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THESIS

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Determination of Distribution of Aimpoints Against a Moving Target

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ABSTRACT

The determination of the optimal distribution of aimpoints is examined for weapons that fire fragmenting projectiles against mobile targets. A finite difference approximation which reduces the problem to a mathematical programming problem is developed. Computational considerations for this nonlinear programming problem are discussed.

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I. NATURE OF THE PROBLEM

Consider a situation where an artillery battery or naval gunfire support ship desires to fire at a moving land target. If the original position, speed, and direction of the target were known, the guns could be aimed so that the projectiles would impact at a calculated future location of the target. This future location is fairly easily calculated in this simple situation. The calculations become more difficult whenever any of the factors of original position, speed and direction are unknown. In these cases a probabilistic determination of future target location becomes necessary. It is then possible to state only that the target is more or less likely, in a probabilistic sense, to be at a certain location than at another.

This paper will examine the problem of determining the optimal distribution of aimpoints for a weapon which fires a projectile against a moving target whose speed is known but whose original position and the direction of movement are not.

A model for the location of a target under the above circumstances was developed by B. Koopman in 1946 (Ref. 1), and is described herein. Although formulated in a Naval setting, Koopman's model equally well applies to a moving target that has been detected by a forward observer; the exact position of the target being unknown. All that is known is that the target is more likely to be at a point 0 than at any other point. The target may not be at 0, however, but only within a short distance of 0, all points the same distance r from 0 being equally

likely. The probability that the target is in an area dA at a distance r from the target is defined as P(r)dA. Koopman assumed that the situation could be approximated by the circular normal distribution,

$$P(r) = \frac{1}{2\pi s^2} \exp\left\{-r^2/2s^2\right\}, \tag{1}$$

where s^2 is the variance in the original target location.

The speed of the target, u, is assumed to be known but the direction of the target movement is unknown, all directions being equally likely. After t units of time the situation may be as shown in Figure 1.

