frac{\ell'}{2}} \frac{2^2(\ell' J' \ell' J'; \frac{1}{2} L)}{(2\ell' + 1)} P_L(\cos \theta)d\Omega \tag{B6}
\]

where \( (\sigma^{J'}_{r\ell' \frac{1}{2}})_\text{exp} \) is the measured reaction cross section.

The optical model indicates a maximum in the \( s \) wave strength function at \( A \approx 56 \), and the analysis of Tucker, et al., shows that \( s \) wave dominance of the outgoing neutrons is consistent with the measured inelastic cross section in Fe\(^{56}\). Knowing that the excited state spin is \( 2^+ \), we conclude \( \ell' = 2 \) and \( J' \) is either \( 5/2 \) or \( 3/2 \). For \( J' = 3/2; \)

\[
\Delta d\sigma_{\alpha;\alpha} = -\frac{(\sigma^{3/2}_{r\ell' \frac{1}{2}})_\text{exp}}{4\pi} \left\{ \frac{2 \sin^2 \delta_{3/2}}{\sin 2 \delta_{3/2}} \right\} [1 + P_2(\cos \theta)]d\Omega \tag{B7a}
\]
For \( J' = \frac{5}{2} \):
\[
\Delta \sigma_{\alpha;\alpha} = -\frac{\left(\sigma_{r'l'=-2}^{5/2}\right)\exp\left\{2\sin^2 \delta_{22}^{5/2}\sin 2\delta_{22}^{5/2}\right\}}{4\pi} \left[1 + \frac{8}{7} P_2(\cos \theta) + \frac{6}{7} P_4(\cos \theta)\right]
\] (B7b)

Using Eq. (16), we obtain the expected cusp effects in the total cross section. For \( J' = \frac{3}{2} \):
\[
\Delta \sigma_T = \left(\sigma_{r'l'=-2}^{3/2}\right)\exp\left\{\cos 2\delta_{22}^{3/2} - \sin 2\delta_{22}^{3/2}\right\}
\] (B8a)

For \( J' = \frac{5}{2} \):
\[
\Delta \sigma_T = \left(\sigma_{r'l'=-2}^{5/2}\right)\exp\left\{\cos 2\delta_{22}^{5/2} - \sin 2\delta_{22}^{5/2}\right\}
\] (B8b)

Since we have assumed isolated resonance behavior in the cross sections in this case, we can write the phases \( \delta \) as explicit functions of energy. Defining \( \epsilon_J = (E - E_0)/(\Gamma_J/2) \), where \( E_0 \) is the resonance energy and \( \Gamma_J \) is the full width at half maximum, we find
\[
2 \sin^2 \delta_{J'} = 2/(1 + \epsilon^2) , \\
\cos 2 \delta_{J'} = -(1 - \epsilon^2)/(1 + \epsilon^2) , \\
- \sin 2 \delta_{J'} = 2 \epsilon/(1 + \epsilon^2),
\] (B9)

where \( \epsilon \) ranges from \(-\infty\) to \(+\infty\) across the resonance.

Below-threshold effects are possible in both the differential elastic and total cross sections if the special assumptions are valid. Unfortunately, this cannot be checked in the present experiment, since
another resonance feature appears just below threshold, whose spin is unknown.

The above-threshold feature can be analyzed for possible cusp effects within the framework of these assumptions, by using Eqs. (B7) with Eqs. (B9) substituted.

\[
\Delta \frac{d\sigma}{d\Omega} = - \frac{(\sigma_{r\ell' = 2})}{4\pi} \exp \left( \frac{2}{1 + \epsilon^2} \right) \\
\times \left[ (1 + P_2(\cos \theta)), \text{ if } J' = \frac{3}{2} \right. \\
\left. (1 + (8/7) P_2(\cos \theta) + (6/7) P_4(\cos \theta)), \text{ if } J' = \frac{5}{2} \right] d\Omega
\]

\[
\Delta \sigma_T = (\sigma_{r\ell' = 2}) \exp \left( \frac{\epsilon^2 - 1}{1 + \epsilon^2} \right)
\]

Equations (B10) and (B11) show an interesting property. The cusp term in expressions for elastic cross sections (differential and integrated) is negative over the entire resonance. The total cross section expression is negative for \(-1 < \epsilon < 1\), and positive for \(\epsilon < -1\) and \(\epsilon > 1\). This can be interpreted simply as an increase in the total cross section resonance width due to the reaction. However, the experimental neutron energy spread (\(\approx 7\) kev) was too broad to allow measurement of such a difference in the widths of the elastic and total cross section resonance features.

