On the Properties of Multiple-Valued Functions that are Symmetric in Both Variable Values and Labels

Butler, Jon T.; Sasao, Tsutomu
Monterey, California. Naval Postgraduate School

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<thead>
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<th><strong>Author(s)</strong></th>
<th>Butler, Jon T.</th>
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On the Properties of Multiple-Valued Functions that are Symmetric in Both Variable Values and Labels

by

Jon T. Butler
Tsutomu Sasao

December 1997

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On the Properties of Multiple-Valued Functions that are Symmetric in Both Variable Values and Labels

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Functions that are symmetric in both variable labels and variable values are important for use as benchmarks. We present the properties of such functions, showing that they are isomorphic to partitions on n (the number of variables) with no part greater than r (the number of logic values). From this, we do an enumeration. Further, we derive lower bounds, upper bounds, and exact values for the number of prime implicants in the minimal sum-of-products expressions for certain subclasses of these functions.

Multiple-valued logic, benchmark function, computer-aided design
On the properties of multiple-valued functions that are symmetric in both variable values and labels

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Abstract:

Functions that are symmetric in both variable labels and variable values are important for use as benchmarks. We present the properties of such functions, showing that they are isomorphic to partitions on \( n \) (the number of variables) with no part greater than \( r \) (the number of logic values). From this, we do an enumeration. Further, we derive lower bounds, upper bounds, and exact values for the number of prime implicants in the minimal sum-of-products expressions for certain subclasses of these functions.

1. Introduction

A variable/value symmetric function remains unchanged if

1. the variables are permuted and/or
2. the logic values of all variables are permuted.

In the binary case, for example, the Exclusive OR function, \( f(x_1, x_2) = x_1 \oplus x_2 \), is symmetric. Thus, if the variables are interchanged, the function remains unchanged. This can be expressed as \( f(x_1, x_2) = x_2 \oplus x_1 = x_1 \oplus x_2 \). If the logic values of all variables are interchanged, the function is also unchanged. This can be expressed as \( f(x_1, x_2) = \overline{x_1} \oplus \overline{x_2} = x_1 \oplus x_2 \). Thus, the Exclusive OR function is variable/value symmetric.

We are motivated to study multiple-valued variable/value symmetric functions because of the discovery [7] of a class of two-valued variable/value symmetric functions that defeat simplification algorithms which produce irredundant sum-of-products expressions. Indeed, the Minato-Morreale [3,4] algorithm does the worst that it can do on such functions. That is, it produces the largest number of prime implicants in irredundant sum-of-products expressions for specific variable/value symmetric functions. The problem has a significant practical implication. The ratio of the number of prime implicants in functions derived from variable/value symmetric functions for the worst case is an arbitrarily large constant times the number of prime implicants in the best case, as \( n \), the number of variables increases without bound. Thus, the Minato-Morreale algorithm may not only do poorly; it can do very poorly. This suggests the importance of understanding...
such functions, since most algorithms, like the Minato-Morreale algorithm, produce irredundant sum-of-products expressions.

Variable/value symmetric functions have another significant practical implication. They are easy to generate (by computer), and they are tractably analyzed. Yet, they are difficult for current algorithms to minimize. Thus, they are ideal benchmark functions [6].

2. Background

Definition 2.1: \( f(x_1, x_2, \ldots, x_n) \) is a variable symmetric function iff \( f(x_1, x_2, \ldots, x_n) \) is unchanged by a permutation of the variables.

Definition 2.2: \( f(x_1, x_2, \ldots, x_n) \) is a value symmetric function iff \( f(x_1, x_2, \ldots, x_n) \) is unchanged by a permutation of the logic values of all variables.

Definition 2.3: \( f(x_1, x_2, \ldots, x_n) \) is a variable/value symmetric function iff \( f(x_1, x_2, \ldots, x_n) \) is a variable symmetric function and a value symmetric function.

There is a long history of work on variable symmetric functions (also called symmetric functions or totally symmetric functions), including multiple-valued variable symmetric functions [2,5,8]. We know of no formal study of variable/value symmetric functions.

Example 2.1: The constant functions are trivial variable/value symmetric functions. The truth table below shows two nontrivial variable symmetric functions on two three-valued variables. Of these, \( f_2 \) is variable/value symmetric.

Table 2.1. Examples of 2-variable 3-valued variable/value symmetric functions.

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( f_1 )</th>
<th>( f_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Observation 2.1: A symmetric function is characterized by a set \( S \) of alpha vectors each of the form \( [\alpha_0, \alpha_1, \ldots, \alpha_{r-1}] \), where \( \alpha_i \) is the number of variables that have logic
value \( i \). Specifically, for all assignments of values to the variables that correspond to a single alpha vector in \( S \), the function has one value, 0, 1, \ldots, or \( r-1 \).

**Example 2.2:** The alpha vectors of the functions in Table 2.1 are shown in Table 2.2 below.

**Table 2.2.** Alpha vectors of the functions shown in Table 2.1.

<table>
<thead>
<tr>
<th>( \alpha_0 )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( f_1 )</th>
<th>( f_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

**Observation 2.2:** Since \( a + a + \ldots + \alpha_{r-1} = n \), each alpha vector corresponds to an \( r \)-part composition on integer \( n \), where each part is non-negative. For each alpha vector, the function value can be chosen in \( r \) ways. Thus, the number of symmetric functions is \( \binom{n-r}{r-1} \), where \( \binom{n+r-1}{r-1} \) is the number of compositions on \( n \) into \( r \) parts, where each part is 0, 1, 2, \ldots. Specifically, a composition can be chosen by selecting \( r \) objects from \( n \) with repetition. This can be done in \( \binom{n+r-1}{r-1} \) ways.

