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SAMUEL C. COLWELL

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by

Samuel C. Colwell, III

Submitted in partial fulfillment of
the requirements for the degree of

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Monterey, California

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ABSTRACT

During the past several years at the United States Naval Postgraduate School there has been much interest in obtaining an efficient method for making a time schedule for classes. A mathematical model for a simplified version of this scheduling problem was devised by several faculty members, and this paper is a study of this model.

This paper, while not offering a general solution to the simplified scheduling problem, does provide some insight into the problem and suggests areas for future study that may lead to a general solution.

The paper is presented in four parts, the first being an explanation of the problem in terms of Boolean algebra. The second part restates the problem in terms of graph theory, showing that this problem is the same as the problem of finding the chromatic number of a given graph. The third part is an attempt to gain insight into a solution of this problem by an exhaustive study of all graphs of order six and less, which are tabulated along with certain of their attributes. The fourth part is a study of certain random graphs of higher order. Among other things this study uses the digital computer to find the number of complete subgraphs of every order within each graph examined.

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1. Introduction

During the past several years at the U. S. Naval Postgraduate School in Monterey, California, there has been much attention given to the problem of class scheduling. It is a tremendous undertaking to create a time schedule such that each student can take the courses specified for him without conflict, where the instructor and certain facilities are also specified in advance. It was an attempt to make a mathematical model of this scheduling problem that led several faculty members to formulate a simplified version of the scheduling problem which is the subject investigated in this paper.

This problem was suggested to me by Dr. E. J. Stewart, Professor of Mathematics at the Naval Postgraduate School, in his course in Boolean algebra in the fall of 1962. It was Dr. Stewart who first drew my attention to such concepts as Boolean vector, conflict matrix, etc., which resulted in the definitions given in Section 2. The subsequent investigation of the topic was directed by Professor W. R. Church, Chairman of the Department of Mathematics at the Naval Postgraduate School. The fact that the partition problem as originally stated is related to a problem in graph theory, along with many other concepts in the paper, can be attributed to Dr. Church. Although it cannot be said that this paper solves the problem investigated, it has presented some insight into the problem and suggests areas for future study that may lead to a solution.

The author assumes that the reader has a knowledge of elementary set theory, modern algebra, and Boolean algebra. Although it will be noticed that most of the material in this paper is given in terms of graph theory, the material in this area is virtually self-contained and requires no previous knowledge on the part of the reader.

The paper itself is divided into four parts, the first being an explanation of the problem in terms of Boolean algebra. The second part restates the problem in terms of graph theory, showing that this problem is the same as the original problem. The third part is an attempt to gain insight into a solution of this problem by an exhaustive study of all graphs of order less than or equal to 6, and the fourth part is a study of certain graphs of higher order.

Throughout the paper, the numbers in the brackets following specific definitions and statements are keyed to the references in the bibliography.

I would like to express my gratitude for the encouragement, guidance, and inspiration which Professor W. R. Church provided and without which this paper would not have been possible.

2. Statement of the Problem.

In order to clarify the statement of the problem, it will first be necessary to make the following definitions.

Definition 1: An m-dimensional row vector, of which every element is a 0 or a 1 is called a Boolean vector.

Definition 2: A Boolean matrix, A, is a rectangular array of elements a_{ij}, each of which can have only the value 0 or 1. Notice that Boolean vectors are 1 x m Boolean matrices.

Definition 3: The product of two Boolean matrices is a Boolean matrix which is obtained by ordinary matrix multiplication, except that the addition and multiplication are logical addition and multiplication respectively. i.e., If A and B are Boolean matrices conformable for multiplication, then $AB = C$, where $C_{ij} = \sum_k a_{ik} \cdot b_{kj}$

Definition 4: The complement of a Boolean matrix A, denoted by A', is defined as: $A' = C$, where $C_{ij} = a'_{ij}$, and a'_{ij} is 0 or 1 accordingly as a_{ij} is 1 or 0.

Definition 5: The sum of two Boolean matrices A and B is defined as: $A + B = C$, where $C_{ij} = a_{ij} + b_{ij}$.

Definition 6: The transpose of a Boolean matrix A, denoted A^t, is formed by interchanging rows and columns of A. i.e., $A^t = C$, where $C_{ij} = a_{ji}$.

Definition 7: Given a Boolean matrix A, then AA^t is called the conflict matrix of A.

If $X_1 = (a_1, a_2, \ldots, a_m)$, where $a_i = 0$ or 1

$X_2 = (b_1, b_2, \ldots, b_m)$, where $b_i = 0$ or 1

then $X_1 + X_2 = (a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m)$ and

$X_1 X_2^t = (a_1 b_1, a_2 b_2, \ldots, a_m b_m)$.

The combination of Boolean vectors $X_1 X_2$ will be denoted by $X_1 \cdot X_2$, and will be called the logical product of X_1 and X_2. The combination $X_1 + X_2$ will be called the logical sum of X_1 and X_2.

Definition 8: Two Boolean vectors X_1 and X_2 are said to conflict if $X_1 \cdot X_2 \neq (0,0,\ldots,0)$. Otherwise X_1 and X_2 are said to be compatible.

Definition 9: (Huntington's Postulates, 1904 [23]). A class of elements B together with two binary operations (+) and (•) is a Boolean algebra if and only if the following postulates hold:

P_1 : The operations (+) and (•) are commutative.

P_2 : There exists in B distinct identity elements 0 and 1 relative to the operations (+) and (•) respectively.

P_3 : Each operation is distributive over the other.

P_4 : For every a in B there exists an a' in B such that $a + a' = 1$ and $a \cdot a' = 0$.

Definition 10: Two Boolean matrices A and B are equal (A = B) if and only if $a_{ij} = b_{ij}$.

Definition 11: If C is an n x n Boolean matrix (c_{ij}), then an elementary transformation E_{ij} is the interchange of rows i and j, along with their corresponding columns.

Definition 12: Two conflict matrices A and B are said to be equivalent if $E(A) = B$, where $E = \prod E_{k_i s_j}$. If the matrices are not equivalent, they are said to be unequivalent.

It has been well known for many years [6] that the set of Boolean vectors of dimension m is isomorphic with a Boolean algebra having a finite number of elements. This Boolean algebra will be said to be of dimension m and is at order 2^m. The Boolean vectors thus constitute a convenient representation for a finite Boolean algebra, and in the remainder of this paper the arbitrarily given set of elements of a Boolean algebra mentioned in the title will be assumed to be given as Boolean vectors.

In the scheduling problem the faculty members, the student groups, and the special facilities required (such as laboratories) are listed in a specific but arbitrary order. The dimension of the Boolean vectors, which correspond to classes, is the total number of these items. One of the Boolean vectors is constructed by placing a 1 in the i^{th} position of the vector if the i^{th} item (faculty member, student group, or facility) is required for the class. Among the features of the scheduling problem which are not included in the problem with which this paper deals is the fact that a class must be scheduled for several meetings on different days of the week, and that some classes (e.g., laboratories) require two or more consecutive standard time periods.

In the light of previous discussion, I can now make a statement of the problem, which will be hereafter referred to as the "partition problem."

Given a set of elements of a Boolean algebra $X = [x_1, x_2, \ldots, x_m]$, where $[x_i]$ is an n-dimensional Boolean vector

($i=1, 2, \ldots, m$), then I wish to partition the set X into a minimum number of subsets $\{P_1, P_2, \ldots, P_s\}$ such that every element of each subset is compatible with every other element in that subset. I use the word "partition" in the commonly accepted manner. i.e., P is a partition of X if P divides X into subsets which are disjoint and exhaustive.

A given set of Boolean vectors may contain some redundant restrictions as far as the partition problem is concerned. These redundant restrictions can be reduced in two different ways, and eliminated in a third. Regard the columns c_i as elements of a Boolean algebra of dimension n. The first reduction, resulting in the reduced matrix, is accomplished by deleting column c_j if, for any two columns c_i and c_j, $c_i + c_j = c_i$. The second modification, resulting in the contracted matrix, is accomplished by replacing in the reduced matrix, blocks of columns (where possible) with single columns yielding the same information as to conflicts, and deleting columns having one element.

If a collection of n m-dimensional Boolean vectors were arranged in a column, one would have an nxm Boolean matrix. This matrix could be reduced to one usually having fewer columns, yet yielding the same information as regards the compatible partitioning problem. For example, if the below given set of vectors, M, were to be partitioned, then c_1 could be eliminated since it has no elements and will contribut nothing.

$$\begin{array}{cc} & \begin{array}{ccccccc} C_1 & C_2 & & \cdots & & & C_7 \end{array} \\ M = & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

I can reduce M by eliminating columns C_3 and C_7, according to the above criteria, leaving me with the matrix M_R', where

$$M_R' = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

If I rearrange the columns in the natural order I have the reduced matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

If one wanted the contracted matrix, then, after eliminating C_3 and C_4, one could replace $C_5 C_6 C_7$ by the single column

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

which gives the same conflicts. One would then write

$$M_C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

where M_C is the contracted matrix. Since M_C or M_R has all of the information with respect to the partitioning problem as had the original matrix M, one could replace M by M_C or M_R in attempting to solve the problem.