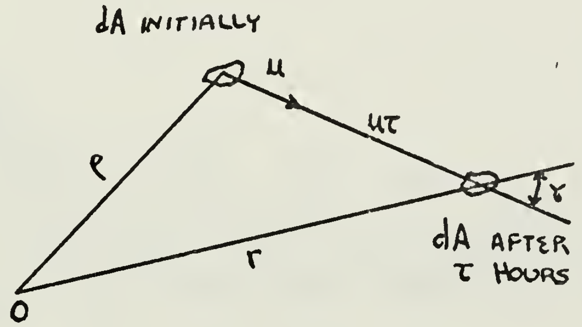


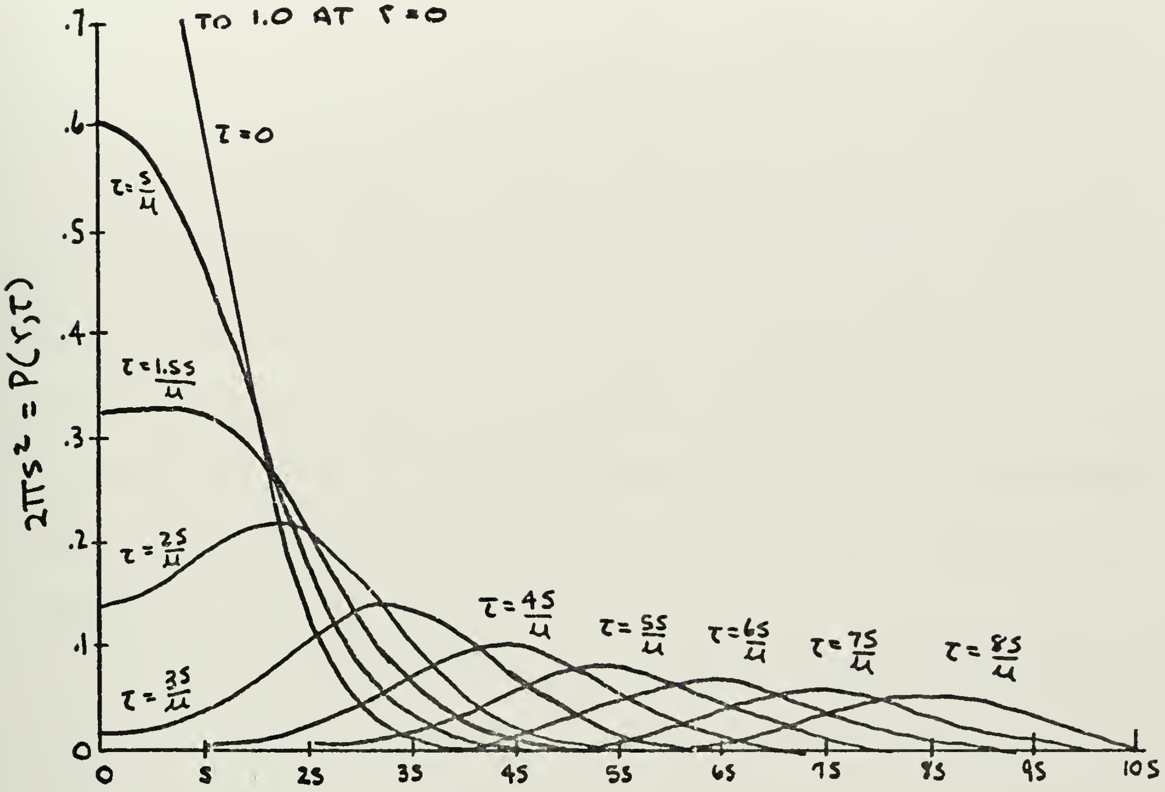
Figure 1. Entry of target into dA

Koopman determined the distribution of moving targets about a point 0 to be

$$P(r,t) = \frac{1}{2\pi s^2} \exp\left\{-(r^2+u^2t^2)/2s^2\right\} I_o\left(\frac{rut}{s^2}\right) \tag{2}$$

where I_o is the modified Bessel function of the first kind and zero order. The graph of P(r,t) for different t is shown in Figure 2.

Observe that the probability spreads outward with time so that the target is most likely to be in an expanding ring about 0.



Distance r from 0.

Figure 2. The distribution of moving targets about a point 0.

II. MATHEMATICAL MODELS OF WEAPONS FIRING FRAGMENTING PROJECTILES

It was necessary to examine the mathematical formulation of a weapon that fires fragmenting projectiles in order to determine the distribution of aimpoints at a moving target. The basic models for such weapons usually include four probability distributions.

The probability that a target located at coordinates x_t, y_t is killed by a round impacting at coordinates x,y is defined as $P_L(x_t-x,y_t-y)$. This probability may be considered as the lethality function. The lethality function is the conditional probability that a target at x_y, y_t, is killed given that a round impacts at x,y. It may take several forms of which exponential, linear and "cookie-cutter" are familiar. Reference 2 discusses lethality functions.

The distribution of impact points about the point of aim is defined as $P_I(x,y)$. This distribution is caused by meterological effects on the projectile and ballistic effects inherent in the weapons system. Some models have incorporated the distribution of impact points about the point of aim into the lethality function. It is then possible to define the probability of killing a target located at coordinates x_t, y_t with a round aimed at coordinates x_a, y_a. This probability is defined as $P_K(x_t-x_a,y_t-y_a)$.

The probability distribution of aimpoints for a target is dependent upon the nature of the target, its movement and location, and the type of weapons system employed. It is defined as $P_A(x_a,y_a)$.

The probability distribution of the location of the target is defined as $P_T(x_t,y_t)$.

These distributions have been combined to model weapons systems. This paper examines three of these models to illustrate the modeling techniques and to provide background for examining the problem of determining the distribution of aimpoints of a weapon firing fragmenting projectiles against a moving target. Although a different notation is used in each of the original source documents, a standard notation has been adopted for purposes of presentation in this thesis.

A. THE GROVES MODEL

This model was developed by Groves in Ref. 3. It is a simple model that does not consider a distribution of aimpoint; i.e. the aimpoint is fixed. If $P_L(x_t-x,y_t-y)$ is the lethality function representing the probability that a target located at coordinates x_t, y_t is killed by a round impacting at coordinates x,y, and $P_I(x,y)$ is the distribution of the impact points; then the probability of killing a target at x_t, y_t by a round aimed at coordinates x_a, y_a is

$$P_K(x_t-x_a,y_t-y_a) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_L(x_t-x,y_t-y)P_I(x,y)\,dx\,dy. \qquad (3)$$

The probability that a target survives one round is then $1-P_K(x_t-x_a,y_t-y_a)$, and the probability of surviving N rounds all aimed at the same aimpoint is $(1-P_K(x_t-x_a,y_t-y_a))^N$. The probability that the target is killed by any of the N rounds is $1-(1-P_K(x_t-x_a,y_t-y_a))^N$. This expression can then be integrated over the area of the target, T, to obtain the expected fractional kill, $\overline{K}$

$$\overline{K} = \iint_T (1-(1-P_K(x_t-x_a,y_t-y_a))^N)\,dx_t\,dy_t. \qquad (4)$$

The substitution of equation (3) into (4) yields

$$\bar{K} = \iint_T (1-(1-\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_L(x_t-x,\ y_t-y)P_I(x,y)\ dx\ dy)^N)\ dx_t\ dy_t. \quad (5)$$

Groves further reduces this expression to one that is more suitable for hand-computing the expected fractional kill. It is noted that the simplicity of this model is a result of the fixed aimpoint as seen by comparing the Groves model with the Weiss and the Breaux-Mohler models.