**Observation 2.3:** Since a variable/value symmetric function is a symmetric function in which the variable logic values can be permuted without changing the function, a variable/value symmetric function corresponds to an \( r \)-part partition on integer \( n \), where each part is 0, 1, 2, \ldots. For each partition, the function value can be chosen in \( r \) ways. Thus, the number of variable/value symmetric functions is \( P(n,r) \), where \( P(n,r) \) is the number of partitions on \( n \) into \( r \) parts.

**Observation 2.4:** A variable/value symmetric function is characterized by a set \( S \) of beta vectors each of the form \( [\beta_0, \beta_1, \ldots, \beta_{r-1}] \), where \( \beta_i \) is the number of variables that have logic value \( i \), and \( \beta_i \geq \beta_{i+1} \geq 0 \). Specifically, for all assignments of values to the variables that correspond to a single beta vector or any permutation of the vector, the function has one value, 0, 1, \ldots, or \( r-1 \). Note that the index, \( i \), of \( \beta_i \) does not represent a specific logic value, as in the case of alpha vectors.
Example 2.3: The beta vectors of the function $f_2$ in Table 2.2 are shown in Table 2.3 below.

<table>
<thead>
<tr>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$f_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

In this example, the function is 2 if two of the variables are 0, two are 1, or two are 2. Further, the function is 1 if one variable is 0 and one is 1, one variable is 0 and one is 2, one variable is 1 and one is 2. We can represent the beta vectors of Table 2.3 by a Ferrar’s Graph as shown in Fig. 2.1 below. Here, a part of size $i$ in the beta vector is represented as a line of $i$ (large) dots.

Figure 2.1. Ferrar’s diagram of the beta vectors of the function $f_2$ shown in Table 2.1.

Fig. 2.1 shows that a beta vector is a partition of $n$ into $r$ or fewer parts. Note that a Ferrar’s Graph rotated 90 degrees is also a Ferrar’s Graph. This shows that a partition of $n$ into $r$ or fewer parts corresponds to a partition of $n$ with no part greater than $r$. This allows us to make the following statement.

Lemma 2.1: The number of $n$-variable $r$-valued variable/value symmetric functions is $r^{g_{n,r}}$, where $g_{n,r}$ is the number of partitions on $n$ with no part greater than $r$.

It should be noted that $r^{g_{n,r}}$ includes functions on fewer than $n$ variables, since one choice among those enumerated is all logic values the same; i.e., a constant function. We can calculate $g_{n,r}$ as follows. Let $G_r(x)$ be the ordinary generating function for the number of partitions on $n$ with no part greater than $r$. Specifically,

$$G_r(x) = g_{0,r} + g_{1,r}x + g_{2,r}x^2 + \ldots + g_{n,r}x^n + \ldots,$$

where $g_{n,r}$ is the number of partitions on $n$ with no part greater than $r$. By Lemma 1, we can express $G_r(x)$ as

$$G_r(x) = (1 + x + x^2 + \ldots)(1 + x^3 + x^4 + \ldots) \cdots (1 + x^r + x^{2r} + \ldots),$$
where each polynomial factor represents the number of ways the parts, 1, 2, ..., and r, of various values can occur. From this, we can express $G_r(x)$ as

$$G_r(x) = \frac{1}{(1-x)(1-x^2) \cdots (1-x^r)}.$$ 

Using a polynomial package (in our case, MACSYMA), we can derive $G_r(x)$ for various $r$.

For example,

$$G_2(x) = 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + 5x^8 + 5x^9 + 6x^{10} + \cdots$$

$$G_3(x) = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + 10x^8 + 12x^9 + 14x^{10} + 16x^{11} + 19x^{12} + \cdots$$

$$G_4(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + 15x^8 + 18x^9 + 23x^{10} + 27x^{11} + 34x^{12} + \cdots$$

$$G_5(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 10x^6 + 13x^7 + 18x^8 + 23x^9 + 30x^{10} + 37x^{11} + 47x^{12} + \cdots$$

$$G_6(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 14x^7 + 20x^8 + 26x^9 + 35x^{10} + 44x^{11} + 58x^{12} + \cdots$$

3. Properties of variable/value symmetric functions

**Lemma 3.1:** A value symmetric function on two $r$-valued variables is variable symmetric, for $r \geq 2$.

**Proof:** Since there are two variables, there are two possibilities; the variable values are the same or they are different. If they are the same, interchanging them leaves the function unchanged. If they are different, since it is value symmetric, any permutation of the logic values leaves the function unchanged, including permutations that interchange the two values. Since interchanging the two values leaves the function unchanged, it is variable symmetric.

Q.E.D.

Lemma 3.1 does not extend to value symmetric functions on three or more variables. For example, $f(x_1, x_2, x_3) = \overline{x_1}x_2x_3 \lor x_1x_2\overline{x_3}$ is value symmetric but not variable symmetric. From the observation in the proof of Lemma 3.1, we can conclude the following.

**Corollary 3.1:** The number of two variable $r$-valued functions that are value symmetric (and variable symmetric) is $r^2$ for all $r \geq 2$.

Alternatively, one can show that the coefficient of $x^2$ in $G_r(x)$ is 2 for all $r \geq 2$. 

5
We now consider a special type of variable/value symmetric function.

**Definition 3.1:** Let \( ST_r(n,k) \) be an \( n \)-variable function that is 1 when there are at least \( k \) variables that are 0, \( k \) that are 1, \( \ldots \), and \( k \) that are \( r-1 \). \( ST_r(n,k) = 0 \) otherwise. Specifically, it is symmetric because its value depends only on the number of variables that are 0 and 1. \( ST_r(n,k) \) is variable/value symmetric because the function specification is the same for all logic values; i.e. there is symmetry among the logic values.