Another operation that could be performed on M is a process called <u>normalizing</u> the matrix. In normalizing, one replaces

every column of the matrix which has more than two elements by a set of columns having exactly two elements, and yielding the same conflicts as the original column. This is, in effect, just the opposite process of contraction, which is mentioned above. If one then eliminates all columns having less than two elements, in addition to eliminating all duplicated columns, and arranges them in the natural order, one is left with a matrix having exactly two elements per column (one conflict), and these conflicts will be precisely those that are in the original matrix.

For example, consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

After eliminating C_3 and C_4 , C_1 is the only remaining column with more than two elements. Therefore I replace C_1 by the set

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

yielding the matrix

$$\begin{array}{cccccc} C_1' & C_2' & \cdots & & & C_6' \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

Since $C_1' = C_6'$ I can eliminate C_6' , being left with the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} .$$

Rearranging columns we get the normalized matrix M_N , where

$$M_N = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Outside the obvious advantage of having a smaller matrix to consider, in the case of reduction or contraction, or of presenting the matrix with one conflict per column, in the case of normalization, there appears to be no practical value in performing a reduction, contraction, or a normalization of a matrix representing a collection of Boolean vectors. The reason for this lies in the fact that the original matrix, the reduced matrix, the contracted matrix, and the normalized matrix are all equivalent as far as the compatible partitioning problem is concerned, and will all yield the same conflict matrix. This will be explained more fully in the next paragraph.

Starting with a collection of n Boolean vectors of dimension m, one can arrange these in any fixed arbitrary order to form a Boolean matrix A, where A is n x m. If this matrix is multiplied by its transpose, A^t, then one has the conflict matrix C (Definition 7), where C is an n x n symmetric matrix with ones down the main diagonal. If $C_{ij} = 1$, then clearly the vector comprising row i of matrix A and that vector comprising row j of matrix A will have a product which does not equal (0,0,...,0), (i.e., conflict), and therefore could not be placed together in any compatible partition of the original set of vectors. If $C_{ij} = 0$, then vectors i and j can be placed together in a partition.

Because of this fact, it is apparent that the conflict matrix made from the matrix of a set of Boolean vectors contains

all the information that the original set of vectors contain, as applied to the problem of finding a minimum compatible partition of this set. Therefore, the remainder of this paper will be concerned with an investigation of conflict matrices.

It is worthwhile to notice that the conflict matrix obtained from the original set of Boolean vectors will be the same as the conflict matrix for the reduced, contracted, or normalized set of vectors. Going the other way, it is easily seen that the conflict matrix will determine a unique normalized set of vectors if the information in it is used in the natural order. Also, the conflict matrix will determine (in the same sense) a unique contracted set of vectors, as can be seen from the discussion of genus vectors in Section 5.

If one knows the minimum number of subsets into which the set can be partitioned, it will always be possible to make a partition using this number. The most satisfactory technique discovered so far will be illustrated in the following example.

Suppose one has 20 Boolean vectors labeled a through t, and it is known that the minimum number of subsets in a compatible partition is 6. From the original set of vectors or the conflict matrix (Figure 1), one can pick out a maximum number of vectors that are not compatible. In this example the maximum number will be 6 (say d, f, g, m, p, s), which is the same as the known minimum number of subsets in a compatible partition. If it is not possible to find 6 vectors such that any two are not compatible, then find as many as possible, and then arbitrarily add vectors until you reach a total of 6.

We then make out a table, using these 6 vectors at the top, and placing the remaining vectors down the side (Figure 2). To fill out the table we place a one in the $\alpha\beta$ position if vector α (row) is compatible with vector β (column), and then sum the number of ones in each row. This information is obtainable from the conflict matrix. One can then put the 6 non-compatible elements at the top of 6 columns (see Figure 3), and place the elements which are the rows of Figure 2 into the column where they will be compatible with every other element already in the column. This is done in ascending order with respect to the row sums of Figure 2. For example, row h has a sum of one, hence h will only be compatible with d, therefore first place h in the column under d. Continuing in this manner, placing each element where it must go to be compatible with the other elements, one will eventually put every element into one of the six columns, where the elements in each column are compatible. It may be necessary to do this several different ways before a partition can be made, but in the problems investigated so far it could usually be done the first time.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t
a	1	0	1	0	1	0	1	0	0	0	0	0	1	1	1	1	0	0	0	1
b	0	1	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	1
c	1	0	1	0	1	1	1	0	1	0	1	0	1	0	1	0	0	0	0	1
d	0	0	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	1	0
e	1	0	1	0	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0
f	0	1	1	1	1	1	1	1	1	0	1	0	1	1	0	1	0	1	1	0
g	1	0	1	1	0	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1
h	0	1	0	0	1	1	1	1	0	1	1	0	1	1	0	1	1	1	1	1
i	0	0	1	1	0	1	0	0	1	0	1	0	1	0	0	0	0	1	0	0
j	0	0	0	0	0	0	0	1	0	1	0	0	1	0	1	1	1	1	1	1
k	0	0	1	1	1	1	0	1	1	0	1	0	0	1	0	1	0	0	1	0
l	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	0	0
m	1	1	1	1	1	1	1	1	1	1	0	0	1	0	1	1	1	1	1	1
n	1	0	0	0	1	1	1	1	0	0	1	0	0	1	0	1	0	0	0	0
o	1	0	1	0	0	0	1	0	0	1	0	1	1	0	1	0	0	1	1	1
p	1	0	0	1	0	1	1	1	0	1	1	0	1	1	0	0	0	0	1	0
q	0	0	0	0	0	0	1	1	0	1	0	0	1	0	0	0	1	1	0	0
r	0	0	0	1	0	1	1	1	1	1	0	1	1	0	1	0	1	1	1	1
s	0	0	0	1	0	1	1	1	0	1	1	0	1	0	1	1	0	1	1	1
t	1	1	1	0	0	0	1	1	0	1	0	0	1	0	1	0	0	1	1	1

Figure 1

	d	f	g	m	p	s	
a	1	1				1	3
b	1		1		1	1	4
c	1				1	1	3
e	1				1	1	3
h	1						1
i			1		1	1	3
j	1	1	1				3
k			1	1			2
l	1	1		1	1	1	5
n	1			1		1	3
o	1	1			1		3
q	1	1			1	1	4
r		1			1		2
t	1				1		2

Figure 2

d	f	g	m	p	s
h	r	k	l	t	c
o	a	b	n	e	q
		j		i	

Figure 3

3. The Partition Problem in Terms of Graph Theory.

In performing the investigation reported in this paper, it has been noticed that there exists a body of knowledge that is very closely related to the study of conflict matrices. The body of knowledge is graph theory, and the succeeding definitions, theorems, and conclusions will show this relation and make use of the terminology and several known results of graph theory.

A graph $G = (X, \Gamma)$ consists of a set X of elements and a function $\Gamma : X \rightarrow X$. The elements of X will be represented by points in the plane, and if (x,y) are two points such that y is in $\Gamma(x)$, then there exists a bond between x and y. Hence, elements of X are vertices, while the pairs (x,y) are bonds [2]. For the purposes of this paper, a bond connecting x and y will be denoted xy instead of (x,y). This will eliminate a possible source of ambiguity that could arise later in the paper.

The <u>order</u> of a graph G is the total number of vertices in G, and the <u>degree</u> of any vertex V_i of G is the number of edges connecting V_i with other vertices in G. A graph is said to be <u>linear</u> if all of its edges are straight lines (no loops).[22]

<u>Definition 13</u>: Two graphs G and G' are <u>isomorphic</u> when there exists a one-one correspondence between their vertex sets V and V' such that corresponding vertices are joined by edges in one of them only if they are also joined in the other. [18]

<u>Definition 14</u>: The graph $U = U(V)$ is called a <u>complete graph</u> if the edges of U are all pairs of possible associations for two different vertices x and y in V. [18]

Definition 15: An incidence matrix, $A = [a_{ij}]$, is a symmetric matrix of n rows and n columns for a graph of n vertices where:

$$a_{ij} = \begin{cases} 1 & \text{if there exists a bond between vertex i and vertex j} \\ 0 & \text{otherwise} \end{cases} \quad [18]$$

Definition 16: A graph H is called a subgraph of the graph G when the vertex set V(H) of H is contained in the vertex set V(G) of G and all edges of H are edges in G. [18]. It can be seen that the subgraphs of a given graph can all be obtained by omitting one or more vertices and the edges involving these vertices.

Definition 17: Given a graph G on n points, there exists a unique complimentary graph $\bar{G}$ on these n points, consisting of all possible edges connecting the n points except those belonging to G. [18]

If we agree that a vertex is always bonded with itself, then ones will always be in the main diagonal of the incidence matrix. Clearly, the incidence matrix is equivalent to the graph in the sense that each is completely determined by the other. [22]

From the previous definitions one can see that the incidence matrix in graph theory is exactly the same thing as the conflict matrix made from a set of Boolean vectors. Therefore, the problem of finding a minimum partition with compatible elements can be thought of as a problem in graph theory.

Theorem I: A minimum compatible partition of a set of Boolean vectors will be greater than or equal to the order of a maximum complete subgraph contained in the graph represented by the incidence (conflict) matrix.