In conclusion, we remark that the below-threshold feature (see Fig. 14, for example) which we have described as "mirrored" across threshold is just the kind of cusp effect one might expect from the theory. We were disappointed that our cusp analysis did not demonstrate this feature to be a cusp anomaly.
APPENDIX C

A PROOF

We wish to show that for the case of many resonances in the averaging interval one finds under certain conditions

\[
\langle \sigma_{r'l'} J' \rangle 2 \sin \delta^J_{l'} \left\{ \begin{array}{c}
\sin(2\delta^J_{l'} - \delta^J_l) \\
\cos(2\delta^J_{l'} - \delta^J_l)
\end{array} \right\}
\]

\[= \left\{ \begin{array}{c}
\langle \sigma_{r'l'} \rangle \\
0
\end{array} \right\} \delta^K_{l'l'} \delta^K_{J'J'}
\]

(\text{Cl})

where \( \delta^K \) is the Kronecker delta. Expanding the left side of (Cl), we obtain

\[
\langle \sigma_{r'l'} J' \rangle 2 \sin \delta^J_{l'} \left\{ \begin{array}{c}
\sin(2\delta^J_{l'} - \delta^J_l) \\
\cos(2\delta^J_{l'} - \delta^J_l)
\end{array} \right\}
\]

\[= \langle \sigma_{r'l'} \rangle \left\{ \begin{array}{c}
\sin 2\delta^J_{l'} \sin 2\delta^J_l - \cos 2\delta^J_{l'} 2 \sin^2 \delta^J_l \\
\cos 2\delta^J_{l'} \sin 2\delta^J_l + \sin 2\delta^J_{l'} 2 \sin^2 \delta^J_l
\end{array} \right\}
\]

(\text{C2})

For simplicity, let

\[
A \equiv \sigma_{r'l'} \left\{ \begin{array}{c}
\sin 2\delta^J_{l'} \\
\cos 2\delta^J_{l'}
\end{array} \right\}
\]

\[
B \equiv \sigma_{r'l'} \left\{ \begin{array}{c}
- \cos 2\delta^J_{l'} \\
\sin 2\delta^J_{l'}
\end{array} \right\}
\]

\[
a \equiv \sin 2\delta^J_l
\]

\[
b \equiv 2 \sin^2 \delta^J_l
\]
Then expression (C2) becomes
\[ = \langle Aa + Bb \rangle \] (C3)

We now assume, for \( l \neq l' \) and/or \( J \neq J' \) and within an averaging interval, that the quantities \( A, B, a, \) and \( b \) may be written as the sum of a mean value and a fluctuating part; for example, \( A = A^0 + \Delta A \), where \( \langle \Delta A \rangle \equiv 0 \). Thus, expression (C3) may be written
\[ \langle A^0a^0 + B^0b^0 + A^0\Delta a + a^0\Delta A + B^0\Delta b + b^0\Delta B + \Delta A \Delta a + \Delta B \Delta b \rangle \] (C4)

Meyerhof showed in Ref. 7 that
\[ \langle \sigma_{rl'} \left\{ \begin{array}{c} \cos 2\delta_{J',l'} \\ \sin 2\delta_{J',l'} \end{array} \right\} \rangle \equiv \langle B \rangle \approx 0 \] (20)

Also
\[ \langle \sigma_{rl'} \left\{ \begin{array}{c} \sin 2\delta_{J',l'} \\ \cos 2\delta_{J',l'} \end{array} \right\} \rangle \equiv \langle A \rangle \approx 0 \]

on the same basis as the proof of (20). Therefore, \( A^0 = B^0 \approx 0 \) and expression (C4) is reduced to
\[ \langle \Delta A \Delta a + \Delta B \Delta b \rangle \] (C5)

where we have also used \( \langle a^0\Delta A \rangle = \langle b^0\Delta B \rangle \equiv 0 \).