**Example 3.1:** When \( r = 2 \), \( ST_r(n,k) \) is a two valued logic function that is 1 iff there are one or more 0’s and one or more 1’s among the variable values. Algebraically, this can be expressed as \( ST_2(3,1) = (x_1 \lor x_2 \lor \ldots \lor x_n) (x_1^1 \lor x_2^1 \lor \ldots \lor x_n^1) \). In the case of \( ST_2(3,1) \), where the variables are ternary, \( ST_2(3,1) \) is the OR of six minterms, in which each minterm represents a way to permute 0, 1, and 2.

**Example 3.2:** The beta vectors for \( ST_2(6,1), ST_2(6,2), \) and \( ST_2(6,3) \) are shown in Table 3.4 below. The beta vectors for \( ST_3(6,1) \) and \( ST_3(6,2) \) are shown in Table 3.5 below.

**Table 3.4.** Beta vectors for \( ST_2(6,1), ST_2(6,2), \) and \( ST_2(6,3) \)

<table>
<thead>
<tr>
<th>( \beta_0 )</th>
<th>( \beta_1 )</th>
<th>( ST_2(6,1) )</th>
<th>( ST_2(6,2) )</th>
<th>( ST_2(6,3) )</th>
<th>( ST_3(6,1) )</th>
<th>( ST_3(6,2) )</th>
<th>( ST_3(6,3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>12</td>
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<td>0</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>30</td>
<td>0</td>
</tr>
<tr>
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<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>20</td>
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</tbody>
</table>

**Table 3.5.** Beta vectors for \( ST_3(6,1) \) and \( ST_3(6,2) \)

<table>
<thead>
<tr>
<th>( \beta_0 )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( ST_3(6,1) )</th>
<th>( ST_3(6,2) )</th>
<th>( ST_3(6,1) )</th>
<th>( ST_3(6,2) )</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
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<td>90</td>
</tr>
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</tr>
</tbody>
</table>

**Total** 540 90 27/1 1/1
Tables 3.4 and 3.5 also show the number of minterms associated with each beta vector. The totals of the number of minterms is shown, as well. Thus, for $ST_2(6,1)$, $ST_2(6,2)$, and $ST_2(6,3)$, the total number of minterms covered is 62, 50, and 20, respectively (out of 64). For $ST_3(6,1)$, and $ST_3(6,2)$, the total number of minterms covered is 540 and 90, respectively (out of 729).

Also shown in Tables 3.4 and 3.5 are the number of minterms each prime implicant covers of the type of minterm shown and the number of prime implicants that cover a specific single minterm of the type shown. So, for example, the minterms of $ST_2(6,1)$, associated with beta vector $[5,1]$ have the property that each prime implicant covers 2 minterms with that beta vector, and, conversely, each minterm with that beta vector is covered by 5 prime implicants. Thus, the corresponding entry is $2/5$. We examine this in depth now.

**Definition 3.2:** Literal $x^S = 1$ if $x \in S$; otherwise $x^S = 0$.

**Example 3.3:** If $x$ is a 3-valued variable, then $x^{\{0,2\}} = 1$ if $x$ is 0 or 2 and $x^{\{0,2\}} = 0$ if $x$ is 1.

For convenience, if $S$ consists of exactly one element, then we omit $\{\}$. Thus, $x^{\{i\}}$ is written as $x^i$. Note that $x^{\{0,1,\ldots,n\}}$ is a constant 1.

**Definition 3.3:** Let $f(x_1,x_2,\ldots,x_n)$ be a two-valued function with $r$-valued variables. $P = x_1^S x_2^S \ldots x_n^S$ is an implicant of $f$, if $f$ is 1 whenever $x_1 \in S_1, x_2 \in S_2, \ldots, x_n \in S_n$. $P$ is a prime implicant (PI) of $f$ if $P'$ is not an implicant off, where $P'$ is the same as $P$ except some $S_i$ is replaced by $S_i'$, where $S_i' \supset S_i$.

$P$ is a prime implicant if all literal sets $S_i$ are as large as they can be.

**Example 3.4:** $x_1^0 x_2^1 x_3^1 = \overline{x_1} x_2^1 x_3$ is an implicant of $ST_2(3,1)$ because, when $x_1 x_2 x_3$ is 1 (when $x_1 = 0, x_2 = 1$, and $x_3 = 1$), $ST_2(3,1)$ is also 1. However, it is not a prime implicant, because $x_1^0 x_2^{\{0,1\}} x_3^1 = x_1^0 x_3^1$ is an implicant of $ST_2(3,1)$. However, $x_1^0 x_3^1$ is a prime implicant because enlarging any of its literal sets causes it not to be an implicant.

**Observation 3.5:** $x_1^0 x_2^1$ is a prime implicant of $ST_2(n,1)$, where $i_1,i_2 \in \{1,2,\ldots,n\}$ and $i_1 \neq i_2$. Also, $x_1^0 x_2^1 \ldots x_r^{i_r-1}$ is a prime implicant of $ST_2(n,1)$, such that $i_1,i_2,\ldots,i_r \in \{1,2,\ldots,n\}$ and all $i_j$ are distinct. Further, $x_1^0 x_2^0 \ldots x_r^0 x_r^{i_1} \ldots x_r^{i_k-1} \ldots x_k^{i_k} \ldots x_n^{i_n}$ is a prime implicant of $ST_i(n,k)$, such that $i_1,i_2,\ldots,i_k \in \{1,2,\ldots,n\}$ and all $i_j$ are distinct.
Theorem 3.1: $ST_r(n,k)$ has
\[
\frac{n!}{(k!)^r(n-kr)!}
\]
prime implicants.