To see this, notice that if one has a graph of order n, with a maximum complete subgraph of order s, this implies that there exists a set of s points (vertices), A, which are connected (bonded) with every other point of the set A. This means that none of these s points can go into a set of a compatible partition with any of the other points in A, which implies that the minimum number of subsets in the partition of the original set of Boolean vectors will be equal to or greater than the cardinal number of A, which is s.

To state the partition problem in graph terminology, we first need to make further observations and definitions. We note that the graphs in question are undirected with single edges, and no loops (linear). For the purposes of this paper the word "graph" will imply these conditions.

If G is a graph, then G is <u>k-colorable</u> when there exists a decomposition of its vertices into k disjoint classes K_1, K_2, ..., K_k, $V = \sum_j K_j$, $K_i \cdot K_j = \emptyset$ such that the vertices in each class are compatible, which means that they are not bonded with other vertices in the class. This decomposition is a <u>k-coloration</u> of G. If each vertex of class K_i is colored with the i^{th} color, then each vertex of the graph is colored in such a manner that the endpoints of an edge always have different colors. One can also represent the colors of the classes K_i by the integers 1, 2, ..., k, and introduce a <u>color function</u> f such that $f(V_i) = i$, V_i in K_i. [18].

The smallest number k=k(G) of classes in any k-coloration is the chromatic number of G, and G is said to be k-chromatic. Under these conditions the decomposition described above will be a chromatic decomposition of V. [18]

Using this terminology, the partition problem can now be stated: Given a graph G, where G is k-colorable, find the chromatic number of G and a chromatic decomposition of V. This will be referred to as the "coloration" problem.

One immediately observes that a complete graph on n vertices (i.e., of order n) has the chromatic number n, and that Theorem I can be restated.

Theorem II: When G contains a complete subgraph U on m vertices, then $k(G) \geq m$.

It can also be shown [3] that if given a graph G, where we define $q = \max n\ (\Gamma x)$, then q+1 colors are sufficient to color the graph.

This, along with Theorem II, gives us an upper and a lower bound for the chromatic number of any graph.

4. An Exhaustive Study of Graphs of Order Six and Less.

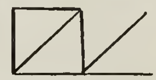
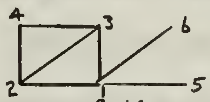
The solution of the coloration problem, in general, appears to be quite difficult. Therefore I propose to examine in detail all of the graphs of order six and less to see if any patterns develop or any inferences can be made to indicate possible areas for future study.

Associated with each graphs an ordered pair of numbers (a, b), called the genus of the graph, where a is the order and b is the number of edges of the graph. For example, (6,5) is the genus of a graph on six points having five edges. In general, different graphs may have the same genus, therefore the genus does not determine a unique graph. Also associated with every graph of order n is an n-tuple $(a_1, a_2, \ldots, a_n)$ called the type of the graph, in which a_i is the degree of the i^{th} vertex of the graph.

Two graphs on n given points are said to be isomorphic if their incidence matrices are equivalent. Therefore every graph has an isomorphism number associated with it, where the isomorphism number of a graph is the total number of graphs in the set consisting of the given graph and its isomorphs. There are several different ways to obtain the isomorphism number of a graph G.

It can be shown [6] that if h is the number of isomorphisms in a set $\{X\}$, then $h = \frac{n!}{k}$, where k is the order of $G_n(X)$, the group of degree n which leaves X unchanged. Although this method is easily applied to some graphs, for others it is quite difficult to determine k.

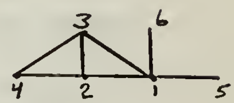
Another more useful procedure to find the isomorphism number of a graph is to determine, by combinatorial methods, the number of different ways that a graph can be labeled.

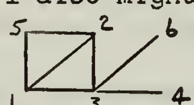
For example, consider the graph on 6 points with seven edges, i.e., a graph with genus (6,7), given by the figure  .
If I label the point with the greatest degree as 1, the next largest degree is 2, and so on, the graph would be labeled as shown : 

.

The type of the graph will be (4, 3, 3, 2, 1, 1) and the edges can be given by the pairs 12, 13, 15, 16, 23, 24, 34 . Notice that these edges are also determined by the incidence matrix

$$
\begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\left[\begin{array}{cccccc}
1 & 1 & 1 & & 1 & 1 \\
1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & & \\
 & 1 & 1 & 1 & & \\
1 & & & & 1 & \\
1 & & & & & 1
\end{array}\right]
\end{array}
$$

I could bend this graph into a different shape, such as 

but it is still the same graph. I also might label the points differently, getting an isomorph 

. The incidence matrix for the graph labeled in this manner would be different, but equivalent to the incidence matrix of the graph labeled the previous way. The type of the graph labeled in this manner would be a permutation of the type of the graph labeled previously. This fact is obvious if one notices that the type of graph is simply the row sums of the incidence matrix with the elements in the main diagonal changed from 1 to 0.

Applying the first method, one can determine the number of isomorph,h, of this graph by determining the order, k, of the permutation group on the n points which leaves the set of edges unchanged, and dividing this number into n! . In this example $n = b$ and $k=4$, therefore $h = \frac{6!}{4} = \frac{720}{4} = 180$.

One can also determine this number by combinatorial methods. i.e., ${}_6C_5 \cdot 60/2 = 180$.

In finding all of the non-isomorphic graphs of order 6 and less, I was aided by a table in Riordan's Combinatorial Analysis, [21] which is reproduced as Table I in this paper. Notice that this table lists the number of non-isomorphic graphs within each genus. As an aid in finding the number of isomorphs of each graph, the following theorems were used.

Theorem III: The total number of graphs of genus (n,a) on n given points, denoted by N(n,a), is the binomial coefficient $\binom{n(n-1)/2}{a}$.

To see this, notice that the incidence matrix for any graph of genus (n,a) will have $n(n-1)/2$ spaces above the main diagonal to be filled by a ones. The number of different ways of doing this, of which each different way corresponds to a different graph on the n given points, is given by the binomial coefficient $\binom{n(n-1)/2}{a}$.

Theorem IV: The total number of graphs of order n, denoted N(n), is $2^{n(n-1)/2}$.

This is clear since $N(n) = \sum_{a=0}^{n} N(n,a)$, and it is well known that the sum of these binomial coefficients is $2^{n(n-1)/2}$.

For example, if $n = 6$, then the total number of possible edges is $6.5/2 = 15$. Therefore the number of graphs of order 6 is given by $2^{15} = 32{,}768$. The number of graphs of genus (6,0) is ${}_{15}C_0 = 1$; of (6,1) is ${}_{15}C_1 = 15$; of (6,2) is ${}_{15}C_2 = 105$; and so on.

Theorem V: If a graph G_0 on n points, where each point is of degree greater than 0, has an isomorphism number N_0, then the same graph on n + k points, G_k, (That graph obtained by adding k vertices but no edges) has an isomorphism number $N_0 \cdot \binom{n+k}{k}$.

This can be seen by noticing that $N_0 = n!/s$ [6], where s is the order of the permutation group which leaves the edges of G_0 unchanged. If we add k points but no edges, then for the same reason the isomorphism number (N_k) of G_k is $(n+k)!/r$. However $r = s \cdot k!$ since the permutation group which leaves the edges unchanged on the n+k points of G_k is an intransitive group which contains two transitive sets. One of these two sets is the n points in G_0 and the other is the k added points of degree 0 in G_k. Thus the permutations of G_k consist of each of the s permutations that leave the edges of G_0 unchanged, combined with each of the permutations of the symmetric group on the k additional points. If we take the ratio $N_k/N_0 = [(n+k)!/s \cdot k!] \cdot [s/n!] = (n+k)!/n! \cdot k! = \binom{n+k}{k}$. Hence $N_k = N_0 \cdot \binom{n+k}{k}$, and the theorem is established.

Graphs of order n can be obtained from graphs of order n - k by adding k vertices in every way and eliminating duplicates. Likewise, the isomorphism number, p, of a graph G of order n can be

obtained from a graph G' of order n - k, which has a known isomorphism number p', if $G' \subset G$. This is done by multiplying p' times the number of different ways that one can add the k edges which form G, to G'.

Appendix I lists all of the non-isomorphic graphs of order 6 by edges, giving isomorphism number, type, order of the maximum complete subgraph and the chromatic number, plus other information that will be explained in Section 5. The graphs are listed by descending types within each genus. It is not difficult to list in advance of constructing the graphs of a given order all n-tuples which can occur as types. It is interesting to note that for each n-tuple so listed, at least one graph actually occurs up through order 6. However, there is not a one to one correspondence between different types and different graphs. As can be seen from Appendix I, there are 15 cases when this occurs. In listing the graphs it was necessary to find only the first half by the methods previously mentioned, since one can find the complementary graphs to these by adding edges to the unconnected vertices of a graph, and deleting the edges which connect the vertices of that graph. Each graph will thus have a unique complementary graph, with the same isomorphism number. If G is any graph of order n, then it is easily seen that the sum of the type for G and the reversed type for $\overline{G}$ will always add to the type for the complete graph on n points.