B. THE BREAUX-MOHLER MODEL

This model was published by Breaux and Mohler in Ref. 4. The model contains all of those distributions discussed earlier. It is used by Breaux and Mohler to compute the expected fraction of a target damaged by a salvo of fragmenting projectiles all aimed at the same aimpoint. They define the probability that a target located at coordinates x_t, y_t is damaged by a round impacting at coordinates x,y as $P_L(x_t-x,y_t-y)$. $P_I(x,y)$ is the density function describing the distribution of impact points x,y about the aimpoint x_a, y_a. It is assumed that all N rounds of a salvo are aimed at the same aimpoint. The probability a target survives N rounds is

$$(1-\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_L(x_t-x,y_t-y)P_I(x,y)\ dx\ dy)^N, \quad (6)$$

and the probability of damage over all impact points is

$$1-(1-\int_{-\infty}^{\infty} \int_{\infty}^{\infty} P_L(x_t-x,y_t-y)P_I(x,y)dx\ dy)^N. \quad (7)$$

Breaux and Mohler assume that the target is distributed over the target area, T, as $P_T(x_t,y_t)$, and the aimpoint itself is a random variable

distributed as $P_A(x_a,y_a)$. The expected fraction of the target damaged is determined to be

$$\bar{f} = \iint_T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[1-\left(1-\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_L(x_t-x,y_t-y)P_I(x,y)dx\,dy\right)^N\right] P_T(x_t,y_t)$$

$$\cdot\, P_A(x_a,y_a)dx_t dy_t\, dx_a dy_a. \tag{8}$$

This expression is reduced using a binomial expansion to.

$$\bar{f} = \sum_{j=1}^{N} (-1)^{j+1}\binom{N}{j} \iint_T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_L(x_t-x,y_t-y)P_I(x,y)dx\,dy\right]^j$$

$$\cdot P_T(x_t,y_t)P_A(x_a,y_a)dx_t\, dy_t\, dx_a\, dy_a. \tag{9}$$

Breaux and Mohler further reduce equation (8) using Jacobi polynomials to produce an expression that can be used for determining the expected fractional kill.

C. THE WEISS MODEL

This model was developed by Weiss in Ref. 5. It will be discussed more thoroughly than the Groves and Breaux-Mohler models because the model is used later in the paper. Weiss assumes a target of n men that is distributed as $P_T(x_t,y_t)$. The probability that a man is in the small area $dx_t\, dy_t$ is $P_T(x_t,y_t)dx_t dy_t$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_T(x_t,y_t)dx_t dy_t = 1. \tag{10}$$

The expected number of targets in a small area is $nP_T(x_t,y_t)dx_t dy_t$. The probability that a round aimed at the aimpoint x_a, y_a will kill a target is $P_K(x_t-x_a,y_t-y_a)$. Weiss assumes that N rounds are fired at the target and the ith round has an aimpoint x_{ai}, y_{ai}. The probability

that a target at x_t, y_t survives all N rounds is

$$q(x_t-x_a,y_t-y_a) = \prod_{i=1}^{N} \left(1-P_K(x_t-x_{ai},y_t-y_{ai})\right), \tag{11}$$

and the probability that a target is killed is

$$1-q(x_t-x_a,y_t-y_a). \tag{12}$$

The probability that there is a target at x_t, y_t and that is killed is

$$P_T(x_t,y_t)\left(1-q(x_t-x_a,y_t-y_a)\right). \tag{13}$$

The expected number killed (K) is

$$\begin{aligned} K &= n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)\left(1-q(x_t-x_a,y_t-y_a)\right)dx_t dy_t, \\ &= n\int_{-\infty}^{\infty}\int_{\infty}^{\infty} P_T(x_t,y_t)dx_t dy_t - n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)q(x_t-x_a,y_t-y_a)dx_t dy_t, \\ &= n\left[1-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)q(x_t-x_a,y_t-y_a)dx_t dy_t\right]. \end{aligned} \tag{14}$$

The expected fraction surviving Φ is $\frac{n-K}{n}$, or

$$\begin{aligned} \Phi &= \frac{n}{n} - \frac{n}{n}\left[1-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)q(x_t-x_a,y_t-y_a)\ dx_t dy_t\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)q(x_t-x_a,y_t-y_a)\ dx_t\ dy_t. \end{aligned} \tag{15}$$

If each round has the same aimpoint, then

$$\Phi(x_a,y_a) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)(1-P_K(x_t-x_a,y_t-y_a))^N dx_t dy_t, \tag{16}$$

and using a binomial expansion

$$\begin{aligned} \Phi(x_a,y_a) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t) \sum_{j=0}^{N}(-1)^N\binom{N}{j}P_K(x_t-x_a,y_t-y_a)^j dx_t dy_t, \\ &= \sum_{j=0}^{N}(-1)^N\binom{N}{j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)P_K(x_t-x_a,y_t-y_a)^j\ dx_t dy_t. \end{aligned} \tag{17}$$

Assuming that the aimpoints themselves are distributed as $P_A(x_a,y_a)$, the expected fraction of targets surviving is

$$\Phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_A(x_a,y_a)\Phi(x_a,y_a)dx_a\, dy_a, \tag{18}$$

$$\Phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_A(x_a,y_a) \sum_{j=0}^{N} (-1)^N \binom{N}{j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_T(x_t,y_t)P_K(x_t-x_a,y_t-y_a)^j \cdot dx_t dy_t\, dx_a dy_a. \tag{19}$$

D. COMPARISON OF BREAUX-MOHLER AND WEISS MODELS

It is easily shown that the Breaux-Mohler and Weiss models give the same results. Substituting equation (3) into equation (8) gives the expected fraction damaged as

$$\bar{f} = \iint_T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[1-(1-P_K(x_t-x_a,y_t-y_a))^N\right]P_T(x_t,y_t)P_A(x_a,y_a)dx_t dy_t dx_a dy_a. \tag{20}$$

Substituting equation (16) into equation (18) gives the expected fraction of targets surviving as

$$\Phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_A(x_a,y_a) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_T(x_t,y_t)(1-P_K(x_t-x_a,y_t-y_a))^N dx_t dy_t dx_a dy_a \tag{21}$$

It is clear that these models are simply complements of each other.

