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# Dividing lines for backlash in the phase plane. 

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# DIVIDNG LNES FOR BACKIASH IN THE PHASE PLANE 

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DIVINING LINES FOR BACKLASH
tn the phase plane

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WILLIAM J. LUTKENHOUSE

## DIVIIING LTNES FOR BACKLASH <br> IN THE PHASE PLANE

by<br>W'illiam J. Lutikenhouse iieutenant, United States Navy

Submitted in nartial fulfillment of the requirements for the depree of

MASTER OF SCIENCE
IN
ELECTRICAL EVGINEERING

United States Naval Postrraduate School
Monterey, California
1959

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This work is accented as fulfilling the thesis requirements for the degree of MASTER OF SCIENCE
IN
ELECTRTCAL ENGINEERING
from the
United States Naval Postpraduate School

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                ABSTRACT
    Tre whse plnne, or displacerent versus velocity plrne, while
rel tivel: now to the en&ineering fielo, is particilarly suited to
furnishine comprehensive graplical cisoloy of nonlineorities operatine
under certain conditions.
    Althourh a succeasive phase plane a,plic tion is required to dis-
plry a system of order freater than two, a one picture phase plane
present-tion is acequate for rredicting the stribility and transient
periormance of a second orcor systen.
    Backlash effects are invostigetod by phose plono techniques in this
thesis.
    The writer ::ishes to express his appreciation to Dr. Goorce J. Thaler,
\sigmaithout whose assistence and encourgmenent, this thesis mould not have been
miNten.
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29. Backlash

Backlash is a real engineering problem. It exists in all gear trains to a greater or lesser extent and in multiple gear arrangements, is the sum of the backlash errors of the individual gears. Dependent upon where the servo meclanism error is measured, backlash contributes either to the steady state error, or instability of the system. Back lash is not always detrimental to system performance for when controlled, can oe used to introduce a slight "dither" to a system, whicl may be used to provide $\%$ od lubrication and freedom from stictional effects.

I'he second order system consisting of motor and load is treated as two linear systems oderating sequentially (a) during the period when the backlash is taken up and the load is being driven by the motor and (0) when the load is driftino separately in the backlash region with the motor beinf controlled by an error siqnal generated from output measured at either load shaft or motor shaft. Botr. locations of outout measurement are considered.

## 2. Dividing Lines

System response may be predicted on the phase plane by the loci of four dividing lines in conjunction with system isoclines: (a) system separation line (b) velocity and disolacement of system without load at time of recombination (c) velocity and displacement of load at time of recombination (d) velocity and displacement of recombined system after nomentun conservation conditions have been satisfied.

This study treats all recombinations as bein $i n e l a s t i c$. When an error exists in a system, and the system is correctins with the backlash taken un, a deceleratin to rque must eventually be applied to the motor. At some time afterward, providing viscous friction is not infinite at the load, and that coulomb friction of load $c$ an be neplected, the velocities of the motor and load become different, mechanical contact is lost and separation occurs. This line of separation is a straicht line consisting of that isocline or constant slope line at which the load trajectory and the traiectory of the system with load removed, have a common sloje. It may be called the "Separation dividina line". Past the separation dividing line, the system overates in the backlash region durinf which time the load drifts sevarately, with undiminished velocity if its viscous friction is nerliaiole or with a constant deceleration oresuming all friction is viscous. Tre remainder of the systern acts as a stable system with decreased inertia and spirals in towards a stable focus if system output is measured at motor shat. If the system output is measured at the drifting load, the system does not behave stably in the backlash region but continues to drive in the reverse direction until the backlash has taken up and system outout completely
corrected as to displacement. From this it may be seen that when the system output is measured at the drifting load, backlash between motor and load may make an overdamped system oscillatory.

The initial position of system velocity and displacement on the separation dividing line determines two points on the phase plane, load velocity and displacement at instant of recombination, and system without load velocity and displacement at instant of recombination. The loci of these two noints constitute dividing lines listed in (b) and (c) above. wach pair of corresponding points for load and system without load, velocity and displacement at recombination, in turn determine a point on the locus for recombined system velocity and displacement. This final noint is obtained from satisfying the conservation of momentum considerations where individual monentums are defined for load and system without load. Trajectories for the recombined system originate from this final dividing line and the system retains its orifinal characteristics until the next separation dividing line is reached, after which, separation and recombination occur as in the previous half cycle.

If in response to a stev displacement, the system attains a auiescent condition, with or wi thout steady state displacement error, tre system is said to be stable.

If the final state of the system is one of a constant amolitude oscillation, the system is said to have a limit cycle. The ma mitude of this limit cycle may be determined by any of the possible non linearities of the system, dead zone, etc. but in this tresis, only limit cycles resulting from backlash are considered.
3. Output measured at motor shaft, load having viscous friction.

Consider first, tho system in which the output measuring device is mounted on the motor shaft. The backlash existing in the gear train between motor and load is outside of the feedback loop as show in "Fig.I". The closed loop is completely linear. Ultimate stability with a steady state displacement error less than or equal to the amount of the backlash is the result. The drifting load can at most store energy and since the load does not determine the output measurement, it does not demand more power from the source than is required. The combined viscous friction of the load and the system without load represent energy dissinations which insure stability.


Figure (I)
When the backlash is taken up, the differential equations for the system are:
(1) $\frac{1}{N}\left(N_{m}^{2}+V_{L}\right) \ddot{\theta}_{0}+\frac{1}{N}\left(N^{2} f m+f_{L}\right) \dot{\theta}_{0}=T$
(2) $T=K, I$
(3) $V=R I+K_{2} \dot{\theta}_{n}$
(4) $I=\left(\frac{V}{R}-\frac{K 2}{R} N \dot{\theta}_{0}\right)$
(5) $V=K_{3}\left(\theta_{R}-\theta_{0}\right)$
(6)

$$
\begin{aligned}
\frac{1}{N}\left(N^{2} V_{m}+S_{L}\right) \ddot{\theta}_{0} & +\frac{1}{N}\left(N_{m}^{2}+f_{L}+\frac{K_{1} K_{2} N^{2}}{R}\right) \dot{\theta}_{0} \\
& +\frac{K_{1} K_{3} \theta_{0}}{R}
\end{aligned}
$$

Where

$$
\begin{aligned}
& K_{1}=\text { motor torque constant } \frac{f_{t} \quad b_{s} b_{e}}{a m p} \quad N_{m}+\frac{K_{1} K_{2}}{R} N=\underset{\text { systerion }}{\text { friction }} \\
& K_{2}=\text { motor generator constant } \frac{\text { volts }}{\text { rad. }} / \mathrm{sec} . \quad=F_{m} \text { without lour } \\
& \begin{array}{ll}
\mathrm{R}_{3}=\text { error measurement constant } \frac{\text { volts }}{\text { nad. }} \\
\mathrm{R}=\text { armature resistance }
\end{array} \quad \frac{\mathrm{K}_{1} \mathrm{~K}_{3}}{\mathrm{R}}=\mathrm{K} \\
& \mathbb{N}=\text { gear ratio } \\
& \frac{\mathrm{rad}, \text { motor }}{\text { raft. output }} \\
& J_{\mathrm{m}}=\underset{\text { without load }}{\text { inertia }} \underset{\text { made }}{\text { rad. } / \mathrm{Ibsec}_{0}} 2 \\
& \theta_{R}=\text { ordered displacement } \quad \theta_{L}=\text { displacement of load } \\
& \theta_{0}=\text { displacement of } \\
& \text { combined system } \\
& \theta_{m}=\text { displacement of } \\
& \text { system without load }
\end{aligned}
$$

Letting $N=I \quad \theta_{R}=1$
From equation (6), to obtain the equation for the isoclines of the combined system:
(n) $\ddot{\theta}_{0}+\left(\frac{F_{m}+F_{2}}{J_{m}+v_{2}}\right) \dot{\theta}_{0}=\left(\frac{k}{m_{m}+v_{2}}\right)\left(1-\theta_{0}\right)$

Letting
Letting

When the system with load removed constitutes a closed system, an equation for the isoclines may be written as follows:


$$
\begin{align*}
& \hat{G} \cdot \pi  \tag{10}\\
& \frac{\theta}{2}=\frac{\theta m m}{6}
\end{align*}
$$

Under deceleration conditions, when the velocity of the system without load is equal to the velocity of the load drifting separately, separation occurs and the backlash becomes operative. The isocline for the drifting load is determined by the friction and inertia of the load.
(11) $\quad \underset{L}{ } \dot{\theta}_{a}+F_{L} \hat{\theta}_{2}=0$

$$
\begin{equation*}
\frac{\ddot{\theta}_{B}}{\theta_{1}}=-\frac{F_{2}}{J_{2}}=\eta_{0} \tag{12}
\end{equation*}
$$

To prove that the system remains combined until it reaches the isocline of the system without load which has the same slope as the load trajectory. consider the following, assuming that $\quad 27,=C$ as shown in "Fig. ?":
 Figure (2)

$$
\begin{equation*}
\frac{\dot{\theta}_{0}}{\theta_{0}}=\frac{K}{F_{m}+F_{L}}\left(\frac{1}{\theta_{0}}-1\right) \tag{13}
\end{equation*}
$$

$$
\begin{equation*}
\frac{\dot{\theta}_{m}}{\theta_{m}}=\frac{M}{F_{m}}\left(\frac{i}{\theta_{m}}-1\right) \tag{14}
\end{equation*}
$$

where

$$
\theta_{n c}=\theta_{m o}
$$

at time of separation
Since

$$
\left|\frac{\dot{\theta}_{n}}{\theta_{n}}\right| \geqslant\left|\frac{\dot{\theta}_{c}}{\theta_{c}}\right|
$$

if separation is attempted when 2 , for the combined system isocline is equal to $\overbrace{3}$ for the drifting load, the motor will accelerate, keening the system combined until 72 for the system without load is common with $7 / 3$ of the drifting load.