\( \Delta A \) and \( \Delta B \) are functions of \( \delta_{J',l'} \), while \( \Delta a \) and \( \Delta b \) refer only to \( \delta_{J,l} \). For \( l \neq l' \) and/or \( J \neq J' \), the fluctuation over the averaging interval in functions of \( \delta_{J,l} \) are uncorrelated to those in functions of \( \delta_{J',l'} \); thus, expression (C5) is equal to zero, and we have proved Eq. (Cl) for this case.
For $J = J'$ and $l = l'$ the LHS of Eq. (C1) reduces to

$$\langle \sigma_{rl'}^{J'} \left\{ \begin{array}{c} 2 \sin^2 \delta_{l'}^{J'} \\ \sin 2 \delta_{l'}^{J'} \end{array} \right\} \rangle,$$

and we may use the result proved in Ref. 7,

$$\sigma_{rl'}^{Jl} \left\{ \begin{array}{c} 2 \sin^2 \delta_{l'}^{J'} \\ \sin 2 \delta_{l'}^{J'} \end{array} \right\} = \left\{ \begin{array}{c} \langle \sigma_{rl'}^{J'} \rangle \\ 0 \end{array} \right\}$$

which completes the proof of Eq. (C1).
A space charge saturation gamma-ray suppression preamplifier was used to distinguish neutron induced from gamma-ray induced pulses in the detector. The circuit is shown in Fig. 22. It is basically that of Owens, modified to include more external controls, a diode clipper on the nonlinear output, and an integrating network for the remaining positive overshoot. The external control on resistor $R_1$ was suggested by the work of Funsten and Cobb.

Certain organic scintillators emit light under particle bombardment, which is composed of an initial spike of fluorescence with about $50 \mu$ sec decay time, followed by one or more slower components of greater than $200 \mu$ sec decay times. Ion recombination may account for the slow components. The ratio of the amplitude of the initial spike to that of the first slow component is a function of the type of particle causing the fluorescence.

The circuit in Fig. 22 is designed to produce linear voltage pulses, i.e., voltage pulses which are proportional to the amplitude of the initial fluorescence, at dynode 10. The remainder of the resistor chain is arranged deliberately to produce space-charge saturation in the region between dynode 14 and the anode, during the initial spike. When the slow-component cascade electrons arrive in the region of space-charge saturation they are sufficiently repelled to produce a
Fig. 22. Circuit diagram of a space charge saturation gamma ray suppression preamplifier. The four externally controlled elements are marked "FOC," "ACC," "R₁," and "DYN 14." "LIN" means linear pulse output. "γ supp" means the gamma ray suppression (nonlinear) output.
SPACE CHARGE SATURATION / SUPPRESSION PREAMPLIFIER

RCA 6810A

100K
2W

120K
2W

1M

Focus

"FOC"

33K
2W

22K
2W

22K
2W

22K
2W

22K
2W

22K
2W

22K
2W

22K
2W

22K
2W

27K
2W

39K
2W

150K
2W

120K
2W

HV + "R"

50K

10K

.01

.005

12K
2W

1M

"ACC"

2nd grid

100K

"LIN"

11

.005

3KV

13

200

\( \mu F \)

240\( \Omega \)

2K

11

.01

3KV

10K

3.92A

200

\( \mu F \)

240\( \Omega \)

2K

All to right of is a standard 5965 anode follower

All capacitors are .005\( \mu F \), 3KV unless otherwise noted.
positive overshoot pulse at dynode $l_4$. Thus, the height of the positive overshoot is a measure of the ratio of slow component to initial spike amplitude and it is possible to use this effect to distinguish the type of particle which initiated the fluorescence.

One is particularly interested in distinguishing neutron-initiated pulses (i.e., from recoil protons) from pulses caused by gamma-rays (i.e., due to electrons), and the method has proved useful for this purpose. Our particular circuit is arranged such that the negative pulse at dynode $l_4$, caused by the initial spike for either incident neutrons or gamma-rays, is clipped by a diode. The remaining positive overshoot is, of course, no longer a linear effect, and we have chosen to integrate this pulse in an attempt to maximize the distinction between neutron and gamma-ray initiated pulses.

