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Theoretical design of reinforcing rings for circular cutouts in flat plates in tension.

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Thesis
S669

A THEORETICAL DESIGN

OF

REINFORCING RINGS FOR CIRCULAR CUTOUTS

IN

FLAT PLATES IN TENSION

A THESIS

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PREFACE

In the field of aircraft design, the question of weight versus structural strength has always been important. In recent years, with the trend toward high speed and high performance in modern aircraft, it is of the utmost importance that the weight factor be made as small as practicable.

It is therefore necessary that the structural engineer use every means at his command for obtaining the necessary strength with the least amount of weight, and this c are must be exercised in every phase of design, down to the smallest detail.

The subject of this thesis was chosen with this thought in mind as the design of reinforcing rings around circular cutouts is still largely a "cut and try" matter.

The writer wishes to express appreciation to his adviser, Professor J. A. Wise, whose advice and assistance was most valuable.

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SUMMARY

This is a presentation of a theoretical analysis conducted for the purpose of designing reinforcing rings for circular cutouts in flat plates in tension. In this report, the general case of a flat plate under a uniaxial tension load was assumed, together with a circular cutout symmetrically placed and reinforced with a circular ring. Expressions for the radial and tangential stress in the ring and the flat plate were derived in terms of the loading on the plate, the dimensions of the ring and plate, and Poisson's ratio.

Experimental data was taken from a test specimen which was constructed and loaded so as to agree with the assumptions made in the theoretical analysis. A comparison of the test data and that obtained from the theoretical analysis showed excellent agreement and attests to the validity of the theoretical solution.

INTRODUCTION

The trend in modern aviation is toward high speed and maximum performance in flight, especially with regard to military aircraft. Along with this trend go the stringent demands for structural designs that reduce the weight factor while still meeting the necessary strength and space requirements.

The design of reinforcing rings around cutouts in flat surfaces for various loading conditions represents one of the many fields where a simple and accurate method of design would save much time and effort and would represent a worthwhile savings in weight.

The need for openings in the stress carrying skin of aircraft for maintenance access, doors, windows, lights, and retractable landing gear have presented many difficult problems to the structural engineer. Usually, these holes are reinforced with metal rings--doubler plates-- riveted to the sheet, but little is known as to how well such a reinforcement approaches the ideal in reducing the stress concentration around the hole while at the same time adding least weight to the structure.

With this thought in mind, a theoretical analysis of a flat plate under a tension load was undertaken. The flat plate was assumed to have a circular cutout, symmetrically placed and reinforced with a circular ring. Expressions for the radial and tangential stress in the ring and the flat plate were derived in terms of the loading on the plate, the dimensions of the ring and plate, and Poisson's ratio.

THEORETICAL ANALYSIS

I. GENERAL DISCUSSION OF METHOD USED.

When a small circular hole is made in a plate submitted to a uniform tensile stress, a high stress concentration occurs at the edges of the hole located at ninety degrees from the direction of the tension load. If the diameter of the hole is less than about one-fifth the width of the plate, the conditions for a plate of infinite width and with the load applied at infinity are approached, and exact theory shows that the tensile stress at the above mentioned points is three times that of the loading. From this theory, it can be seen that the stress concentration is of a very localized character and is confined to the immediate vicinity of the hole.

Since failure will first occur at these points, it is necessary in the design of a ring to reduce the stress concentration in the plate in those areas. It is therefore pertinent that expressions for the radial and tangential stress in the ring and the plate be developed in terms of the loading and the dimensions of the plate and ring. This was the basis of the theoretical analysis as made in this report.

In this analysis it was assumed that the reinforcement is an integral part of the sheet and that the stresses do not vary across the plate thickness. Previous studies indicate that these assumptions are reasonably valid and give good results outside the reinforced area. It was found that measured and computed reinforcement strains agreed best in the case of a reinforcing ring fastened with two concentric rows of rivets.

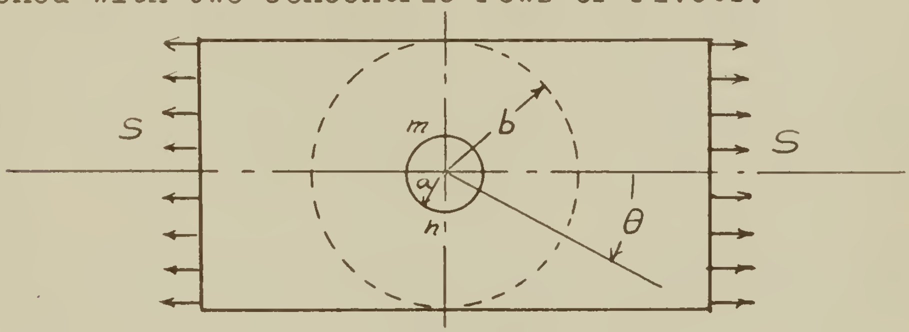


FIG. 1.

Let Fig. 1 represent a plate, with a small hole in the middle, submitted to a uniform tension of magnitude S as shown. From Saint-Venant's principle, the stress concentration that occurs around the hole at m and n will be negligible at distances which are large compared to the diameter of the hole.

As developed in reference (a), and considering the portion of the plate within a concentric circle of radius b, large with respect to radius a, the stresses at radius b are essentially the same as in

a plate without the hole and thus are given by

$$(\sigma_r)_{r=b} = S \cos^2\theta = \frac{1}{2} S \, (1 + \cos 2\theta)$$

$$(\tau_{r\theta})_{r=b} = -\frac{1}{2} S \sin 2\theta$$

These forces, acting on the outer circumference of the ficticious ring at r = b give a stress distribution within the ring which can be regarded as consisting of two parts. The first is due to the constant component $\frac{1}{2}$ S of the normal forces. The remaining part consists of the normal forces $\frac{1}{2}$ S cos 2θ, together with the shearing forces -- $\frac{1}{2}$ S sin 2θ.

As is evident later, a stress function is needed for the solution of the stresses due to the above mentioned forces and it would be very difficult to find a single stress function for representing both parts. Therefore for simplification of the problem, the radial and tangential stresses are worked out separately for the forces as divided above and then are added to give the final results for the flat plate with a reinforcing ring. This is carried out in Parts III and IV of this section.

Part II of this section consists mainly of a check on the validity of the basic theory used in this analysis. It takes the equations for the stresses

in a flat plate with a hole in the middle, as obtained from reference (a), and further develops these equations to obtain expressions for radial and tangential displacement which were then checked experimentally to substantiate the general theory.

Briefly the equations for the stresses in the flat plate are developed as follows.

For the stress distribution symmetrical about the center due to constant component $\frac{1}{2}$ S of the normal forces, the shearing stress $\tau_{r\theta}$ vanishes, leaving the radial and tangential stress as follows:

$$\sigma_r = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right)$$

For the remaining part, consisting of the normal forces $\frac{1}{2}$ S cos 2 θ, together with the shearing forces -$\frac{1}{2}$ S sin 2θ, there is produced a stress which may be derived from a stress function of the form $\phi = f(r)$ cos 2 θ.