To find the loci for the dividing lines:
(a) system without load velocity and displacement at recombination,
(b) load velocity and displacement at time of recombination, the equations of the individual systems are solved as Linear equations, using initial conditions determined from $\epsilon_{0} \dot{\theta}_{0}$ on the separation dividing lines, and final conditions determined by the amount of backlash to be taken up.
for the combined system:


For the system with load removed:
(16) $\operatorname{In}_{m} \ddot{\theta}_{m}+F_{m} \dot{\theta}_{m}+K \hat{T} \theta=K \theta_{p}$

For the load drifting separately
(17)

$$
\rho_{L} \ddot{\theta}_{L}+F_{L} \dot{\theta}_{2}=0
$$

At separation

$$
\theta_{m o}=\theta_{h 0}=\theta_{00}=\text { displacement of system with/ wi thou }
$$

$\dot{\theta}_{\mathrm{mo}}=\dot{\theta}_{\mathrm{LO}}=\dot{\theta}_{\mathrm{o}}=$ velocity of system with/without load
Putting these conditions into the system wi th load removed and solving yields:
(18) $\theta_{m}\left(s^{2}+\frac{\bar{m}_{m}}{\sqrt{m}} 5+\frac{K}{\sqrt{m}}\right)=\frac{F_{1} \theta_{p}}{s m}+5 \theta_{m o}$

$$
+\frac{F_{m}}{1 m} \theta_{m o}+\theta_{m o}
$$

Since the isoclines are straight lines, $\frac{\dot{\theta} \operatorname{\theta inc}}{\operatorname{tar}}=\operatorname{ta}$
where is the angle the common load, system without load isocline makes with the positive $\theta_{0}>\theta_{R}$ axis. $\theta_{m o}$ may be expressed in terms of Eirro as
(19) $\theta_{m o}=\theta_{p}+\frac{\theta_{m o}}{\operatorname{Tan}}$ Equation (18) then reduces to
(20) $\theta_{m}=\frac{K \theta_{p}}{s\left(s^{2}+\frac{F_{m}}{J_{m}} s+\frac{K}{J_{m}}\right)}$

$$
+\frac{\left(\theta_{p}+\frac{\dot{\theta}_{m c}}{\tan \phi}\right)\left(s+\frac{F_{m}}{v_{m}}\right)+\dot{\theta}_{m 0}}{\left(s^{2}+F_{m} / J_{m} s+K / U_{m}\right)}
$$

(21)

$$
\dot{\theta}_{m}=\frac{\operatorname{\theta }_{m}}{\left(s^{2}+\frac{\sigma_{m}}{V_{m}} s+\frac{K}{V_{m}}\right)}+\frac{s\left(\theta_{R}+\frac{\dot{\theta}_{m 0}}{T_{2} \varnothing}\right)\left(s+\frac{F_{m}}{V m}\right)+s \dot{\theta}_{m 0}}{\left(s^{2}+\frac{F_{m}}{V_{m}}+\frac{K}{N m}\right)}
$$

For the load only, imposing initial coitions yields:
(22) $\theta_{L}=\left(\frac{\left.\theta_{R}+\frac{\dot{\theta}_{L O}}{\tan \phi}\right)\left(5+\frac{F_{L}}{J_{L}}\right)+\dot{\theta}_{L O}}{S\left(S+F_{L} / L_{L}\right)}\right.$
(23)

$$
\dot{\theta}_{L}=\frac{\dot{\theta}_{L O}}{\left(s+F_{L} / \Omega_{L}\right)}
$$

Wien recombination occurs, neglecting elastic bounce of gen teeth,
$\dot{\theta} m$ and $\dot{\theta}_{L}$ change instentrnoously in accordance with the law of conservation of monentun, to satisfy the following equation:
$(24) \quad\left(J_{m}+J_{L}\right) \dot{\theta}_{0}=I_{m} \dot{\theta}_{m}+V_{L} \dot{\theta}_{L}$

The loci of the divicine lines for $\theta_{m}, \theta_{m}, \theta_{L}, \theta_{L}$,
$\theta_{0}, \dot{\theta}_{0}$
at recombination are obtained as follows:
Equations (20) and (22) are solved as a function of time and plotted with an arbitrary ordinate scale, for example let 1 in. ordinate displacement $=1$ rad., as in "Fig. 3 ", which is representative of any system where backlash is outsi de the feedback loop.


Figure (3)

To find time ( $t$ ) at which .2 rad. backlash is taken up given an initial condition of $\dot{\theta}_{\hat{A} C}=.5$, proceed as follows: Since above figure was plotted using $\dot{\theta}_{20}$ as a parameter with 1 rad. $=1$ in., let ordinate $=$ scale of 1 rad $=2 \mathrm{in} .=1 / 5 . \quad .2$ rad. backlash represented by separation between $\theta_{\kappa}$ and $\theta_{n}$, is now equal to 14 in. Wit! dividers, fit .4 in. to time ( $t$ ) for recombination. ( $t$ ) is found to be .95 sec


To find time (t) at which . 2 rad. backlash is taken up given an initial condition of $Q_{0}=.2$, let ordinate scale be $1 \mathrm{rad} .=5 \mathrm{in}$. or 1 in. =. ? rad. Backlash of .? rad is now equal to $l$ in. with dividers fit 1 in. between $\theta_{L}$ and $\theta_{\boldsymbol{m}}$, read ( $t$ ) on atreissa equal to 1.45 sec .


Figure (5)

To find time ( $t$ ) at which . 2 rad backlash is taken up given an initial condition of $\dot{\theta}_{\alpha 0}=$. I, let ordinate scale be 1 rad. $=1$ ) in. or 1 in. $=$. l rad. Backlash of . 2 rad is now equal to 2 in. with dividers fit 2 in. between $\epsilon_{L}$ and $\theta_{m}$, read ( $t$ ) on abscissa at 2.05 sec .

Complete loci of dividing lines are obtained by substituting values of $(t)$ obtained from initial conditions of $\hat{\theta}_{\text {LO }}$, into equations (20), (21), (27), ana (23). Resulting values of $\Theta_{n}, \dot{\theta_{m}}, \Theta_{L}, \dot{\epsilon}_{L}$ are then plotted on the phase plane. $\Theta_{0}$ and $\dot{\theta}_{0}$ are obtained by substituting values of $\dot{\theta}$ and $\dot{\theta}_{2}$, from equations (21) and (23) into equation ( 2 L ), for corresponding times.

In the case where output is measured at the motor shaft, upon recombination is equal to the displacement of $\Theta_{m}$. when systern output is measured at the load, $\theta_{0}$ upon recombination is equal to the displacement of $\theta_{L}$ 。
4. Stability

A spot chock of stability may be mode at any particular recombination point by treating the phase plane trajectories as logarithmic spirals with trensformations s follow:
For the combined system let $J_{m}+J_{L}=J, F_{m}+F_{L}=F$
(25) $\sqrt{\dot{\theta}_{0}}+F \dot{\theta}_{0}+K \theta_{0}=K \theta_{R}$
(26) $\frac{\ddot{\theta}_{0}}{\dot{\theta}_{0}}=\frac{\frac{K \theta_{R}}{J}-\frac{K \theta_{0}}{V}}{\dot{\theta}_{0}}-\frac{F}{V}$
(27) Let $\dot{\theta}_{0}=u\left(\theta_{p}-\theta_{0}\right)$
(28) $d \dot{\theta}_{0}=\left(\theta_{p}-\theta_{0}\right) d u-4 d \theta_{0}$
(29) $\left(\theta_{p}-\theta_{0}\right) d u=u+\frac{K / U}{u}-F / \omega$
(30) $\int \frac{d \theta_{0}}{\left(\theta_{R}-\theta_{0}\right)}=\int \frac{u d u}{u^{2}-F / J_{u}+K / J}$

(34) $-\operatorname{n}\left(\theta_{p}-\theta_{0}\right)=\sqrt{\left(\frac{\left.\sqrt{K / J-\frac{F^{2}}{4 U^{2}}} \tan z+\frac{F}{2 J}\right)\left(\sqrt{K / J-\frac{F^{2}}{4 J^{2}}} \sec ^{2} z d z\right)}{\left(\frac{K}{J}-\frac{F^{2}}{4 J^{2}}\right) \sec ^{2} z}\right.}$

$$
=\operatorname{lncos} z+\frac{F / 2 v+C}{\sqrt{\frac{K}{J}-\frac{F^{2}}{4 J^{2}}} \sqrt{\sqrt{\frac{K}{J}}-\frac{F^{2}}{4}}}
$$

(35) $\quad \cos z=\sqrt{\frac{K / J-F^{2} / 4 J^{2}}{4^{2}-F / J u^{4}+K / J}}$

$$
\begin{aligned}
(36)-\ln \left(\theta_{R}-\theta_{0}\right)= & -\ln \sqrt{\frac{K / J-F 2 / 4 J^{2}}{4^{2}-F / U}}+\quad+K+J
\end{aligned}
$$

$$
+\frac{F / 2 J}{\sqrt{K} / V^{F^{2}} \frac{4 V^{2}}{2}}
$$

(37)

$$
\begin{aligned}
& \dot{\theta}_{0}^{2}-F / J \dot{\theta}_{0}\left(\theta_{R}-\theta_{0}\right)+K / J\left(\theta_{R}-\theta_{0}\right)^{2} \\
= & C^{2}\left(K / J-\frac{F^{2}}{4 J^{2}}\right) e^{\frac{-F / 2 U}{\sqrt{k / J-F^{2} / J^{2}}} \arctan \frac{\dot{\theta}_{0}-F}{\left(\theta_{R}-\theta_{0}\right) \sqrt{k / J-\sigma^{2} / 4 J^{2}}}\left(\theta_{R}-\theta_{0}\right)}
\end{aligned}
$$

To determine if a particular recombination point on a trajectory is approaching stability (i.e., the recombined trajectory is closer to the strible focus than prior to separation), it is necessary to compare values of $C$ obtained from the coordinates of the trajectory where separation occurs and from the coordinates, where recombination occurs, the conservation of momentum criteria having boon first satisfied.
(38)


If $\mathrm{C}\left(\theta_{c}, \dot{\theta}_{c}\right)$ recombined $>\boldsymbol{C}\left(\theta_{i}, \dot{\epsilon}_{i}\right)$ unseparated, the trajectory will diverge into a limit cycle. This comparison of values of $C$ is illustrated in Fig. K. This method is applied to test limit cycles developed in cases III and IV.


Figure (6)
5. Output measured at load, load having viscous friction.

When the backlash is placed inside the feedback loop, i.e., ven the output is measured at the load, the configuration of the system is as illustrated in Fig. 7.


In the combined region, behavior of the system is identical to that of the system described in section throe; however, the system with load removed, as it is when in the backlash region, is no longer a closed system. The system with load removed is driven open loop and attempts to correct. the error as usual. It is prevented from doing so by the existence of beciclash. In this attempt, the inertia of the system with load removed develops a momentum which in an underdamped system always causes instability and a consequent limit cycle, but for the presence of coulomb friction. Eau Lion (9) reduces to

$$
\begin{equation*}
U_{m} \ddot{\theta}_{m}+F_{m} \dot{\theta}_{m}=K\left(\theta_{R}-\theta_{L}\right) \tag{39}
\end{equation*}
$$

Since from equation (22), $\theta_{L}$ in La Place form is equal to

$$
\frac{\left(\theta_{R}+\frac{\dot{\theta}_{L 0}}{\tan \phi}\right)\left(5+F_{L / L}\right)+\dot{\theta}_{L 0}}{S\left(5+F_{L} / U_{L}\right)}
$$

equation (20) is reduced to
(40) $\theta_{m}=\frac{K / u_{m}\left(-\frac{\dot{\theta}_{m 0}}{F_{a n} \phi}\right)}{s^{2}\left(s+F_{m} / S_{m}\right)}-\frac{K / s_{m} \dot{\theta}_{m 0}}{s^{2}\left(s+F_{m} / /_{m}\right)\left(s+F_{k} / S_{k}\right)}$

$$
+\frac{\dot{\theta}_{m 0}}{s(s+F \mathrm{~m} / \mathrm{Jm})}+\frac{1+\frac{\dot{\theta}_{m o}}{\tan \phi}}{s}
$$

(41)

$$
\dot{\theta}_{m}=\frac{k / s_{m}\left(-\frac{\dot{\theta}_{m 0}}{F_{n} \phi}\right)}{s\left(s+F_{m} / S_{m}\right)}-\frac{k / S_{m} \dot{\theta}_{m o}}{s\left(s+\frac{F_{m}}{S_{m}}\right)\left(s+\frac{K_{u}}{V_{n}}\right)}+\frac{\dot{\theta}_{m 0}}{\left(s+\frac{F_{m}}{\sqrt{m}}\right)}
$$

Since the system with load removed is not suitable for the isocline method, a time solution of equations ( 1,0 ) and ( 47 ) is required.

