Calhoun: The NPS Institutional Archive
DSpace Repository

Reports and Technical Reports

All Technical Reports Collection

1997-04-01

Assessment of damage aggregation for weapons salvos

Jayachandran, Toke

Monterey, California. Naval Postgraduate School

<https://hdl.handle.net/10945/15330>

Downloaded from NPS Archive: Calhoun

<http://www.nps.edu/library>

Calhoun is the Naval Postgraduate School's public access digital repository for research materials and institutional publications created by the NPS community. Calhoun is named for Professor of Mathematics Guy K. Calhoun, NPS's first appointed -- and published -- scholarly author.

Dudley Knox Library / Naval Postgraduate School
411 Dyer Road / 1 University Circle
Monterey, California USA 93943

NAVAL POSTGRADUATE SCHOOL Monterey, California

Assessment of Damage Aggregation for Weapons Salvos

by

Toke Jayachandran
I. Bert Russak

April 1997

Approved for public release; distribution is unlimited.

Prepared for: Naval Postgraduate School
Monterey, California 93943

19970710 024

DTIC QUALITY INSPECTED

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA 93943

Rear Admiral M. J. Evans
Superintendent

Richard Elster
Provost

This report was prepared in conjunction with research conducted for Naval Postgraduate School and funded by the Naval Postgraduate School in Monterey, California.

Reproduction of all or **part** of this report is authorized.

This report was prepared by:

Toke Jayachandran
Professor of Mathematics

I. Bert Russak
Professor of Mathematics

Reviewed by:

WALTER M. WOODS
Chairman

Released by:

DAVID W. NETZER
Associate Provost and Dean of Research

REPORT DOCUMENTATION PAGE

form Approved
OMB NO. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE April 1997		3. REPORT TYPE AND DATES COVERED Technical Report Sept 96 - Mar 97	
4. TITLE AND SUBTITLE Assessment of Damage Aggregation For Weapons Salvos				5. FUNDING NUMBERS N/A	
6. AUTHOR(S) Toke Jayachandran and I. Bert Russak					
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Postgraduate School Monterey, CA 93943-5000				8. PERFORMING ORGANIZATION REPORT NUMBER NPS-MA-97-003	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Naval Postgraduate School Monterey, CA 93943				10. SPONSORING/MONITORING AGENCY REPORT NUMBER	
11. SUPPLEMENTARY NOTES The views expressed in this report are those of the authors and do not reflect the official policy or position of the Dept of Defense of the United State.					
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited.				12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) Estimation of the expected proportion of an area target destroyed by a salvo of weapons is an important problem in weapons systems effectiveness studies. Analysts sometimes use an approximate formula called the "empirical rule" for this purpose. Esary [1], [2] demonstrated that this rule consistently over estimates the parameter of interest. In this report, we propose a modification to the empirical rule that appears to provide significantly improved estimates of the cumulative damage.					
14. SUBJECT TERMS estimation, weapons salvo, damage assessment, empirical rule				15. NUMBER OF PAGES 10	
				16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT unclassified	20. LIMITATION OF ABSTRACT		

ASSESSMENT OF DAMAGE AGGREGATION FOR WEAPONS SALVOS

Toke Jayachandran and I.Bert Russak

1. INTRODUCTION. Estimation of the expected proportion of an area target damaged by a salvo of weapons is an important problem in weapons systems effectiveness studies. Analysts sometimes use an approximate extant formula, referred to herein as the "empirical rule", for this purpose. The genesis and the rationale underlying this formula are not readily available. Esary [1] , [2] introduced a "proportional effects model" for which the empirical formula is a first-order approximation to the expected cumulative damage to the target. Assessment of this damage exactly would require a knowledge of the probability distribution of the number of weapons from the salvo that will impact on the target. Esary investigated the accuracy of the empirical rule approximation for three choices of the weapons impact distribution. His conclusion was that the empirical rule consistently overestimates the cumulative damage for all possible impact distributions and that the approximation can be quite poor for many realistic situations. The advantage in using the empirical rule is that one can compute, approximately, the cumulative damage using only the mean or expected number of weapons that will hit the target, without knowing the actual distribution of hits. In this report, we propose modifications to the empirical rule that use in addition to the mean, the variance of the hit distribution. These modifications result in significantly improved measures of the expected cumulative damage to the target.