The graphs of order 2, 3, 4, and 5 are not included in Appendix I because these graphs are included in those of order 6. Likewise, a pictorial representation for each graph is not included, since this is easily produced from the edges.

Out of a total of 156 non-isomorphic graphs of order 6, note that there exists 4 cases where the chromatic number differs from the order of the maximum complete subgraph. Likewise, in the 34 graphs of order 5, there exists one such case. There are no such cases for graphs of order less than 5. In each of these 5 cases, if k is the order of the maximum complete subgraph, then the chromatic number is k+1. Berge [3] proves that a graph is 2-chromatic if and only if it contains no cycles of uneven length. The graphs where this theorem applies are easily identified from Appendix I. The probability that a random graph on 6 given points will have a chromatic number greater than the order of the maximum complete subgraph is $(72 + 360 + 180 + 72)/2^{15} = 0.0287$. The same probability computed for a random graph on 5 points is $12/2^{10} = 0.0120$.

The probability that a random graph of order ≤ 6 and of a given genus has a particular chromatic number can be found from Appendix I, and is listed in Appendix II, along with the mean and standard deviation of the probability distribution corresponding to each genus. These means are plotted following Appendix II, with the abscissa corresponding to the genus, and the ordinate corresponding to the mean. Notice that this function is monotone increasing, and also that the slope of a smooth curve connecting these points increases with increasing abscissa.

5. A Study of Some Random Graphs of Order 6.

It is apparent that in order to make any further progress in the study of the coloration problem, one must be able to investigate graphs of a larger order. Since this would be quite a tedious operation if done by hand, even for graphs of order 7 or 8, one is naturally led to the large-scale digital computer. For reasons stated at the end of Section 3, we devised a method to find the maximum complete subgraph of selected graphs of order $\leqslant 48$. We do this by finding the complete subgraphs of order 3, and then using these to find the complete subgraphs of order 4, and so on until we have the maximum complete subgraphs. The program is restricted to graphs of order $\leqslant 48$ since the 48 bit positions in a word in the CDC 1604 computer were used to record the presence of a complete subgraph of a certain order. Also, the main memory places a restriction on us by not allowing the number of complete subgraphs of any order to exceed 15,000.

This program could have been written so that the complete subgraphs of order i ($i=1, 2, \ldots, k$) which are contained in any complete subgraph of order greater than i, are discarded. If all of the remaining subgraphs are listed in a matrix form, then one has essentially the contracted matrix mentioned in Section 2, the only exception being that this matrix may contain complete subgraphs of order 1 (single elements in a column), while in the contracted form these were eliminated.

Utilizing the CDC 1604 in this manner, we have so far randomly produced 25 graphs of order 20 and 11 of order 40.We had hoped to then find

the chromatic number of these graphs, and gather some statistical data relating the order of the maximum complete subgraph of a random graph to the chromatic number. However, we were diverted from this study by an unexpected property which all of these graphs possess. To explain this property I will define another N-tuple associated with each graph, which I will call the genus vector of the graph.

Definition 18: The genus vector of a graph G is the N-tuple $[a_1, a_2, a_3, \ldots, a_N]$, where a_1 is the number of complete subgraphs on 1 point (the number of vertices), a_2 is the number of complete subgraphs on 2 points (the number of edges), and so on until a_N, which is the number of complete subgraphs on N points, where N is the order of a maximum complete subgraph of G.

Notice that (a_1, a_2), obtained from the first two elements of the genus vector, is the genus of G. The genus vector for each of the graphs of order 6 is listed in Appendix I, and the conditional probabilities for obtaining a particular genus vector, given the genus, for all graphs of order 4, 5, and 6 are computed and are listed in Appendix III.

To produce the 36 graphs investigated so far, we punched an IBM Data Card for each position in the incidence matrix (i.e., for each position in a 20 x 20 or 40 x 40 matrix), and on each of these cards we also punched a series of random numbers from a random number table. The cards were then mixed in a card sorter according to the random numbers, and a certain number of cards were selected, where each card corresponded to one bit in an

incidence matrix. Using this method we were able to make up graphs, random in the described sense, with a fixed percentage of edges. For example, a graph of order 20 would have a 20 x 20 incidence matrix, and the total possible number of edges would be 20.19/2 = 190. If we wanted a graph "5/10 full" we would arbitrarily select 95 cards from the card sorter.

For this investigation we made up 25 graphs of order 20, where 5 were 3/10 full (57 edges), 5 were 4/10 full (76 edges), and so on until we got 5 which were 7/10 full (133 edges). We did the same thing for graphs of order 40, except we only made up 2 within each percentage of edges, and the range was from 2/10 full to 6/10 full, with one graph 6.5/10 full due to machine limitations.

When each of the above graphs was run through the computer we obtained the genus vector for that graph, plus a listing of the elements in each maximum complete subgraph. We accumulated the components of the genus vector and computed the cumulative probabilities corresponding to each component. This sample cumulative distribution function was plotted on normal probability paper, with the order of the subgraphs being the ordinates, and the cumulative probabilities being the abscissas. The unexpected property mentioned earlier is the fact that in every case the plot was almost linear. I will illustrate this property by a typical example.

The 20 x 20 incidence matrix in Figure 4 was produced by the above methods, and after being run through the computer we

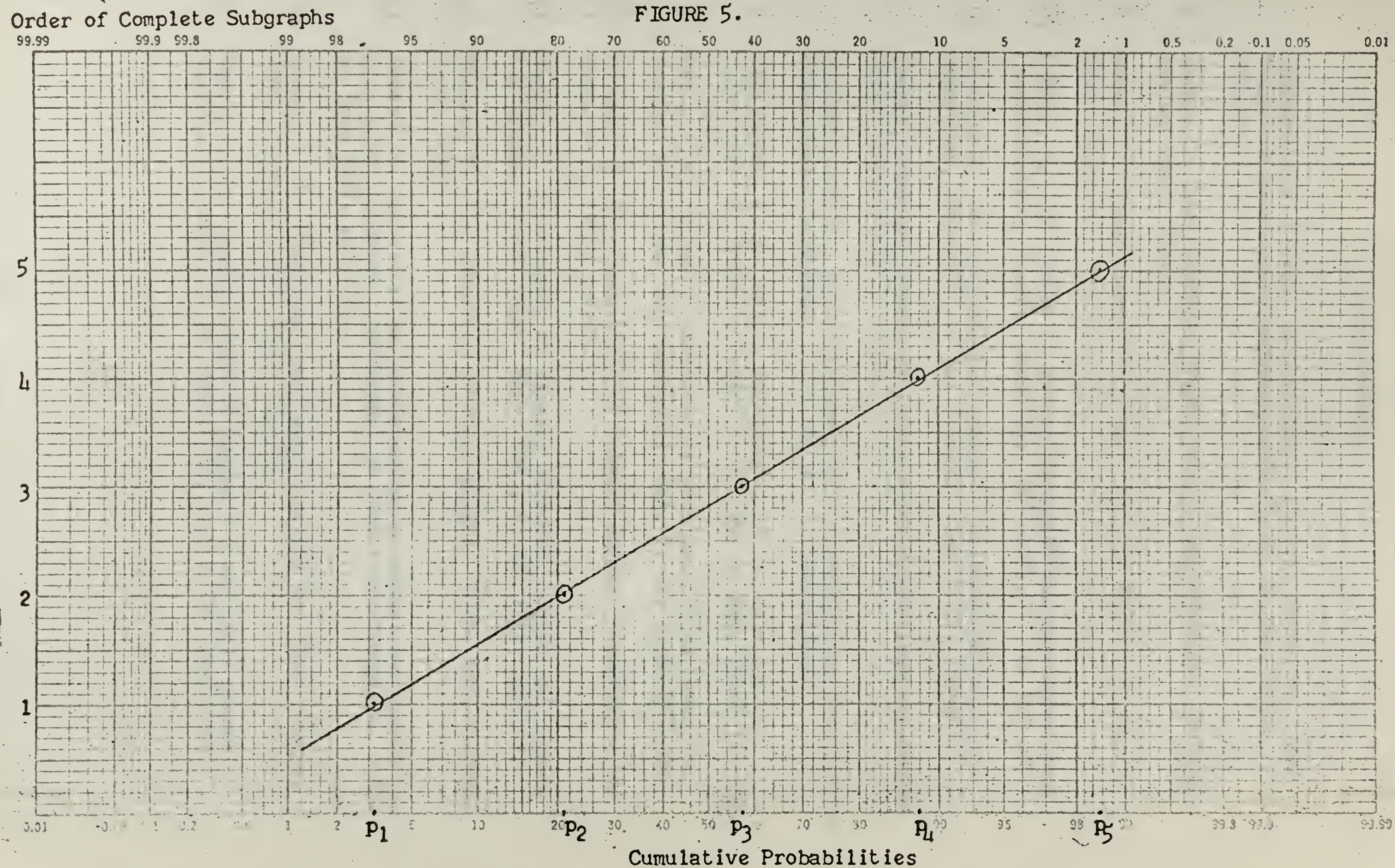
obtained the genus vector $[20, 95, 134, 58, 9]$. If we let P_i denote the cumulative probability of having a complete subgraph of order i or less, then we compute: $P_1 = 0.03164$; $P_2 = 0.21361$; $P_3 = 0.57595$; $P_4 = 0.87975$; $P_5 = 0.98576$. This information is plotted on normal probability paper in Figure 5, with abscissa and ordinate as described above. As yet, we have been unable to determine a reason why the points in Figure 5 and in all of the other cases lie very nearly in a straight line. The genus vectors of the sample of 25 graphs of order 20 and 11 graphs of order 40 are listed in Appendix IV, along with the mean and standard deviation of each sample cumulative distribution function, as read from the plot on the normal probability paper.