The circuit was effective in gamma suppression for neutron energies greater than about 0.3 Mev; however, some care must be taken in biasing out gamma-ray induced pulses and to select neutron caused pulses, particularly when working at low neutron energies.

An example of the nonlinear spectrum obtained at 1.7-Mev neutron energy is given in Fig. 23. A nonlinear spectrum of gamma-ray initiated pulses is also shown for comparison. These gamma-rays were produced under Van de Graaff machine operating conditions with the proton beam incident upon a stopper in front of the tritium target.

While working at low neutron energies ($<0.5$ Mev) we found that very small dynode $l_4$ voltages were necessary to provide good gamma-ray
Fig. 23. Nonlinear pulse spectra obtained at $E_n = 1.7$ Mev from a space charge saturation gamma ray suppression preamplifier. The detector crystal was in the $0^\circ$ neutron beam direction. The solid circles show the nonlinear spectrum with the proton beam on the $T^3$ target ($T(p,n)He^3$). Open circles show the nonlinear spectrum with the proton beam blocked by a tantalum beam stopper in front of the $T^3$ target.
suppression. This produced smaller nonlinear output pulses, which could be compensated by increasing the high voltage on the photomultiplier, but which eventually represented an essential limitation on the circuit function because of the increasing influence of tube noise.
APPENDIX E

MULTIPLE SCATTERING AND FINITE SCATTERER CORRECTIONS

The slab scatterers used in the experiment were between $l/5$ and $l/2$ neutron mean free paths in thickness. Thus a significant number of multiply scattered neutrons were detected. Cross section data must be corrected for this effect. A second correction was required because the detection crystals were close enough to the slab scatterer to detect once-scattered neutrons over an angular range of about ±8 degrees in polar angle with respect to the line of crystal-scatterer centers. One can see this by examining Fig. 1 which shows the scatterer and detectors in typical positions. Because the detectors were sampling the differential cross section over this angular spread, peaks in the experimental angular distribution were reduced and valleys filled. We will refer to the correction for this effect as a "finite scatterer correction."

We were not prepared to make the computer analysis necessary for obtaining accurate corrections because for cusp analysis a first order estimate was considered satisfactory.

The measurements of differential cross sections versus energy were made for only one or two angles at a time. Hence, the experimental points were correlated in energy but not in angle. The absolute error in the (uncorrected) measured cross section was approximately ±15 percent (see Error Analysis) and was independently established for each
angle. Therefore, when one cross plots to present the angular distribution for different energies, the points determining the shape of the angular distribution are separately uncertain by ±15 percent.

The shape of the differential scattering cross section is, of course, essential to a determination of the multiple scattering correction. We used first order multiple scattering and finite scatterer corrections in this experiment in two ways. We corrected the optical model prediction of Moldauer\textsuperscript{17} for the angular distribution of cerium and compared the results with experimental data (see Fig. 8). We also corrected the calculated "energy-averaged" cusp effects before applying them to the experimental data for Ce, Ba, and Zr.

In the Fe experiments, the structure in the data did not justify any detailed estimate of the corrections, short of a full computer study on the angular and energy dependence of multiple scattering. On the basis of characteristic scatterer dimensions we estimated an upper limit of 20 percent for the corrections in the case of iron.

In the lithium experiments, the scatterer was 0.17 neutron mean free paths in characteristic dimension. This dimension is small in comparison with other scatterer thicknesses typically used in this experiment. The angular distribution is peaked backwards and the lab neutron energy loss for back scattering is from 30 to 40 percent of incident energy (below the detection bias of the experiment). All of these factors, particularly the last, contribute to an estimate that the correction for multiple scattering in this case is small. The
general agreement in magnitude between the measured lithium angular distribution and other work is shown in Fig. 9 and supports this conclusion.

It remains to discuss in detail the first order multiple scattering and finite scatterer corrections for Ce, Ba, and Zr. We used a procedure and notation presented by Nauta. Three assumptions underlie the calculation: first, the ratio $S_{K+1}/S_K$ is independent of $K$, the order of scattering, where $S_K$ is the probability of scattering elastically $K$ times before leaving the scatterer; second, the angular distribution of multiply scattered neutrons is the same as in an infinite medium of the same material; and third, there is no loss of lab energy upon scattering. The scatterer is characterized by a parameter $\theta$ which is the fraction of neutrons which, having been once scattered elastically, have undergone at least one more interaction.

