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THESIS

A CONTINUOUS REVIEW INVENTORY ORDERING POLICY
WITH AN EXPEDITED SHIPPING TIME OPTION

by

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ABSTRACT

A continuous review policy for ordering inventory stockage items with the option of expediting the shipping time is formulated. Demand is assumed to have a Poisson distribution with a stationary demand rate. Inventory holding costs, ordering costs, shortage costs and expediting costs are postulated. The measure of effectiveness is the minimization of a linear combination of these costs. The optimal policy is determined analytically through the use of first differences. An iterative computational procedure is recommended for obtaining the optimal order quantity, reorder point and expediting level. Analysis of the first differences indicates the conditions under which there is a solution and a simple numerical test for these conditions is developed. A numerical example is given.

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I. INTRODUCTION

This thesis proposes a model of a continuous review policy for ordering inventory stock items in which the inventory manager has the option of expediting the shipping time of any particular order. He is able to exercise this option at any point during the procurement leadtime until the order is actually shipped. Thus the model divides procurement leadtime into two parts. The first is the production leadtime, and the second is the shipping leadtime. The circumstance which will trigger the expediting decision is that demand during the production leadtime has exceeded a specified amount. The reason for expediting, of course, is to protect against backorders.

The decision to expedite an order is quite common in actual operations. One simple example is that of a parts clerk in a local body and fender repair shop picking up his telephone and asking his supplier to ship by bus instead of by rail. A slightly more sophisticated example is the use of urgency-of-need codes by requisitioners in the armed forces under the Uniform Military Issue Priority System. In the latter case, a requisitioner located overseas can choose among codes which impose delivery deadlines on the national inventory control points which may vary from one week to 11 weeks. For some overseas locations, only a shipment by air will satisfy the one week requirement.

Common though the expediting situation is, most of the analytic models which deal with expediting are of the periodic review variety. A literature search conducted in July, 1969 at the document branch of the Army Logistics Management Center, Fort Lee, Virginia and an informal search conducted at The Army Inventory Research Office in Philadelphia yielded only one analytic, continuous review, expediting model. This is the model developed by Allen and D'Esopo [1].

There is a closely related continuous review model developed by Morey [2]. This model treats the problem of protecting against backorders by supplementing the routine sealift deliveries with emergency airlift of additional special deliveries.

It was the work done by Allen and D'Esopo which served as a basis for the model developed in this thesis. The two major differences between the two models will be explained before detailed formulation and analysis is begun.

The first difference is in the treatment of one of the cost parameters. Allen and D'Esopo caused the unit price of each item in an expedited order to be increased in order to reflect the fact that the cost of expediting would partly depend upon the size of the order. However, they failed to consider this increase in unit price when they determined inventory holding costs. Under the present forulation, the dependence of the expediting cost upon the order size is accounted for by incrementing the cost of transporting each item in an expedited order. In both models the mathematical analysis involving this parameter is the same. The difference is conceptual.

The second, and basic, difference is in the treatment of procurement leadtime. In their model a decision to expedite will result in the order being delivered a fixed length of time after the decision is made. Under the present formulation, a decision to expedite will result in the fast shipment mode of transportation being used after the production phase of the procurement leadtime is completed. Thus two completely different leadtime situations are being modeled.

II. THE MODEL

This section treats the formulation of the model considering in turn the ordering policy, the assumptions and costs, and the cost equation.

A. THE ORDERING POLICY

The following ordering policy is considered:

When the on hand inventory level is reduced to r, order an amount Q. If during the production leadtime inventory is further reduced to a level X, called the expediting level, then expedite the outstanding order by specifying the fast means of transportation.

This will yield a three parameter continuous review policy (Q,r,X) in which the decision variables are the order quantity, the reorder point and the expediting level. The measure of effectiveness will be to minimize the average annual variable cost. The cost expression will be a linear combination of ordering costs, inventory holding costs, expediting costs, and shortage costs.

B. ASSUMPTIONS

1. Demands have a Poisson distribution with a stationary demand rate.

2. Procurement leadtime, a two-valued random variable, is composed of:

   a. T_p = Production leadtime -- a constant, and either

b. T_R = Fast shipping time -- a constant,

or

c. T_L = Slow shipping time -- a constant.

3. No more than one order is outstanding at any time.

4. When an order arrives, it is sufficient to raise the on hand inventory level above r.

5. Both r and X are non-negative.

Assumption 3 is necessary in order to make the model mathematically tractable. If more than one order can be outstanding, the question of which of these orders should be the one expedited arises. Also the behavior of the on hand inventory level becomes quite difficult to describe. This assumption is valid if the reorder quantity is much larger than the expected demand during the procurement leadtime.

Assumption 4 is concomitant with Assumption 3. If it were not made, it would be possible that a replenishment would not raise the inventory level above r. Then the on hand inventory would never again be reduced to r -- being always below r -- and no more reorders would be made.

Assumption 5 is made only for mathematical simplicity. It should be noted, however, that current military inventory management practices will not allow the level of service implied by a negative reorder point.

C. COSTS

A = Ordering cost.

I = Holding cost rate per dollar cost of each item per year.

C = Unit cost.

A' = Increment to order cost for expediting.

α = Increment to order cost for each unit expedited.

π = Cost for each backorder.

K = Average annual variable cost.

D. NOTATION

$p(z,T)$ = Prob(Z=z) where Z has a Poisson distribution with parameter λT.