As developed in detail in Part IV of this section this finally results in

$$\sigma_r = \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right) \cos 2\theta$$

$$\sigma_\theta = -\frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right) \cos 2\theta$$

$$\tau_{r\theta} = -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right) \sin 2\theta$$

Adding the stresses produced by the uniform tension $\frac{1}{2}$ S gives the final expressions for the stresses in a flat plate under a tension load, with a hole in the middle.

$$\sigma_r = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

These are the equations which are used in Part II of this section.

THEORETICAL ANALYSIS

II. EXPERIMENTAL CHECK OF BASIC THEORY.

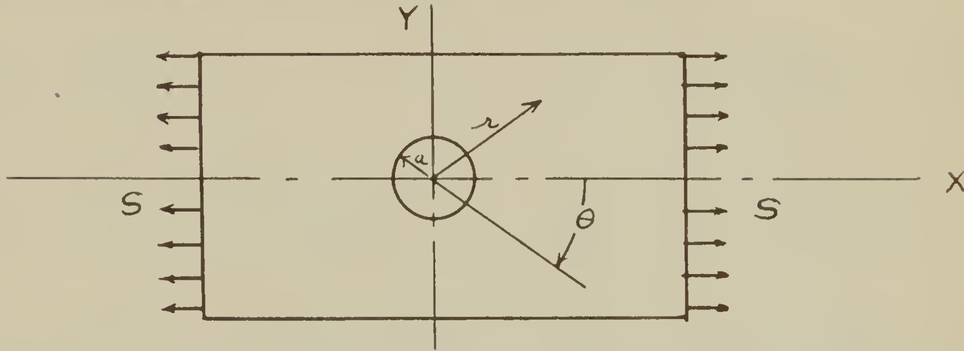


Fig. 2.

For the above figure, the stresses were given in Part I as,

$$\sigma_r = \frac{S}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{S}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{S}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{S}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{S}{2}\left(1-\frac{3a^4}{r^4}+2\frac{a^2}{r^2}\right)\sin 2\theta$$

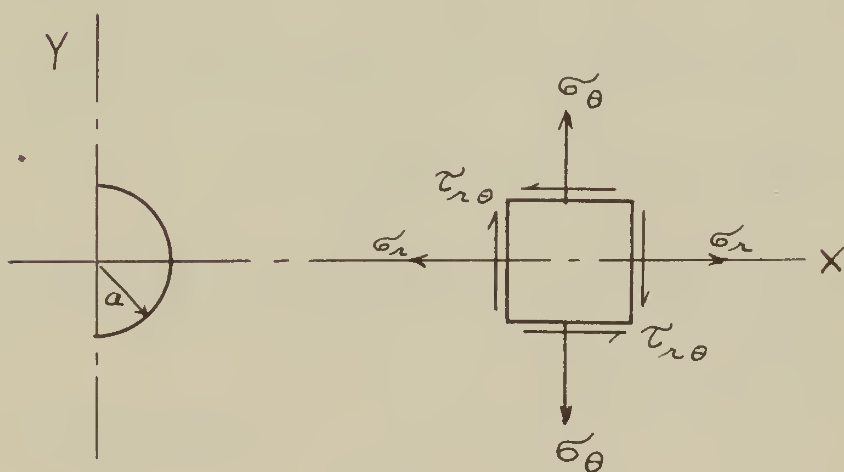


Fig. 3.

Let

u = radial displacement

v = tangential displacement

For this distribution of stress, the correspon-ding displacements are obtained by applying the basic relationships given below:

$$\epsilon_r = \frac{du}{dr} \qquad \epsilon_r = \frac{1}{E}\left(\sigma_r - \mu\,\sigma_\theta\right)$$

$$\epsilon_\theta = \frac{u}{r} + \frac{dv}{r\,d\theta} \qquad \epsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \mu\,\sigma_r\right)$$

$$\gamma_{r\theta} = \frac{du}{r\,d\theta} + \frac{dv}{dr} - \frac{v}{r} \qquad \gamma_{r\theta} = \frac{1}{G}\,\tau_{r\theta}$$

$$G = \frac{E}{2\,(1+\mu)}$$

From the above equations:

$$\frac{du}{dr} = \frac{1}{E}\left(\sigma_r - \mu\,\sigma_\theta\right)$$

$$\frac{du}{dr} = \frac{1}{E}\left\{\left[\frac{S}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{S}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta\right] - \mu\left[\frac{S}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{S}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]\right\}$$

$$\frac{du}{dr} = \frac{S}{2E}\left\{(1-\mu) - \frac{a^2}{r^2}(1+\mu) + (1+\mu)\cos 2\theta - \frac{4a^2}{r^2}\cos 2\theta + (1+\mu)\frac{3a^4}{r^4}\cos 2\theta\right\}$$

Integrating

$$u = \frac{S}{2E}\left\{ r(1-\mu) + \frac{a^2}{r}(1+\mu) + r(1+\mu)\cos 2\theta + \frac{4a^2}{r}\cos 2\theta - (1+\mu)\frac{a^4}{r^3}\cos 2\theta \right\} + f(\theta)$$

in which $f(\theta)$ is a function of θ only.

$$\epsilon_\theta = \frac{u}{r} + \frac{dv}{r\,d\theta} = \frac{1}{E}\left(\sigma_\theta - \mu\,\sigma_r\right)$$

$$\frac{dv}{d\theta} = \frac{r}{E}\left(\sigma_\theta - \mu\,\sigma_r\right) - u$$

Substituting values for σ_θ and u, and simplifying gives

$$\frac{dv}{d\theta} = -\frac{Sr}{E}\left[(1+\mu) + 2(1-\mu)\frac{a^2}{r^2} + (1+\mu)\frac{a^4}{r^4}\right]\cos 2\theta - f(\theta)$$

Integrating

$$v = -\frac{Sr}{2E}\left[(1+\mu) + 2(1-\mu)\frac{a^2}{r^2} + (1+\mu)\frac{a^4}{r^4}\right]\sin 2\theta - \int f(\theta)\,d\theta + F(r)$$

where F (r) is a function of r only.

Substituting the above expressions for u and v in the shear equation

$$\gamma_{r\theta} = \frac{du}{r\,d\theta} + \frac{dv}{dr} - \frac{v}{r}$$

$$\frac{du}{r\,d\theta} = \frac{S}{2E}\left\{-2(1+\mu)\sin 2\theta - \frac{8a^2}{r^2}\sin 2\theta + 2(1+\mu)\frac{a^4}{r^4}\sin 2\theta\right\} + \frac{f'(\theta)}{r}$$

$$\frac{dv}{dr} = -\frac{S}{2E}\left\{(1+\mu) - 2(1-\mu)\frac{a^2}{r^2} - 3(1+\mu)\frac{a^4}{r^4}\right\}\sin 2\theta + F'(r)$$

$$\gamma_{r\theta} = \frac{S}{2E}\left\{-2(1+\mu) - \frac{8a^2}{r^2} + 2(1+\mu)\frac{a^4}{r^4}\right\}\sin 2\theta + \frac{f'(\theta)}{r}$$

$$+\frac{S}{2E}\left\{-(1+\mu) + 2(1-\mu)\frac{a^2}{r^2} + 3(1+\mu)\frac{a^4}{r^4}\right\}\sin 2\theta + F'(r)$$