2. PROBLEM DESCRIPTION. A salvo of n weapons is launched at an area target. The number of weapons that will actually hit the target is a random variable N whose possible values are $0, 1, 2, \dots, n$. Each hit will result in damaging a certain proportion of the target. The cumulative damage to the target is then a random variable D which depends on N the random number of weapons that impact the target. The quantity of interest is $E(D)$ the expected (or mean) cumulative damage. To estimate $E(D)$ exactly, the probability distribution of N the number of weapons that impact the target, the target configuration and the damage aggregation scenario for multiple hits are needed. However, in practice analysts are known to use the empirical rule

$$E(D) = 1 - (1-d)^{E(N)} \quad (1)$$

where d is the damage proportion for a single hit and $E(N)$ is the expected number of weapons out a salvo of n weapons that hit the target. As mentioned earlier, the genesis of this formula is not well documented. Esary [1] considered the following “proportional effects” model that can be used to provide a rationale for the empirical rule. The premise of this model is that a proportion d of a pristine target is damaged by the first hit and each additional hit damages the same proportion d of that part of the target not previously damaged. Then, it can be shown that if N weapons hit the target, the cumulative damage is

$$D(N) = 1 - (1-d)^N \quad (2)$$

and $E[D(N)]$, or simply $E(D)$, is the expected cumulative damage to the target. Replacing N with $E(N)$ in $D(N)$ will result in the empirical rule $E(D)$ which will be different from $E(D)$, in general. To evaluate $E(D)$ exactly, the probability distribution of N , the number of weapon hits is needed. Esary proved that the empirical rule consistently over estimates the expected cumulative damage i.e., $E(D) > E(D)$ for every possible hit distribution. If it is assumed that the distribution is Binomial i.e., if each of the n weapons hits the target independently with the same probability p , the expected cumulative damage

$$E_B(D) = 1 - (1-pd)^n \quad (3)$$

and the estimate using the empirical rule for this case is

$$E_B(D) = 1 - (1-D)^{np} \quad (4)$$

For a Poisson (sometimes used as an approximation to the Binomial) hit distribution with parameter $\lambda=np$,

$$E_P(D) = 1 - e^{-\lambda d} \quad (5)$$

and $E_P(D)$ will be the same as for the binomial distribution. The corresponding formulas for a Uniform hit distribution are

$$E_U(D) = 1 - (1-d)^{n+1}/(n+1)d \quad (6)$$

$$E_U(D) = 1 - (1-d)^{n/2}. \quad (7)$$

For the Binomial and the Poisson distributions, the exact values of $E(D)$ were compared with the corresponding approximate values for the empirical rule, for various choices of n , p and d , to demonstrate the magnitudes of the errors in the approximation. The resulting graphs are presented in figures 1-16. It can be seen that the two measures are reasonably close for small values of d ($0 < d < .5$) and they diverge markedly as d approaches 1 for all choices of n and p . However, the disparity decreases as n the number of weapons in the salvo increases.

3. A SECOND-ORDER TAYLOR APPROXIMATION. The Taylor approximating polynomial of degree two about the point $X = \mu_X$ for a function $f(X)$ of a random variable X with $\mu_X = E(X)$ is

$$f(X) = D(\mu_X) + f'(\mu_X)(X - \mu_X) + f''(\mu_X)(X - \mu_X)^2/2 \quad (8)$$

and taking expectations on both sides

$$E[f(X)] = f(\mu_X) + f''(\mu_X) \sigma_X^2/2 \quad (9)$$

where $\sigma_X^2 = V(X)$ the variance of the random variable X . Applying this result to $D(N)$ in equation (2) above, we get the second-order Taylor approximation to $E[D(N)]$,

$$\begin{aligned} T[D(N)] &= D(\mu_N) + D''(\mu_N) \sigma_N^2/2 \\ &= 1 - \{1 + [\ln(1-d)]^2 \sigma_N^2/2\} (1-d)^{\mu_N} \end{aligned} \quad (10)$$

where $\ln$ is the natural logarithm function and μ_N , σ_N^2 are the mean and the variance of the random variable N . We propose $T[D(N)]$, the Taylor approximation as a better alternative to the empirical rule in equation (1). Whereas only μ_N is needed to use the empirical rule, both μ_N and σ_N^2 are required for the use of this second-order approximation.

Graphs of the Taylor approximation are included in figures 1-16; they show significant improvements in the approximation relative to the empirical rule, for all choices of the parameters n, p, d . However, for

larger values of d ($d > .8$) the Taylor approximation errors are still a little on the high side (albeit smaller than that for the empirical rule), especially for small values of n the number of weapons in the salvo. To remedy this problem, we propose an ad hoc correction, determined through trial and error, to the Taylor approximation, to further reduce the error at the high values of d . Specifically, we suggest the modified formula

$$M[D(N)] = T[D(N)] - D^4(1-P^2)/2(N+1) , \quad (11)$$

Approximation error plots for the modified formula are also shown in figures 1-16. The modification appears to work well, in most cases, as can be seen from the closeness of the graphs to those for the exact formula. The exceptions are for the case of the Uniform hit distribution which is a flat distribution i.e., its graph is rectangular in shape. The Uniform distribution as a model for the number of hits in a salvo of n weapons is not very realistic since it would imply that the probability of 0 hits or 1 hit or 2 hits . . . or n hits is exactly the same. The distribution was included in this study to indicate the behavior of the approximations in the case of a relatively flat hit distribution.