The estimated means (from the sample shown) of the probability distributions for the order of the maximum complete subgraph within each genus of graphs of order 20 and 40 can be easily found from Appendix IV. A plot of these means is not included since the sample size is so small, and therefore the estimated means could be considerably different if based on larger samples. However, based on the sample we have, a plot is quite similar to the plot of the means for graphs of order 5 and 6, which would be almost identical with that shown following Appendix II.

FIGURE 4

1	0	1	0	1	0	0	0	0	1	1	0	0	1	0	0	1	1	1	0
0	1	1	0	1	1	0	1	0	0	0	1	1	1	0	0	0	0	0	0
1	1	1	1	0	0	0	0	1	1	0	1	1	0	1	0	0	1	0	1
0	0	1	1	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0
1	1	0	0	1	1	0	0	1	1	0	1	0	0	1	1	1	1	0	1
0	1	0	1	1	1	0	0	1	0	0	0	1	0	1	1	1	0	0	1
0	0	0	0	0	0	1	1	1	1	0	1	0	0	0	1	1	0	1	0
0	1	0	0	0	0	1	1	1	1	1	0	1	1	1	1	0	1	1	1
0	0	1	0	1	1	1	1	1	0	0	1	1	0	0	0	1	0	1	0
1	0	1	0	1	0	1	1	0	1	0	0	0	1	1	0	0	0	1	1
1	0	0	0	0	0	0	1	0	0	1	1	1	1	0	0	1	1	0	1
0	1	1	0	1	0	1	0	1	0	1	1	0	0	0	0	0	1	1	0
0	1	1	0	0	1	0	1	1	0	1	0	1	1	1	1	1	0	1	1
1	1	0	0	0	0	0	1	0	1	1	0	1	1	1	0	1	0	0	1
0	0	1	1	1	1	0	1	0	1	0	0	1	1	1	0	1	1	0	1
0	0	0	1	1	1	1	1	0	0	0	0	1	0	0	1	1	1	0	1
1	0	0	1	1	1	1	0	1	0	1	0	1	1	1	1	1	1	0	1
1	0	1	1	1	0	0	1	0	0	1	1	0	0	1	1	1	1	1	0
1	0	0	0	0	0	1	1	1	1	0	1	1	0	0	0	0	1	1	1
0	0	1	0	1	1	0	1	0	1	1	0	1	1	1	1	1	0	1	1

FIGURE 5.



Order of Complete Subgraphs
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
5
4
3
2
1
P_1
P_2
P_3
P_4
P_5
Cumulative Probabilities

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TABLE I

THE NUMBER OF LINEAR GRAPHS WITH n POINTS AND k LINES

k / n	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
2		1	2	2	2	2	2	2
3		1	3	4	5	5	5	5
4			2	6	9	10	11	11
5			1	6	15	21	24	25
6			1	6	21	41	56	63
7				4	24	65	115	148
8				2	24	97	221	345
9				1	21	131	402	771
10				1	15	148	663	1637
11					9	148	980	3252
12					5	131	1312	5995
13					2	97	1557	10120
14					1	65	1646	15615
15					1	41	1557	21933
16						21	1312	27987
17						10	980	32403
18						5	663	34040
19						2	402	32403
20						1	221	27987
21						1	115	21933
22							56	15615
23							24	10120
24							11	5995
25							5	3252
26							2	1637
27							1	771
28							1	⋮
Total	2	4	11	34	156	1044	12346	274668

APPENDIX I

Non-Isomorphic Graphs of Order 6

Column A = Edges
Column B = Isomorphism Number (when more than one number is listed, the numbers are the isomorphism numbers for the corresponding graphs on 6 points, 5 points, etc., in reverse order.)
Column C = Type
Column D = Genus Vector
Column E = Order of Maximum Complete Subgraph
Column F = Chromatic Number

A	B	C	D	E	F
None	1,1,1,1,1, 1	0,0,0,0,0,0	6	1	1
12	1,3,6,10, 15	1,1,0,0,0,0	6,1	2	2
12,13	3,12,30, 60	2,1,1,0,0,0	6,2	2	2
12,34	3,15, 45	1,1,1,1,0,0	6,2	2	2
12,13,14	4,20, 60	3,1,1,1,0,0	6,3	2	2
12,13,23	1,4,10, 20	2,2,2,0,0,0	6,3,1	3	3
12,13,24	12,60,180	2,2,1,1,0,0	6,3	2	2
14,15,23	30,180	2,1,1,1,1,0	6,3	2	2
12,34,56	15	1,1,1,1,1,1	6,3	2	2
12,13,14,15	5, 30	4,1,1,1,1,0	6,4	2	2
12,13,14,23	12,60,180	3,2,2,1,0,0	6,4,1	3	3
12,14,15,23	60,360	3,2,1,1,1,0	6,4	2	2
12,13,14,56	60	3,1,1,1,1,1	6,4	2	2
12,13,24,34	3,15, 45	2,2,2,2,0,0	6,4	2	2
12,13,23,45	10, 60	2,2,2,1,1,0	6,4,1	3	3
13,14,23,25	60,360	2,2,2,1,1,0	6,4	2	2
13,14,25,26	90	2,2,1,1,1,1	6,4	2	2
12,13,24,56	180	2,2,1,1,1,1	6,4	2	2

A	B	C	D	E	F
12,13,14,15,16	6	5,1,1,1,1,1	6,5	2	2
12,13,14,15,23	30,180	4,2,2,1,1,0	6,5,1	3	3
12,13,14,15,26	120	4,2,1,1,1,1	6,5	2	2
12,13,14,23,24	6,30, 90	3,3,2,2,0,0	6,5,2	3	3
12,13,14,23,25	60,360	3,3,2,1,1,0	6,5,1	3	3
12,13,14,25,26	90	3,3,1,1,1,1	6,5	2	2
12,13,14,23,45	60,360	3,2,2,2,1,0	6,5,1	3	3
13,14,15,23,24	60,360	3,2,2,2,1,0	6,5	2	2
12,13,14,23,56	180	3,2,2,1,1,1	6,5,1	3	3
12,13,14,25,36	360	3,2,2,1,1,1	6,5	2	2
12,14,15,23,36	360	3,2,2,1,1,1	6,5	2	2
12,14,25,34,35	12, 72	2,2,2,2,2,0	6,5	2	3
12,13,24,34,56	45	2,2,2,2,1,1	6,5	2	2
12,13,23,45,46	60	2,2,2,2,1,1	6,5,1	3	3
12,13,25,34,46	360	2,2,2,2,1,1	6,5	2	2
12,13,14,15,16,23	60	5,2,2,1,1,1	6,6,1	3	3
12,13,14,15,23,24	60,360	4,3,2,2,1,0	6,6,2	3	3
12,13,15,16,23,24	360	4,3,2,1,1,1	6,6,1	3	3
12,13,14,15,23,45	15, 90	4,2,2,2,2,0	6,6,2	3	3
12,13,14,15,23,46	360	4,2,2,2,1,1	6,6,1	3	3
12,13,15,16,24,34	180	4,2,2,2,1,1	6,6	2	2
12,13,14,23,24,34	1, 5, 15	3,3,3,3,0,0	6,6,4,1	4	4
12,13,14,23,24,35	60,360	3,3,3,2,1,0	6,6,2	3	3
12,13,14,23,25,36	120	3,3,3,1,1,1	6,6,1	3	3
13,14,15,23,24,25	10, 60	3,3,2,2,2,0	6,6	2	2
12,13,14,23,25,45	60,360	3,3,2,2,2,0	6,6,1	3	3
12,13,14,23,25,46	720	3,3,2,2,1,1	6,6,1	3	3
12,13,15,24,26,34	360	3,3,2,2,1,1	6,6	2	2
12,13,14,23,24,56	90	3,3,2,2,1,1	6,6,2	3	3
13,14,15,23,24,26	180	3,3,2,2,1,1	6,6	2	2
12,13,14,25,26,34	180	3,3,2,2,1,1	6,6,1	3	3