$$+\frac{S}{2E}\left\{(1+\mu) + 2(1-\mu)\frac{a^2}{r^2} + (1+\mu)\frac{a^4}{r^4}\right\}\sin 2\theta$$

$$+\frac{1}{r}\int f(\theta)\,d\theta - \frac{F(r)}{r}$$

$$r\,\gamma_{r\theta} = \int f(\theta)\,d\theta + f'(\theta) + r\,F'(r) - F(r) + $$

$$+\frac{Sr}{2E}\sin 2\theta\begin{bmatrix}-2(1+\mu) - 8\frac{a^2}{r^2} + 2(1+\mu)\frac{a^4}{r^4}\\ -(1+\mu) + 2(1-\mu)\frac{a^2}{r^2} + 3(1+\mu)\frac{a^4}{r^4}\\ +(1+\mu) + 2(1-\mu)\frac{a^2}{r^2} + (1+\mu)\frac{a^4}{r^4}\end{bmatrix}$$

$$\gamma_{r\theta} = \frac{1}{G}\,\tau_{r\theta}$$

$$\gamma_{r\theta} = \frac{1}{G}\left[-\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + 2\frac{a^2}{r^2}\right)\sin 2\theta\right]$$

$$\frac{1}{G} = \frac{2(1+\mu)}{E}$$

$$r\,\gamma_{r\theta} = -\frac{Sr}{E}\sin 2\theta\left[(1+\mu) - 3(1+\mu)\frac{a^4}{r^4} + 2(1+\mu)\frac{a^2}{r^2}\right]$$

$$\int f(\theta)\, d\theta + f'(\theta) + r F'(r) - F(r) =$$

$$- \frac{Sr}{2E} \begin{bmatrix} -2(1+\mu) & -\ 8\frac{a^2}{r^2} & +\ 2(1+\mu)\frac{a^4}{r^4} \\ -(1+\mu) & +\ 2(1-\mu)\frac{a^2}{r^2} & +\ 3(1+\mu)\frac{a^4}{r^4} \\ +(1+\mu) & +\ 2(1-\mu)\frac{a^2}{r^2} & +\ (1+\mu)\frac{a^4}{r^4} \\ +2(1+\mu) & +\ 4(1+\mu)\frac{a^2}{r^2} & -\ 6(1+\mu)\frac{a^4}{r^4} \end{bmatrix} \sin 2$$

Since the quantity in parenthesis = o,

$$\int f(\theta)\, d\theta + f'(\theta) + r F'(r) - F(r) = 0$$

This equation is satisfied by putting

$$F(r) = A r$$

$$f(\theta) = B \sin\theta + C \cos\theta$$

in which A, B, and C are arbitrary constants to be determined from the conditions of restraint.

Therefore:

$$u = \frac{Sr}{2E}\left[(1-\mu) + (1+\mu)\frac{a^2}{r^2} + (1+\mu)\cos 2\theta + \frac{4a^2}{r^2}\cos 2\theta - (1+\mu)\frac{a^4}{r^4}\cos 2\theta\right] + B\sin\theta + C\cos\theta$$

$$v = -\frac{Sr}{2E}\left[(1+\mu) + 2(1-\mu)\frac{a^2}{r^2} + (1+\mu)\frac{a^4}{r^4}\right]\sin 2\theta + B\cos\theta - C\sin\theta + Ar$$

The conditions of restraint are:

(1.) At $\theta = 90^\circ$, $v = o$

(2.) At $\theta = 0^\circ$, $v = o$

(3.) At $\theta = 0^\circ$ & 90°, $\frac{dv}{dr} = 0$

From condition (1.)

$$o = -C + Ar$$

$$C = Ar$$

From condition (2.)

$$o = B + Ar$$

$$B = -Ar$$

From condition (3.)

$$\frac{dv}{dr} = o = -\frac{S}{2E}\left[(1+\mu) - 2(1-\mu)\frac{a^2}{r^2} - 3(1+\mu)\frac{a^4}{r^4}\right]\sin 2\theta + A$$

$$A = o$$

$$B = o$$

$$C = o$$

The final equations for u and v are

$$u = \frac{Sr}{2E}\left[(1-\mu) + (1+\mu)\frac{a^2}{r^2} + (1+\mu)\cos 2\theta + 4\frac{a^2}{r^2}\cos 2\theta - (1+\mu)\frac{a^4}{r^4}\cos 2\theta\right]$$

$$v = -\frac{Sr}{2E}\left[(1+\mu) + 2(1-\mu)\frac{a^2}{r^2} + (1+\mu)\frac{a^4}{r^4}\right]\sin 2\theta$$

At $\theta = 0^\circ$ and $r = a$

$$u = \frac{3Sa}{E}$$

At $\theta = 90^\circ$ and $r = a$

$$u = -\frac{Sa}{E}$$

To find θ for $u = o$ at $r = a$

$$0 = \frac{Sa}{E}\left(1 + 2\cos 2\theta\right)$$

$$\cos 2\theta = -\tfrac{1}{2}$$

$$\cos 120^\circ = -\tfrac{1}{2}$$

$$\theta = 60^\circ$$

Using the above equation for radial displacement (u), a plot of the displacement was made as shown in Fig. 4 for a loading of eight hundred pounds on the test specimen as indicated in the same figure. An experimental check using the Huggenberger Tensometer was also made on the test specimen and gave the experimental values as shown. The results indicated that the basic theory used is sufficiently valid for purposes of this analysis.

A PLOT OF THEORETICAL AND EXPERIMENTAL RADIAL DISPLACEMENT AROUND THE HOLE WITH TEST SPECIMEN UNDER A TENSION LOAD OF 800 LBS.