A blot of $t$ versus $\theta_{m}$ and $\theta_{L}$ result in a graph similar to tip. 8 and is used as previously described in section three to give values of ( $t$ ) for computation of $\theta_{m}, \dot{\theta}_{m}, \theta_{L}, \dot{\theta}_{L}, \theta_{0}, \dot{\theta}_{0}$ dividing lines.


Figure ( 8 )
6. Case I. Output measured at motor shaft.

Given system
Combined
(42) $/ \ddot{\theta}_{0}+.8 \dot{\theta}_{0}+\theta_{0}=1$

System with load separated
(43) $.6 \ddot{\theta}_{m}+.48 \dot{\theta}_{m}+\theta_{m}=1$

Load
(44) $.4 \ddot{\theta}_{L}+.32 \dot{\theta}_{L}=0$

Backlash . 2 rad.
Isocline at which load separates from system is determined from load equation.
(45) $\quad \frac{\dot{\theta}_{L}}{\dot{\theta}_{L}}=7,3=-\frac{.32}{4}=-.8$

To find isocline of combined system letting $\theta_{R}=1$
(46) $2,+8=\frac{1-\theta_{0}}{\dot{\theta}_{0}}$

Computed values of $\arctan \frac{\dot{\theta}_{0}}{1-\theta_{0}}$ for various values of $N_{0}$ are as listed in table one.

Table (1)

| N, | $\operatorname{Arctan} \frac{\dot{\theta}_{0}}{1-\theta_{0}}$ | N, | Arctan | $\frac{\dot{\theta}_{0}}{1-\theta_{0}}$ |
| :---: | :---: | :---: | :---: | :---: |
| 10 | $-5.30^{\circ}$ | - . 2 | $-50^{\circ}$ |  |
| 7 | - $7.31{ }^{\circ}$ | -. 3 | - 63.4 | 。 |
| 5 | $-9.78^{\circ}$ | - . 5 | - 73.3 |  |
| 3 | $-14.75^{\circ}$ | $-.7$ | -84.3 |  |
| 2 | $-19.65^{\circ}$ | -. 8 | 90 |  |
| 1.5 | $-23.5^{\circ}$ | - 1 | 78.7 |  |
| 1 | $-29.05^{\circ}$ | - 1.2 | 68.2 | $\bigcirc$ |
| . 7 | $-33.7^{\circ}$ | - 1.5 | 55 |  |
| .5 | $-37.6^{\circ}$ | - 1.7 | 48 |  |
| - 3 | $-42.25^{\circ}$ | - 2 | 39.8 | $\bigcirc$ |
| . 2 | $-45^{\circ}$ | - 2.5 | 30.45 | - |
| . 1 | $-48^{\circ}$ | - 3 | 24.45 | - |
| 0 | -51.35 | - 5 | 13.39 | $\bigcirc$ |
| . 1 | -55 | - 7 | 9.15 |  |
|  |  | -10 | 6.20 | - |

To find isoclines of system wi th load separated
(47) $\eta_{z}+.8=\frac{1.66\left(1-\theta_{m}\right)}{\theta_{m}}$

Table (2)


To obtain loci for $\theta_{m} \dot{\theta}_{m}, \theta_{\alpha} \dot{\theta}_{l}$ dividing lines, it is necessary to solve equations for $\theta_{m}, \theta_{L}$ using initial conditions determined from separation dividing line of $72=-8$
(48) $\ddot{\theta}_{m}+8 \dot{\theta}_{m}+1.666 \Theta_{n}=1.666$

In LaPlace form
(49)

$$
\theta_{m}=\frac{\frac{1.666}{5}+5 \theta_{m 0}+8 \theta_{m 0}+\operatorname{\theta imo}_{m o}}{\left(s^{2}+.85+1.666\right)}
$$

Foots of quadratic factor are $-.4 \pm, 1.225$
(50)

$$
\theta_{m}=\frac{1666}{5(5+.4 \pm 11.225)}+\frac{\theta_{0.0}(5+.8)+\dot{\theta}_{m 0}}{(5+.4 \pm j 1.225)}
$$

(51)

$$
\theta_{m}=1+\frac{\dot{\theta}_{m 0}}{1.225} e^{-.4 t} \sin 1.225 t
$$

(52) $\dot{\theta_{m}}=-\frac{\dot{\theta}_{m 0}}{1.225} e^{--.4 t} \sin 1225 t+\dot{\theta}_{m 0} e^{-.4 t} \cos 1.225 t$

For load separately
In La Place Form
(53) $\theta_{L}=\frac{\dot{\theta}_{m 0}}{5(5+.8)}+\frac{\theta_{m 0}}{5}$
(54) $\theta_{L}=\frac{\dot{\theta}_{m 0}}{\cdot 8}\left(1-e^{-8 t}\right)+\theta_{m 0}$
(55) $\dot{\theta}_{L}=\dot{\theta} \dot{\theta}_{0} e^{-.8 t}$
equations (51) and (5) are evaluated at times indicated from factors listed in Table three. These equations are then plotted for $\dot{\theta}_{m o f}$.I to produce a graph similar to Fig. 3., from which values of time for recombination are obtained for various initial values of $\dot{\theta}$ mo.

| $t$$.1$ | Taule 3 |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\sin 1.225 t$ | $e^{-4 t}$ | $e^{-8 t}$ | $\theta m$ | $\theta_{L}$ |
|  | . 1222 | $.950{ }^{4}$ | .9232 | - 0096 | . 07096 |
| . 2 | . 2425 | . 9231 | . 854.5 | . 0183 | . 0183 |
| . 4 | . 1.706 | . 8521 | . 7260 | $.732^{\circ}$ | . 03425 |
| . $¢$ | . 7706 | .7966 | . 8170 | . 0431 | . 04775 |
| . 8 | . 8305 | . 7260 | .5275 | . 0493 | . 259 |
| 1.0 | . 9408 | . 6700 | .1 .500 | . 0515 | . 06888 |
| 1.2 | . 9949 | . 6188 | . 3830 | . 2503 | . 277 |
| 1.5 | . 96442 | .5400 | . 3015 | . 04332 | . 0473 |
| 1. ${ }^{2}$ | . 8070 | . 4965 | . 2370 | . 0321 | . 0955 |
| 2.0 | . 8377 | . 4490 | . 2020 | . 0234 | . 0955 |
| 2.2 | .4275 | . 4145 | . 1720 | . 0145 | .1035 |
| 2.3 | . 3175 | . 3985 | . 1590 | . 0103 | . 105 |
| $? .5$ | . 3750 | .3630 | . 1355 | . 0023 | . 10 ? |
| 2.6 | -. 04.36 | .3540 | .1250 | -. 0013 | .1795 |
| 2.8 | -. 289 | . 3260 | .1736 | -. 0077 | . 1115 |
| 3.2 | -. 270 | .2780 | . 1777 | -. 0159 | . 1153 |
| 3.5 | -. 910 | .2460 | - 7 (6) 8 | $-.0183$ | .2175 |
| 3.9 | -. 0975 | . 21.00 | . 2140 | -. 0171 | . 1194 |

From plotting and scaling as described in section three, it was determined that .2 rad. backlash was tak en up at times irdicated in Table four correspondine to initial separation values of $\dot{\Theta}_{m c}$ as indicated. Values of $\theta_{m} \dot{\theta}_{m} \theta_{c} \dot{\theta}_{c}$ were obtained from substitution of values of $\dot{\theta}_{m o}$ and $t$, in equations (51) (52), (54) and (55). Values oi $\dot{\theta}_{0}$ were obtained from substituting values of $\dot{\theta}_{m}$ and $\dot{\theta}_{L}$ into equation (21).

$$
\text { Table } 4
$$

| $\dot{\theta}_{m 0}$ | $t$ | $\theta_{m}$ | $\dot{\theta}_{m}$ | $\theta_{L}$ | $\dot{\theta}_{L}$ | $\dot{\theta}_{0}$ recombined |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| .9 | 1.1 | 1.460 | $-.050 L_{1}$ | 1.660 | .372 | .1150 |
| .8 | 1.175 | 1.406 | -.0245 | 1.607 | .3135 | .0648 |
| .7 | 1.23 | 1.348 | -.1114 | 1.548 | .262 | .0379 |
| .5 | 1.32 | 1.288 | -.1304 | 1.489 | .208 | .0055 |
| .5 | 1.44 | 1.226 | -.1417 | 1.426 | .159 | -.0213 |
| .4 | 1.59 | 1.161 | -.1422 | 1.360 | .112 | -.0405 |
| .3 | 1.825 | 1.080 | -.1264 | 1.289 | .3697 | -.0331 |
| .2 | 2.35 | 1.013 | -.0812 | 1.213 | .0298 | -.0368 |
| .13 | 2.7 | -.0605 | -.0195 | 1.1436 | .0151 | -.2057 |

Iividing lines were plotted on the piase plane of the system shown in Fig. 9. Phase trajectories were formed by standard ohase plane technioues. The transient response of the system as obtained from the phase plane by methods outlined in Control System Synthesis by Truxal, pare 629, is shown in Fic. 10. Values of displacement versus time for transient response are listed in Taole five.

$l_{1, ~}$ (

Table 5

| $t$ | $\theta_{0}$ | $\theta_{m}$ |
| :---: | :---: | :---: |
| .258 | .026 |  |
| .520 | .1 |  |
| .759 | .2 |  |
| 1.133 | .4 |  |
| 1.472 | .6 |  |
| 1.805 | .8 |  |
| 2.221 | 1.0 |  |

$\theta_{L}$

$\theta_{m}$
.9929
 .9929 5.810 .986
.986
$6.012 .975^{\circ} .9747$
6.210
$.96{ }^{8} 1$
.9648
(.LIL)
.9635
.9564
6.61
.9619
.9491
2.2211 .0

Separation occurs at $\dot{\theta}_{10} .42$

| 2.321 | 1.0393 | 1.0393 |
| :--- | :--- | :--- |
| 2.421 | 1.075 | 1.075 |
| 2.621 | 1.134 | 1.14 |
| 3.021 | 1.202 | 1.242 |
| 3.221 | 1.211 | 1.282 |
| 3.421 | 1.206 | 1.315 |
| 3.721 | 1.177 | 1.358 |

Recombination occurs
$4.350 \quad 1.130$
$4.837 \quad 1.08$
5.5451 .01
5.611 .00

Separstion occurs at $\dot{\theta}_{10}-.074$

From Fig. 9 it may be seen that the system as described is stable and contains a residual steady state displacenent error $\leqslant$ the amount of the backlash. A steo displacement inout of 1.0 is seen to separate
into load and system wi thout load at $\theta_{0}=1, \dot{\theta}_{L 0}=.142$, at point A. The load is seen to follow a constant deceleration isocline of -.8 while the system without the load converges towards the stable focus. Recombination occurs when the system without load is at $B$ and the load is at $B^{\prime}$. The momentum balance causes the recombined trajectory to originate from point $C$. Separation reoccurs at point $D$ and recombination does not occur thereafter since the output measuring device is inside the $\perp$ imits of the backlash. The load comes to rest with -. 091 rad. residual error.