$P(z,T) = \text{Prob}(Z \geq z) = \sum_{j=z}^{\infty} p(j,T)$.

λ = Mean annual demand rate.

Q = Order quantity.

r = Reorder level.

X = Expediting level.

θ = Expected number of orders per year = λ/Q.

W = Expected number of orders which are expedited.

U = Expected number of units which are expedited = WQ.

S = Shortages per cycle.

E(S) = Expected shortages per cycle.

T_s = "Slow" cycle length = $T_p + T_L$.

Y_j = Demand during time period T_j, e.g., Y_p = demand during production leadtime T_p.

E. DEVELOPMENT OF THE COST EQUATION

1. Inventory Holding Cost

The holding cost is proportional to the units of stock held per unit time, i.e., it is proportional to the area under the on hand inventory curve calculated over a one year time period. Note that this area is numerically equal to the average inventory level for a year.

Given that no expediting has occurred, i.e., given that $Y_p < r-X$, the conditional expectation for Y_p is

$$E(Y_p | Y_p < r-X) = y_p' = (1/Prob(Y_p < r-X)) \sum_{j=0}^{r-X-1} j \; Prob(Y_p = j).$$

Figure 1 is a typical illustration of the behavior of the on hand inventory level over a cycle in which no expediting has occurred. The safety level, s, is $(r - y_p' - \lambda T_L)$. The area for one cycle is

$$1/2[(Q+s) + s] \; [(Q+s)/\lambda + T_p + T_L],$$

which can be written as

$$[Q/2 + r - (y_p'+\lambda T_L)] \; [(Q-y_p')/\lambda + T_p].$$

Multiply this area by λ/Q to get the average inventory

$$[Q/2 + r - (y_p'+\lambda T_L)] \; [Q - y_p' + \lambda T_p]/Q.$$

Given that expediting has occurred, i.e., given that $Y_p \geq r - X$, the conditional expectation for Y_p is

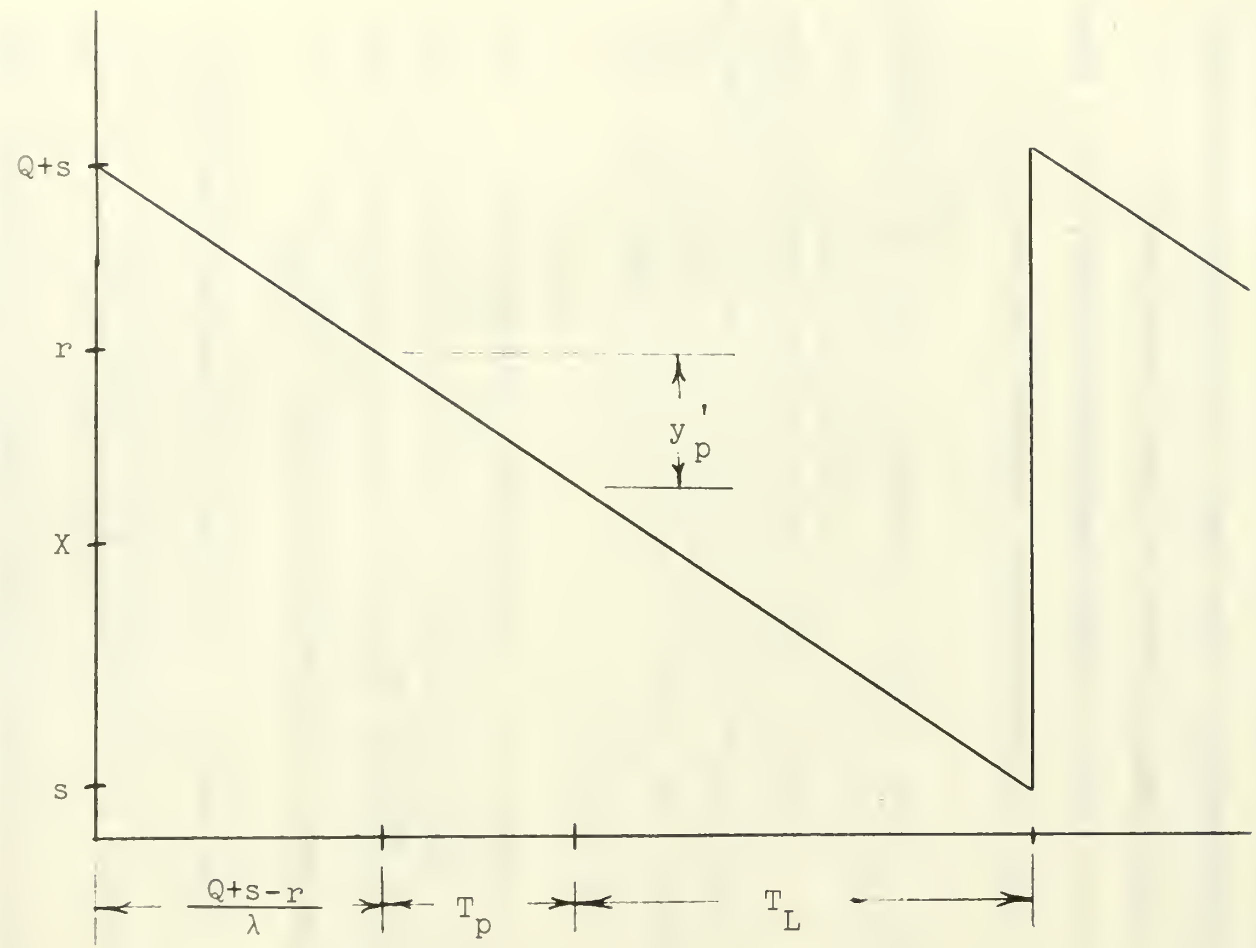

Figure 1. Typical Cycle in Which There is (No) Expediting.