Proof: The prime implicants of $ST_r(n,k)$ have the form
\[
x_1^0 x_2^0 \ldots x_k^0 x_{k+1}^1 x_{k+2}^1 \ldots x_{r+x}^1 \ldots x_{(r-1)+k}^1 x_{(r-1)+2k}^1 \ldots x_{r-1}^1.
\]
Specifically, this expression is 1 when all $k$ variables in a specific set $(\{x_1, x_2, \ldots, x_k\})$ are 0, when all $k$ variables in another specific set $(\{x_{k+1}, x_{k+2}, \ldots, x_r\})$ are 1, ... and when all $k$ variables in another specific set $(\{x_{(r-1)+k}, x_{(r-1)+2k}, \ldots, x_{r-1}\})$ are $r-1$. On the contrary, if some variable has fewer than $k$ representatives, then the corresponding product term is 1 when fewer than $k$ of those variables are 1, and thus the function has this property, contradicting the fact that the function is $ST_r(n,k)$. If some variable has more than $k$ representatives, then deleting literals so there are exactly $k$ representatives creates an implicant of the function that is implied by the original implicant; thus, the original implicant was not prime.

There are $n!/(k!)^r(n-kr)!$ implicants of this form. Specifically, each implicant corresponds to a permutation of $r+1$ objects, where $r$ of the objects (the logic values) each occur as $k$ identical objects and one object (variables not included) occur as $n-kr$ identical objects.

Q.E.D.

Example 3.5: Consider the functions in Tables 3.4 and 3.5. From Theorem 3.1, $ST_2(6,1)$, $ST_2(6,2)$, and $ST_2(6,3)$ have a total of 30, 90, and 20 prime implicants, respectively, while $ST_3(6,1)$, and $ST_3(6,2)$ have a total of 120 and 90 prime implicants, respectively.

Lemma 3.2: Each prime implicant in $ST_r(n,k)$ covers $r^{n-rk}$ minterms.

Proof: The prime implicants of $ST_r(n,k)$ have the form
\[
x_1^0 x_2^0 \ldots x_k^0 x_{k+1}^1 x_{k+2}^1 \ldots x_{r+x}^1 \ldots x_{(r-1)+k}^1 x_{(r-1)+2k}^1 \ldots x_{r-1}^1.
\]
There are $n-rk$ variables not included, whose values can each be chosen in $r$ ways. Each different specification represents a minterm covered by the prime implicant.

Q.E.D.
Example 3.6: Consider the functions in Tables 3.4 and 3.5. From Lemma 3.2, the prime implicants of $ST_2(6,1)$, $ST_2(6,2)$, $ST_2(6,3)$, $ST_3(6,1)$, and $ST_3(6,2)$ cover, $2^2=4$, $2^0=1$, $3^3=27$, and $3^0=1$ minterms, respectively. This is shown in Tables 3.4 and 3.5 in the last three column of the last row. So, for example, the $2^4 = 16$ minterms covered by prime implicants of $ST_2(6,1)$ are distributed to minterms associated with beta vectors $[5,1]$, $[4,2]$, and $[3,3]$ as 2, 8, and 6 respectively.

The following lemma represents an observations about sum-of-products expressions for $ST_d(n,k)$ functions that will be useful later when one already has one sum-of-products expression.

Lemma 3.3: Let $F$ be a sum-of-products expression for $ST_d(n,k)$. Let $F'$ be a sum-of-products expression derived from $F$ by permuting all variable labels $j$ in $x_j$ according to permutation $\pi_{variable}: \{1,2,\ldots, n\} \rightarrow \{1,2,\ldots, n\}$ and by permuting all logic values $i$ of $x_j$ according to permutation $\pi_{value}: \{0,1,\ldots, r-1\} \rightarrow \{0,1,\ldots, r-1\}$. Then, $F'$ is a sum-of-products expression for $ST_d(n,k)$.

Proof: The statement follows from the fact that $ST_d(n,k)$ is unchanged by a permutation of variable labels and logic values.

Q.E.D.

Lemma 3.3 simply states that, although $ST_d(n,k)$ is unchanged by a permutation of logic values and variable labels, a sum-of-products expression for $ST_d(n,k)$ may be changed.

Lemma 3.4: Every prime implicant of $ST_d(n,k)$ occurs in an MSOP of $ST_d(n,k)$.

Proof: Consider any prime implicant $P$ of $ST_d(n,k)$. Consider a prime implicant $P'$ in an MSOP of $ST_d(n,k)$. By a permutation of variable labels and logic values, we can convert $P'$ into $P$. Applying this permutation to all prime implicants in the MSOP yields an MSOP that contains $P$.

Q.E.D.

We are interested in the number of prime implicants in the minimal sum-of-products expression (MSOP) expression of $ST_d(n,k)$. Tables 3.4 and 3.5 provide an insight on this. For example, in Table 3.4, we see that for $ST_2(6,1)$, there are 12 minterms associated with beta vector $[5,1]$ and (from the third column from the left) each prime implicant covers 2 minterms in this group. It follows that at least $\lceil 12 / 2 \rceil = 6$ prime implicants are needed in the MSOP of $ST_2(6,1)$. From beta vector $[4,2]$, at least $\lceil 30 / 8 \rceil = 4$ prime implicants are needed, and from beta vector $[3,3]$, at least $\lceil 20 / 6 \rceil = 4$ prime implicants are needed. Beta vector $[5,1]$ provides the greatest restriction, and we have a lower bound on the number of prime implicants for $ST_2(6,1)$ of 6 prime implicants. In a similar manner, we can calculate a lower bound on the number of prime implicants of
\[ ST_2(6,2), ST_3(6.3), ST_2(6,1), \text{ and } ST_3(6,2) \text{ as } 15, 20, 30, \text{ and } 90, \text{ respectively.} \]