Neglecting third and higher order scattering, the necessary formulas are:

$$d\sigma_{\text{corr}} = f_{\text{m.s.}}(\theta) d\sigma_{\text{exp}} = [1 + (S_2^{\text{el}}(\theta)/S_1^{\text{el}}(\theta))]^{-1} d\sigma_{\text{exp}}(\theta) \quad (E1)$$

where

$$S_2^{\text{el}}(\theta)/S_1^{\text{el}}(\theta) = \left(\frac{2\pi \theta}{\sigma_T} \right) \sum_{i=0}^{\infty} \frac{[2/(2i + 1)](a_i)^2 P_i(\cos \theta)}{\int_{-1}^{1} d\sigma_{\text{el}} P_i(\mu) d\mu}$$

and where $d\sigma_{\text{corr}}$ is the true differential cross section; $d\sigma_{\text{exp}}$ is the measured differential cross section; $\sigma_T$ is the total cross section;
and \( d\sigma_{el} = \sigma_{el} (S_{el}^{el}(\theta))/S_{el}^{el} \), with \( S_{el}^{el} = \int S_{el}^{el}(\theta)d\Omega \). The last definition emphasizes the fact that these equations are circularly related and that approximations of some kind are necessary to obtain a solution. Our procedure was to substitute an optical model angular distribution\(^{17}\) for \( d\sigma_{el} \); thus we calculated a correction to the optical model prediction for comparison with experiment. Satisfactory agreement between the corrected optical model distributions and experiment was considered justification for applying these corrections to the comparison of predicted cusp effects with experiment. This general procedure is analogous to that usually adopted in solving Eq. (El); one assumes a function \( d\sigma_{el} \) and by iteration of Eq. (El) forces it to converge to \( d\sigma_{corr} \). We stopped this process at a first iteration.

In essence, the method separates that part of the multiple scattering which depends upon the angular distribution characteristic of the scattering material from a part which depends upon the size, shape, and aspect (with respect to the beam axis) of the scatterer. The latter dependence is contained in the parameter \( \delta \).

Lacking a better estimate of \( \delta \), we assumed that the quantity \( \varphi_{1}/\varphi_{11} \) in the detailed work of A. Langsdorf, Jr., R. O. Lane, and J. E. Monahan\(^{37}\) is \( 1 - \delta \), where \( \varphi_{1} \) is the emergent singly scattered flux per unit solid angle and \( \varphi_{11} \) is the unit idealized scattered flux per unit solid angle. The parameter \( \delta \) is dependent upon the angle of scattering at which the cross section is measured because slab scatterers are extremely asymmetric. In the work of Nauta,\(^{36}\) who used ring scatterers, \( \delta \) could be assumed independent of angle.
Accordingly, we used the values of $\varphi_1/\varphi_{11}$ given in Fig. 9 of Ref. 37 to obtain $a$ for each scatterer and for each angle of interest. In most cases extrapolation of the curves was necessary. These curves are entered with a particular value of $h$, the scatterer thickness, in units of neutron mean free path.

Our scatterer thicknesses, calculated at the neutron energies of interest, were: for Zr, $h = 0.32$; for Ce, $h = 0.46$; and for Ba, $h = 0.55$.

The moments $a_i$ of the optical model angular distribution were included to $a_4$. Multiple scattering corrections as a function of angle were then computed using Eq. (1) and the appropriate values of $a$. The degree to which corrected optical model predictions agreed with experimental data is indicated in Fig. 8. We estimated the overall uncertainty in the absolute multiple scattering corrections resulting from the assumptions and approximations in the foregoing treatment to be ±30 percent. Expressed as the fraction $f_{m.s.}$, which is convenient for modifying data, the uncertainty in multiple scattering corrections adds to the uncertainty obtained in the experimental error analysis. The uncertainty in $f_{m.s.}$ varies with scatterer and detection angle, as indicated in Table III.