III. THE DETERMINATION OF NECESSARY AND SUFFICIENT CONDITIONS

The mathematical models described in chapter II can be used to determine necessary and sufficient conditions for determining the aimpoint distribution that maximizes the expected fraction of the

target destroyed or minimizes the expected fraction of survivors, depending upon the model used. Reference 6 discusses mathematical models of hit probabilities and the techniques of maximizing salvo kill probabilities. Weiss (Ref. 7) attacks this problem using techniques of Svesnikov (Ref. 8) and Morse and Kimball (Ref. 9). His approach to the problem of maximizing the salvo kill probability (or minimizing the fraction of survivors) required the application of the calculus of variations. This application of the calculus of variations can be understood by examining the problem considered by Morse and Kimball in Ref. 9. Their treatment is presented herein to facilitate the understanding of Weiss' development of the necessary and sufficient conditions contained in Appendix A.

A. THE MORSE AND KIMBALL PROBLEM

Morse and Kimball (Ref. 9) examined the problem of determining the firing pattern that maximizes the probability of at least one hit on a target when N rounds are fired in a single salvo. An approximate solution is determined for large patterns.

Morse and Kimball define the probability that a projectile will hit the small area element dxdy as f(x,y)dxdy. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = N, \tag{22}$$

where N is the number of rounds fired. The function f(x,y) is considered to be the pattern density function. Extensive changes in the hit probability can be obtained by changing the firing pattern. The expected number of lethal hits on a target located at coordinates x,y is L f(x,y),

where L is the lethal area of the target. Morse and Kimball approximate the expected number of hits on the target using the Poisson probability distribution. The probability of at least one hit is $1\text{-}\exp\left\{Lf(x,y)\right\}$, (23) which is the probability of destroying the target. The total probability of destroying the target is then

$$P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1-e^{-Lf(x,y)})f_p(x,y)dxdy, \tag{24}$$

where f_p is the probability density for aiming the pattern, usually the normal density. The problem then is to determine the function f(x,y) which maximizes P subject to equation (22). The problem is maximize

$$P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1-e^{-Lf(x,y)})f_p(x,y)dxdy,$$

$$\text{Subject to} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = N, \tag{25}$$

$$f(x,y) \geq 0.$$

It is the method of solution of this problem by Morse and Kimball that is interesting. They consider a pertubation of the function f(x,y); that is, an increase of f(x,y) by a small amount δ at the point x_1,y_1, and a decrease of f(x,y) by the same amount δ at x_2,y_2. The constraint remains satisfied while the objective function changes by an amount

$$\left(e^{-Lf(x_1,y_1)}f_p(x_1,y_1) - e^{-Lf(x_2,y_2)}f_p(x_2,y_2)\right)\delta \; dxdy.$$

Suppose $f(x_1,y_1)$ and $f(x_2,y_2)$ are > 0; then if

$$e^{-Lf(x_1,y_1)}f_p(x_1,y_1) > e^{-Lf(x_2,y_2)}f_p(x_2,y_2),$$

P can be increased by a $\delta > 0$. Conversely, if

$$e^{-Lf(x_1,y_1)}f_p(x_1,y_1) < e^{-Lf(x_2,y_2)}f_p(x_2,y_2),$$

then P can be increased by $\delta < 0$. Hence, for that function $f(x,y)$ which maximizes P, it must be the case that

$$e^{-Lf(x_1,y_1)} f_p(x_1,y_1) = e^{-Lf(x_2,y_2)} f_p(x_2,y_2) \qquad (26)$$

for all points x,y where $f(x,y) > 0$. For all such points

$$e^{-Lf(x,y)} f_p(x,y) = c > 0. \qquad (27)$$

Now, instead of x_2,y_2 a point x_3,y_3 where $f(x,y)=0$ is considered. If $f(x_1,y_1)$ is decreased by δ (which now must be positive) and $f(x_3,y_3)$ is increased by δ, then the increase in the objective function is

$$\left(f_p(x_3,y_3) - e^{-Lf(x_1,y_1)} f_p(x_1 y_1)\right) \delta \; dxdy, \qquad (28)$$

which equals $\left(f_p(x_3,y_3) - c\right) \delta \; dxdy$. This implies a positive increase in P is possible if $f_p(x_3,y_3) > c$. Hence, $f(x,y)$ cannot equal 0 unless $f_p(x,y) \leq c$. It is seen that the solution is

$$\begin{aligned} \delta(x,y) &= \frac{1}{L} \operatorname{Ln} \left(\frac{f_p(x,y)}{c}\right) && \text{if } f_p(x,y) > c, \\ &= 0 && \text{if } f_p(x,y) \leq c. \end{aligned} \qquad (29)$$

The unknown constant c is determined from the constraint

$$N = \iint_{f_p(x,y)>c} \frac{1}{L} \operatorname{Ln}\left(\frac{f\;(x,y)}{c}\right) dx\, dy. \qquad (30)$$

The set of conditions of equations (29) form a set of necessary and sufficient conditions for determining that pattern that maximizes the total probability of destroying the target.