In general, we can state

**Theorem 3.2:** A lower bound on the number of prime implicants in the MSOP of \( ST_r(n, 1) \) is

\[
\frac{n!}{(n-r+1)!}
\]

**Proof:** Consider a set \( S \) of minterms corresponding to beta vector \([n-r+1, 1, 1, \ldots, 1]\). The number of minterms in \( S \) is \( rn!/(n-r+1)! \), when \( n > r \) and \( n! \) when \( n = r \). If \( n > r \), each prime implicant, \( x_1^0 x_1^1 \ldots x_1^{r-1} \), covers \( r \) minterms in \( S \), since there are \( r \) ways to choose the variable values not in \( x_1^i \). If \( n = r \), there are no other values to specify. Thus, for both \( n > r \) and \( n = r \), at least \( n!/(n-r+1)! \) prime implicants are needed.

Q.E.D.

Consider now an upper bound on the number of prime implicants in an MSOP for \( ST_r(n, 1) \). Any sum-of-products expression for \( ST_r(n, 1) \) has the form

\[
x_n^0 F_0 \lor x_n^1 F_1 \lor \ldots \lor x_n^{r-1} F_{r-1} \lor F,
\]

where \( F_i \) and \( F \) are sum-of-products expressions on \( x_1, x_2, \ldots, x_{n-1} \). Each contains literals of the form \( x_i^k \), where \( 0 \leq k \leq r-1 \). In the case of \( F_i \), \( k \neq i \), while \( F \) is unrestricted. That is, both contain all variables except \( x_r \). \( F \) contains literals of the form \( x_r^k \), for all \( k \in \{0, 1, \ldots, r-1\} \), while \( F_i \) does not contain the literal \( x_r^i \). For example, an MSOP for \( ST_4(4, 1) \) can be written as

\[
ST_2(3,1) = x_1^0 x_2^0 x_1^1 \lor x_1^0 x_1^1 \lor x_1^0 x_2^1,
\]

which can be verified by a Karnaugh Map. From Lemma 3.3, it follows that by permuting the logic values (in this case, complementation) or by permuting the variables, we can obtain another MSOP.

\[
ST_2(3,1) = x_2^0 x_1^1 \lor x_2^0 x_1^1 \lor x_2^0 x_2^1.
\]

As another example, an MSOP for \( ST_3(4, 1) \) can be written as

\[
ST_3(4,1) = x_1^0 (x_1^1 x_2^2 \lor x_1^0 x_1^1 x_2^2) \lor x_1^1 (x_3^0 x_1^2 \lor x_2^0 x_2^2 \lor x_1^0 x_2^3) \\lor x_1^2 (x_1^0 x_1^2 \lor x_1^0 x_1^2) \lor (x_2^0 x_2^2 \lor x_2^0 x_2^2 \lor x_2^0 x_2^2).
\]

It is straightforward (but tedious) to verify that this is indeed is a sum-of-products expression for \( ST_3(4, 1) \). To verify that it is minimal requires some insight.

**Lemma 3.5** If \( x_n^0 F_0 \lor x_n^1 F_1 \lor \ldots \lor x_n^{r-1} F_{r-1} \lor F \) is an MSOP of \( ST_{r-1}(n-1, 1) \), then

1. \( F_i \) is an MSOP of \( ST_{r-1}(n-1, 1) \) on logic values \( \{0, 1, \ldots, r-1\} \setminus \{i\} \), and
2. \( F \) is an MSOP of \( ST_r(n-1,1) \) with (consensus) terms removed that are also covered in \( x:F, v \ldots VX', \llap{~}F, \llap{~}F \).

**Proof:** When \( ST_r(n,1) = F_i \) and \( F_i \) is \( ST_r(n-1,1) \) on logic values \( \{0,1,\ldots,r-1\}-\{i\} \). The expression for \( F_i \) must be an MSOP of \( ST_r(n-1,1) \); otherwise, it can be replaced by an MSOP, thus reducing the products in the expression for \( ST_r(n,1) \), contradicting the statement that \( ST_r(n-1,1) \) is an MSOP. Consider a minterm \( m \) covered by \( ST_r(n-1,1) \) which has at least one 0, one 1, \ldots, and one \( r-1 \) among \( x_1, \ldots, x_{n-1} \). Then, \( m \) is covered by a prime implicant that is in \( F \) or is a consensus term among \( x:F, v \ldots VX', \llap{~}F, \llap{~}F \) or both. Among all terms must be an MSOP for \( ST_r(n-1,1) \); otherwise \( x:F, v \ldots VX', \llap{~}F, \llap{~}F \) is not an irredundant SOP for \( ST_r(n,1) \). Property 2 follows from this observation.

Q.E.D.

If we add the consensus terms to \( F \), forming \( F' \), we obtain the following expression.

\[
ST_3(4,1) = x^0_3(x^1_1x^2_3 \lor x^1_3x^2_1) \lor x^1_4(x^0_2x^2_1 \lor x^2_2x^0_1x^1_3) \\
\lor x^2_4(x^0_1x^1_3 \lor x^0_3x^1_2 \lor x^0_2x^1_1) \lor (x^0_1x^1_2x^2_3 \lor x^2_1x^0_3 \lor x^0_2x^1_1x^2_3 \lor x^1_1x^2_2x^0_3 \lor x^1_2x^3 \lor x^2_3x^1x^0_2 \lor x^0_1x^1_3x^2_2x^1_1)
\]

The last three terms are each redundant. \( F \) is an MSOP for \( ST_3(3,1) \); it is the OR of minterms representing all 6 ways to permute three logic values among three variables. In this expression, the \( F_i \) terms are minimal, but \( F' \) is not. Therefore, this expression would be an MSOP if \( F \) has as few terms as possible, which means that the largest number of terms of \( F \) should be covered by the \( F_i \) terms. However, no more than three such terms can be covered collectively by the \( F_i \) terms. It follows that the original expression (with the last three terms removed) is an MSOP.