We now consider the calculation of finite scatterer corrections. At each mean polar angle, $\theta_o$, once scattered neutrons are detected over an angle spread of $\theta_o \pm \Delta \theta$. The magnitude of $2 \Delta \theta$ was determined graphically from the geometry of the experiment, see Fig. 1, and was found to vary from 10 to 35 degrees. The angular spread is a
function of $\theta_0$ because the scatterer-to-detector distance varied depending upon the experimental situation (counting rate, background to effect ratio) and also because the projected area of the scatterer, normal to the scatterer-detector line of centers, varies with $\theta_0$.

As in the correction for multiple scattering, the procedure was to correct an optical model prediction for experimental effects. The finite scatterer correction is defined by

$$d\sigma_{\text{corr}} = f_{\text{f.s.}}(\theta) d\sigma_{\text{exp}}$$

(E2)

$f_{\text{f.s.}}(\theta)$ was computed using

$$f_{\text{f.s.}}(\theta) = d\sigma_{\text{opt}} / \left( \frac{1}{N} \sum_{n=1}^{N} d\sigma_{\text{opt}}(\sin \theta_n / \sin \theta_0) \right)$$

(E3)

where $d\sigma_{\text{opt}}(\theta_0)$ is the optical model prediction at mean angle $\theta_0$, and the denominator represents an average of the optical model distribution over $N$ equal divisions of the spread $2 \Delta \theta$ weighted by the solid angle subtended by the nth angle division. $N$ was selected to give a representative average and varied from 5 to 8 depending upon the size of $2 \Delta \theta$.

The results of this calculation are given in Table III. The magnitude of typical finite scatterer corrections is indicated on the angular distribution shown in Fig. 8, where it can be seen that the correction is small except in the valley near $\theta_0 = 90^\circ$. 
TABLE III. Multiple Scattering and Finite Scatterer Corrections

<table>
<thead>
<tr>
<th></th>
<th>$\theta_0$</th>
<th>$f_{\text{m.s.}}$</th>
<th>$f_{\text{f.s.}}$</th>
<th>Uncertainty in $f_{\text{m.s.}} \times f_{\text{f.s.}}$ (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cerium</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>30°</td>
<td>0.95</td>
<td>1.04</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>50°</td>
<td>0.87</td>
<td>0.98</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>70°</td>
<td>0.80</td>
<td>0.92</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>90°</td>
<td>0.75</td>
<td>0.90</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>110°</td>
<td>0.78</td>
<td>0.99</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>130°</td>
<td>0.80</td>
<td>1.00</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>150°</td>
<td>0.85</td>
<td>1.05</td>
<td>6</td>
</tr>
<tr>
<td><strong>Barium</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>90°</td>
<td>0.65</td>
<td>0.90</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>125°</td>
<td>0.70</td>
<td>1.00</td>
<td>14</td>
</tr>
<tr>
<td><strong>Zirconium</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>60°</td>
<td>0.90</td>
<td>~1.0</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>130°</td>
<td>0.85</td>
<td>~1.0</td>
<td>6</td>
</tr>
<tr>
<td><strong>Iron</strong></td>
<td>All</td>
<td>0.8-1.0</td>
<td>~1.0</td>
<td>~10</td>
</tr>
<tr>
<td><strong>Lithium</strong></td>
<td>All</td>
<td>~1.0</td>
<td>1.0</td>
<td>~5</td>
</tr>
</tbody>
</table>

The factors in this table may be applied to the experimental cross section data presented in this report, by using

$$d\sigma_{\text{corr}} = f_{\text{m.s.}} \times f_{\text{f.s.}} \times d\sigma_{\text{exp}}.$$
LIST OF REFERENCES

1. E. P. Wigner, Phys. Rev. 73, 1002 (1948).
   Meyerhof (6) has noted that \((2l + 1)\) factor in Eq. (3.2) of this paper should be deleted if \(\sigma_{\text{inel}}\) is the measured inelastic cross section.
17. P. A. Moldauer (private communication).


37. A. Langsdorf, R. O. Lane, and J. E. Monahan, Phys. Rev. 107, 1087 (1957), Fig. 4; also Report ANL-5567 Rev. (1961) Fig. 9.
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