$$E(Y_p | Y_p \geq r-X) = y_p'' = (1/\text{Prob}(Y_p \geq r-X)) \sum_{j=r-X}^{\infty} j\,\text{Prob}(Y_p = j).$$

Figure 2 is a typical illustration of the on hand inventory level over a cycle in which expediting has occurred. The safety level is $(r - y_p'' - \lambda T_R)$. The area for one cycle is

$$[Q/2 + r - (y_p''+\lambda T_R)]\ [(Q-y_p'')/\lambda + T_p],$$

and the average inventory level is

$$[Q/2 + r - (y_p''+\lambda T_R)]\ [(Q - y_p'' + \lambda T_p)/Q].$$

Define k' and k" as follows:

$$k' = \text{Prob}(Y_p < r-X-1)/\text{Prob}(Y_p < r-X);$$

$$k'' = \text{Prob}(Y_p \geq r-X-1)/\text{Prob}(Y_p \geq r-X).$$

As demand is Poisson with parameter λT, y_p' equals $k'\lambda T_p$, and y_p'' is equal to $k''\lambda T_p$.

Therefore, the expected value of the average inventory is

$$[Q/2 + r - \lambda(k'T_p+T_L)]\ [Q - \lambda T_p(k'-1)]\ \text{Prob}(Y_p < r-X)/Q$$

$$+\ [Q/2 + r - \lambda(k''T_p+T_R)][Q - \lambda T_p(k''-1)]\ \text{Prob}(Y_p \geq r-X)/Q,$$

which equals

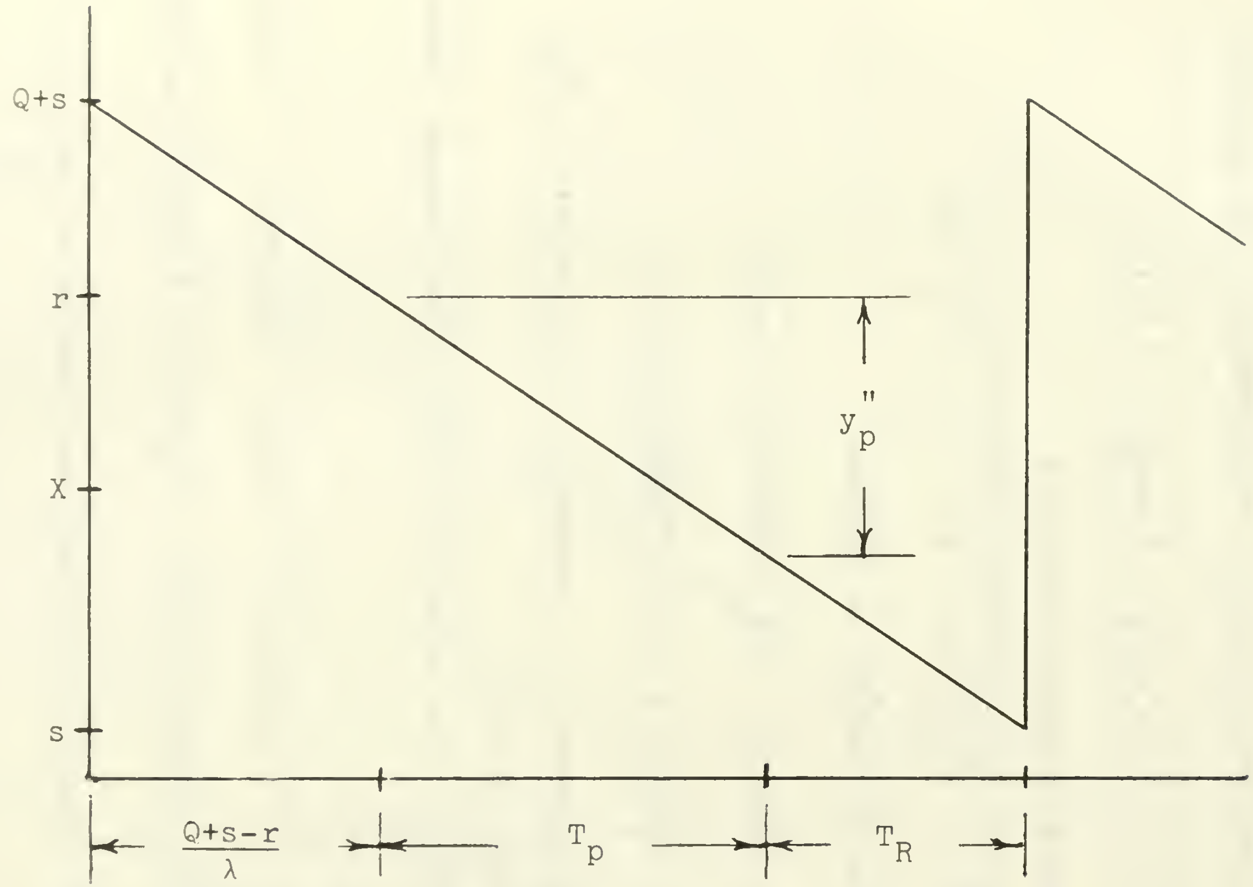

Figure 2. Typical Cycle in Which There is Expediting.