The two conditions of Lemma 3.6 are necessary as the proof shows. They are not sufficient. That is, the irredundant sum-of-products expression for \( ST_3(4,1) \) satisfies

\[
ST_3(4,1) = x^0_4(x^1_1x^2_3 \lor x^1_3x^2_1) \lor x^1_4(x^0_2x^2_1 \lor x^0_2x^0_1x^1_3) \\
\lor x^2_4(x^0_1x^1_3 \lor x^0_3x^1_2 \lor x^0_2x^1_1) \lor (x^0_1x^1_2x^2_3 \lor x^2_1x^0_3 \lor x^0_2x^1_1x^2_3 \lor x^1_1x^2_2x^0_3 \lor x^1_2x^3 \lor x^2_3x^1x^0_2 \lor x^0_1x^1_3x^2_2x^1_1)
\]

the two conditions, but it is not minimal. This is the above equation with the \( F_i \) term modified. Specifically, too few (none) product terms in \( F \) appear as consensus terms covered by the \( F_i \) terms. Fig. 3.1 below shows a graph representation of the \( F_i \) terms in both expressions. In this figure, an arc represents a prime implicant in an \( F_i \) term.
For example, \( x_3^0x_1^0 \) is represented as an arc labeled by the logic values 01 from node 3 representing \( x_3 \) to node 2, representing \( x_2 \). The fact that \( x_3^0x_1^0 \lor x_2^0x_1^0 \lor x_1^0x_1^0 \) is an MSOP of \( ST_3(2,1) \) is seen in the graph by the cycle of arcs labeled 01 from \( x_3 \) to \( x_2 \), \( x_2 \) to \( x_1 \), and \( x_1 \) to \( x_3 \). Similarly, the arcs labeled 12 form a cycle, as do the arcs labeled 20. A consensus term occurs also as a cycle. For example in Fig. 3.1a, there is a consensus term \( x_3x_1x_2 \) associated with the 01 arc from 1 to 3, the 12 arc from 3 to 2, and a 20 arc from 2 to 1. Notice in Fig. 3.1b, there are triples of arcs of this type and thus no consensus terms.

How can we produce an MSOPs for \( ST(n, 1) \)?

**Algorithm:** Produce an MSOP for \( ST(n, 1) \) given an MSOP for \( ST(n-1, 1) \).

1. Set \( F \) of \( ST(n, 1) \) equal to the MSOP of \( ST(n-1, 1) \).
2. Form the largest set of prime implicants in \( F \) as consensus terms and form \( F_0, F_1, \ldots, F_{r-1} \). Remove these terms from \( F \).

We illustrate by an example. Form an MSOP of \( ST_3(5, 1) \) from an MSOP of \( ST_3(4, 1) \) as follows. Specifically, use the MSOP shown in (2). Consider the consensus terms to remove. \( F \) will be an MSOP on 4 variables and 2 logic values. Fig. 3.2 below shows that we can create at most two consensus terms, \( x_3^0x_1^0x_3^0x_1^0 \lor x_2^0x_1^0x_1^0 \). Indeed, this figure was generated by first drawing the largest number of consensus terms and then completing it so that the \( F_i \) terms formed MSOPs. The SOP so formed is

\[
ST_i(4, 1) = x_3^0(x_1^0x_4^0 \lor x_1^1x_2^0 \lor x_1^1x_3^0 \lor x_2^1x_3^0) \lor x_1^1(x_2^0x_4^0 \lor x_4^0x_2^0 \lor x_2^0x_1^0 \lor x_1^0x_4^0) \\
\lor x_2^0(x_4^0x_3^0 \lor x_4^0x_1^0 \lor x_1^0x_3^0 \lor x_1^0x_4^0) \\
\lor x_3^0x_1^0x_2^0 \lor x_2^0x_1^0x_3^0 \lor x_1^0x_2^0x_3^0 \lor x_2^0x_3^0x_4^0 \lor x_3^0x_1^0x_2^0 \lor x_1^0x_3^0x_4^0 \\
\lor x_1^0x_2^0x_3^0 \lor x_2^0x_3^0x_4^0 \lor x_3^0x_1^0x_2^0 \lor x_1^0x_3^0x_4^0 \\
\lor x_3^0x_1^0x_2^0 \lor x_1^0x_3^0x_4^0 \lor x_2^0x_3^0x_4^0 \lor x_3^0x_1^0x_2^0 \lor x_1^0x_3^0x_4^0 \\
\lor x_3^0x_1^0x_2^0 \lor x_1^0x_3^0x_4^0 \\
\lor x_3^0x_1^0x_2^0 \lor x_1^0x_3^0x_4^0
\]

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This proves that the MSOP for $ST_3(5,1)$ has $3 \times 4 + 12 - 2 = 22$ prime implicants, which is two more than the lower bound given in Theorem 3.2.

![Graph representation of the $F_i$ in the formation of an MSOP of $ST_3(5,1)$](image)

Figure 3.2. Graph representation of the $F_i$ in the formation of an MSOP of $ST_3(5,1)$

For specific values of $n$ and $r (=n-1)$, we can determine the exact number of prime implicants in the MSOP. Figure 3.3 below shows a graph representation of the sum-of-products expression of $ST_2(3,1)$. Here, nodes represent prime implicants of $ST_2(3,1)$ and edges represent minterms covered by prime implicants. Specifically, each minterm is covered by two prime implicants (e.g. minterm 011 is covered by 0*1 and 01*), which correspond to the two nodes (0*1 and 01*) incident to the edge (011). The problem of finding an MSOP is identical to the problem of finding a set $S$ with the fewest nodes such that each edge is incident to at least one node in $S$. From Figure 3.3, it can be seen that there are two sets each with nodes that satisfy this criteria. These represent two MSOPs.