A	B	C	D	E	F
12,13,14,23,45,56	360	3,2,2,2,2,1	6,6,1	3	3
12,13,15,24,34,56	360	3,2,2,2,2,1	6,6	2	2
12,13,16,25,34,45	360	3,2,2,2,2,1	6,6	2	3
12,13,23,45,46,56	10	2,2,2,2,2,2	6,6,2	3	3
12,13,24,35,46,56	60	2,2,2,2,2,2	6,6	2	2
12,13,14,15,16,23,24	180	5,3,2,2,1,1	6,7,2	3	3
12,13,14,15,16,23,45	90	5,2,2,2,2,1	6,7,2	3	3
12,13,14,15,23,24,25	10, 60	4,4,2,2,2,0	6,7,3	3	3
12,13,14,15,23,24,26	180	4,4,2,2,1,1	6,7,2	3	3
12,13,14,15,23,24,34	20,120	4,3,3,3,1,0	6,7,4,1	4	4
12,13,14,15,23,25,34	60,360	4,3,3,2,2,0	6,7,3	3	3
12,13,14,15,23,24,36	720	4,3,3,2,1,1	6,7,2	3	3
12,13,15,16,23,24,34	180	4,3,3,2,1,1	6,7,2	3	3
12,13,14,15,23,24,56	360	4,3,2,2,2,1	6,7,2	3	3
12,13,14,15,23,26,45	360	4,3,2,2,2,1	6,7,2	3	3
13,14,15,16,23,24,25	120	4,3,2,2,2,1	6,7	2	2
12,13,14,16,23,25,45	720	4,3,2,2,2,1	6,7,1	3	3
12,13,14,15,23,46,56	180	4,2,2,2,2,2	6,7,1	3	3
13,14,15,23,24,25,34	30,180	3,3,3,3,2,0	6,7,2	3	3
12,13,14,23,24,34,56	15	3,3,3,3,1,1	6,7,4,1	4	4
12,13,14,23,24,35,46	180	3,3,3,3,1,1	6,7,2	3	3
12,13,14,23,24,35,56	360	3,3,3,2,2,1	6,7,2	3	3
12,13,16,23,24,35,45	360	3,3,3,2,2,1	6,7,1	3	3
13,14,15,23,24,25,56	180	3,3,3,2,2,1	6,7	2	2
12,13,14,24,25,35,36	720	3,3,3,2,2,1	6,7,1	3	3
12,13,14,25,26,35,46	180	3,3,2,2,2,2	6,7	2	2
12,13,14,23,26,45,56	360	3,3,2,2,2,2	6,7,1	3	3
12,13,14,25,26,34,56	90	3,3,2,2,2,2	6,7,2	3	3
13,14,15,23,24,26,56	180	3,3,2,2,2,2	6,7	2	3

A	B	C	D	E	F
12,13,14,15,16,23,24,25	120	5,4,2,2,2,1	6,8,3	3	3
12,13,14,15,16,23,24,34	60	5,3,3,3,1,1	6,8,4,1	4	4
12,13,14,15,16,23,24,35	360	5,3,3,2,2,1	6,8,3	3	3
12,13,14,15,16,24,25,36	180	5,3,2,2,2,2	6,8,3	3	3
12,13,14,15,23,24,25,34	30,180	4,4,3,3,2,0	6,8,5,1	4	4
12,13,14,15,23,24,26,34	180	4,4,3,3,1,1	6,8,4,1	4	4
12,13,14,15,23,24,25,36	180	4,4,3,2,2,1	6,8,3	3	3
12,13,15,16,23,24,25,34	720	4,4,3,2,2,1	6,8,3	3	3
12,14,15,16,23,25,26,34	180	4,4,2,2,2,2	6,8,2	3	3
13,14,15,16,23,24,25,26	15	4,4,2,2,2,2	6,8	2	2
12,13,14,15,24,25,34,35	15, 90	4,3,3,3,3,0	6,8,4	3	3
12,13,14,15,24,26,34,35	720	4,3,3,3,2,1	6,8,3	3	3
12,13,14,15,23,24,34,56	120	4,3,3,3,2,1	6,8,4,1	4	4
12,13,14,16,24,25,34,35	360	4,3,3,3,2,1	6,8,2	3	3
13,14,15,16,23,24,25,34	360	4,3,3,3,2,1	6,8,2	3	3
12,13,15,16,23,26,34,45	720	4,3,3,2,2,2	6,8,2	3	3
13,14,15,16,23,24,34,56	180	4,3,3,2,2,2	6,8,3	3	3
12,13,14,15,25,26,34,36	360	4,3,3,2,2,2	6,8,2	3	3
13,14,15,16,23,25,26,34	360	4,3,3,2,2,2	6,8,1	3	3
12,13,14,23,25,35,45,46	180	3,3,3,3,3,1	6,8,2	3	3
13,14,16,23,24,25,34,56	180	3,3,3,3,2,2	6,8,2	3	3
13,14,16,23,24,26,35,45	90	3,3,3,3,2,2	6,8	2	2
13,14,16,24,25,26,34,35	360	3,3,3,3,2,2	6,8,1	3	3
12,14,16,23,26,34,35,45	180	3,3,3,3,2,2	6,8,2	3	3
12,13,14,15,16,23,24,25,26	15	5,5,2,2,2,2	6,9,4	3	3
12,13,14,15,16,23,24,25,34	360	5,4,3,3,2,1	6,9,5,1	4	4
12,13,14,15,16,23,24,25,36	360	5,4,3,2,2,2	6,9,4	3	3
12,13,14,15,16,23,24,35,45	90	5,3,3,3,3,1	6,9,4	3	3
12,13,14,15,16,23,24,34,56	60	5,3,3,3,2,2	6,9,5,1	4	4
12,13,14,15,16,24,26,34,35	360	5,3,3,3,2,2	6,9,4	3	3
12,13,14,15,23,24,25,34,35	10, 60	4,4,4,3,3,0	6,9,7,2	4	4
12,13,14,15,23,24,25,34,36	360	4,4,4,3,2,1	6,9,5,1	4	4
12,13,14,15,23,24,26,35,36	120	4,4,4,2,2,2	6,9,4	3	3
12,14,15,16,23,24,25,34,35	360	4,4,3,3,3,1	6,9,4	3	3