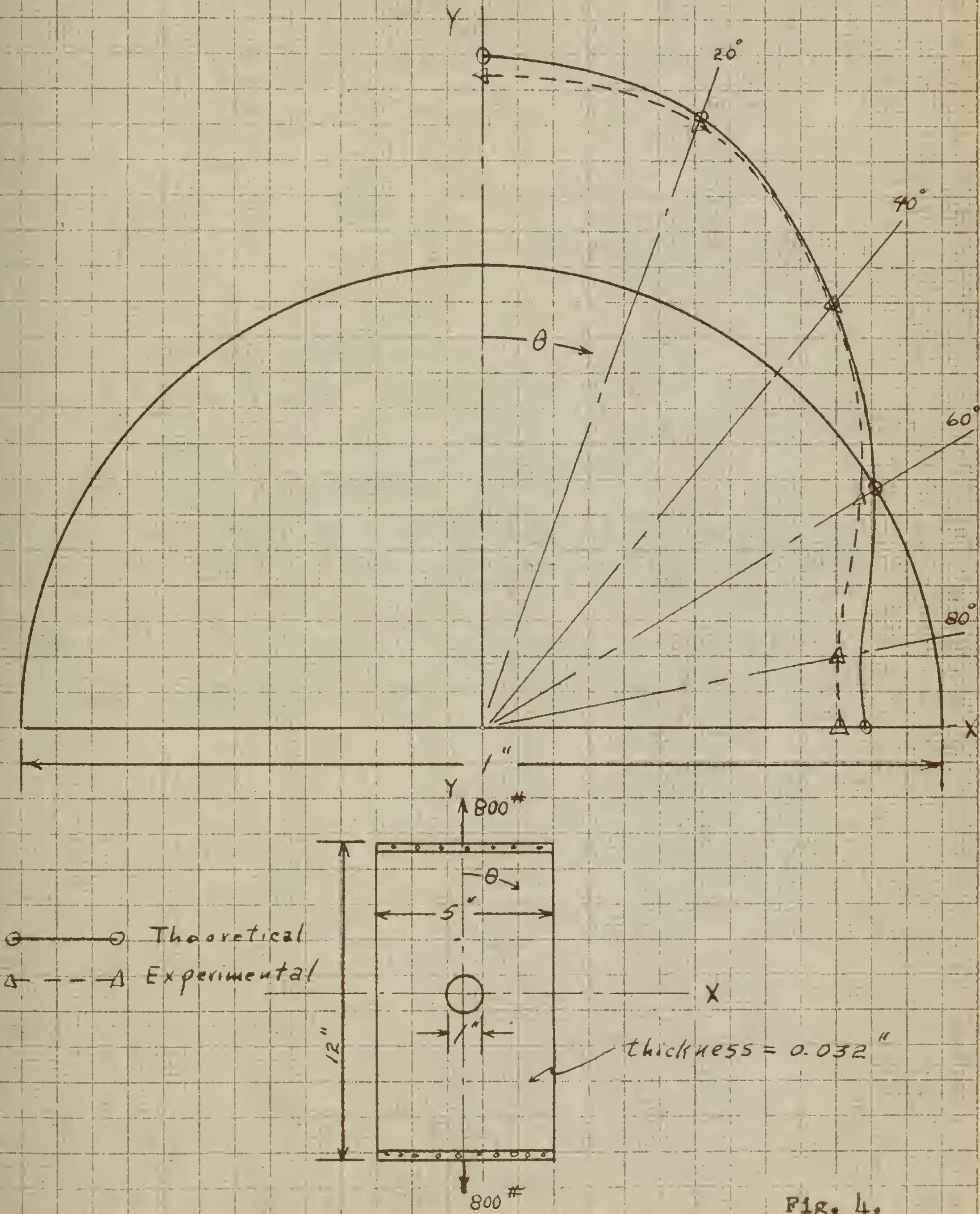


Fig. 4.

THEORETICAL ANALYSIS

III. DETERMINATION OF RADIAL AND TANGENTIAL STRESS DUE TO THE CONSTANT COMPONENT $\frac{1}{2}$ S OF THE NORMAL FORCES.

Let

t_i = thickness in region of ring.

t_o = thickness of flat plate (outside ring).

To obtain the radial stress (σ_r) and the tangential stress (σ_θ) in both regions (t_i + t_o) due to the constant component $\frac{1}{2}$ S of the normal forces, the procedure is as follows:

$\tau_{r\theta}$ = 0, since the stress distribution is symmetrical, the stress components do not depend on θ and are functions of r only.

This is also evident from the polar equation of equilibrium, $\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r\, \partial \theta}$

The displacement (u) is a function of the radius only, because of symmetry.

Since

$$\epsilon_r = \frac{du}{dr}$$

$$\epsilon_\theta = \frac{2\pi(r+u) - 2\pi r}{2\pi r} = \frac{u}{r}$$

Then

$$\sigma_r = E\left(\epsilon_r + \mu\,\epsilon_\theta\right) = E\left(\frac{du}{dr} + \mu\frac{u}{r}\right)$$

$$\sigma_\theta = E\left(\epsilon_\theta + \mu\,\epsilon_r\right) = E\left(\frac{u}{r} + \mu\frac{du}{dr}\right)$$

Considering the equilibrium of a small element cutout from the plate by the radial sections O_c and O_b, perpendicular to the plate, and by two cylindrical surfaces a d and b c with radii r and r + dr, normal to the plate,

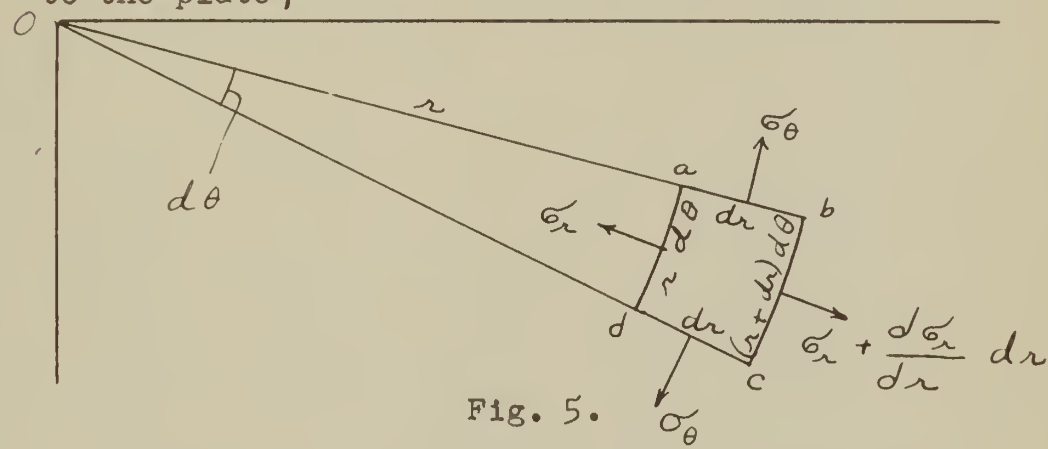


Fig. 5.

σ_θ remains constant because of symmetry,

$\tau_{r\theta} = 0$ (from previous considerations)

Summing the forces in the radial direction

$$\left(\sigma_r + \frac{d\sigma_r}{dr}dr\right)\left(r + dr\right)d\theta - \sigma_r\, r\, d\theta - 2\sigma_\theta\frac{d\theta}{2}dr = 0$$

Neglecting second order quantities and cancelling out dr $d\theta$ gives

$$\sigma_r + r\frac{d\sigma_r}{dr} - \sigma_\theta = 0$$

Substituting σ_r and σ_θ in the above equation,

$$E\left\{\frac{du}{dr} + \mu\frac{u}{r} + r\left(\frac{d^2u}{dr^2} + \frac{\mu}{r}\frac{du}{dr} - \frac{\mu}{r^2}u\right) - \frac{u}{r} - \mu\frac{du}{dr}\right\} = 0$$

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} - \frac{u}{r} = 0$$

Multiplying through by r,

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = 0$$

To solve this ordinary differential equation,

Let $r = e^t$

$$\frac{dr}{dt} = e^t$$

$$\frac{dt}{dr} = \frac{1}{e^t} = \frac{1}{r}$$

$$\frac{du}{dr} = \frac{du}{dt}\cdot\frac{dt}{dr} = \frac{1}{r}\frac{du}{dt}$$

$$r\frac{du}{dr} = \frac{du}{dt} = Du$$

where $D = \frac{d}{dt}$

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{d}{dr}\left(\frac{du}{dt}\right) = \frac{d}{dt}\left(\frac{du}{dt}\right)\frac{dt}{dr}$$