![Graph representation of prime implicants and minterms for $ST_2(3,1)$](image)

Figure 3.3. Graph representation of prime implicants and minterms for $ST_2(3,1)$.

**Theorem 3.3:** The number of prime implicants in an MSOP of $ST_{n-1}(n,1)$ is
\[
\frac{n!}{2}
\]
Proof: Consider a graph $G(V,E)$, where $V$ is the set of prime implicants of $ST_{n-1}(n,1)$ and $E$ is the set of minterms of $ST_{n-1}(n,1)$. Each minterm $m$ has the form $x_1^i x_2^i ... x_n^i$, where each logic value $\{0,1, ... ,n-2\}$ occurs as one $i_j$, except one (repeated) logic value that occurs twice. Let the repeated logic value be $i_j$ and $i_k$. The two prime implicants that cover $m$ have the form 1) $x_1^j x_2^j ... x_n^j$, where $x_i^j$ is omitted (i.e. replaced by $x_i^{(0,1,...,n-2)} = 1$) and 2) $x_1^i x_2^i ... x_n^i$, where $x_i^i$ is omitted (i.e. replaced by $x_i^{(0,1,...,n-2)} = 1$). Note that there are $n!$ prime implicants; each corresponds to a permutation on the elements $\{0,1, ... ,n-2, *\}$, where $0, 1, ... , n-2$ are the $n-1$ logic values whose position determines the literal with that logic value and * represents the missing variable.

We now show that $G$ is bipartite. From this, it follows that the nodes in the part with the fewest nodes represents an MSOP of $ST_{n-1}(n,1)$. However, from Theorem 3.2, a lower bound on this number is $n!/2$. Since the total number of nodes is $n!$, and both parts have no less than $n!/2$ prime implicants, it follows that both have exactly this number. Therefore, there are two MSOPs of $ST_{n-1}(n,1)$ each with $n!/2$ prime implicants.

Consider a minterm $m$ and the two prime implicants, $P_1$ and $P_2$, that cover $m$. Let $P_1$ and $P_2$ have the representation shown below in Fig. 3.4. Specifically, $P_1$ is represented as $i_0i_1 ... i_{j-1} * i_{j+1} ... i_{n-1}$, a compact representation of $x_1^{i_0} x_2^{i_1} ... x_n^{i_{n-1}}$. Since $i_0i_1 ... i_{j-1} * i_{j+1} ... i_{n-1}$ is a permutation of the elements $\{0,1, ... ,n-2, *\}$, it can be represented as a directed graph (the cycle

![Figure 3.4. Cycle structure of prime implicants, $P_1$ and $P_2$, which both cover the same minterm $m$.](image-url)
structure representation) where the nodes correspond to elements of \(\{0, 1, \ldots, n-2, \ast\}\), and the arcs are \(0 \rightarrow i_0, 1 \rightarrow i_1, \ldots, i_s \rightarrow \ldots, n-1 \rightarrow i_{n-1}\) for \(i_j \in \{0, 1, \ldots, n-2, \ast\}\). Similarly, let \(P_1\) be represented as \(i_0 i_1 \ldots i_t \ast i_{t+1} \ldots i_{n-1}\). The minterm covered by \(P_1\) and \(P_2\) is therefore \(i_0 i_1 \ldots i_{t+1} i_t \ldots i_{t+1} i_t i_{t+1} \ldots i_{n-1}\), where \(i_t = i_{t+1}\) Consider the cycle structure representation of \(P_1\). There are two possibilities 1) \(i_t = i_{t+1}\) are in the same cycle or 2) \(i_t = i_{t+1}\) are in different cycles. In Fig. 3.4, the case where \(i_t = i_{t+1}\) are in the same cycle is shown. Note that the cycle structure of \(P_2\) is the same as that of \(P_1\) except that the arc from \(s\) to \(i_t = i_{t+1}\) is replaced by an arc from \(s\) to \(i_t = i_0\), while the arc from \(t\) to \(i_t = i_{t+1}\) is replaced by and arc from \(t\) to \(i_t = i_{t+1}\). As a result, the single cycle of \(P_1\) becomes two cycles in \(P_2\). This is shown in Fig. 3.4. Conversely, if \(i_t = i_{t+1}\) are in different cycles in \(P_1\), it can be seen that these two cycles become one in \(P_2\). From this, it follows that \(G\) is bipartite, where one part corresponds to all permutations with an even number of cycles and the other part corresponds to all permutations with an odd number of cycles.

Q.E.D.

From the above discussion, we can state

**Corollary 3.2:** \(ST_{n-1}(n, 1)\) has two MSOPs.

We can now prove the following:

**Lemma 3.7:** An upper bound on the number of product terms in MSOP for \(ST_r(n, 1)\) for \(2 \leq r \leq 6\) is

<table>
<thead>
<tr>
<th>(r)</th>
<th>Upper Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(n)</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{3}{2}n^2 - \frac{3}{2}n - 3)</td>
</tr>
<tr>
<td>4</td>
<td>(2n^3 - 6n^2 - 8n + 24)</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{5}{2}n^4 - 15n^3 + \frac{5}{2}n^2 - 32n - 330)</td>
</tr>
</tbody>
</table>

**Proof:** Let \(U(n,r)\) be an upper bound on the number of prime implicants in the MSOP of \(ST_r(n,1)\). From our previous observation on the decomposition \(x_n^0 F_0 \lor x_n^1 F_1 \lor \ldots \lor x_{r-1}^n F_{r-1} \lor F\) of \(ST_r(n,1)\), \(F_i\) must be an MSOP of \(ST_{r-1}(n-1,1)\), while \(F\) is an MSOP on \(ST_r(n-1,1)\) less consensus terms. Thus, a recursion relation for \(U(n,r)\) is

\[
U(n,r) = r \cdot U(n-1,r-1) + U(n-1,r).
\]
Using $U(n,2) = n$ and $U(n,n) = n!$, one can recursively solve for the closed form upper bounds shown in Table 3.6.