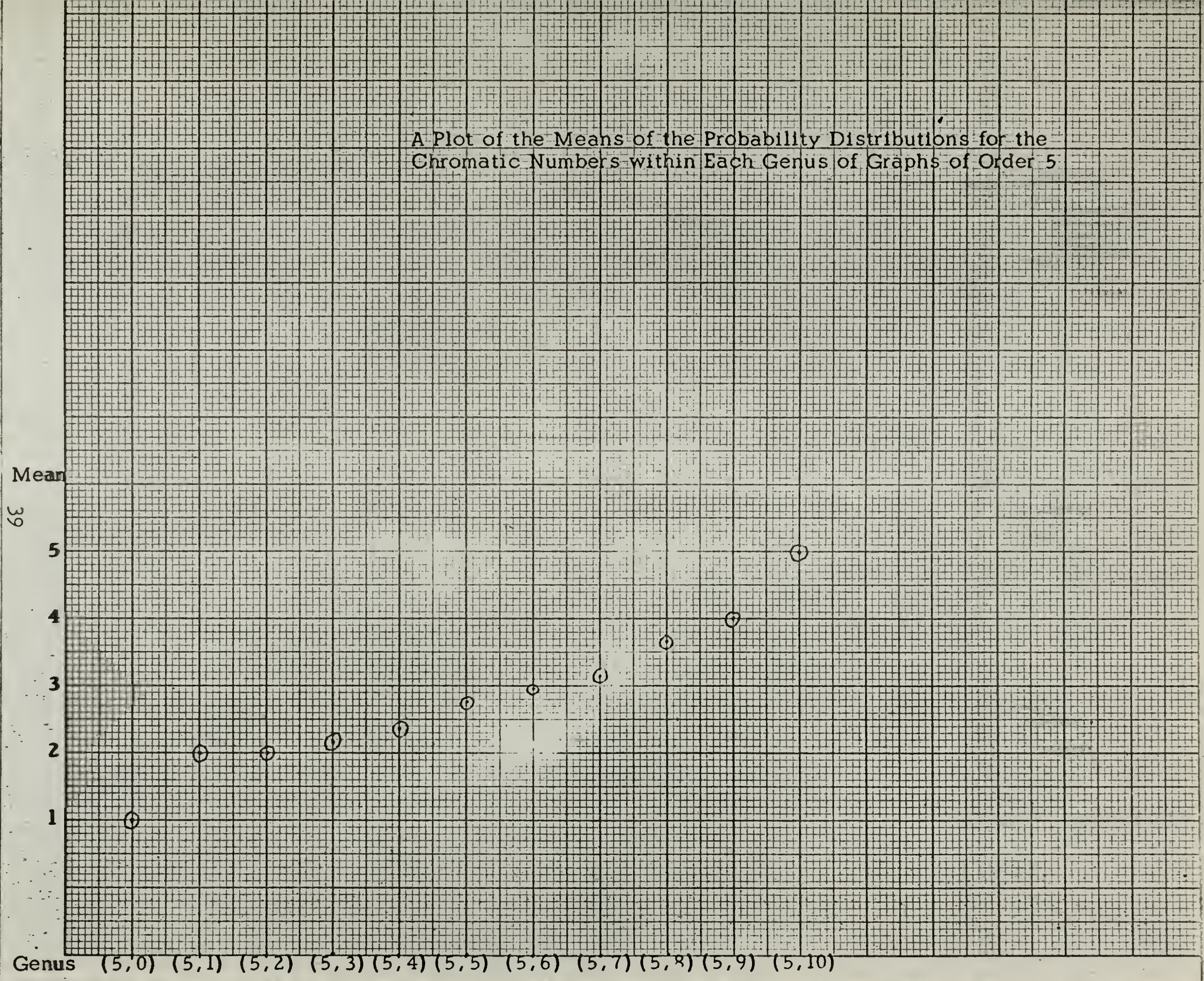
A	B	C	D	E	F
12,13,14,15,23,24,25,36,45	180	4,4,3,3,3,1	6,9,5,1	4	4
12,14,15,16,23,24,26,34,35	720	4,4,3,3,2,2	6,9,3	3	3
12,13,14,16,23,24,25,36,45	360	4,4,3,3,2,2	6,9,4	3	3
13,14,15,16,23,24,25,26,34	90	4,4,3,3,2,2	6,9,2	3	3
12,13,14,16,23,24,25,34,56	180	4,4,3,3,2,2	6,9,4,1	4	4
12,13,14,16,23,24,26,35,45	180	4,4,3,3,2,2	6,9,3	3	3
13,14,15,16,24,25,26,34,35	360	4,3,3,3,3,2	6,9,2	3	3
13,14,15,16,23,24,25,36,45	360	4,3,3,3,3,2	6,9,3	3	3
12,13,14,15,24,26,35,36,45	360	4,3,3,3,3,2	6,9,3	3	3
14,15,16,24,25,26,34,35,36	10	3,3,3,3,3,3	6,9	2	2
14,15,16,23,25,26,34,36,45	60	3,3,3,3,3,3	6,9,2	3	3
12,13,14,15,16,23,24,25,26,34	90	5,5,3,3,2,2	6,10,6,1	4	4
12,13,14,15,16,23,24,25,34,35	180	5,4,4,3,3,1	6,10,7,2	4	4
12,13,14,15,16,23,24,25,34,36	360	5,4,4,3,2,2	6,10,6,1	4	4
12,13,14,15,16,24,25,26,34,35	360	5,4,3,3,3,2	6,10,5	3	3
12,13,14,15,16,23,24,25,34,56	360	5,4,3,3,3,2	6,10,6,1	4	4
12,13,14,15,16,23,24,35,46,56	72	5,3,3,3,3,3	6,10,5	3	4
12,13,14,15,23,24,25,34,35,45	1, 6	4,4,4,4,4,0	6,10,10,5,1	5	5
12,13,14,15,23,24,25,34,35,46	120	4,4,4,4,3,1	6,10,7,2	4	4
12,13,14,15,23,24,25,34,36,46	90	4,4,4,4,2,2	6,10,6,1	4	4
13,14,15,16,23,24,25,26,34,35	180	4,4,4,3,3,2	6,10,4	3	3
12,13,15,16,23,24,26,34,35,45	360	4,4,4,3,3,2	6,10,5	3	3
12,13,15,16,23,24,25,34,35,46	360	4,4,4,3,3,2	6,10,5,1	4	4
13,14,15,16,23,24,25,26,36,45	45	4,4,3,3,3,3	6,10,4	3	3
12,14,15,16,24,25,26,34,35,36	60	4,4,3,3,3,3	6,10,3	3	3
12,14,15,16,23,24,26,35,36,45	360	4,4,3,3,3,3	6,10,4	3	3
12,13,14,15,16,23,24,25,26,34,35	180	5,5,4,3,3,2	6,11,8,2	4	4
12,13,14,15,16,23,24,25,26,35,36	45	5,5,3,3,3,3	6,11,8,2	4	4
12,13,14,15,16,23,24,25,34,35,45	30	5,4,4,4,4,1	6,11,10,5,1	5	5
12,13,14,15,16,23,24,25,34,35,46	360	5,4,4,4,3,2	6,11,8,2	4	4
12,13,14,15,16,24,25,26,34,35,36	60	5,4,4,3,3,3	6,11,6	3	3
12,13,14,15,16,23,24,26,34,35,56	360	5,4,4,3,3,3	6,11,7,1	4	4
12,13,14,15,23,24,25,34,35,46,56	60	4,4,4,4,4,2	6,11,7,2	4	4
12,13,14,16,23,24,26,34,35,45,56	90	4,4,4,4,3,3	6,11,6,1	4	4
13,14,15,16,23,24,25,26,34,36,45	180	4,4,4,4,3,3	6,11,6	3	3

A	B	C	D	E	F
12,13,14,15,16,23,24,25,26,34,35,36	20	5,5,5,3,3,3	6,12,10,3	4	4
12,13,14,15,16,23,24,25,26,34,35,45	60	5,5,4,4,4,2	6,12,11,5,1	5	5
12,13,14,15,16,23,24,25,26,34,36,45	180	5,5,4,4,3,3	6,12,10,3	4	4
12,13,14,15,16,23,24,25,34,36,46,56	180	5,4,4,4,4,3	6,12,9,2	4	4
12,13,14,15,23,24,26,35,36,45,46,56	15	4,4,4,4,4,4	6,12,8	3	3
12,13,14,15,16,23,24,25,34,35,36,45,56	60	5,5,5,4,4,3	6,13,13,6,1	5	5
12,13,14,15,16,23,24,25,26,34,36,45,56	45	5,5,4,4,4,4	6,13,12,4	4	4
12,13,14,15,16,23,24,25,26,34,35,36,45,46	15	5,5,5,5,4,4	6,14,16,9,2	5	5
12,13,14,15,16,23,24,25,26,34,35,36,45,46,56	1	5,5,5,5,5,5	6,15,20,15,6,1	6	6

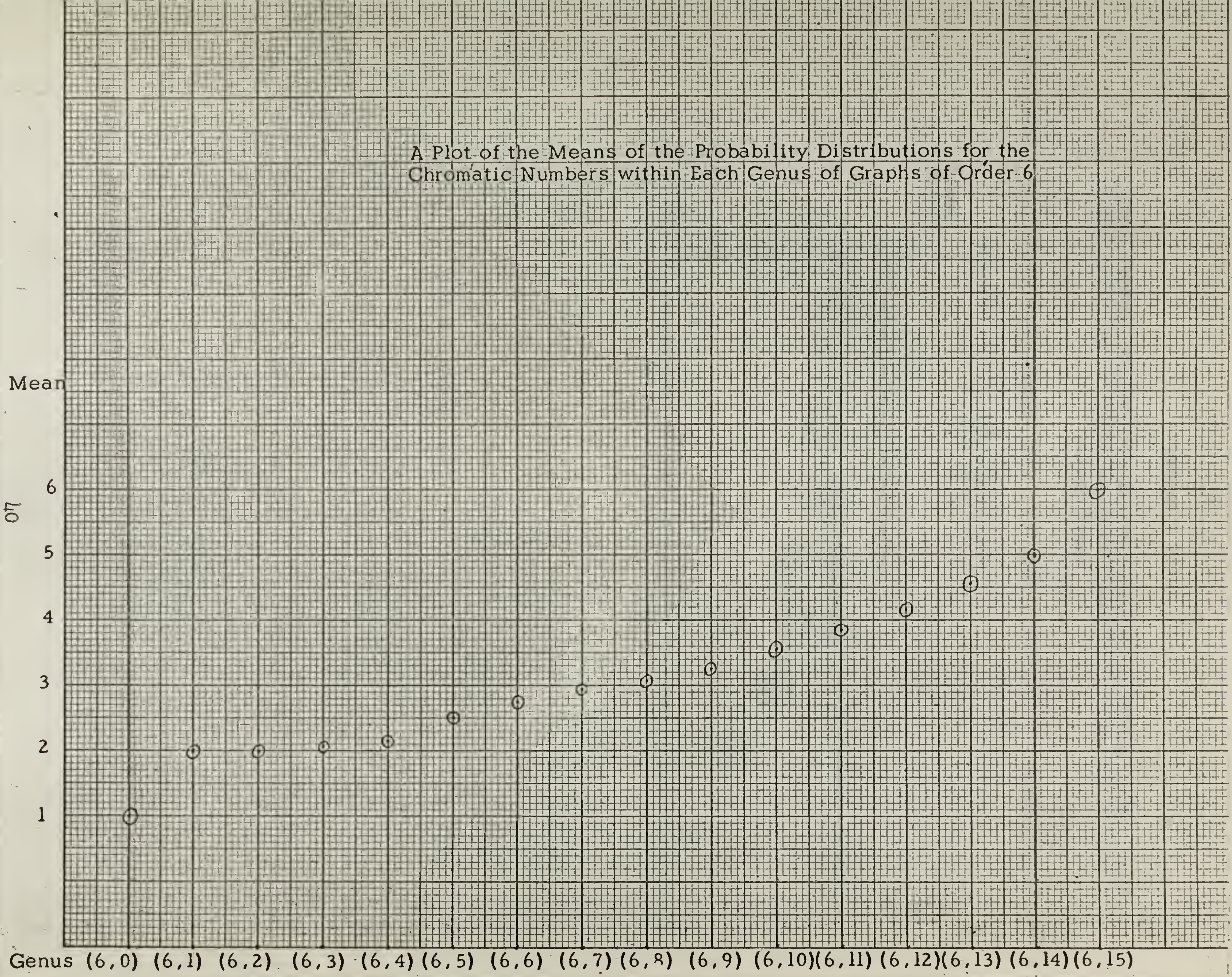
APPENDIX II

The Probabilities of having a Particular Chromatic Number, Given Genus, Plus μ and σ for the Distribution.