A sten displacement input oripinating at $\theta_{0}=/, \dot{\theta}_{c}=1$ is seen to separate at $\Theta_{0}=1$, recombine when the system without load is at $E$ and the load is at $E$. The recombined trajectory is seen to originate from $F$, reseparate at $G$, recombine at $H, H^{\prime}$, I and separate again at J where the outout detector is once again inside the backlash region so that further correction is impossible.

Fig. 10 describes the transient response to a steo displacement input of 1.0 rad. From the fipure it may be seen that system resonant frequency changes between combined and uncomrined systems, and that velocity changes abruptly upon recombination.

Fig. 11 is a plot of $\Theta_{m}, \theta_{L}$ at separation.
7. Case II. Output measured at motor shaft.

Load possesses no viscous friction.
Given system.
Combined

$$
\begin{equation*}
1 \ddot{\theta}_{0}+2 \dot{\theta}_{0}+\theta_{0}=1 \tag{56}
\end{equation*}
$$

System as closed loop with load separated.

$$
\begin{equation*}
.5 \ddot{\theta}_{m}+2 \dot{\theta}_{m}+\theta_{m}= \tag{57}
\end{equation*}
$$

Load separately

$$
\begin{equation*}
5 \ddot{\theta}_{R}=0 \tag{58}
\end{equation*}
$$

$$
3
$$

Backlash . 3 rad .
Isocline at which load separates from system is determined of the isocline equation for the load.

- $5 \ddot{\theta}_{2}$
$0 \dot{\theta}_{L}=0$

$$
\begin{equation*}
7_{3}=0 \tag{58}
\end{equation*}
$$

To find the isocline for the combined system, letting $\theta_{p}=1$

$$
\begin{equation*}
\eta,: r \cdot 2=\frac{1-\theta_{0}}{\dot{\theta}_{0}} \tag{59}
\end{equation*}
$$

Slopes of the isocline are given in Table six.


To find isoclines for the system with load separated
(50) $\eta_{2}+4=2\left(\frac{1-\theta_{m}}{\theta_{m}}\right)$

Slopes of the isocline are given in Table seven.

Table 7


To obtain dividing lines for $\theta_{m} \dot{\theta}_{m}, \theta_{L} \dot{\theta}_{L}, \theta_{0} \dot{\theta}_{c}$ at recombination,
(57) $\ddot{\theta} m+4 \dot{\theta_{m}}+2 \theta_{m}=z$

In Laplace form
(61) $\theta_{m}=\frac{2 / s+\dot{\theta}_{m 0}+5 \theta_{m c}+.4 \theta_{m o}}{\left(s^{2}+.4 s+2\right)}$

Roots of quadratic factor are $-.2 \pm j 1.4$

$$
(h 2) \cdot \theta_{m}=1+.715 \dot{\theta}_{m 0} e^{-.2 t} \sin (1.4 t-\phi)
$$

where $\varnothing=1$. $\varnothing 25^{\circ}$
$\left((6) \dot{\theta}_{m}=-.1430 \dot{E}_{n c} e^{-.2 t} \sin (1.4 t-\notin)\right.$

$$
+\dot{\theta}_{m 0} e^{--.2 t} \cos (1.4 t-\phi)
$$

For load separately
(64)
(65)

$$
\theta_{L}=\frac{\dot{\theta}_{L 0}}{5^{2}}+\frac{\theta_{L C}}{5}
$$

$$
\text { 连 } 2
$$

Substituting values of ( $t$ ) in factors of equation (62) results in value of Table eight.


When the terms of Table eight are plotted in accordance with equations (62) and (65), for $\dot{\theta}_{m 0}=1$ figure 12 is obtained.

From figure twelve, the times of recombination for .3 rad. back 1 ash are determined for various values of $\dot{\theta}_{m}$ as described in section 3, thus describing dividing lines for $\theta_{m} \dot{\theta}_{m}, \theta_{L} \dot{\theta}_{L}, \theta_{c} \dot{\theta}_{c}$ at recombination. Values determined are as listed in Table nine. Values of $\theta_{m} \dot{\Theta}_{m}, \theta_{k} \dot{\epsilon}_{2}$ were determined fro equations $(\leqslant 2)$, ( $\left.\ell 3\right)$ arid (65). $\dot{\theta}$ 。 values were determined from equation (2l).


The transient response as determined from the phase plans is tabulated in Table ten.:



Figure thirteen is a phase slane portrait of the system wen it is subjected to a step in put of . It rad. The dividine linos are shown to have particularly unusual confimurations for tris system characterized by no viscous friction in the load, and slipht dampine. The $\theta_{m} \dot{\theta}_{m}$ dividing line is seen to spiral into the stable focus much. like a trajectory. The $\theta_{0} \dot{\theta}_{0}$ dividing line for recombination is seen to consist
of a series of loons, the to s of wrich renre ont max mum nerey fed back from the load unor recombination, and the nottoms of wim renresent minimum encrey fed back. Tt is pl eromenon is atiributable to th e existence of a natural period for senaration wheh permits recomination velocities to add vectorially, increasing the displacarent of the syston imnediately after reconnination.

The phase trajectcry is seen to senarate load fro syster at moi t A. Fecombination occurs at point $H$, and the new $\theta_{0} \dot{\theta}_{0}$ is sear to tat DLice at $\bar{U}$, a point of relatively low eneryy feedback. The system separates arain at $I$ and recomoines aeain when $\theta m$ is at: the $\theta_{0} \Leftrightarrow$ for th is recumbination occurs at $\hat{H}$, apain a relatively low energy reeriback, and from this convergence, it is seen that $\theta_{0} \theta_{0}$ loons aopro ior tre stable focus, rakine the system ultimately staple. As in sase?, the outbut error of the system will be the magnitude of the backlash.

Firure fourteen Dictures the syster: beravior in rasponse to a sta innut of "not. tron tris it may be seen that the $\epsilon_{0} \hat{\theta}_{0}$ pos tion unon recmbiration is at the ton of the first l.000, renresenting a maximm feed-back of cnerey from the load. The load semarates frow the syster at $A$, recombines at $R$ to produce a $\theta_{0} \theta_{0}$ for the recombined sys om at $u$. The syster seoarates again at ard recombines when $\Theta_{m}$ is at , resuiting in a recombined $\theta_{0} \dot{\theta}_{c}$ at . is oefore, it may ro geer that the system is ultimately stable, however it tares the sy.ter a longer period to settle from a sten of . oat than from a ster nf. . I'.

Firure fifteen shows the system transient performance to a step of . If and . 9 á. Uncf apain, this illustrates the more osci latary perforriace induced in the systen by the srabler inout, when conditions are favorable for load momentun reinforcement unori recombination.
(A)

 Ty $5 \% \mathrm{~m}$..th Load Seprialed $.5 \ddot{C m}_{m}+2 \mathrm{C}_{m}+\epsilon_{m}=1$ La ad ${ }^{\prime} \ddot{\theta}_{k}=c \quad$. Back lash

$$
5 \ddot{e}_{1}=c
$$

Response \% step input. 16 Separation Line-Lood from fy stern: Velocity and Displacement of uncombined sysifern immediately, prior to recombination Velocity and Displacement of system at resembinalion, Lead drifting separately combined system. Trajectory system Trajectory with Load separated

* Dividing Lines an ivy partially divan to simislify picture


5


Combineá Sysiern
$1 \ddot{c}_{r}+\ddot{\theta_{c}}+\dot{e}_{i c}=1$
System with Loo' separated
$.5 \ddot{e}_{n}+\overrightarrow{\varepsilon_{r}} \dot{e}_{r r}+$ Err $_{r}=1$
Load $5 \ddot{\theta}$ - j Bucḱlash́
$.5 e_{2}=0$

Response to step input os
$\qquad$ Separation Line Lana' form System hisiscotinuricement of uncombined syr stem immeáively, prier te recombination ' cf $5 y$ st cm at recerrhination Lead drifting separately -...................ad ding Lines crim, partially drown to simplify picture

Combined System Trajectory Sister Trajectory witín Lad Separated


Case III. utput measured at load.
Load possesses no viscous friction. Inertia distribution is equalized between load and system without load.

Given system . 3 Backlash
Combined.
(bR) $/ \dot{\theta}_{0}+, R \dot{\theta}_{0}$ r' $\theta_{0}=1$
System as open loop with load separated.
(hl) $5 \dot{\theta}_{m}+2 \dot{\theta}_{m}=1-\theta_{L}$
Load separately
(68) $\quad 5 \dot{\theta}_{c}=0$

Slones for the isocline of the combined system are as tabulated in
Table six. Equation (40) and (4I) are applicable to system when oneratina in the backlash region, and reduce to

$$
(60) \theta_{m}=\frac{13.5 \dot{\theta}_{m 0}}{5^{2}}-\frac{5 \dot{\theta}_{m 0}}{5^{3}}-31.25 \dot{\theta}_{m 0}\left(\frac{1}{5}+\frac{1}{5+.4}\right)
$$

$$
+\frac{1-2 \dot{\theta_{m o}}}{5}
$$

(7))

$$
\begin{aligned}
\theta_{m}= & 13 \dot{5} \cdot \dot{\theta}_{m o t} t-2.5 \dot{\theta}_{m 0} t t^{2}-31.25\left(\dot{\theta}_{m c}\right)\left(1-e^{-.4 t}\right) \\
& t / 2 \dot{\theta}_{m o} \\
\dot{\theta}_{m}= & \frac{13.5 \dot{\theta}_{m o}}{5}-\frac{5 \dot{\theta}_{m 0}}{52}+\frac{31.255 \dot{\theta}_{m c}}{(5+.4)}
\end{aligned}
$$

(72)

$$
\dot{\theta}_{m}=13.5 \dot{\theta} \dot{\theta}_{m o}\left(1-e^{-.4 t)+\dot{\theta}_{m o} e^{-.4 t}}\right.
$$

- $5 \dot{\theta}_{m o} t$
(73) $\theta_{L}=\operatorname{\theta }_{20} t+\theta_{L O}$

Substituting values of ( $t$ ) in factors of equat ors (73) and ! 13 results in data tabulated in Table eleven.

Table 11


Thire sicter is a rrant. of vaıues tabulated in table elever. when scaled for various viluos of $\dot{\theta}_{\text {mo }}$ on the Severation dividing lines, vatue of ( $t$ ) are obiained as described in section 3, which are the times required for recombination to occur for a snecific $\dot{\theta}_{\text {moc }}$. These values are listed with corresponding vaiues of $\dot{\epsilon}_{m}$ in Tabie twelve, witr factors from equations(7)) and (72).