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = \frac{d^2u}{dt^2}\cdot\frac{1}{r}$$

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} = \frac{d^2u}{dt^2}$$

$$r^2\frac{d^2u}{dr^2} = \frac{d^2u}{dt^2} - \frac{du}{dt} = D^2u - Du$$

$$r^2 \frac{d^2 u}{d r^2} = D(D-1) u$$

Therefore

$$r^2 \frac{d^2 u}{d r^2} + r \frac{d u}{d r} - u = 0$$

may be written as

$$\left[D(D-1) + D - 1 \right] u = 0$$

$$\left[D^2 - 1 \right] u = 0$$

$$\text{Let} \quad u = e^{mt}$$

$$m^2 - 1 = 0$$

$$m = +1, -1$$

$$u = A e^{t} + B e^{-t} = A e^{t} + \frac{B}{e^{t}}$$

Since $r = e^{t}$

$$u = Ar + \frac{B}{r}$$

To check on the correctness of this formula, consider a flat plate as follows with a constant thickness:

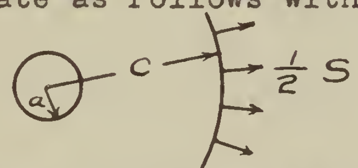


Fig. 6.

with boundary conditions as follows:

When $r = a$, $\sigma_r = 0$

When $r = C$, $\sigma_r = \frac{S}{2}$

Using the original equation for σ_r,

$$\sigma_r = E\left[\frac{du}{dr} + \mu \frac{u}{r} \right]$$

$$\frac{du}{dr} = A - \frac{B}{r^2}$$

$$\sigma_r = E\left[A - \frac{B}{r^2} + \frac{\mu}{r}\left(A r + \frac{B}{r}\right)\right]$$

$$\sigma_r = E\left[A - \frac{B}{r^2} + \mu A + \mu \frac{B}{r^2}\right]$$

$$\sigma_r = E\left[(1+\mu) A - (1-\mu) \frac{B}{r^2}\right]$$

Substituting boundary conditions in the above equation:

$$0 = E\left[(1+\mu) A - (1-\mu) \frac{B}{a^2}\right]$$

$$\frac{S}{2} = E\left[(1+\mu) A - (1-\mu) \frac{B}{c^2}\right]$$

$$B = \left(\frac{1+\mu}{1-\mu}\right) a^2 A$$

$$\frac{S}{2} = E\left[(1+\mu) A - (1-\mu) \frac{1+\mu}{1-\mu} \frac{a^2}{c^2} A\right]$$

$$A = \frac{S/2E}{(1+\mu)\left(1 - \frac{a^2}{c^2}\right)}$$

$$\sigma_r = E\left[\frac{S}{2E\left(1 - \frac{a^2}{c^2}\right)} - \frac{a^2}{r^2} \frac{S}{2E\left(1 - \frac{a^2}{c^2}\right)}\right]$$

$$\sigma_r = \frac{S}{2}\left[\frac{1 - a^2/r^2}{1 - a^2/c^2}\right]$$

For $c = \infty$

$\sigma_r = \frac{S}{2}\left[1 - \frac{a^2}{r^2}\right]$ for a flat plate of constant thickness with a hole at the center.

On page fifty-five of reference (a), the following formula is given for a hollow cylinder submitted

to uniform pressure on the inner and outer surfaces:

$$\sigma_r = \frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \cdot \frac{1}{r^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

In the above case:

$$p_i = 0$$

$$p_o = -\frac{S}{2} \quad \text{(positive for compression)}$$

$$\sigma_r = \frac{a^2 b^2 (-S/2)}{b^2 - a^2} \cdot \frac{1}{r^2} + \frac{S/2 \; b^2}{b^2 - a^2}$$

$$\sigma_r = \frac{S}{2} \left[\frac{-a^2 b^2 + b^2 r^2}{(b^2 - a^2) r^2} \right]$$

Multiplying numerator and denominator by $\frac{1}{b^2}$:

$$\sigma_r = \frac{S}{2} \left[\frac{r^2 - a^2}{\left(1 - \frac{a^2}{b^2}\right) r^2} \right]$$

At b = ∞

$$\sigma_r = \frac{S}{2} \left[\frac{r^2 - a^2}{r^2} \right]$$

$$\sigma_r = \frac{S}{2} \left[1 - \frac{a^2}{r^2} \right]$$

This gives an exact check on the formula previously developed:

$$u = Ar + \frac{B}{r}$$

For the case of a flat plate with a different

thickness around the hole as follows:

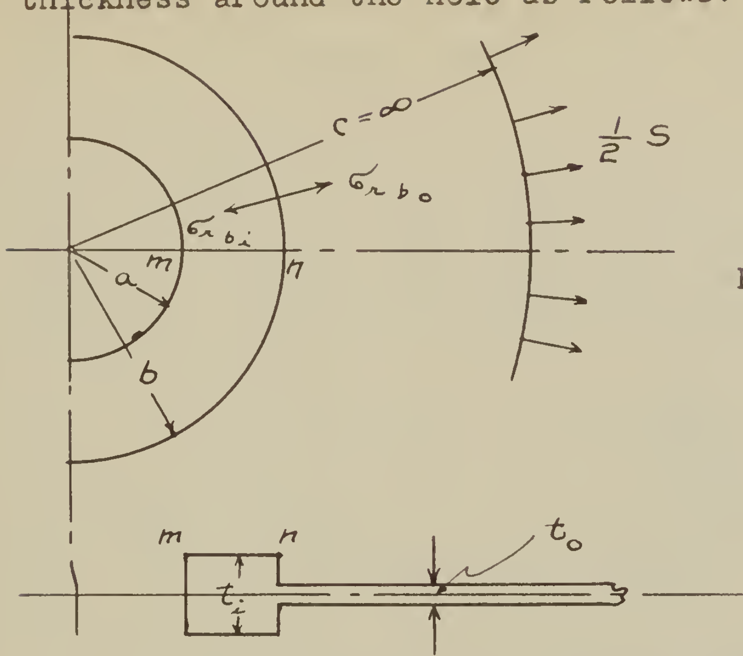


Fig. 7.

At the boundary (b), the designation is

i - inside

o - outside

as indicated in the above figure.

$$u_i = A_i r + B_i/r$$

$$u_o = A_o r + B_o/r$$

Boundary conditions are as follows:

(1.) At r = a, $\sigma_r = 0$

(2.) At r = C, $\sigma_r = \frac{S}{2}$

(3.) At r = b, $u_i = u_o$

(4.) At r = b, $\sigma_{r_{b_i}} = \sigma_{r_{b_o}} \frac{t_o}{t_i}$

Using the formulas previously developed in this section

$$\sigma_r = E\left[(1+\mu)\, A - (1-\mu)\, \frac{B}{r^2}\right]$$

and

$$u = Ar + \frac{B}{r}$$

the above boundary conditions give the following equations, assuming the modulus of elasticity (E) to be a constant.