$$Q/2 + r - \lambda T_p - \lambda T_L + \lambda(T_L - T_R)\mathrm{Prob}(Y_p \geq r-X)$$

$$+ (\lambda T_p/Q)(-\lambda T_L + \lambda(T_L - T_R)\mathrm{Prob}(Y_p \geq r-X)$$

$$- (\lambda T_p)^2/Q + (\lambda T_L \lambda T_p/Q)\mathrm{Prob}(Y_p > r-X-1)$$

$$+ (\lambda T_R \lambda T_p/Q)\mathrm{Prob}(Y_p \geq r-X-1)$$

$$+ (\lambda T_p)^2 k'\mathrm{Prob}(Y_p < r-X-1)/Q$$

$$+ (\lambda T_p)^2 k''\mathrm{Prob}(Y_p \geq r-X-1)/Q.$$

This expression can be simplified by adding and subtracting the following four terms:

$$\lambda T_L \lambda T_p \mathrm{Prob}(Y_p = r-X-1)/Q, \quad -\lambda T_R \lambda T_p \mathrm{Prob}(Y_p = r-X-1)/Q,$$

$$(\lambda T_p)^2 k'\mathrm{Prob}(Y_p = r-X-1)/Q, \quad -(\lambda T_p)^2 k''\mathrm{Prob}(Y_p = r-X-1),$$

to obtain as the expected average inventory level the following expression:

$$Q/2 + r - \lambda T_s + \lambda(T_L - T_R)\mathrm{Prob}(Y_p \geq r-X)$$

$$+ (r-X)\lambda[T_p(k''-k') - T_L + T_R]\ \mathrm{Prob}(Y_p = r-X)/Q.$$

The average annual inventory holding cost is obtained by multiplying the expected average inventory level by IC.

2. Shortage Costs

Shortages can occur whether or not expediting has occurred. Given that there was no expediting, i.e., given that demand $Y_p < r-X$, then the quantity short, or backordered, will be equal to $(Y_p + Y_L - r)$ if r is less than the total quantity demanded, and it will be equal to zero otherwise. In this case, the conditional expectation for shortages is given by:

$$E(S|Y_p < r-X) = \sum_{z=r-y}^{\infty} (z+y-r)\ p(z,T_L).$$

Given that there was expediting, the quantity short will be equal to $(Y_p + Y_R - r)$ if r is less than the total quantity demanded, and it will be equal to zero otherwise. Considering first the situation in which demand Y_p is greater than or equal to (r-X) but less than r, the conditional expectation for shortages is given by:

$$E(S|r-X \leq Y_p < r) = \sum_{z=r-y}^{\infty} (z+y-r)\ p(z,T_R).$$

If the total quantity demanded is greater than or equal to r, the conditional expectation for shortages is given by:

$$E(S|Y_p \geq r) = \sum_{z=0}^{\infty} (z+y-r)\ p(z,T_R).$$

To find the expected number of shortages per cycle, multiply the conditional expectations for shortages by their respective probabilities:

$$E(S) = \sum_{y=0}^{r-X-1} \sum_{z=r-y}^{\infty} (z+y-r)\, p(z,T_L)\, p(y,T_p)$$

$$+ \sum_{y=r-X}^{r} \sum_{z=r-y}^{\infty} (z+y-r)\, p(z,T_R)\, p(y,T_p)$$

$$+ \sum_{y=r+1}^{\infty} (\lambda T_R+y-r)\, p(y,T_p).$$

The expected annual shortage cost is found by obtaining the product of the expected number of shortages per cycle, $E(S)$, the expected number of cycles per year, θ, and the cost per shortage, π, that is,

$$E(\text{Shortage Cost}) = \pi E(S)\theta.$$

3. Other Cost Elements and the Cost Equation

The other elements of the cost equation can be found in a straight-forward manner. The expected number of orders per year, θ, is equal to λ/Q. The expected number of orders expedited per year, W, is equal to θ $\text{Prob}(Y_p \geq r-X)$, and the expected number of items expedited per year, U, is equal to WQ. So the expected annual cost of placing orders is $A\theta$, and the expected annual cost of expediting is $(A'W + \alpha U)$. Therefore, the expression for average, annual, variable cost, K, is

$$K = \theta A + \pi E(S)\theta + A'W + \alpha U$$

$$+ IC(Q/2 + r - \lambda T_s + \lambda(T_L-T_R)\ \text{Prob}(Y_p \geq r-X)$$

$$+ IC\lambda(r-X)[T_p(k''-k') - T_L+T_R]\ \text{Prob}(Y_p = r-X)/Q.$$

III. SOLUTION AND ANALYSIS

A. SOLUTION OF THE COST EQUATION

The average, annual cost, K, is a function of three variables: Q, r, and X. This relation can be written as $K = K(Q,r,X)$. This function is not continuous, nor is it necessarily convex; therefore, none of the minimization techniques of differential calculus are applicable. However, the method of first differences can be used. When using this method, either "backward" or "forward" differences can be used. It is convenient to use a "backward" difference for Q because the result is directly analagous to that of Hadley and Whitin [3]. On the other hand, it is convenient to use a "forward" difference for r because this will facilitate computer program coding. For the following discussion, it will be assumed that $K(Q,r,X)$ has only a global minimum. A graphical illustration of when such an assumption is appropriate is given in Figure 3 (page 27).