Tabl c 12

| $\dot{\theta}_{m_{0}}$ | $t$ | $e^{-.4 t}$ | $\left(1-e^{-.4 t}\right)$ | $-25 t^{2}$ | 13.57 | $\begin{aligned} & -3125 \\ & \left(1-e^{-47}\right) \end{aligned}$ | $-.2$ | $\theta \mathrm{O}_{-1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | . 98 | . 676 | . 324 | - 2.4 | 13.21 | -17.12 | -. ? | . 15 |
| . 5 | 1.27 | . 502 | .308 | $-4.245$ | 17.15 | $-12.1 .2$ | -. 2 | . 243 |
| . 4 | 1.37 | . 579 | . 1.22 | -4.77 | 18.5 | -1?.? | -. 2 | . ${ }^{1}$ |
| . 3 | 1.5? | . 544 | . 1.56 | - 5.77 | 23.52 | $-1 i 1.2 f$ | -. 2 | . 357 |
| . 2 | 1.75 | .1965 | . 5035 | $-7.6 .3$ | 23.6 | -15.71 | - | . 21 ? |
| . 1 | 2.25 | . 4.36 | - 59 4 | -12.6.8 | 30.4 | -18.55 | -.? | -.172 |
| $.0^{\circ}$ | 2.41 | .381 | . 519 | -14.5 | 32.:2 | $-0.37$ | -.? | $-.1 ?$ |
| .) ${ }^{4}$ | 2.68 | . 342 | . 658 | $-2.9$ | 36.? | -7).55 | -. ${ }^{\text {P }}$ | -. |
| . 24 | 3.1 | . 200 | . 710 | - ? 4.0 | $41 . ?$ | $-22.20$ | -.? | -. |
| . 02 | 4.7 | . 232 | . 798 | $-40.0$ | 54 | $-24.0$ | -.? | -. ${ }^{\text {? } 2}$ |

```
Table l2 continued
```

| $\dot{\theta}_{m o}$ | t | $\frac{13.5}{\left(1-e^{-. .4 t}\right)}$ | $-5 t$ | $\theta m$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | . 98 | 4.375 | -4.9 | . 151 |
| . 5 | 1.27 | 5.375 | -6. 35 | -. 1865 |
| . 4 | 1.37 | 5.7 | $-6.85$ | -. 22888 |
| . 3 | 1.52 | 6.16 | -7. 6 | -. 2698 |
| .? | 2.75 | \%.10 | -8. 75 | -. 2927 |
| . 1 | 2.25 | 8.01 | -11.25 | -. 2834 |
| . 28 | 2.41 | ?. 35 | -12.05 | -. 271 |
| . 06 | 2.48 | 8.88 | -13.4 | -. 2507 |
| . 01 | 3.1 | 9.58 | -15.5 | -. 225 |
| . 02 | 4.9 | 10.78 | -20.0 | -. 180 |

ro: equation (24) it is determined that corresponcing values of $\theta_{0}, \dot{\theta}_{0} \quad$ for recombination are as listed in Table thirteen.

Table 13

| $\dot{\theta}_{m 0}$ | $\dot{\theta}_{0}$ | $\theta_{0}$ |
| :---: | :---: | :---: |
| .1 | .5755 | .78 |
| .5 | .1567 | .535 |

.4 .0858 .459
.3 . 3156.396
. 2 -.0464 . 310
.1 -.0918 . 205
$.08 \quad-.0955 \quad .1769$
$.06-.0953 \quad .1488$
.04 -. 0927 . 116
$.02-.2803$. 276

Fipure seventeen is the phase plane blot of tre system wi th pertinent dividing lines.

It is seen that a step displacemert innut of 1.145 spirals into a limit cycle with a constant amnlitude of 1.030 while a sten disolacement of .057 diverges into the identical limit cycle.

When the system is oderating in a limit cycle, separation occurs at $\because$ recombination occurs when $\theta_{m} \dot{\theta}_{m}$ is at $B$ and $\theta_{L}, \dot{\theta}_{L}$ is at B.' The recombined phase trajectory originates from point $C$ on the $\theta_{0} \dot{\theta}_{0}$ dividing line and the system separates apain at $I$.

Applying stability criteria of section 4 yields the following: At separation, $\dot{\theta}_{0}=.45, \theta_{p}-\theta_{0}=.046$ in Limit cyche. iubstituting into equation (39), $C=.524^{2}$

Upon recombiration in limit cycle, $\dot{\theta}_{0}=.12, \theta_{p}-\theta_{0}=-.5$ ubstitutine into equation ( 38 ), $\quad C=.51$.


?. Case IV. output measured at load.
Load possesses no viscous friction and has sweater part of inertia.
Given system . 3 rad backlash
$(74) / \dot{\theta}_{0}+\cdot 2 \dot{\theta}_{0}+\theta_{0}=1$
System as open loon with: load separated
(75) . $2 \ddot{\theta}_{m}+.2 \dot{\theta}_{m}=1-\Theta_{2}$

Load separately
(76)

- $8 \ddot{\theta}_{2}=$ $\qquad$

Slopes of the isoclines of the combined system are as tabulated in Table six.
equations (LO) and (LI) are applicable to system when operating in the backlash region, and reduce to
(77) $\theta_{m}=-25 \dot{\theta}_{m 0} t^{2}+6 \dot{\theta}_{m o t}$

$$
-5 \dot{\theta}_{\operatorname{mo}}\left(1-c^{-t}\right)+1-\cdot 2 \dot{\theta}_{m o}
$$

$$
\begin{equation*}
\dot{\theta} m=-5 \dot{\theta}_{m 0} t+6 \dot{\theta_{m o}} \tag{78}
\end{equation*}
$$

(70)

$$
\text { - } 5 \dot{\theta}_{r o} e^{-t}
$$

$$
\theta_{L}=\theta_{\angle 0}+t+\theta_{\angle 0}
$$

substituting values of ( $t$ ) in factors of equations (77) and ( $7^{\circ}$ )
result: in data tabulated in Table fourteen.

Laule 14

| $t$ | $e^{-t}$ | e | $-2.5$ | 67 | (1-e | -. 2 | $\theta_{m}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 2 | . 819 | .191 | -. 11 | 1.2 |  | -. 2 | -. 005 |
| . 4 | . 670 | . 330 | - . 4 | 2.4 | -1.65 | -. 2 | .15 |
| . 6 | . 549 | . 451 | - .9 | 3.6 | $-2.26$ | -. 2 | .25 |
| 1.3 | . 368 | . 832 | - 2.5 | 6 | -3.16 | -. ? | .14 |
| 1.2 | . 302 | . 698 | - 3.6 | 7.2 | -3.49 | -. 2 | -. 09 |
| 1.4 | . 247 | . 753 | $-4.9$ | $\because \cdot 4$ | -3.76 | -. 2 | -. 46 |
| 1.6 | - 232 | . 798 | - 6.4 | 0.6 | -3.99 | -. 2 | -. 99 |
| . 9 | . 166 | . 934 | - 4.1 | 10.4 | -4.14 | -. 2 | -1.66 |
| 2.0 | . 136 | . $8 \mathrm{Cl}_{4}$ | -10.0 | 12.0 | -4.31 | -. 2 | --3.51 |
| 2.2 | . 111 | . 289 | -12.2 | 13.2 | $-1.44$ | -.? | $-3.64$ |
| 2.4 | . 291 | . 909 | $-14.4$ | 14.4 | -4.54 | -. 2 | $-4.74$ |
| 2.6 | . 274 | . 227 | -16.9 | 15.6 | $-4.13$ | -. 2 | - 6.13 |
| 2.4 | . 263 | . 239 | $-19.6$ | 16.9 | -4. 68 | -.? | -7.59 |
| 3.) | . 25 | . 950 | -22.5 | 14.7 | -1.75 | -. ? | -0.45 |
| 3.2 | . 241 | . 950 | -25.6 | $10 . ?$ | -4.79 | -.? | -21.39 |
| 3.4 | . 3335 | . 967 | $-2^{2} .9$ | 20.4 | $-4.43$ | -. 2 | -13.53 |
| 3.6 | . 027 | . 973 | -32.4 | 21.6 | $-4.86$ | -.? | $-1 \cdots .86$ |
| $3 . r$ | . 0323 | . 078 | -36.0 | 22.8 | -4.99 | -. 2 | $-15.29$ |
| 4.0 | . 21 M 4 | .98? | -40 | 24 | -1.90 | -. 2 | -21.1 |

Table lis conti nued


Fifure eighteen is a gramn of equations (77) anc (79) Iimes of recombination are obtained from anprodriate ordinste scaliry of ïir. I for vario:s valius of $\dot{\theta}$ mo as was done ir. Case ITT. Talues of ( $t$ ), correspending to $\dot{\theta}$ mo and factors of time solutions for equations (77) and (70) are listed in Table 15.


'rom equation (2 24 ) values of $\theta_{0}, \dot{\theta}_{0}$ are Listed in Table sixteen for recombination

| $\theta_{m 0}$ | $\theta_{0}$ | $\dot{\theta}_{0}$ |
| :--- | :---: | :---: |
| 1 | .78 | .162 |
| .8 | .68 | .5904 |
| .6 | .552 | .4085 |
| .4 | .424 | .2372 |
| .2 | .272 | .0756 |
| .1 | .175 | .0077 |
| .08 | .15 | -.004 |
| .06 | .1266 | -.01374 |
| .01 | .1092 | -.02364 |
| .02 | .1750 | -.02568 |

Q ure nineteen is the phase Dlane portrait oi the system. The mai-
$\therefore$. $e$ of the limit cycle in this case is 1.15 ) wich considerine the neference in inertia distribution between cases III and If, is not anpreciably areater than the $\mathbf{I} .030$ marnitude of case III.

1 steo displacement inrut of . 69 is seen to spiral into the 1 i ir cyclo while a step of .435 is seen to spiral out toward the same limjt cycle.
when the system is perfornine 3 limit cycle, it is sef to sevarate at $A$. The backlash of .3 rad. is not taken us unt 1 the sy. tem wh th Inad removed has reached 'a and the load has reached 3'. Recombi atior occurs and the recombined trajuctory oriminates from $C$. iemaration reoccurs on the separation dividing line of point $\therefore$.

Applying the stability criteria of section 4 to the limit cycle of case IV yields the following.

For equation (38), the applicable constants are
$K=1$
$F=.2$
$J=1$
At separation $\theta_{0}=.1495$

$$
\theta_{R}-\theta_{0}=.095
$$

Substituting in equation (38)
(80)

$$
\begin{aligned}
& c=\sqrt{(.495)^{2}-(.2)(.795)(.095)+(.095)^{2}} \\
& =.556
\end{aligned}
$$

son recombination,

$$
\begin{aligned}
\dot{\theta}_{0} & =.315 \\
\theta_{R}-\theta_{0} & =-.1 .85
\end{aligned}
$$

Substituting in equation (38)
$(81) \quad c=\sqrt{\frac{(.315)^{2}-(5)(-.485)+(-.485)^{2}}{1-\frac{.2^{2}}{4}}}$

$=$.618, which represents only fris agreement.
$\epsilon$
Case IV
c....... during reparation

$+2$

10. Case V. Outout measured at Load.

Load possesses viscous friction. Inertia equally divided between load and system with load separated.