Weiss' analysis of the problem of minimizing the fraction of targets surviving a salvo of N rounds is presented in detail in Appendix A. The original paper (Ref. 7) is greatly condensed and the entire development is presented in this paper to facilitate the understanding of the techniques used in the solution.

IV. THE SOLUTION

A. ANOTHER REPRESENTATION OF THE PROBLEM

Weiss' coverage problem discussed in Appendix 1 can be written as

$$\text{Minimize} \quad \Phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_T(x_t,y_t)\; e^{-u(x_t,y_t)} dx_t\; dy_t;$$

Subject to

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_A(x_a,y_a)\; dx_a\; dy_a = N \tag{31}$$

$$P_A(x_a,y_a) \geq 0$$

where

$$u(x_t,y_t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_A(x_a,y_a) \operatorname{Ln} q(x_t-x_a,y_t-y_a) dx_a dy_a.$$

$P_T(x_t,y_t)$ and $P_A(x_a,y_a)$ are the target and aimpoint distributions described earlier. It is noted that in addition to the constraints above, it is trivially true that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_T(x_t,y_t) = 1. \tag{32}$$

The solution to this problem is determined in Appendix 1 to be

$$\begin{aligned} -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_T(x_t,y_t)\; e^{-u(x_t,y_t)} \operatorname{Ln} q(x_t-x_{a1},y_t-y_{a1}) dx_t dy_t &= C \quad \text{if } P_A(x_a,y_a)>0 \\ &> C \quad \text{if } P_A(x_a,y_a)=0 \end{aligned} \tag{33}$$

where $q(x_t-x_{a1},y_t-y_{a1})$ relates to the lethality function. It is not possible to simply substitute other than trivial target distributions or lethality functions into the necessary and sufficient conditions of equations (33) and determine the distribution of aimpoints that

minimizes the number of survivors. However, a solution to these equations may be obtained by using a finite difference approximation to the integrals. This paper assumes that the solution to the finite difference approximation converges to the solution of the original problem. The proof of this convergence, however, is beyond the scope of this paper. It is realized that finite difference techniques may guarantee solution for specific values but usually preclude the seeing of relationships between equation parameters.

Equation (31) can be approximated as

$$\begin{aligned} &\text{Minimize} \quad \sum_{x_t=1}^{k} \sum_{y_t=1}^{k} P_T(x_t,y_t) e^{-u(x_t,y_t)}, \\ &\text{Subject to} \quad \sum_{x_a=1}^{k} \sum_{y_a=1}^{k} P_A(x_a,y_a) = N, \\ &\qquad\qquad P_A(x_a,y_a) \geq 0, \end{aligned} \tag{34}$$

$$\text{where } u(x_t,y_t) = \sum_{x_a=1}^{k} \sum_{y_a=1}^{k} P_A(x_a,y_a) \text{Lnq}(x_t-x_a,\ y_t-y_a)$$

and the upper limit of summation, k, is the number of increments in the approximation. Equations (33) can similiarly be approximated by

$$\begin{aligned} -\sum_{x_t=1}^{k} \sum_{y_t=1}^{k} P_T(x_t,y_t)\, e^{-u(x_t,y_t)} \text{Lnq}(x_t-x_{a1},y_t-y_{a1}) &= C \quad \text{if } P_A(x_a,y_a)>0 \\ &> C \quad \text{if } P_A(x_a,y_a)=0. \end{aligned} \tag{35}$$

The corresponding finite approximation to equation (32) is

$$\sum_{x_t=1}^{k} \sum_{y_t=1}^{k} P_T(x_t,y_t) = 1. \tag{36}$$

It is noted that equations (35) are the derivatives with respect to $P_A(x_a,y_a)$ of the objective function of equation (34) when the expression

for $u(x_t, y_t)$ is substituted into the objective function. Equations (35) can be written as

$$\frac{\partial \Phi}{\partial P_A(x_a, y_a)} \begin{array}{ll} = C & \text{if } P_A(x_a, y_a) > 0 \\ > C & \text{if } P_A(x_a, y_a) = 0 \end{array} \qquad (37)$$

It is noted that the objective function is convex in $P_A(x_a, y_a)$ and that equations (35) and (37) are simply the results of Gibbs Lemma[1] as applied to equations (34).

A solution to equations (35) can be obtained with a computer by dividing the target area into cells sufficiently small for approximation of the small areas $dx_t dy_t$ and $dx_a dy_a$. To simplify the computer procedure, the following notations will be used:

P_i = The expected fraction of the target in cell i; it is analogous to the target distribution.

y_j = The proportion of the total rounds aimed at cell j; it is analogous to the aimpoint distribution $P_A(x_a, y_a)$;

b_{ij} = The probability that a target in cell i is killed by a round aimed at cell j; it is analogous to the lethality function $q(x_t - x_a, y_t - y_a)$.