Define $\Delta K(Q)$, the first "backward" difference for Q as follows:

$$\Delta K(Q) = K(Q,r,X) - K(Q-1,r,X).$$

If, for a given r and X, a local minimum does exist, then at that minimum $\Delta K(Q)$ must be less than or equal to zero. Thus an optimal value for Q, say Q^*, is the largest value of Q such that $\Delta K(Q)$ is less than or equal to zero. Therefore, Q^* is the largest value of Q such that:

$$Q(Q-1) < (2\lambda/IC)\ [A + \pi E(S) + A'\ P(r-X,T_p)$$

$$+ IC(r-X)(T_p(k''-k') - T_L+T_R)\ p(r-X,T_p)]. \quad (i)$$

Similarly, define $\Delta K(r)$, the first "forward" difference for r as follows:

$$\Delta K(r) = K(Q,r+1,X) - K(Q,r,X).$$

If, for a given Q and X, a local minimum does exist, then $\Delta K(r)$ must be greater than or equal to zero at that minimum. Thus an optimal value for r, say r^*, would be the smallest value of r such that $\Delta K(r)$ is greater than or equal to zero. Therefore, r^* will be the smallest value of r such that the following inequality holds:

$$(\lambda/Q)\left\{\pi\left[\sum_{y=0}^{r-X-1}((r-y)p(r-y+1,T_L) - \lambda T_L p(r-y,T_L))p(y,T_p)\right.\right.$$

$$+ (\lambda T_L p(X+1,T_L)+(\lambda T_L-(X+1))P(X+2,T_L))p(r-X,T_p)$$

$$- (\lambda T_R p(X,T_R) + (\lambda T_R-(X))P(X+1,T_R))p(r-X,T_p)$$

$$+ \sum_{y=r-X+1}^{r}((r-y)p(r-y+1,T_R) - \lambda T_R p(r-y,T_R))p(y,T_p)$$

$$- p(r+1,T_p) - IC(r-X+1)T_L p(r-X+1,T_p)$$

$$+ IC(r-X+1)T_R p(r-X+1,T_p)$$

$$+ IC(r-X)T_L p(r-X,T_p) - IC(r-X)T_R p(r-X,T_p)$$

$$+ \frac{IC(r+1-X)^2(p(r+1-X,T_p))^2}{\lambda P(r+1-X,T_p)(1-P(r+1-X,T_p))}$$

$$\left.\left. - \frac{IC(r-X)^2(p(r-X,T_p))^2}{\lambda P(r-X,T_p)(1-P(r-X,T_p))}\right] -A'p(r-X,T_p)\right\}$$

$$+ IC(1 - \lambda(T_L-T_R)p(r-X,T_p))-\alpha\lambda p(r-X,T_p) \geq 0. \quad \text{(ii)}$$

The first difference for X could be taken, and then a possible method of solution would be to use an iterative procedure to find values of Q, r, and X which would satisfy the first difference requirements. This method is not recommended as there is no guarantee of convergence.

A better method is to specify values for X, and then use an iterative procedure to find values of Q and r which will satisfy the first difference requirements for the given X. Then choose that set of values, $\{Q^*(X), r^*(X), X^*\}$, which gives the minimum, average, annual cost. The next section is a discussion of the conditions under which a solution may be expected.

B. ANALYSIS OF THE FIRST DIFFERENCE INEQUALITIES

For any given X, the optimal values of Q and r must satisfy both of the first difference inequalities. If these inequalities were approximated by being taken as equations, they would describe a pair of curves in the Qr plane. Then Q^* and r^* would lie at an intersection of these two curves. This approach will be taken in the following analysis of the first differences.

1. The First Difference For Q

Define f(r), g(r), and h(r) as follows:

$$f(r) = \pi E(S),$$

$$g(r) = IC(r-X)(T_p(k''-k')-T_L T_R)\ p(r-X,T_p),$$

$$h(r) = A'P(r-X,T_p).$$

Then Q^* is the largest value of Q such that

$$Q(Q-1) < (2\lambda/IC)(A+f(r)+g(r)+h(r)).$$

For large values of Q^*, say $Q^* > 20$, Q^* may be approximated by

$$Q^* = \sqrt{(2\lambda/IC)(A+f(r)+g(r)+h(r))}\ . \qquad \text{(iii)}$$

Treating Q^* as continuous, for the moment, the above expression describes a curve in the Qr plane. The graph of this curve is sketched in Figure 3 on page 27, and is labeled (iii).

The behavior of Q^* as r varies from X to infinity can be derived as follows:

a. $$\lim_{r\to\infty} f(r) < \lim_{r\to\infty} \sum_{y=0}^{\infty} \sum_{z=r-y}^{\infty} (z+y-r)p(z,T_L)p(y,T_p).$$

This is true because the left-hand side of the above inequality is the expected number of shortages annually when some of the leadtimes have been shortened through expediting, and the right-hand side is the expected number of shortages when no expediting has occurred in any cycle.