Given system . 3 rad. backlash.
(By) / $\ddot{\theta}_{0}+.4 \dot{\theta}_{0}+\theta_{0}=1$
System as open loop with load separated
(83) $.5 \dot{\theta}_{m}+.1 \dot{\theta}_{m}=1-\Theta_{L}$

Load separately
(PL) $.5 \ddot{\theta}_{L}+.3 \dot{\theta}_{L}=0$
To obtain isoclines for the combined system.
$(85) 72+.4=\frac{\left(1-\theta_{0}\right)}{\dot{\theta}_{0}}$
(1.5) $\Theta_{0}=1-\dot{\theta}_{0}(N,+.4)$
computed values of arctan $\frac{\dot{\theta}_{0}}{1-\Theta_{0}}$ for various values of $N$, are as listed
in Table seventeen.

## Table 17



To obtain the isocline of the system without load, wire separation of the load from the system occurs, solve for $\ddot{\theta}_{L} / \dot{\theta}_{L}$ (37) $\quad \ddot{\theta}_{L} / \dot{\theta}_{L}=7_{3}=-.6$

At instant of separation,
(88)

$$
\frac{\ddot{\theta}_{m}}{\dot{\theta}_{m}}+\cdot 2=\frac{2\left(1-\theta_{2}\right)}{\dot{\theta}_{m}}
$$

(89) $\eta_{2}+\cdot 2=\frac{2\left(1-\theta_{m 0}\right)}{\theta_{m}}$
(90) $\frac{\dot{\theta}_{m 0}}{1-\theta_{m r}}=\frac{2}{\eta_{2}+.2}$

Letting N2 $=$ NO $=-.6$
(01) arctan $\frac{\dot{\theta}_{m 0}}{\left(1-\theta_{m 0}\right)}=72.7^{\circ} \quad \phi=78.7^{\circ}$

Equation (40) reduces to
(92) $\theta_{m}=1+1263 \dot{\theta}_{m 0}-140 \dot{\theta}_{m 0} e^{-.2 t}$

$$
t 13.9 \dot{\theta} m_{0} e^{-.6 t}-13.66 \dot{\theta}_{m o t}
$$

Equation (LI) reduces to
193) $\dot{\theta}_{m}=-18.66 \dot{\theta}_{m 0}+28 \dot{\theta}_{m_{0}} e^{-.2 T}$

$$
-8.33 \quad \dot{\sin } \mathrm{~m}^{-.6 t}
$$

Figure 20 is a plot of $\theta_{m} \theta_{L}$ during separation. Times of recombination are obtained from this plot as described in section three. Table eightteen is a tabulation of equations (92) and (03)

Tabse 1 K


Tabie 1 cor tinued


To obtain the $\Theta_{L} \dot{\theta}_{\alpha}$ dividing line, equation (2?) reciuces to (04) $\theta_{L}=1+2.2 \dot{\theta}_{m_{0}}+1.66 \dot{\theta}_{m 0}\left(1-e^{-.6 t}\right)$ Values of equation (9L) are tabulated in mable nineteen

Table 19



Table 20 continued

| . | $28 e^{-.2 t}$ | $-8.33 e^{-.6 t}-18.66$ | 0.97 |  |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 22.59 | -4.37 | -18.66 | -.14 |
| .8 | 22.2 | -4.18 | -18.66 | -.511 |
| .6 | 21.65 | -3.88 | -18.66 | -.533 |
| .4 | 20.92 | $-3.486-18.66$ | -.49 |  |
| .2 | 19.5 | $-2.812-18.66$ | -.384 |  |
| .1 | 17.51 | $-2.03-18.66$ | -.319 |  |
| .08 | 14.81 | $-1.805-18.66$ | -.292 |  |
| .06 | 15.85 | $-1.506-18.66$ | -.259 |  |
| .04 | 14.47 | -1.15 | -18.66 | -.2133 |
| .03 | 13.35 | $-.909-18.66$ | -.1865 |  |


kquation (24) yields values for $\dot{\theta}_{0}$ upon recombiration, as tabuls of in lable twenty-one.

| Er po $_{1}$ | $\theta_{0}$ |
| :--- | ---: |
| .8 | .0425 |
| .6 | -.055 |
| .4 | -.1265 |
| .2 | -.16125 |
| .1 | -.1582 |
| .08 | -.1468 |
| .06 | -.12413 |
| .04 | -.1039 |
| .03 | -.0916 |

Fig. 21 is the phase plane portrait of $\theta_{0} \dot{\theta}_{0}, \theta_{L} \dot{\theta}_{L}$
It is seen that in response to a stem input of.$f ?$, the system separates
at Point $A$, recombines when $C_{2} \dot{O}_{2}$ is at point 3 , and once the momentum balance has been satisfied, $D_{0}$ for the recombined system originates from point $C$, reseparating at point D. This trajectory spirals in towards a limit cycle of mamitode .5 ? while a step input of .23 is seen to spiral outwards into the same limit cycle described by points E, F, G and H .

Case V
Om $\theta_{1} \quad$ during separation
 $\dot{\theta}_{r a c}=1$
$\bar{\epsilon}_{1} \ldots,+\cdots$ $\qquad$ -

11. Case VI.
(output measured at load.
Load possesses viscous friction. Inertia equally divided between load and system with. load separated
riven system
(95) / $\ddot{\theta}_{0}+4 \dot{\theta}_{0}+\theta_{c}=1$

System as open loop with load separated. . 3 rad backlash
(0, $.5 \ddot{\theta}_{m}+.3 \dot{\theta}_{m}=1-\theta_{L}$
Load separately
(07) $\quad 5 \ddot{\theta}_{L}+. / \dot{\theta}_{L}=0$

Isocline for the combined system are identical with those of case V ,
listed in Table seventeen.
To obtain the isocline of the system without load, where separation of the load from system occurs, solve for
(9\%)

$$
\frac{\ddot{\theta}_{L}}{\dot{\theta}_{L}}=7 P_{3}=-2
$$

At the instant of separation
(30) $\frac{\ddot{\theta}_{m}}{\dot{\theta}_{m}}+\cdot \sigma=\frac{2\left(1-\theta_{m o}\right)}{\dot{\theta}_{m 0}}$
(1.00)

$$
P_{2}+\cdot 6=2\left(\frac{\left.1-\theta m_{0}\right)}{\dot{\theta}_{m}}\right.
$$

(101) $\frac{\dot{\theta}_{m 0}}{1-\hat{\theta}_{m 0}}=\frac{z}{\partial_{2}-6}$

Letting $N_{2}=\sqrt{3}=-.2$
$\arctan \frac{\dot{\theta}_{m c}}{-\theta_{m 0}}=78.7^{\circ} \varnothing=101.3^{\circ}$
Lquation 40 reduces to
(102) $\theta_{m}=-16 \dot{\theta}_{m 0} t+111.455 \dot{\theta}_{m 0}+13345 \dot{\theta}_{m c} e^{-6 t}$ - 125 Amos $e^{-.2 t}+1$

Equation 4 reduces to
(103) $\dot{\theta}_{m}=-16 \dot{\theta}_{m 0}-8 \dot{\theta}_{m 0} e^{-.6 t}+25 \dot{\theta}_{m 0} e^{-.2 t}$ Table twenty-two ists values of $\theta_{m} \dot{\theta}_{m}$ versus ( $t$ )

able 22 continued


To ont in the $\theta_{L} \dot{\theta}_{L}$ nuvidine lino, e un, ion (22) reduces to (104) $\theta_{L}=\frac{1-.2 \dot{\theta}_{m 0}}{5}+\frac{\dot{\theta}_{m 0}}{5(5+. z)}$
(105) $\theta_{L}=1-2 \dot{\theta}_{m o}+5 \dot{\theta_{m o}}\left(1-e^{-.2 t}\right)$ Values of $\theta_{L}$ versus ( $t$ ) are tabulated in Table 23

Table 23


Figure 22 is a graph of equations (102) and (105). By appropriate ordinate scaling as described in section three, recombination times mere computed as listed in Table 24 , for corresponding values of $\dot{\theta}_{m o}$ Values of $\theta_{m} \dot{\theta}_{m} \theta_{L} \dot{\theta}_{L}$ for dividing lines re listed in Table 2/4.

## Table 24



## Table 24 continued

| $\dot{\theta}_{\text {mo }}$ | $25 e^{-.2 t}$ | $-8 e^{-.6 t}$ | -16 | $\dot{\theta}_{\text {m }}$ |
| :--- | :--- | :--- | :--- | :--- |
| 1 | 20.08 | -4.135 | -16 | -.055 |
| .8 | 10.8 | -3.04 | -16 | -.112 |
| .6 | 19.35 | -3.73 | -16 | -.222 |
| .14 | 19.8 | -3.41 | -16 | -.2144 |
| .2 | 17.2 | -2.67 | -15 | -.28 |
| .1 | 15.4 | -1.87 | -16 | -.247 |
| .08 | 11.75 | -1.65 | -16 | -.232 |
| .26 | 13.2 | -1.175 | -16 | -.238 |
| .24 | 12.32 | -.965 | -16 | -.151 |
| .03 | 11.3 | -.0735 | -16 | -.113 |


| $\dot{\theta}$ | - | $\dot{\theta}_{\alpha}$ | $\theta_{4}-1$ |
| :---: | :---: | :---: | :---: |
| 1 | 1.1 | . 803 | . 785 |
| . $R$ | 1.175 | . 632 | . 675 |
| . 5 | 1.275 | . 465 | . 555 |
| . 4 | 1.425 | . 30 | . 1.1555 |
| . 2 | 1.875 | . 1375 | . 272 |
| .1 | 2.425 | .0816 | . 172 |
| . 08 | 2.64 | .047 | . 14.9 |
| . 25 | 3.2 | .0316 | . 1292 |
| . 34 | 3.525 | . 01975 | . 0933 |
| . 03 | 3.98 | . 01352 | . 0761 |

$\therefore$ ppiying equation (2 $2 L_{+}$) to satisfy the orinciple of conservation of momentum yields the following values of $\dot{\theta}_{0}$ unon recombination comresvonding to values of $\dot{\theta}_{n \rightarrow 0}$ uxon separaticn, listed in Table twenty-five.

$$
\text { Table } 25
$$

| $\dot{\theta}_{\text {mo }}$ | $\dot{\theta}_{0}$ |
| :---: | :---: |
| 1 | . .374 |
| .8 | .260 |
| .6 | .1216 |
| .4 | .029 |
| .2 | -.07125 |
| .1 | -.0027 |
| .08 | -.10325 |
| .06 | -.0656 |
| .04 | -.0497 |

Fifure 23 is the phase plane presentation of $\theta_{0}, \dot{\theta}_{B}, \theta_{L}, \dot{\theta}_{L}$ as a combined system and when operating in the backlash region.

A step input of .55 is seen to separate the load from the system at
A. Recombination takes place when the load is at $B$, and the balancing of momentum between the load and the system without load, causes $\theta_{0}$, $\dot{\theta}_{0}$ of the recombined system to originate from point C. Reseparation occurs at point $D$ and the trajectory is seen to spiral into a limit cycle defined by points $E, F, G$ and $H$.