(1.) $0 = E\left[(1+\mu)\, A_i - (1-\mu)\, \frac{B_i}{a^2}\right]$

(2.) $\frac{S}{2} = E\left[(1+\mu)\, A_o - (1-\mu)\, \frac{B_o}{c^2}\right]$

(3.) $A_i b + \frac{B_i}{b} = A_o b + \frac{B_o}{b}$

(4.) $t_i\left[(1+\mu) A_i - (1-\mu)\, \frac{B_i}{b^2}\right] = t_o\left[(1+\mu) A_o - (1-\mu)\, \frac{B_o}{b^2}\right]$

These equations may be solved by the method of determinates for the constants A_i, B_i, A_o, and B_o.

$$A_i \text{ (numerator)} = -\frac{t_o S (1-\mu)}{a^2 b E}$$

$$B_i \text{ (numerator)} = -\frac{t_o S (1+\mu)}{b E}$$

$$A_o \text{ (numerator)} = \frac{S(1-\mu)}{2\, a^2 b^3 E}\left\{\begin{array}{l} a^2(1+\mu)\left[t_i - t_o\right] \\ -b^2 t_i (1+\mu) - b^2 t_o (1-\mu) \end{array}\right\}$$

$$B_o \text{ (numerator)} = \frac{S(1+\mu)}{2\, a^2 b E}\left\{\begin{array}{l} b^2(1-\mu)\left[t_i - t_o\right] \\ -a^2 t_i (1-\mu) - a^2 t_o (1+\mu) \end{array}\right\}$$

$$\text{Denominator} = \frac{(1-\mu^2)}{a^2b^3c^2}\left\{\begin{matrix}(a^2-b^2)\,t_i\left[c^2(1+\mu)+b^2(1-\mu)\right]\\+(b^2-c^2)\,t_o\left[b^2(1-\mu)+a^2(1+\mu)\right]\end{matrix}\right\}$$

The validity of considering the load applied at infinity has previously been proven in section II, and this is now done for the value of c as it appears in the denominator of the four constants listed above.

$$\text{Denominator} = \frac{(1-\mu^2)}{a^2b^3}\left\{\begin{matrix}(a^2-b^2)\,t_i\left[(1+\mu)+\frac{b^2}{c^2}(1-\mu)\right]\\+\left(\frac{b^2}{c^2}-1\right)t_o\left[b^2(1-\mu)+a^2(1+\mu)\right]\end{matrix}\right\}$$

When $c \to \infty$

$$\text{Denominator} = \frac{(1-\mu^2)}{a^2b^3}\left\{\begin{matrix}(a^2-b^2)\,t_i\,(1+\mu)\\-\,t_o\left[b^2(1-\mu)+a^2(1+\mu)\right]\end{matrix}\right\}$$

$$\text{Denominator} = \frac{(1-\mu^2)}{a^2b^3}\left\{\begin{matrix}a^2(1+\mu)(t_i-t_o)\\-b^2t_i(1+\mu)-b^2t_o(1-\mu)\end{matrix}\right\}$$

Combining the numerators and denominator give the constants:

$$A_i = -\frac{b^2t_oS}{E\,(1+\mu)}\left[\frac{1}{a^2(1+\mu)(t_i-t_o)-b^2t_i(1+\mu)-b^2t_o(1-\mu)}\right]$$

$$B_i = -\frac{a^2b^2t_oS}{E\,(1-\mu)}\left[\frac{1}{a^2(1+\mu)(t_i-t_o)-b^2t_i(1+\mu)-b^2t_o(1-\mu)}\right]$$

$$A_o = \frac{S}{2E(1+\mu)}$$

$$B_o = \frac{b^2 S}{2E(1-\mu)} \left[\frac{b^2(1-\mu)[t_i - t_o] - a^2 t_i (1-\mu) - a^2 t_o (1+\mu)}{a^2(1+\mu)[t_i - t_o] - b^2 t_i (1+\mu) - b^2 t_o (1-\mu)} \right]$$

Considering previous formulas developed:

$$\sigma_r = E\left[\frac{du}{dr} + \mu \frac{u}{r}\right]$$

$$\sigma_\theta = E\left[\frac{u}{r} + \mu \frac{du}{dr}\right]$$

$$u = Ar + \frac{B}{r}$$

$$\frac{du}{dr} = A - \frac{B}{r^2}$$

By proper substitution, the above gives:

(1.) $$\sigma_r = E\left[(1+\mu) A - (1-\mu)\frac{B}{r^2}\right]$$

(2.) $$\sigma_\theta = E\left[(1+\mu) A + (1-\mu)\frac{B}{r^2}\right]$$

(3.) $$u = Ar + \frac{B}{r}$$

Therefore for the radial and tangential stresses due to the constant component ($\frac{1}{2}$ S) of the normal forces, the following equations apply in which the above constants A_i, B_i, A_o and B_o are to be used,

$$\sigma_{r_i} = E\left[(1+\mu)\ A_i - (1-\mu)\ \frac{B_i}{r^2}\right]$$

$$\sigma_{r_o} = E\left[(1+\mu)\ A_o - (1-\mu)\ \frac{B_o}{r^2}\right]$$

$$\sigma_{\theta_i} = E\left[(1+\mu)\ A_i + (1-\mu)\ \frac{B_i}{r^2}\right]$$

$$\sigma_{\theta_o} = E\left[(1+\mu)\ A_o + (1-\mu)\ \frac{B_o}{r^2}\right]$$

Also,

$$u_i = A_i\, r + \frac{B_i}{r}$$

$$u_o = A_o\, r + \frac{B_o}{r}$$

THEORETICAL ANALYSIS

IV. DETERMINATION OF RADIAL AND TANGENTIAL STRESS DUE TO THE NORMAL FORCES ($\frac{1}{2}$ S cos 2θ) AND THE SHEARING FORCES ($-\frac{1}{2}$ S sin 2θ).

Neglecting body forces, the equations of equilibrium for an element, expressed in polar coordinates are:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} = 0$$

These equations are satisfied by taking:

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}.$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r\,\partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

where ϕ is a function of r and θ and is known as a stress function.

As developed in reference (a), the stresses developed from the normal forces ($\frac{1}{2}$ S cos 2θ) and the shearing forces ($-\frac{1}{2}$ S sin 2θ) may be derived from a stress function of the form

$$\phi = f(r)\cos 2\theta .$$

Substituting this into the compatibility equation in polar co-ordinates

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0$$

gives the ordinary differential equation,

$$\frac{d^4 f}{dr^4} + \frac{2}{r}\frac{d^3 f}{dr^3} - \frac{9}{r^2}\frac{d^2 f}{dr^2} + \frac{9}{r^3}\frac{df}{dr} = 0$$

or

$$r^4 \frac{d^4 f}{dr^4} + 2r^3 \frac{d^3 f}{dr^3} - 9r^2 \frac{d^2 f}{dr^2} + 9r\frac{df}{dr} = 0$$

Solving this equation in the same manner as used in Section III gives

$$\left[D^4 - 4D^3 - 4D^2 + 16D\right] f = 0$$

The roots are +4, +2, and -2

Therefore

$$f = Ar^2 + Br^4 + \frac{C}{r^2} + D$$

and $\phi = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\cos 2\theta$

The corresponding stress components are:

$$\sigma_r = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta$$

To obtain general expressions for radial displacement (u) and tangential displacement (v) for the above conditions, the procedure is as follows.