The limit of the right-hand side is equal to zero; therefore, the limit of f(r) is less than or equal to zero. But no term in f(r) is ever negative; so it must be that the limit of f(r) as r tends to infinity is greater than or equal to zero. The only way that both inequalities can be satisfied is that the limit of f(r) as r tends to infinity is zero.

b. $$\lim_{r\to\infty} g(r) = \frac{IC \lim_{r\to\infty} (r-X)^2(p(r-X,T_p))^2}{P(r-X,T_p)(1-P(r-X,T_p))}.$$

Let $j = r-X$, and let $u = \lambda T_p$. When u is less than or equal to one, the above limit is easily seen to be zero. When u is greater than one, the numerator is of the form

$$j^2 u^{2j} e^{-2u}/j!,$$

and the denominator is

$$\left(\sum_{z=0}^{j} u^z e^{-u}/z!\right)\left(\sum_{z=j}^{\infty} u^z e^{-u}/z!\right) j! \; .$$

Now, as r gets large, j will get large, but

$$\lim_{j\to\infty} j^2 u^{2j} e^{-2u}/j! = 0,$$

and

$$\lim_{j\to\infty} \left(\sum_{z=0}^{j} u^z e^{-u}/z!\right)\left(\sum_{z=j}^{\infty} j! u^z e^{-u}/z!\right) = \infty.$$

Therefore, the limit of g(r) as r tends to infinity is zero.

c. $\lim\limits_{r\to\infty} h(r) = \lim\limits_{r\to\infty} A' \; P(r-X,T_p) = 0.$

d. If r is equal to X, f(r) is the expected number of shortages when the procurement leadtime is always $T_p + T_R$, say it is z'(X). Also g(r) equals zero and h(r) equals A'.

So when r gets large, Q^* tends to $\sqrt{(2\lambda A/IC}$, the Wilson Q, and when r equals X, Q^* equals $\sqrt{(2\lambda/IC)(A + \pi z'(X) + A')}$. Call this latter value $\hat{Q}$. These results agree with those of Hadley and Whitin ([3], p. 170).

2. <u>The First Difference For r</u>.

In order to facilitate examining the behavior of r^* as Q varies over its range, several approximations will be made. First, the following terms will be eliminated from the cost equation:

a. $\sum_{y=r+1}^{\infty} (\lambda T_R + y - r) p(y, T_p)$.

b. $IC(r-X) \, [T_p(k''-k') - T_L + T_R] \, p(r-X, T_p)$.

The first of the above expressions is the expected number of shortages in a cycle in which the demand during production leadtime exceeded r. This value should be relatively small. The second of the above expressions is the product of several terms with the point probability that the demand during production leadtime is exactly equal to (r-X). This product also should be relatively small. Using this approximate cost equation, the first difference for r is

$$
\begin{aligned}
(\lambda/Q) \Bigg\{ \pi \Bigg[& \sum_{y=0}^{r-X-1} ((r-y)p(r-y+1,T_L) - \lambda T_L p(r-y,T_L))p(y,T_p) \\
& + [\lambda T_L p(X+1,T_L) + (\lambda T_L - (X+1))P(X+2,T_L)] \, p(r-X,T_p) \\
& - [\lambda T_R p(X,T_R) + (\lambda T_R - X)P(X+1,T_R)] \, p(r-X,T_p) \\
& + \sum_{y=r-X+1}^{r} ((r-y)p(r-y+1,T_R) - \lambda T_R p(r-y,T_R))p(y,T_p) \\
& + \lambda T_R p(r+1,T_p) \Bigg] - A' p(r-X,T_p) \Bigg\} \\
& + IC(1 - \lambda(T_L - T_R)p(r-X,T_p)) - \alpha\lambda p(r-X,T_p) > 0.
\end{aligned}
$$

Once again the terms which are products involving the point probability that demand during production leadtime is exactly equal to (r-X) will be ignored. The first difference for r then becomes

$$(\lambda/Q)\Bigg\{\pi\Bigg[-\sum_{y=0}^{r-X-1} p(r-y+1,T_L)p(y,T_p) \quad \sum_{y=r-X}^{r} p(r-y+1,T_R)p(y,T_p)$$

$$+\sum_{y=X+2}^{\infty}(P(y,T_L)-P(y,T_R))p(r-X,T_p)+\lambda T_R p(r+1,T_p)\Bigg]$$

$$-A'p(r-X,T_p)\Bigg\}$$

$$+IC-\alpha\lambda p(r-X,T_p)>0.$$

Assuming for the moment that r^* is continuous, the above inequality can be considered to be an equality. Then, for any given X, it also describes a curve in the Qr plane. This curve is sketched in Figure 3 and is labeled (iv).

Now note that the first summation in the above expression is the joint probability that exactly one backorder occurs in a cycle and no expediting occurs, and the second summation is the joint probability that exactly one backorder occurs and that there was expediting. Therefore the sum of these two summations is the probability that exactly one backorder occurs in a cycle. Denote this probability as S(r). Ignoring the last two terms within the braces -- both of these should be quite small -- the equation becomes

$$(\lambda/Q)(-\pi S(r) - A'p(r-X,T_p)) + IC - \alpha\lambda p(r-X,T_p) = 0,$$

which can be written as

$$ICQ/\lambda = S(r) + [(A'+\alpha Q)/\pi]\ p(r-X,T_p). \qquad \text{(iv)}$$

Examination of (iv) reveals that it should behave somewhat like a probability mass function rather than a cumulative distribution function.