1 step input of .13 is seen to spiral outward to the same limit cycle of magnitude . 550



## 2?. case VII

Output measured at load.
Load possesses viscous friction and the treater part of the system inertia.
?even system
(106) $/ \ddot{\theta}_{0}+\cdot 4 \dot{\theta}_{0}+\theta_{0}=1$

System as oven loop witt load separated. . 3 rad. backlash
(107) $\cdot 2 \dot{\theta}_{m}+.1 \dot{\theta}_{m}=1-\theta_{L}$

Load separately
$(10 H) .8 \dot{\theta}_{L}+.3 \dot{\theta}_{L}=0$
isocline of the combined system are identical with those of case $V$, listed in Table seventeen.
do obtain the isocline of the system without load, where separation of the load from the system occurs, solve for
(100) $\frac{\dot{\theta}_{L}}{\dot{\theta}_{L}}=72_{3}=-3 / 8$

At the instant of separation,
(110)

$$
\frac{\dot{\theta}_{m}}{\dot{\theta}_{m}}+.5=\frac{5\left(1-\theta_{m o}\right)}{\dot{\theta}_{m 0}}
$$

(111) $72+.5=5 \frac{(1-\theta m o)}{\dot{\theta}_{m 0}}$
(112) $\frac{\dot{\theta} \text { mo }}{1-\theta_{m 0}}=\frac{5}{7 / 2+.5}$
wetting $N_{2}=N_{3}=-.375$
(113) arctan $\frac{\dot{\theta}_{m 0}}{/-\theta_{m 0}} \arctan 5 \frac{50.56^{\circ} \phi=9.01 .4^{\circ}}{-375+.5}$

Equation ( 40 ) reduces to
(114)

$$
\begin{aligned}
\theta_{m}= & -26.4 \dot{\theta}_{m 0} t+158.475 \dot{\theta}_{m 0} e^{-.5 t} \\
& +125.975 \dot{\theta}_{m 0}-284.475 \dot{\theta}_{m 0} e^{-.375 t}+1
\end{aligned}
$$

Equation (il) reduces to
(115)

$$
\begin{aligned}
\dot{\theta} m=-26.4 & \dot{\theta}_{m o}-79.238 \dot{\theta}_{m o} e^{-.5 t} \\
& +106.638 \dot{\theta}_{m o} e^{-.375 t}
\end{aligned}
$$

Equation 22 reduces to
(115) $\theta_{2}=1 \cdot 025-\dot{\theta}_{m o}+2.668 \dot{\theta}_{m 0}\left(1-\epsilon^{-.375 t}\right.$ equation 23 reduces to
(117) $\dot{\theta}_{L}=\dot{\theta}_{\text {mo }} e^{-.375 t}$

Table twenty-six lists values of equations (114), (115), (11') and (11' for various values of ( $t$ )

Equations (114) and (114) are plotted on Fir. 24 and various times of recombination obtained from the ordinate scaling method described in section 3.


Table continued

| $t$ | $-79.238$ | $106.638$ | -26.4 | -m |
| :---: | :---: | :---: | :---: | :---: |
| . 2 | -71.65 | 08.05 | -2f. 4 | . 90 |
| . 4 | $-44.9$ | 91.9 | -25.4 | . 60 |
| . 6 | -58.58 | 25.1 | $-26.14$ | . 02 |
| . 8 | -53.05 | 79.0 | $-25.4$ | -. 45 |
| 1.0 | $-4{ }^{2} .05$ | 73.4 | $-26.16$ | $-1.05$ |
| 1.2 | $-43.53$ | 68.0 | -26.4 | -1.93 |
| 1.4 | -39.4 | 63.1 | $-2+.4$ | -2.7 |
| 1.6 | -35.62 | 5 L .5 | $-26.4$ | -3.52 |
| 1.8 | -32.2 | 54.45 | $-26.4$ | $-4.15$ |
| 2.0 | -?9.18 | 50.119 | $-26.4$ | -5.09 |
| 2.2 | $-2 ? .4$ | 46.65 | $-25.4$ | -5.15 |
| 2.4 | -23.93 | 43.40 | $-26.4$ | - 6.93 |
| ?. $!$ | -21.56 | 140.35 | -? 2.4 | -7.61 |
| 2.8 | -17.58 | 37.35 | $-26.4$ | -2. 53 |
| 3.0 | $-17.76$ | 34.62 | $-26.4$ | $-9.54$ |
| 3.2 | -16.) | 32.2 | $-25.4$ | -10.2 |
| 3.4 | -14.5 | 29. ${ }^{\text {a }}$ | $-2 F .4$ | -11.1 |
| 3.6 | -13.16 | 27.7 | $-27.4$ | -11. 86 |
| 3.9 | -11.9 | 25.7 | $-2 f .4$ | $-17 \cdot 6$ |
| 4.0 | -17. 79 | 23.9 | $-26.4$ | $-13.29$ |

Table 26 continued

| t | $\begin{gathered} \left(1-e^{-.375 t}\right)\left(1-e^{-.375 t}\right)-.025 \\ .072 \\ .192 \end{gathered}$ |  |  | $\theta_{L}-1$ .167 | $\dot{\theta}_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| . 4 | . 139 | . 371 | -. 025 | . 34.6 | . 961 |
| . 5 | . 201 | . 536 | -. 0225 | . 511 | . 799 |
| . 8 | . 259 | . 6905 | -. 025 | . 6655 | . 741 |
| 1.0 | . 312 | . 8325 | -. 025 | . 8075 | . 688 |
| 1.2 | . 342 | . 9125 | -. 025 | . 2875 | . 638 |
| 1.1 | . 408 | 1.09 | -. 025 | 1.065 | . 592 |
| . | . 451 | 1.205 | -. 025 | 1.180 | .549 |
| 2.9 | .490 | 1.31 | -. 025 | 1.285 | . 510 |
| 〕.) | . 527 | 1.41 | -. 025 | 1.385 | .473 |
| ?. 2 | . 542 | 1.1449 | -. 025 | 1.424 | . 438 |
| 3.1 | . 593 | 1.592 | -. 025 | 1.557 | . 107 |
| 2. 6 | . 622 | 1.66 | -. 025 | 1.635 | . 378 |
| ? 0 | . 650 | 1.735 | -. 025 | 1.710 | . 350 |
| 3.7 | .575 | 1.80 | -. 025 | 1.775 | .325 |
| $3 . ?$ | .498 | 1.362 | -. 025 | 1.837 | . 302 |
| 3.4 | . 720 | 1.92 | -. 025 | 1.895 | . 280 |
| 3, 6 | .740 | 1.975 | -. 025 | 1.950 | . 260 |
| 3.9 | . 759 | 2.022 | -. 025 | 1.997 | . 241 |
| 4.0 | . 775 | 2.075 | -. 025 | 2.050 | . 224 |

Values of $\theta_{m} \dot{\theta}_{m}, \theta_{L} \dot{\theta}_{L}$ for recombination dividing lines are listed in table twenty-seven, for values of $(t)$ obtained from fig. 24.

Table 27


## Table 27 continued



| $\dot{\theta}_{1}$ | $t$ .525 | $\left(i-e^{-.}\right.$ | $\begin{aligned} & 2.66 \\ & 1-e^{2} \\ & .1 .75 \end{aligned}$ | -.023 <br> -. 025 | $\begin{aligned} & \theta_{L}-1 \\ & .4503 \end{aligned}$ | $\dot{\theta}_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 8 | . 587 | . 197 | . 525 | -. 025 | . 1.100 | . 5425 |
| . 5 | . 587 | .22-7 | . 205 | -. 225 | . 31.8 | .1.64 |
| . 4 | . 862 | . 276 | . 736 | -. 0225 | . 2342 | .2293 |
| . 2 | 1.25 | .374 | . 906 | -. 025 | .1942 | . 125,2 |
| . 1 | 1.70 | . 171 | 1.250 | -. 025 | . 1734 | . 2529 |
| . 08 | 1.875 | . 504 | 1.343 | -. 025 | . 1053 | . 0397 |
| . 7 ( | 2.115 | . 547 | 1.46 | -. 025 | . 056 | . $) 272$ |
| . 04 | 2.465 | . 603 | 1.11 | -. 025 | . 3635 | . 11590 |
| . 02 | 3.275 | . 706 | 1.885 | -. 025 | . 0372 | . 205880 |

Equation 24 yields values for $\dot{\theta}_{0}$ recorbination line as ollows:

| $\dot{\theta}_{m 0}$ | $\dot{\theta}_{0}$ |
| :--- | :---: |
| 1 | .085 |
| .2 | .5252 |
| .5 | .371 |
| .4 | . .1553 |
| .2 | -.0151 |
| .1 | -.03425 |
| .08 | -.04 .02 |
| .06 | -.04529 |
| .04 | -.03755 |

Fimure 25 is the whase nane portrait of $\theta_{0} \dot{\theta}_{c} \theta_{L} \dot{E}_{L}$ for the combined and separated system. A sten inout of .59 is seen to semarate load froin system at point:A. The system recomoines when the load has decelerated to point $B$, and satisfaction of the momentum balance arain causes $\theta_{0} \dot{\theta}_{0}$ to originate fron point C. Besenaration occurs at point $D$ and the trajectory is seen to converge to the limit cycle described by points E, $V^{\prime}, C_{7}$ and 11. A step inout of .09 is seen to spiral outward to the same limit cycle of magnitude . 250 .


13. Case VIII

Output measured at load
Load possesses greater part of systems inertia but lesser part of system friction.

Given system.
(118) $/ \dot{\theta}_{0}+.4 \dot{\theta}_{0}+\theta_{0}=1$

System as open loop with load separated. . 3 rad. backlash
(119) $\cdot 2 \dot{\theta}_{m}+\cdot 3 \dot{\theta}_{m}=1-\theta_{2}$

Load separately
(120) $\cdot 8 \ddot{\theta}_{L}+\cdot / \dot{\theta}_{L}=0$

Isoclines of the combined system are identical with those of case $V$, listed in Table seventeen.

Io obtain the isocline of the system without load, where separation of the load from the system occurs, solve for
(121) $\frac{\ddot{\theta}_{L}}{\dot{\theta}_{L}}=7 / 3=-1 / 8$

At the instant of separation
(122) $\frac{\ddot{\theta}_{m}}{\dot{\theta}_{m}}+1.5=\frac{5\left(1-\theta_{m 0}\right)}{\dot{\theta}_{m 0}}$
(123) $72_{2}+1.5=\frac{5(1-\theta \mathrm{mo})}{\dot{\theta} \mathrm{mo}}$
(124) $\frac{\dot{\theta}_{m 0}}{1-\theta_{m 0}}=\frac{5}{72+1.5}$

Letting $\mathrm{N}_{2}=\mathrm{N}_{3}=-.125$
(125) $\arctan \frac{\dot{\theta}_{\mathrm{m}}}{1-\theta_{m 0}}=\arctan \frac{5}{1.375}=74.6^{\circ} \phi=105.4^{\circ}$

Equation (1,0) reduces to
(226)

$$
\begin{aligned}
& \text { on (10) reduces to } \\
& \theta_{m}=-25.75 \dot{\theta}_{m 0 t}+231.581 \dot{\theta}_{m 0}+1 \\
&+1.564 \dot{\theta}_{m 0} e^{-1.5 t}-233.42 e^{-.125 t} \\
& 86
\end{aligned}
$$

Equation (4]) reduces to
(127) $\dot{\theta}_{m}=-2.34 \dot{\theta}_{m 0} e^{-1.5 t}+29.08 \dot{\theta}_{m 0} e^{-.125 t}$ - 25.74 家

Equation (22) reduces to
(125) $\theta_{L}=1-.275 \dot{\theta}_{m 0}+8 \dot{\theta}_{m o}\left(1-e^{-.125 t}\right)$

Equation (23) reduces to
(129) $\dot{\theta}_{L}=\dot{\theta}_{\text {mo }}$

Table twenty-nine is a tabulation of values of equations (126),(128), and (129) for various values of ( $t$ ).