The problem in equation (34) can then be rewritten as

$$\text{Minimize} \quad \Phi = \sum_{i=1}^{k} P_i e^{-\sum_{i=1}^{k} \sum_{j=1}^{k} y_j b_{ij}},$$

$$\text{Subject to} \quad \sum_{i=1}^{k} P_i = 1, \qquad (38)$$

$$\sum_{j=1}^{k} y_j = 1$$

$$P_i, y_j, b_{ij} \geq 0$$

---

[1]Gibbs Lemma is discussed in Ref. 10.

The solution is in terms of the y_j's and k is the total number of cells.

Equations (35) and (37) can be rewritten as

$$\frac{\partial \Phi}{\partial y_j} = -P_i\, b_{ij}\, e^{-y_j b_{ij}} \quad \begin{array}{ll} = C & \text{if } y_j > 0, \\ > C & \text{if } y_j = 0, \end{array} \tag{39}$$

or, equivalently,

$$P_i b_{ij}\, e^{-y_j b_{ij}} \quad \begin{array}{ll} = Y & \text{if } y_j > 0, \\ < Y & \text{if } y_j = 0, \end{array} \tag{40}$$

where $Y = -C$.

The problem thus presented is similar to one presented by Professor John M. Danskin to his Games of Strategy class at the U. S. Naval Postgraduate School in December, 1970.

B. THE COMPUTER SOLUTION TECHNIQUE

A solution technique is to divide an area of one Kilometer square into cells of twenty meters square. The P_i's are then determined for each of the 2,500 cells from the target distribution. To solve the problem of firing at targets obeying Koopmans' moving target distribution described earlier, values of time of impact and speed of the target are inputs and the P_i's are determined from equation (2) for each cell of the simulated target area. The b_{ij}'s are determined from the lethality functions for the weapon system being investigated. For an artillery weapon it is the case that the b_{ij}'s = 0 for all cells j more than, say, 60 meters from cell i. This means the number of machine calculations required for solving the problem in equations (40) would be reduced considerably as the sum over j would be limited to the 48 cells immediately surrounding cell i and, of course, cell i itself.

It is noted that if $b_{ij} = 0$ for all $j \neq i$; that is, the probability that targets outside the aimpoint cell are killed is zero, the solution is

$$y_j = \frac{1}{b_{ij}} \operatorname{Ln} \frac{P_i b_{ij}}{\gamma} \qquad \text{if } P_i b_{ij} > \gamma ,$$

$$= 0 \qquad \text{if } P_i b_{ij} \leq \gamma .$$

γ is determined from the equation

$$\sum_{i=1}^{k} \frac{1}{b_{ij}} \operatorname{Ln} \frac{P_i \, b_{ij}}{\gamma} = 1$$

which is analyzed in Ref. 10.

After the values for the P_i's have been initialized and the lethality function formulas are determined for the cells surrounding each cell i, a beginning set of values for each y_j is initialized. A possible set of y_j's is $y_1=1$, $y_2=\ldots= y_{2500}=0$. It is then necessary to determine if that initialized set of y_j's give a minimum value of Φ and, if not, in which direction to move (i.e. what other vector of y_j's) to produce a lower Φ.

An algorithm has been developed by Professor Danskin that will cause convergence to a vector of y_j's that produce a minimum Φ. The computer program for this algorithm as well as the application of the algorithm to the problem discussed in this paper are presented as a separate thesis for Professor Danskin by Major Paul T. Zmuida, USA.

APPENDIX A

Weiss desires to minimize

$$\Phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_T(x_t,y_t) \prod_{i=1}^{N} q(x_t-x_{ai},y_t-y_{ai})dx_t dy_t, \tag{A-1}$$

Subject to

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_T(x_t,y_t)dx_t dy_t = 1,$$

$$P_T(x_t,y_t) \geq 0,$$

where Φ is the expected fraction of the target surviving, $P_T(x_t,y_t)$ is the target distribution, and $\prod_{i=1}^{N} q(x_t-x_{ai},y_t-y_{ai})$ is the probability a target at coordinates x_t,y_t survives a salvo of N rounds when the ith round is aimed at aimpoint x_{ai},y_{ai}. Weiss wishes to minimize Φ by a proper choice of x_{ai},y_{ai}.