Q.E.D.

From this result, we can state

**Lemma 3.8:** An upper bound on the number of product terms in an MSOP of $ST_r(n,1)$ is $O\left(\frac{r}{2}n^{r-1}\right)$.

**Proof:** From Lemma 3.7, one can recursively solve for the coefficient of the largest term as $O\left(\frac{r}{2}n^{r-1}\right)$.

Q.E.D.

**Lemma 3.9:** For fixed $r$, the number of product terms in an MSOP of $ST_r(n,1)$ is $O(n^{r-1})$.

**Proof:** Compare the upper and lower bounds for the number of product terms in an MSOP of $ST_r(n,1)$.

Q.E.D.

4. Experimental Results

Table 4.1 below shows the results of experiments on various classes of $ST_r(n,1)$ functions. Here, the number of prime implicants, as derived in Theorem 3.1 are shown in the second column labeled **No. of PIs**, the number of prime implicants calculated by a Quine-McCluskey type logic minimizer is shown in the third column labeled **QM**, and the lower bound derived in Theorem 3.2 in the fourth column labeled **LB**. The fifth column, labeled **UB**, shows an upper bound on the number of prime implicants as derived in Lemma 3.9. The last column, labeled **NWS**, shows the number of prime implicants obtained by a heuristic program designed to find an irredundant sum-of-products expression with many prime implicants.
Table 4.1 Number of prime implicants in total, as derived by Quine-McCluskey, lower and upper bounds on prime implicants in an MSOP, and in a large sum-of-products.

<table>
<thead>
<tr>
<th>Function</th>
<th>No. of PIs</th>
<th>QM</th>
<th>LB</th>
<th>UB</th>
<th>NSW</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ST_3(4,1)$</td>
<td>24</td>
<td>**12</td>
<td>*12</td>
<td>24</td>
<td>16</td>
</tr>
<tr>
<td>$ST_3(5,1)$</td>
<td>60</td>
<td>**22</td>
<td>20</td>
<td>37.5</td>
<td>32</td>
</tr>
<tr>
<td>$ST_3(6,1)$</td>
<td>120</td>
<td>33</td>
<td>30</td>
<td>54</td>
<td>50</td>
</tr>
<tr>
<td>$ST_3(7,1)$</td>
<td>210</td>
<td>49</td>
<td>42</td>
<td>73.5</td>
<td>75</td>
</tr>
<tr>
<td>$ST_3(8,1)$</td>
<td>336</td>
<td>70</td>
<td>56</td>
<td>96</td>
<td>98</td>
</tr>
<tr>
<td>$ST_3(9,1)$</td>
<td>504</td>
<td>89</td>
<td>72</td>
<td>121.5</td>
<td></td>
</tr>
<tr>
<td>$ST_4(5,1)$</td>
<td>120</td>
<td>**60</td>
<td>*60</td>
<td>250</td>
<td>96</td>
</tr>
<tr>
<td>$ST_4(6,1)$</td>
<td>360</td>
<td>159</td>
<td>120</td>
<td>432</td>
<td>210</td>
</tr>
<tr>
<td>$ST_4(7,1)$</td>
<td>840</td>
<td>282</td>
<td>210</td>
<td>686</td>
<td></td>
</tr>
<tr>
<td>$ST_4(8,1)$</td>
<td>1,680</td>
<td>490</td>
<td>336</td>
<td>1024</td>
<td></td>
</tr>
<tr>
<td>$ST_5(6,1)$</td>
<td>720</td>
<td>a 455</td>
<td>*360</td>
<td>3240</td>
<td>600</td>
</tr>
<tr>
<td>$ST_5(7,1)$</td>
<td>2,520</td>
<td>b 1,231</td>
<td>832</td>
<td>6002.5</td>
<td></td>
</tr>
<tr>
<td>$ST_6(7,1)$</td>
<td>5,040</td>
<td>3,365</td>
<td>*2,520</td>
<td>50,421</td>
<td>4,320i</td>
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<tr>
<td>$ST_7(8,1)$</td>
<td>40,320</td>
<td>32,760</td>
<td>*20,160</td>
<td>917,504</td>
<td></td>
</tr>
</tbody>
</table>

* Known to be the exact number of prime implicants in an MSOP of $ST_r(n,1)$ by Theorem 3.3. ** Quine-McCluskey obtained a solution known to be an MSOP. a MINI2 obtained a 455 prime implicant solution. b MINI2 obtained a 1,206 prime implicant solution.

4. Concluding Remarks

Functions that are symmetric in both variable labels and variable values are important because they provide tractably analyzed functions that are difficult to minimize. Thus, they are useful as benchmark functions. Yet, as far as we know, there has been no formal study of such functions. In this paper, we have shown that there is a one to one correspondence between such functions and partitions on $n$ with no part greater than $r$. We have shown how many prime implicants there are in each function, and, we have shown upper and lower bounds on the number of prime implicants in minimal sum-of-products expressions of a specific class of variable/value symmetric functions. For another class of variable/value symmetric functions, we have shown the exact number of prime implicants in minimal sum-of-products expressions.

5. Acknowledgments

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References


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