For this distribution of stress, the corres-

ponding displacements are obtained by applying the relationship as follows:

$$\epsilon_r = \frac{du}{dr} \qquad\qquad \epsilon_r = \frac{1}{E}\left(\sigma_r - \mu\,\sigma_\theta\right)$$

$$\epsilon_\theta = \frac{u}{r} + \frac{dv}{r\,d\theta} \qquad\qquad \epsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \mu\,\sigma_r\right)$$

$$\gamma_{r\theta} = \frac{du}{r\,d\theta} + \frac{dv}{dr} - \frac{v}{r} \qquad\qquad \gamma_{r\theta} = \frac{1}{G}\,\tau_{r\theta}$$

$$G = \frac{E}{2\,(1+\mu)}$$

$$\epsilon_r = \frac{du}{dr} = \frac{1}{E}\left(\sigma_r - \mu\,\sigma_\theta\right)$$

$$\epsilon_\theta = \frac{u}{r} + \frac{dv}{r\,d\theta} = \frac{1}{E}\left(\sigma_\theta - \mu\,\sigma_r\right)$$

$$\gamma_{r\theta} = \frac{du}{r\,d\theta} + \frac{dv}{dr} - \frac{v}{r} = \frac{1}{G}\,\tau_{r\theta}$$

$$\epsilon_r = \frac{du}{dr} = \frac{1}{E}\Big\{-\left(2A + \frac{6C}{r^4} + 4\frac{D}{r^2}\right)\cos 2\theta - \mu\left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta\Big\}$$

$$\frac{du}{dr} = \frac{2\cos 2\theta}{E}\left[-A(1+\mu) - \mu\,6Br^2 - \frac{3C}{r^4}(1+\mu) - \frac{2D}{r^2}\right]$$

Integrating,

$$u = \frac{2r\cos 2\theta}{E}\left[-A(1+\mu) - \mu\,2Br^2 + \frac{C}{r^4}(1+\mu) + \frac{2D}{r^2}\right] + f(\theta)$$

in which $f(\theta)$ is a function of θ only.

$$\epsilon_\theta = \frac{u}{r} + \frac{dv}{r\,d\theta} = \frac{1}{E}\left(\sigma_\theta - \mu\,\sigma_r\right)$$

$$\frac{dv}{d\theta} = r\ \epsilon_\theta - u$$

$$\frac{dv}{d\theta} = \frac{r}{E}\left(\sigma_\theta - \mu\ \sigma_r\right) - u$$

Substituting in the above equation gives:

$$\frac{dv}{d\theta} = \frac{2r\cos 2\theta}{E}\left\{\left(A + 6Br^2 + \frac{3C}{r^4}\right) + \mu\left(A + \frac{3C}{r^4} + \frac{2D}{r^2}\right)\right\}$$

$$+ \frac{2r\cos 2\theta}{E}\left\{A(1+\mu) + \mu\ 2Br^2 - \frac{C}{r^4}(1+\mu) - \frac{2D}{r^2}\right\} - f(\theta)$$

$$\frac{dv}{d\theta} = \frac{4r\cos 2\theta}{E}\left\{A(1+\mu) + Br^2(3+\mu) + \frac{C}{r^4}(1+\mu) - \frac{D}{r^2}(1-\mu)\right\} - f(\theta)$$

Integrating,

$$v = \frac{2r\sin 2\theta}{E}\left\{A(1+\mu) + Br^2(3+\mu) + \frac{C}{r^4}(1+\mu) - \frac{D}{r^2}(1-\mu)\right\}$$
$$- \int f(\theta)\ d\theta + F(r)$$

where F (r) is a function of r only.

$$v = \frac{2\sin 2\theta}{E}\left\{A(1+\mu)r + Br^3(3+\mu) + \frac{C}{r^3}(1+\mu) - \frac{D}{r}(1-\mu)\right\}$$
$$- \int f(\theta)\ d\theta + F(r)$$

Substituting the above expressions for u and v in

$$\gamma_{r\theta} = \frac{du}{r\,d\theta} + \frac{dv}{dr} - \frac{v}{r}$$

$$\frac{du}{r\,d\theta} = \frac{4\sin 2\theta}{E}\left\{A(1+\mu) + \mu\ 2Br^2 - \frac{C}{r^4}(1+\mu) - \frac{2D}{r^2}\right\} + \frac{f'(\theta)}{r}$$

$$\frac{dv}{dr} = \frac{2\sin 2\theta}{E}\left\{A(1+\mu) + 3Br^2(3+\mu) - \frac{3C}{r^4}(1+\mu) + \frac{D}{r^2}(1-\mu)\right\}$$
$$+ F'(r)$$

$$\gamma_{r\theta} = \frac{2\sin 2\theta}{E}\left\{2A(1+\mu) + 4Br^2\mu - \frac{2C}{r^4}(1+\mu) - \frac{4D}{r^2}\right\} + \frac{f'(\theta)}{r}$$

$$+ \frac{2\sin 2\theta}{E}\left\{A(1+\mu) + 3Br^2(3+\mu) - \frac{3C}{r^4}(1+\mu) + \frac{D}{r^2}(1-\mu)\right\} + F'(r)$$

$$+ \frac{2\sin 2\theta}{E}\left\{-A(1+\mu) - Br^2(3+\mu) - \frac{C}{r^4}(1+\mu) + \frac{D}{r^2}(1-\mu)\right\}$$

$$+ \frac{1}{r}\int f(\theta)\,d\theta - \frac{F(r)}{r}$$

$$r\gamma_{r\theta} = \int f(\theta)\,d\theta + f'(\theta) + r\,F'(r) - F(r) +$$

$$+ \frac{2r\sin 2\theta}{E}\left[\begin{array}{l} 2A(1+\mu) + 4Br^2\mu - \frac{2C}{r^4}(1+\mu) - \frac{4D}{r^2} \\ A(1+\mu) + 3Br^2(3+\mu) - \frac{3C}{r^4}(1+\mu) + \frac{D}{r^2}(1-\mu) \\ -A(1+\mu) - Br^2(3+\mu) - \frac{C}{r^4}(1+\mu + \frac{D}{r^2}(1-\mu) \end{array}\right]$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

$$\gamma_{r\theta} = \frac{1}{G}\left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta$$

$$G = \frac{E}{2(1+\mu)} \qquad \frac{1}{G} = \frac{2(1+\mu)}{E}$$

$$r\gamma_{r\theta} = \frac{2r\sin 2\theta}{E}\left[2A(1+\mu) + 6Br^2(1+\mu) - \frac{6C}{r^4}(1+\mu) - \frac{2D}{r^2}(1+\mu)\right]$$

$$\int f(\theta)\,d\theta + f'(\theta) + r\,F'(r) - F(r) =$$

$$-\frac{2r\sin 2\theta}{E}\left[\begin{array}{l} -2A(1+\mu) - 6Br^2(1+\mu) + \frac{6C}{r^4}(1+\mu) + \frac{2D}{r^2}(1+\mu) \\ 2A(1+\mu) + 4Br^2\mu - \frac{2C}{r^4}(1+\mu) - \frac{4D}{r^2} \\ A(1+\mu) + 3Br^2(3+\mu) - \frac{3C}{r^4}(1+\mu) + \frac{D}{r^2}(1-\mu) \\ -A(1+\mu) - Br^2(3+\mu) - \frac{C}{r^4}(1+\mu) + \frac{D}{r^2}(1-\mu) \end{array}\right]$$

Since the quantity in brackets is equal to zero,

$$\int f(\theta)\,d\theta + f'(\theta) + r\,F'(r) - F(r) = 0$$

This equation is satisfied by putting

$$F(r) = X\,r$$

$$f(\theta) = Y\sin\theta + Z\cos\theta$$

in which X, Y, and Z are arbitrary constants to be determined from the conditions of restraint.