A new function is defined such that

$$u(x_t,y_t) = -\mathrm{Ln} \prod_{i=1}^{N} q(x_t-x_{ai}, y_t-y_{ai}),$$

$$= -\mathrm{Ln} \left[q(x_t-x_{a1},y_t-y_{a1}) \cdot \ldots \cdot q(x_t-x_{aN},y_t-y_{aN}) \right],$$

$$= -\left[\mathrm{Ln}\, q(x_t-x_{a1},y_t-y_{a1}) + \ldots + \mathrm{Ln}\, q(x_t-x_{aN},y_t-y_{aN}) \right],$$

$$= -\sum_{i=1}^{N} \mathrm{Ln}\, q\,(x_t-x_{ai},y_t-y_{ai}),$$

$$\text{or } -u(x_t,y_t) = \sum_{i=1}^{N} \mathrm{Ln}\, q\,(x_t-x_{ai},y_t-y_{ai}). \tag{A-2}$$

The objective function can then be rewritten as

$$\Phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_T(x_t,y_t)\, e^{-u(x_t,y_t)} dx_t dy_t. \tag{A-3}$$

Weiss then makes an approximation that a distribution function for the aimpoints can be substituted for the exact knowledge of x_{ai}, y_{ai} so that the number of rounds aimed at area $dx_a dy_a$ is $P_A(x_a,y_a)dx_a dy_a$. The total number of rounds is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_A(x_a,y_a)dx_a dy_a = N, \tag{A-4}$$

and since negative rounds cannot be fired,

$$P_A(x_a,y_a) \geq 0. \tag{A-5}$$

Equation (A-2) can then be written as

$$-u(x_t,y_t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_A(x_a,y_a)\text{Ln } q\ (x_t-x_a,y_t-y_a)dx_a dy_a. \tag{A-6}$$

The complete problem was minimize

$$\Phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)\ e^{-u(x_t,y_t)}dx_t,dy_t,$$

subject to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,x_a)dx_t dy_t = 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_A(x_a,y_a)dx_a dy_a = N.$$

$$P_T(x_t,y_t) \geq 0,\ P_A(x_a,y_a) \geq 0$$

Weiss continues his solution using techniques similar to those of Morse and Kimball described earlier. For any arbitrary $P_A(x_a,y_a)$ add a small increment δ to $P_A(x_a,y_a)$ over the interval Δx_a at x_{a1} and Δy at y_{a1}. Now examine the change in Φ. The procedure can be illustrated as shown in Figure 3.

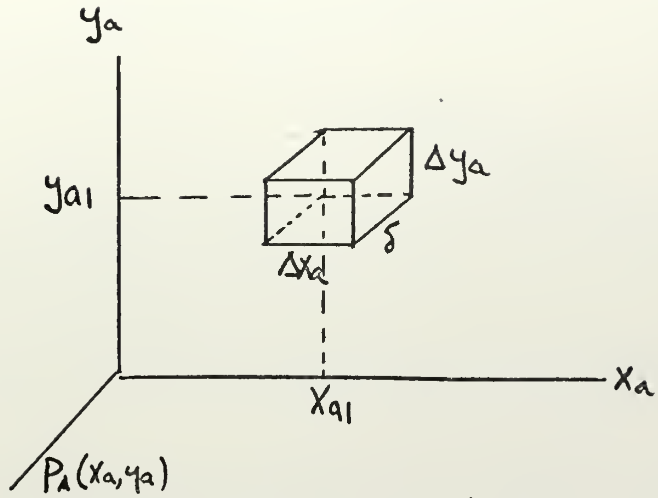


Figure 3. Illustration of small increment δ added to $P_A(x_a,y_a)$ over interval Δx_a at x_{a1} and Δy_a at y_{a1}.

The change in Φ can be expressed as

$$\Phi_{P_A(x_a,y_a)+\Delta P_A(x_a,y_a)} - \Phi_{P_A(x_a,y_a)} = \Delta\Phi . \tag{A-7}$$

From equation (A-3),

$$\Delta\Phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)\ \exp\left\{-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(P_A(x_a,y_a)+\Delta P_A(x_a,y_a)\Big)\mathrm{Ln}\,q(x_t-x_{a1},y_t-y_{a1})\cdot dx_a dy_a\right\}$$
$$-\exp\left\{-\int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} P_A(x_a,y_a)\mathrm{Ln}\ q(x_t-x_{a1},y_t-y_{a1})\ dx_a dy_a\right\}\ dx_t dy_t .$$

$$\Delta\Phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)\ \exp\left\{-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_A(x_a,y_a)\mathrm{Ln}\ q(x_t-x_{a1},y_t-y_{a1})dx_a dy_a\right.$$
$$\left.-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta P_A(x_a,y_a)\ \mathrm{Ln}\ q(x_t-x_{a1},y_t-y_{a1})dx_a dy_a\right\}-\exp\left\{-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_A(x_a,y_a)\right.$$
$$\left.\cdot\mathrm{Ln}\ q(x_t-x_{a1},y_t-y_{a1})dx_a dy_a\right\}\ dx_t dy_t ,$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)\ e^{-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_A(x_a,y_a)\mathrm{Ln}\,q(x_t-x_{a1},y_t-y_{a1})dx_a dy_a}$$
$$\cdot\left[e^{-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta P_A(x_a,y_a)\mathrm{Ln}\ q(x_t-x_{a1},y_t-y_{a1})dx_a dy_a}-1\right]\ dx_t dy_t ,$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)e^{-u(x_t,y_t)}\left[e^{-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta P_A(x_a,y_a)\mathrm{Ln}\, q(x_t-x_{a1},y_t-y_{a1})}\right.$$

$$\left.\cdot dx_a dy_a - 1\right] dx_t dy_t. \qquad \text{(A-8)}$$

Since $P_A(x_t,y_t)=\delta$ and Ln q $(x-x_{a1},y-y_{a1})$ is approximately constant over a very small area $dx_a dy_a$,

$$\Delta\Phi \simeq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)e^{-u(x_t,y_t)}\left[e^{-\delta\Delta x_a\Delta y_a\ \mathrm{Ln}\, q(x_t-x_{a1},y_t-y_{a1})}-1\right]dx_t dy_t. \qquad \text{(A-9)}$$