Figure (26) is a graph of equations (126) and (128). Suitable scaling of the ordinate yields recombination times for various values of $\dot{\theta}_{m 0}$ as listed in Table thirty.
Equation (2/) yields values of $\dot{\theta}_{0}$ upon recombination, as listed in Table thirtu-one.

| $t$ |  | $e^{-.125 t}$ |  | ble 29 231.581 | $\begin{aligned} & -23342 \\ & e^{-.125 t} \end{aligned}$ | -25.74t |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 2 | . 741 | . 9753 | 1.157 | 231.581 | -228 | - 5.15 | -. 1172 |
| . 4 | . 549 | . 951 | . 858 | 231.581 | -222 | $-10.3$ | . 139 |
| . 6 | .407 | . 9278 | . 635 | 231.541 | -21f.1 | $-15.43$ | . 687 |
| . 8 | . 372 | . 905 | . $472 \%$ | $231.5 \% 2$ | -211.3 | $-20.5$ | . 1535 |
| 1.9 | . 272 | . 883 | . 31.75 | 231.541 | -206.0 | -25.74 | . 1885 |
| 1.2 | .146 | . 861 | . 260 | 231.581 | $-201.2$ | -30.86 | -. 219 |
| 1.4 | . 1225 | . 24 | . 192 | 231.541 | -196.0 | $-3 \leftarrow .0$ | -. 227 |
| 1.6 | . 091 | . 219 | . 14222 | 231.591 | -191.0 | $-11.15$ | -. 27 |
| 1.8 | . 267 | . 7987 | . 105 | $231.5^{51}$ | -19t.2 | $-46.3$ | -. 011 |
| 2.0 | . 05 | . 779 | . 0783 | 231.581 | $-181.6$ | -51.5 | $-1.14$ |
| $? .2$ | . 237 | . 760 | . 058 | $231.5^{9} 1$ | -177.? | -56.55 | -?.11 |
| 2.14 | . 2273 | . 741 | . 2427 | 231.581 | $-173$ | - 61.75 | -3.13 |
| 2.6 | .0203 | . 7227 | . 0318 | 231.591 | $-158.6$ | - 4.2 | $-3 .-27$ |
| 2.8 | . 215 | . 705 | . 02345 | 231.581 | $-164.6$ | $-72.0$ | $-.0$ |
| 3.7 | . 011 | . 488 | . $017 ?$ | 231.581 | .160 .3 | $-77.25$ | -5.95? |
| 3.2 | . 7191 | .670 | - 21422 | 231.591 | $-154.3$ | -82.4 | $-7.125$ |
| 3.4 | . $00 \times 1$ | . 654 | . 00955 | 231.581 | -152.6 | $-87.5$ | -5.510 |
| 3.5 | . 20145 | . 538 | . 00705 | $231.5 \times 1$ | $-149.9$ | - -2.6 | -6). 31 ? |
| 3.9 | . 2034 | . 627 | - 105325 | $231.5^{29}$ | $-1145.1$ | -27.4 | -11.314 |
| 1.3 | .0)25 | .607 | . 10392 | 231.582 | $-141.5$ | -103 | -1?.915 |

Table 29 continued


## Table 30



Table 30 continued



Table 31

| $\dot{\theta}_{m o}$ | $\dot{\theta}_{0}$ |
| :---: | :---: |
| 1 | . |
| .8 | .3588 |
| .4 | .3217 |
| .4 | .250 |
| .2 | .182 |
| .1 | .1027 |
| .08 | .0561 |
| .06 | .0356 |
| .04 | .0259 |
| .02 | .0131 |

Fipure 27 is the phase portrait of $\theta_{0} \dot{\theta}_{0}, \theta_{L} \dot{\theta}_{L}$ comvined and separated.

A step input of .265 is seen to separate tre load from the system at point A. A recombination occurs when the load has driftec to point B. $\theta_{0} \dot{\theta}_{0}$ for the recombined syster originate from point C. Kesedaration occurs at point $D$ and the trajectory is seen to converge toward the limit cycle described by E, F, F, H. A steo of . $n^{2}$ is seen to diverge toward the same limit cycle of magnitude . L66.



1L. Conclusions.

By way of recapitulation, the eipht considered cases were governed by the following equations.

Case I

$$
\begin{aligned}
& \ddot{\theta}_{0}+8 \dot{\theta}_{0}+\theta_{0}=1 \\
& 6 \ddot{\theta}_{m}+48 \dot{\theta}_{m}+\theta_{m}=1 \\
& .4 \ddot{\theta}_{L}+.32 \dot{\theta}_{L}=0
\end{aligned}
$$

Case II

$$
\begin{aligned}
& \ddot{\theta}_{0}+2 \dot{\theta}_{0}+\theta_{0}=1 \\
& -5 \ddot{\theta}_{m}+.2 \dot{\theta}_{m}+\theta_{m}=1 \\
& 5 \ddot{\theta}_{2}=0
\end{aligned}
$$

Limit cycle None

None
1.030
.52
.550
1.150
.250
$\cdot 2 \ddot{\theta}_{m}+\cdot 1 \dot{\theta}_{m}=1-\theta_{L}$
. $8 \ddot{\theta}_{L}+.3 \dot{\theta}_{L}=0$

$$
\begin{aligned}
\dot{\theta}_{0}+.4 \dot{\theta}_{0}+\theta_{0} & =1 \\
\cdot 2 \ddot{\theta}_{m}+.3 \dot{\theta}_{m} & =1-\theta_{L} \\
\cdot 8 \ddot{\theta}_{L}+.1 \dot{\theta}_{L} & =0
\end{aligned}
$$

Systems were grouped in the above order sn that similar inertia ratios could be inspected as a group.

When backlash is outside the feedback loon, the system is uitimately stable and does not limit cycle, however it is subject to a residual steady state error in response to a step input, which error may be as great as the magnitude of the backlash.
weer backlash is enclosed in the feed back loon, a limit cycle will invariably result providing the system possesses viscous friction only. The majority of cases considered exhibited $\theta_{L} \dot{\theta}_{L}, \theta_{0} \dot{\theta}_{0}$ lines for recombination which approached the origin from slopes of opposite sign, as in the following example.


Ficuro (23)

It was originally believed that this difference in slopes was a necessary condition for the existence of a limit cycle and that the minimum value of the $\theta_{0} \ddot{\theta}_{0}$. Line was an indication of the marnitude of the limit cycle. iveither of these is the case. The limit cycle was always found to occur outside of the minimum value of the $\theta_{0} \dot{\theta}_{0}$ line. Case VIII showed the following confifuration for the $\theta_{L} \dot{\theta}_{L}, \theta_{0} \dot{\theta}_{0}$ lines in the neighborhood of the origin.

Mis is not inconsistent with oreviously considered cases and is simply an indication that the decrease in momentum for both load and open loop system are equal for decreasing values of $\dot{\theta}$ mo

The position of the system without the load, at recombination, for various initial separation values of $\dot{\theta}_{m o}$ possesses a characteristic downward bow as in the followinf:


Figure (30)

It was or finally believe that there rich t oe some correlation between the existence and ramitude of the limit cycle and the position of minimum value of the $\theta_{m} \dot{\theta}_{m}$ recombination line, however no correlation was found to exist.

The shape and location of this $\theta_{m} \dot{\theta}_{m}$ line varied from system to system with varying frictional and inertia distributions, however it al ways possessed a minimum value and its most characteristic feature was the 10 ward hook in the third quadrant which corresponded to small initial $\dot{\theta}$ mo values for the separated trajectories.

Of most significance are the graphs shown as fy mes (31) and (32). Figure ( 31 ) is a plot of the magnitude of the limit cycle versus the $\frac{\text { Friction of Load }}{\text { Friction of System }}$ ratio
when (a) the inertia of the load $\left(J_{L}\right)$ was equal to the inertia of the system without the load ( $\mathrm{i}_{\mathrm{m}}$ ) and when (b), $\int_{L}=4 \mathrm{Jm}$ The steen upward slope of the magnitude of the limit cycle with decreaseing $F_{L} / F_{0}$ values is of interest, clearly demonstrating that the chance in frictional effects in the load is much more pronounced when the load possesses the greater part of the inertia. This is illustrated by the fact that the $J_{L}=4 \mathrm{Jm}$ line crosses the $V_{L}=J_{m}$ line at low values of $F_{L} / F_{0}$
Figure 3? is a not of the marnituie of the Limit cycle versus the SLUm ratio for three different frictional distributions. This graph presents the same information as figure (31) but in a different manner. frost noteworthy is the charge in slope of the $\frac{\mathrm{J} / \mathrm{Jm} \text { maglim lye }}{\text { line with change in }}$ frictional distribution. when the load possess. th e reater part of the friction, an increase ir the $\mathrm{J} / \mathrm{L} / \mathrm{m}$ ratio $\operatorname{decr}$ as es the limit cycle
hore s en the lo d uoscesses consinormble less demine than the systrm, $n$ incroase in the momiturie of tho limit cycle is tho result of incronsinf the $J_{I} / J_{T a}$ ratio.

The ovorall system carming is seen to have the most effect in movinई the $\frac{\mathrm{J} / \mathrm{Jm}}{\text { ma\%/im. cyc. curves upward or downard while the individual } F_{I} / F_{m} \text { ratios }}$ determine the slope of those curves.

From this, the following generalizations for design may be made for the purpose of minimizing the magnitude of the limit cycle:

If the systern damping is small, $2 f \omega_{n} \leqslant .2$, the $J_{\mathrm{L}} / J_{m}$ ratio
snould be made as small as possible when backlash is included inside the foodback loop.

If the system damping is somerhat greater, $2 \zeta \omega_{n} \geqslant .4$, the $J_{L} / J_{m}$ ratio should be made large, and a further reduction in the limit cycle may be achieved by placing the preponderance of damping in the load. This is quite ieasible, and would be the case when a tachoneter used in feedback is attached directly to the output shaft.


Horo s in en the loc poscessos consinorobly less danpine than the system, $n$ increase in the magniturie of tho limit cucle is tho result of incressine the $J_{V} / J_{\text {ra }}$ ratio.

The overall system carming is seen to have the most effect in moving
 determine the slope of those curves.

From this, the following generalizations for design may be made for the purpose of minimizing the magnitude of the limit cycle:

If the system damping is small, $2 \mathfrak{h}_{\mathrm{h}} \mathrm{n} \leqslant .2$, the $J_{\mathrm{L}} / J_{\mathrm{m}}$ ratio snould be made as small as possible when backlash is included inside the foodback loop.

If the systom damping is somowhat greater, $2 f \omega_{n} \geqslant .4$, the $J_{L} / J_{m}$ ratio should bo made large, and a further reduction in the limit cycle may be achieved by placing the preponderance of damping in the load. This is quite feasible, and would be the case when a tachometer used in feedback is attached directly to the output shaft.