GENUS	CHROMATIC NUMBER 1	2	3	4	5	6	μ	σ
(2,0)	1	0	0	0	0	0	1	0
(2,1)	0	1	0	0	0	0	2	0
(3,0)	1	0	0	0	0	0	1	0
(3,1)	0	1	0	0	0	0	2	0
(3,2)	0	1	0	0	0	0	2	0
(3,3)	0	0	1	0	0	0	3	0
(4,0)	1	0	0	0	0	0	1	0
(4,1)	0	1	0	0	0	0	2	0
(4,2)	0	1	0	0	0	0	2	0
(4,3)	0	0.8	0.2	0	0	0	2.2	0.4
(4,4)	0	0.2	0.8	0	0	0	2.8	0.4
(4,5)	0	0	1	0	0	0	3	0
(4,6)	0	0	0	1	0	0	4	0
(5,0)	1	0	0	0	0	0	1	0
(5,1)	0	1	0	0	0	0	2	0
(5,2)	0	1	0	0	0	0	2	0
(5,3)	0	0.9167	0.8333	0	0	0	2.0831	0.2764
(5,4)	0	0.6667	0.3333	0	0	0	2.3333	0.4713
(5,5)	0	0.2381	0.7619	0	0	0	2.7624	0.4259
(5,6)	0	0.0476	0.9286	0.0238	0	0	2.9762	0.1190
(5,7)	0	0	0.8333	0.1667	0	0	3.1667	0.3727
(5,8)	0	0	0.3333	0.6667	0	0	3.6667	0.4713
(5,9)	0	0	0	1	0	0	4	0
(5,10)	0	0	0	0	1	0	5	0
(6,0)	1	0	0	0	0	0	1	0
(6,1)	0	1	0	0	0	0	2	0
(6,2)	0	1	0	0	0	0	2	0
(6,3)	0	0.9560	0.0440	0	0	0	2.0444	0.2050
(6,4)	0	0.8242	0.1758	0	0	0	2.1762	0.3807
(6,5)	0	0.5664	0.4336	0	0	0	2.4336	0.4956
(6,6)	0	0.2398	0.7572	0.0030	0	0	2.7628	0.4321
(6,7)	0	0.0746	0.9044	0.0210	0	0	2.9460	0.3082
(6,8)	0	0.0163	0.8998	0.0839	0	0	3.0676	0.3053
(6,9)	0	0.0020	0.7582	0.2398	0	0	3.2384	0.4305
(6,10)	0	0	0.4545	0.5435	0.0020	0	3.5475	0.5017
(6,11)	0	0	0.1758	0.8022	0.0220	0	3.8462	0.4173
(6,12)	0	0	0.0330	0.7912	0.1758	0	4.1428	0.4341
(6,13)	0	0	0	0.4286	0.5714	0	4.5714	0.4949
(6,14)	0	0	0	0	1	0	5	0
(6,15)	0	0	0	0	0	1	6	0



A Plot of the Means of the Probability Distributions for the Chromatic Numbers within Each Genus of Graphs of Order 5

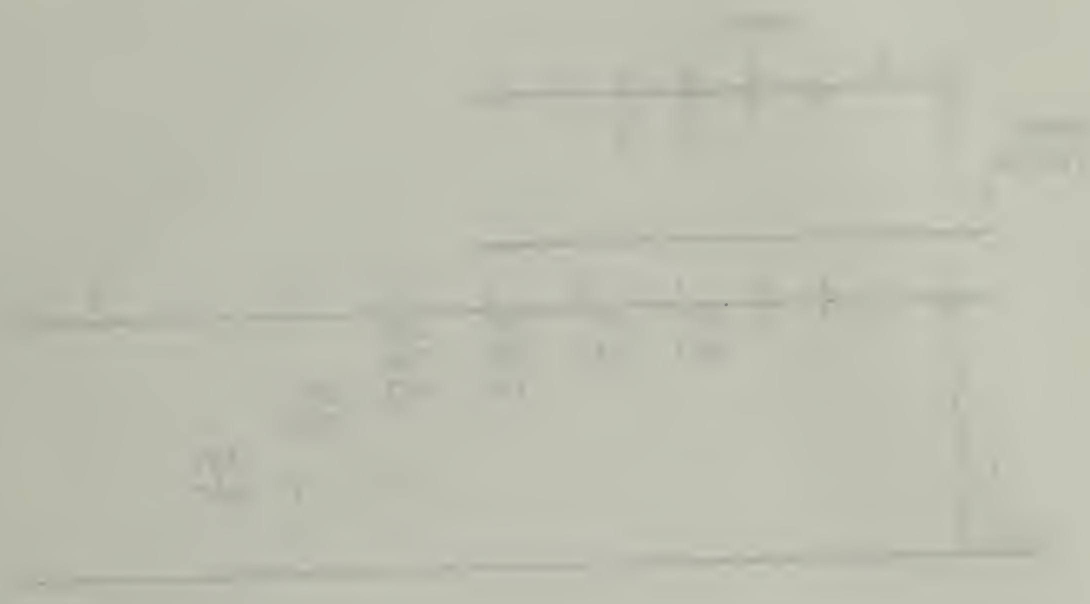

A Plot of the Means of the Probability Distributions for the Chromatic Numbers within Each Genus of Graphs of Order 6
Mean
6
5
4
3
2
1
Genus (6,0) (6,1) (6,2) (6,3) (6,4) (6,5) (6,6) (6,7) (6,8) (6,9) (6,10) (6,11) (6,12) (6,13) (6,14) (6,15)

APPENDIX III

A Table Showing the Conditional Probabilities of Having a Particular Genus Vector for a Random Graph of Order Four, Five, or Six, of Given Genus

	4	0	1	2	3	4	5	6
					Genus			
	0		1	1	.8	.2		
Genus	1				.2	.8		
Vector	2						1	
	4,1							1

5	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	.917	.667	.286	.048				
1				.083	.333	.595	.286				
2						.119	.643	.250			
3								.583			
4									.333		
4,1							.023	.167	.667		
7,2										1	
10,5,1											1



6	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	1	1	.956	.824	.590	.312	.103	.016	.002						
1				.044	.167	.380	.503	.364	.112							
2						.030	.182	.447	.392	.102						
3								.065	.382	.324	.020					
4									.014	.333	.195					
4,1							.003	.021	.060	.036						
5											.264					
5,1									.028	.192	.120					
6												.176				
6,1											.300	.066				
7,1												.264				
7,2										.012	.100	.044				
8													.033			
8,2												.429				
9,2													.396			
10,3													.440			
10, 5,1											.002	.022				
11, 5,1													.132			
12,4														.429		
13, 6,1														.571		
16, 9,2															1	
20,15, 6,1																1

APPENDIX IV

The Genus Vectors for Certain Random Graphs of Orders 20 and 40, and the Mean and Standard Deviations for the Sample Cumulative Distribution Function.

GRAPHS OF ORDER 20

Name	Type 1	2	3	4	5	6	7	8	μ	σ
B	20	57	26	2					2.10	0.80
A	20	57	29	2					2.10	0.86
C	20	57	33	4					2.20	0.85
D	20	57	34	4					2.20	0.85
E	20	57	38	9					2.30	0.95
C	20	76	68	14	1				2.45	0.91
E	20	76	73	17	1				2.45	0.93
A	20	76	70	18	1				2.45	0.93
B	20	76	70	19	2				2.50	0.95
D	20	76	83	25	3				2.60	0.97
D	20	95	131	55	8				2.80	0.98
B	20	95	139	62	8				2.80	1.00
E	20	95	134	58	9				2.85	1.00
A	20	95	140	62	9	1			2.85	1.00
C	20	95	156	96	24	1			3.05	1.05
D	20	114	241	199	59	3			3.30	1.05
A	20	114	241	210	77	10			3.40	1.10
E	20	114	244	219	88	15			3.40	1.15
C	20	114	239	201	77	17	2		3.50	1.15
B	20	114	273	324	201	61	7		3.75	1.21
A	20	133	388	540	354	97	7		3.90	1.20
C	20	133	389	553	384	120	13		3.95	1.20
B	20	133	385	535	364	115	15	1	3.95	1.20
D	20	133	389	543	367	120	19	1	4.00	1.20
E	20	133	379	508	346	122	21	1	4.00	1.20

GRAPHS OF ORDER 40

	1	2	3	4	5	6	7	8	9	μ	σ
A	40	156	80	5						2.15	0.80
B	40	156	75	6						2.15	0.80
A	40	234	278	63	3					2.60	0.90
B	40	234	267	67	4					2.60	0.90
A	40	312	602	293	29	2				2.95	0.90
B	40	312	613	333	53	2				3.05	1.00
B	40	390	1293	1633	802	136	4			3.75	1.10
A	40	390	1242	1483	708	126	5			3.70	1.10
B	40	468	2099	4014	3431	1339	245	17		4.35	1.15
A	40	468	2145	4352	4136	1867	383	26		4.40	1.20
A	40	507	2668	6513	7926	5057	1695	269	13	4.80	1.25