Both $S(r)$ and $p(r-X,T_p)$ are probabilities. As r gets very large, both of these probabilities must go to zero. Thus the right-hand side of the above equation goes to zero as r gets large. The only variable on the left-hand side is Q; therefore, as r gets large, Q must go to zero.

On the other hand, if $r = X$, the equation becomes

$$ICQ/\lambda = S(X) + [(A' + \alpha Q)/\pi]\ p(0,T_p),$$

which, upon solving for Q, becomes

$$Q = [\pi\lambda S(X) + A'p(0,T_p)]/[IC - \alpha\lambda p(0,T_p)] = \hat{Q}'.$$

3. Convergence of the Solution

Figure 3 is an illustration of the relation between Q and r for a given X. The optimal values of Q and r must satisfy both curves. Therefore, if the parameters of the problem are such that $\hat{Q}'$ is greater than $\hat{Q}$, Q^* and r^* will be found at the intersection of the two curves. But, if $\hat{Q}'$ is less than $\hat{Q}$, nothing can be said about Q^* and r^*.

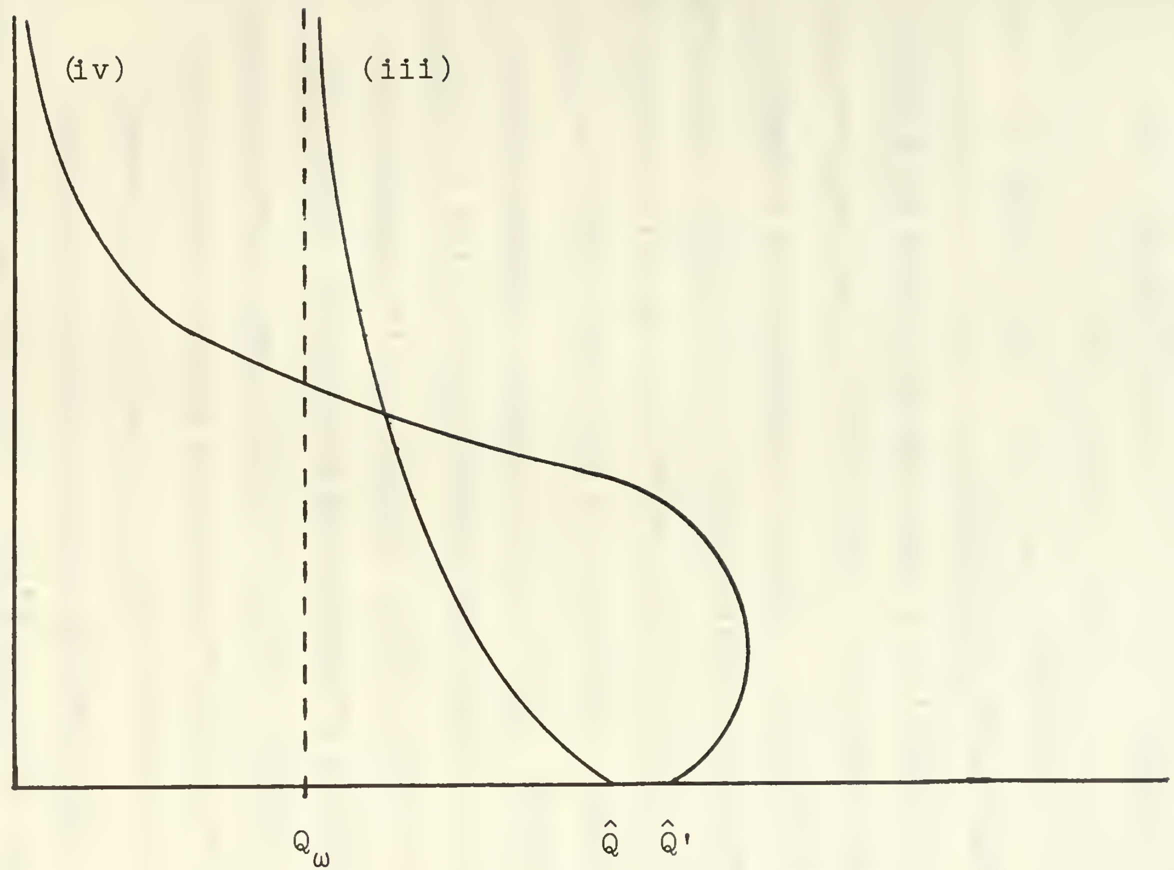

Figure 3. Graph of the First Difference Inequalities for the Optimal Order Quantity and Reorder Point.

Depending upon how far to the right the "bulge" in the curve labeled (iv) goes, either multiple solutions will exist or no solutions will exist. The only way to determine which condition obtains in a specific situation is to plot both curves.