Error percentages ($100 \times (\text{approximate} - \text{exact}) / \text{exact}$) are presented in Tables 1-3 for each of the three approximation formulas viz., Empirical, Taylor and Modified for $n=4,6,8$; $p=.4,.6,.8$; and the three hit distributions, Binomial, Poisson and the Uniform. These tables show more clearly the improvements attained by using the approximations proposed in this paper.

4. CONCLUSIONS. As shown by Esary, the empirical rule grossly overestimates the expected cumulative damage. We proposed two new approximate rules viz., the Taylor approximation and the Modified approximation that consistently provide better results than those for the empirical rule. The modified approximation provides the best measures unless the hit distribution is somewhat flat, i.e, the distribution is close to the uniform distribution; in which case the Taylor approximation appears to be the best choice.

TABLE I
PERCENT ERRORS FOR THE BINOMIAL DISTRIBUTION

N	EMPIRICAL						TAYLOR					MODIFIED				
	d p	.4	.6	.8	.95	.99	.4	.6	.8	.95	.99	.4	.6	.8	.95	.99
4	.4	11	15	18	16	15	2	2	6	12	15	0	0	1	4	5
	.6	6	7	6	4	3	1	2	3	3	3	0	1	0	-2	-4
	.8	2	7	1	0	0	0	1	1	0	0	0	0	-1	-3	-3
6	.4	9	10	9	6	5	0	2	4	5	5	0	1	2	0	0
	.6	4	3	2	1	1	1	1	1	1	1	0	0	-1	-3	-4
	.8	1	1	0	0	0	0	0	0	0	0	0	0	0	-2	-3
8	.4	7	7	4	2	2	1	2	3	2	2	0	1	1	-2	-3
	.6	3	2	1	0	0	0	1	0	0	0	0	0	-1	-3	-3
	.8	1	0	0	0	0	0	0	0	0	0	0	0	-1	-2	-2

TABLE II
PERCENT ERRORS FOR THE POISSON DISTRIBUTION

N	EMPIRICAL						TAYLOR					MODIFIED				
	d P	.4	.6	.8	.95	.99	.4	.6	.8	.95	.99	.4	.6	.8	.95	.99
4	.4	18	25	28	27	26	6	10	15	22	25	6	8	10	14	15
	.6	15	17	15	11	10	9	11	12	11	10	8	10	9	5	3
	.8	12	11	8	5	4	9	10	7	5	4	9	9	6	2	1
6	.4	15	17	15	11	10	6	8	10	10	10	5	7	7	5	3
	.6	10	9	6	3	3	6	6	5	3	3	6	6	3	-1	-2
	.8	7	5	2	1	0	6	4	2	1	1	6	4	-1	-1	-2
8	.4	12	11	8	5	4	5	6	6	5	4	5	5	4	1	0
	.6	7	5	2	1	1	5	4	2	1	1	5	3	1	-2	-3
	.8	4	2	1	0	0	4	2	1	0	0	4	2	0	-1	-2

TABLE III
PERCENT ERRORS FOR THE UNIFORM DISTRIBUTION

N	EMPIRICAL						TAYLOR					MODIFIED				
	d p	.4	.6	.8	.95	.99	.4	.6	.8	.95	.99	.4	.6	.8	.95	.99
4	.4	19	25	28	26	25	1	6	14	24	25	1	4	10	15	15
	.6	19	25	28	26	25	1	6	14	24	25	1	4	11	17	17
	.8	19	25	28	26	25	1	6	14	24	25	1	5	12	20	21
6	.4	20	23	21	18	17	3	9	16	17	17	3	8	13	12	10
	.6	20	23	21	18	17	3	9	16	17	17	3	8	13	13	12
	.8	20	23	21	18	17	3	9	16	17	17	3	8	14	15	14
8	.4	20	20	16	13	13	5	11	14	13	13	4	10	12	9	8
	.6	20	20	16	13	13	5	11	14	13	13	4	10	13	10	9
	.8	20	20	16	13	13	5	11	14	13	13	4	10	13	11	11

REFERENCES

- [1] J.D.Esary. *Studies On Damage Aggregation For Weapons Salvos*, Naval Postgraduate School Technical Report NPS55-90-16, July 1990.
- [2] J.D.Esary. *Studies On Damage Aggregation For Weapons Salvos II*, Naval Postgraduate School Technical Report NPSOR-92-007, December 1991.

Distribution List

	No. of copies
Director Defense Technology Information Center 8725 John J. Kingman Road., STE 0944 Ft Belvpor, VA 22060-6218	2
Research Office Code 09 Naval Postgraduate School Monterey, CA 93943	1
Library Code 013 Naval Postgraduate School Monterey, CA 93943	2
Professor Toke Jayachandran Department of Mathematics Naval Postgraduate School Monterey, CA 93943	3
Chairman, W. Max Woods Department of Mathematics Naval Postgraduate School Monterey, CA 93943	2
Professor I. Bert Russak Department of Mathematics Naval Postgraduate School Monterey, CA 93943	3