If the bracketed term is now expanded in a Taylor's expansion of e and a first order approximation is applied, then

$$\Delta\Phi \simeq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)e^{-u(x_t,y_t)}\left[-\delta\Delta x_a\Delta y_a\ \mathrm{Ln}\, q(x_t-x_{a1},y_t-y_{a1})\right]dx_t dy_t$$

$$\simeq -\delta\Delta x_a\Delta y_a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)e^{-u(x_t,y_t)}\mathrm{Ln}\, q(x_t-x_{a1},y_t-y_{a1})dx_t dy_t.$$

Weiss defines

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)e^{-u(x_t,y_t)}\mathrm{Ln}\, q(x_t-x_{a1},y_t-y_{a1})dx_t dy_t = \rho(x_{a1},y_{a1}),$$

and then

$$\Delta\Phi \simeq \rho(x_{a1},y_{a1})\delta\Delta x_a\Delta y_a. \qquad \text{(A-10)}$$

Now either $P_A(x_a,y_a) > 0$ or $P_A(x_a,y_a) = 0$. That is, either some rounds or no rounds are aimed at the immediate vicinity of x_a,y_a. If $P_A(x_a,y_a) > 0$, δ can be added or subtracted from it; if $P_A(x_a,y_a) = 0$, δ can only be added to it.

Weiss considers two points x_{a1},y_{a1} and x_{a2},y_{a2} where $P_A(x_a,y_a) > 0$: δ is added to $P_A(x_a,y_a)$ at one point and subtracted from the other. The change in Φ can be determined from equation (A-10).

$$\Delta\Phi \simeq \Delta\Phi\,(x_{a1},y_{a1}) - \Delta\Phi\,(x_{a2},y_{a2}),$$

$$\simeq \rho\,(x_{a1},y_{a1})\;\delta\Delta x_a\Delta y_a - \rho\,(x_{a2},y_{a2})\,\delta\Delta x_a\Delta y_a,$$

$$\simeq \left[\rho\,(x_{a1},y_{a1}) - \rho\,(x_{a2},y_{a2})\right]\delta\Delta x_a\Delta y_a. \tag{A-11}$$

If (A-11) is positive then Φ can be reduced further by changing the sign of δ, which it is permissible to do since $P_A(x_a,y_a) > 0$ at both points. It is clear that if

$$\rho\,(x_{a1},y_{a1}) < \rho\,(x_{a2},y_{a2}) \text{ then } \Delta\Phi < 0 \quad \text{if } \delta > 0,$$

$$\rho\,(x_{a1},y_{a1}) > \rho\,(x_{a2},y_{a2}) \text{ then } \Delta\Phi < 0 \quad \text{if } \delta < 0.$$

The conditions for an optimum exists when

$$\rho\,(x_{a1},y_{a1}) = \rho\,(x_{a2},y_{a2}) = \text{constant } C, \tag{A-12}$$

for all points where $P_A(x_a,y_a) > 0$.

Weiss then examines a point x_{a3},y_{a3} where $P_A(x_a,y_a) = 0$. $P_A(x_a,y_a)$ can only be added to at this point. Comparing x_{a3},y_{a3} to the point where $P_A(x_a,y_a) > 0$,

$$\Delta\Phi = \left[\rho\,(x_{a3},y_{a3}) - \rho(x_{a1},y_{a1})\right]\delta\Delta x_a\Delta y_a,$$

which is negative if and only if

$$\rho\,(x_{a3},y_{a3}) < \rho\,(x_{a1},y_{a1}) = C. \tag{A-13}$$

That is to say Φ can be decreased only if $\rho\,(x_{a3},y_{a3}) < C$. At the optimum solution Φ can no longer be decreased.

The original problem is now reduced to solving the following conditions:

$$\rho\,(x_{a1},y_{a1}) \begin{cases} = C & \text{if } P_A(x_a,y_a) > 0 \\ < C & \text{if } P_A(x_a,y_a) = 0, \end{cases} \tag{A-14}$$

where $\rho(x_{a1},y_{a1}) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_T(x_t,y_t)\, e^{-u(x_t,y_t)} \ln q(x_t-x_{a1},y_t-y_{a1})dx_t dy_t$

and $u(x_t,y_t)$ is as defined in equation (A-6).

Weiss has determined necessary and sufficient conditions for the aimpoint distribution to be positive. The problem now has been reduced to solving equations (A-14) which is a difficult problem and to which there exists no general solution in closed form.

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13. ABSTRACT

The determination of the optimal distribution of aimpoints is examined for weapons that fire fragmenting projectiles against mobile targets. A finite difference approximation which reduces the problem to a mathematical programming problem is developed. Computational considerations for this nonlinear programming problem are discussed.

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