C. A NUMERICAL EXAMPLE

1. Computational Procedure

The computational procedure was coded in FORTRAN for an IBM 360 computer. The procedure used was the one recommended previously. That is, rather than attempting to solve three first differences simultaneously, the first differences for Q and r were solved for specific values of X. The procedure consisted of a two-stage search across the cost function. In the first stage, a coarse search grid was used to locate the neighborhood of the minimum cost. In the second stage, the grid of the search was refined within this neighborhood in order to locate the minimum cost exactly. For the first stage search, values of X were chosen which were equally spaced from zero to three standard deviations above the mean of the demand distribution. The spacing (or grid) was one standard deviation. The values of $Q^*(X)$, $r^*(X)$ and the annual cost were calculated at each X. As soon as the minimum cost obtained during the first stage was found, the second stage began. The grid of the search was refined so that values of $Q^*(X)$, $r^*(X)$ and the annual cost were calculated for each X within one standard deviation above and below the first stage minimum. From this second set of calculations was identified the minimum cost and the corresponding values of $\{Q^*(X), r^*(X), X^*\}$.

For any particular X, the admissable Q values were constrained to be the integers between the Wilson Q and five times the demand rate. Admissable values of r were constrained to be the integers between the X specified and a value equal to the demand rate plus three standard deviations. The rationale for these bounds is fairly simple. Reasonable order quantities should be less than five times the annual demand rate; however, they should also be greater than the Wilson Q -- the amount which would be ordered if the model were deterministic. The difference between the order quantity and the lower bound is a rough indication of the degree of uncertainty in the model. As for the bounds on r, the upper bound is that value which provides about 99% protection against a stockout, and the lower bound logically follows from the definition of r and X. The iterative procedure at any value of X was:

a. Calculate the Wilson Q.

b. Use this value of Q and the given X to find r from the first difference for r inequality (ii), i.e., find r from Curve (iv) in Figure 3. Note that this involves a numerical solution.

c. Use this value of r and the given X to find a new value for Q from the first difference for Q inequality (i), i.e., find Q from Curve (iii) in Figure 3.

d. Repeat steps b and c until the values found converge to the intersection of the two curves.

2. Problem Parameters and Results

The parameters used for the example were:

$$\pi = \$4000.0, \qquad A = \$75.00,$$

$$A' = \$\ 5.00, \qquad \alpha = \$\ 0.50,$$

$$C = \$\ 50.00, \qquad I = 0.20,$$

$$\lambda = 50.00, \qquad T_p = 0.25,$$

$$T_R = 0.02, \qquad T_L = 0.08.$$

Optimal values of Q and r and the cost were calculated for X = 0, 7, 14, ... , 70. The set of Q, r, and X in which X equaled seven gave the minimum cost. Next optimal Q and r were calculated for each X within one standard deviation of X = 7. The results are tabulated below:

X	Q	r	Annual Variable Cost
0	30	29	$424.83
1	30	29	423.65
2	31	29	421.64
3	31	28	420.24
4	30	28	418.20
5	31	27	410.87
6	30	27	404.45
7	30	26	397.12
8	30	25	394.18
9	30	25	391.41
10	29	26	388.50
11	29	25	393.52
12	28	26	399.07
13	29	26	401.03
14	29	26	408.41

For this particular example, the minimizing set of the decision variables is $Q^* = 29$, $r^* = 26$, and $X^* = 10$; and the minimum average, annual, variable cost is $388.50. The probability that expediting occurs in a cycle is equal to 0.19, which is the probability that demand during the production leadtime is greater than or equal to 16.

Note that $\hat{Q}$ is equal to 735.39 and that $\hat{Q}'$ is equal to 750.25. Therefore, by the test developed in Section III,B,3, the above solution is the unique, optimal solution to the problem.

IV. CONCLUSION

The purpose of this thesis was to construct a model of a continuous review inventory policy which would allow the option of expediting deliveries. The decision to expedite is quite common in actual operations; yet this is not reflected by most inventory policies in use today.

The model which was developed does allow expediting. It has as its objective the minimization of the average, annual, variable cost. This is also the objective of most of the other infinite horizon continuous review inventory models. In this respect it is similar to these other models, but it goes beyond them because it more closely resembles actual operations.

Naturally, drawbacks exist. Every assumption which was made and every cost which was postulated detract from the resemblance to reality. The use of an iterative computational procedure makes the model impractical for an inventory containing a large number of line items if an optimal (Q,r,X) policy is necessary for each item. Also, as was shown, a solution is not always guaranteed.

However, the state of the art has not progressed much beyond the assumptions of the model and the postulation of costs. And, in those cases where the assumptions seem reasonable and the costs can be found, the model provides a new managerial tool for minimizing the average, annual, variable cost of operating an inventory policy.

LIST OF REFERENCES

1. Allen, S. G. and D'Esopo, D. A., "An Ordering Policy for Stock Items when Delivery Can Be Expedited," Operations Research, v. 16, No. 4, p. 880, July-August, 1968.

2. Morey, R. C., "An Inventory Model with Crisis Provision," Large Scale Provisioning Systems, edited by Ferrier,J., English Universities Press Limited, 1968.

3. Hadley, G. and Whitin, T. M., Analysis of Inventory Systems, Prentice-Hall Inc., 1963.

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13. ABSTRACT

A continuous review policy for ordering inventory stockage items with the option of expediting the shipping time is formulated. Demand is assumed to have a Poisson distribution with a stationary demand rate. Inventory holding costs, ordering costs, shortage costs and expediting costs are postulated. The measure of effectiveness is the minimization of a linear combination of these costs. The optimal policy is determined analytically through the use of first differences. An iterative computational procedure is recommended for obtaining the optimal order quantity, reorder point and expediting level. Analysis of the first differences indicates the conditions under which there is a solution and a simple numerical test for these conditions is developed. A numerical example is given.

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