Substituting these values in the equations for u and v gives:

$$u = \frac{2r\cos 2\theta}{E}\left[-A(1+\mu) - 2Br^2\mu + \frac{C}{r^4}(1+\mu) + \frac{2D}{r^2}\right] + Y\sin\theta + Z\cos\theta$$

$$v = \frac{2r\sin 2\theta}{E}\left[A(1+\mu) + Br^2(3+\mu) + \frac{C}{r^4}(1+\mu) - \frac{D}{r^2}(1-\mu)\right] + Y\cos\theta - Z\sin\theta + X r$$

For the purpose of determining the value of X, Y, and Z, the conditions of restraint are:

(1.) At $\theta = 90^\circ$, $v = 0$

(2.) At $\theta = 0^\circ$, $v = 0$

(3.) At $\theta = 90^\circ$ or 0°, $\frac{dv}{dr} = 0$

From condition (1.)

$$0 = -Z + X r$$

$$Z = X r$$

From condition (2.)

$$0 = Y + X r$$

$$Y = - X r$$

From condition (3.)

$$\frac{dv}{dr} = \frac{2 \sin 2\theta}{E}\left[A(1+\mu) + 3B r^2 (3+\mu) - \frac{3C}{r^4}(1+\mu) + \frac{D}{r^2}(1-\mu)\right] + X$$

$$X = 0$$

$$Y = 0$$

$$Z = 0$$

Therefore

$$u = \frac{2r \cos 2\theta}{E}\left[-A(1+\mu) - 2Br^2\mu + \frac{C}{r^4}(1+\mu) + \frac{2D}{r^2}\right]$$

$$v = \frac{2r \sin 2\theta}{E}\left[A(1+\mu) + Br^2(3+\mu) + \frac{C}{r^4}(1+\mu) - \frac{D}{r^2}(1-\mu)\right]$$

These equations are the general expressions for radial displacements (u) and tangential displacement (v) for the normal forces $\frac{1}{2}$ S cos 2θ together with the shearing forces - $\frac{1}{2}$ S sin 2θ.

As a check on the equations as derived above, the equations for σ_r, σ_θ and $\tau_{r\theta}$ on page 29 can be used to determine the values of the constants of integration from conditions for the outer boundary and from the condition that the edge of the hole is free from external forces.

The boundary conditions are:

(1.) At r = b, $\sigma_r = \frac{1}{2} S \cos 2\theta$

(2.) At r = a, $\sigma_r = 0$

(3.) At r = b, $\tau_{r\theta} = -\frac{1}{2} S \sin 2\theta$

(4.) At r = a, $\tau_{r\theta} = 0$

These conditions give

$$2A + \frac{6C}{b^4} + \frac{4D}{b^2} = -\frac{1}{2} S$$

$$2A + \frac{6C}{a^4} + \frac{4D}{a^2} = 0$$

$$2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2} = -\frac{1}{2} S$$

$$2A + 6Ba^2 - \frac{6C}{a^4} - \frac{2D}{a^2} = 0$$

Solving these equations and putting a/b = 0, ie., assuming an infinitely large plate, gives:

A = -S/4

B = 0

C = $-\frac{a^4}{4} S$

D = $\frac{a^2}{2} S$

Substituting the above values for the constants A, B, C, and D in the equation for u gives the following result:

$$u = \frac{2r\cos 2\theta}{E}\left[\frac{S}{4}(1+\mu) - \frac{a^4 S}{4r^4}(1+\mu) + \frac{a^2}{r^2}S\right]$$

$$u = \frac{Sr}{2E}\left[(1+\mu) - \frac{a^4}{r^4}(1+\mu) + \frac{4a^2}{r^2}\right]\cos 2\theta$$

This expression for u contains the last three terms of the general equation developed for u on page 14 of section II. The other two terms in the general equation are not dependent on θ and are due to the constant component $\frac{1}{2}$ S of the normal forces.

To obtain the other two terms, in the general equation of section II, which are due to the constant component $\frac{1}{2}$ S of the normal forces, proceed as follows:

From section III

$$u = Ar + \frac{B}{r}$$

$$B = \frac{1+\mu}{1-\mu}a^2 A$$

$$A = \frac{S}{2E(1+\mu)\left(1-\frac{a^2}{c^2}\right)}$$

$$u = \frac{Sr}{2E(1+\mu)\left(1-\frac{a^2}{c^2}\right)} + \frac{1+\mu}{1-\mu}\frac{a^2}{r}\frac{S}{2E(1+\mu)\left(1-\frac{a^2}{c^2}\right)}$$

$$u = \frac{Sr}{2E}\left[\frac{1}{(1+\mu)\left(1-\frac{a^2}{c^2}\right)} + \frac{a^2}{r^2(1-\mu)\left(1-\frac{a^2}{c^2}\right)}\right]$$

For $C \to \infty$

$$u = \frac{Sr}{2E}\left[\frac{1}{1+\mu} + \frac{a^2}{r^2(1-\mu)}\right]$$

$$u = \frac{Sr}{2E}\left[\frac{1-\mu}{1-\mu^2} + \frac{a^2}{r^2}\cdot\frac{1+\mu}{1-\mu^2}\right]$$

Since $1-\mu^2 \approx 1$

$$u = \frac{Sr}{2E}\left[(1-\mu) + \frac{a^2}{r^2}(1+\mu)\right]$$

Therefore, using the principle of superposition, and adding the radial displacement due to the constant component $\frac{1}{2}$ S of the normal forces, ie:

$$u = \frac{Sr}{2E}\left[(1-\mu) + \frac{a^2}{r^2}(1+\mu)\right]$$

and that due to the normal forces $\frac{1}{2}$ S cos 2θ together with the shearing forces - $\frac{1}{2}$ S sin 2θ , ie.:

$$u = \frac{Sr}{2E}\left[(1+\mu) - \frac{a^4}{r^4}(1+\mu) + \frac{4a^2}{r^2}\right]\cos 2\theta$$

gives the final expression

$$u = \frac{Sr}{2E}\left[(1-\mu) + (1+\mu)\frac{a^2}{r^2} + (1+\mu)\cos 2\theta + \frac{4a^2}{r^2}\cos 2\theta - (1+\mu)\frac{a^4}{r^4}\cos 2\theta\right]$$

and is identical to that on page 14 which is the formula for u due to all of the loading. This gives a complete check on the validity of the equation for u.

In the same manner, the expression for v can be checked as follows:

$$v = \frac{2r\sin 2\theta}{E}\left[A(1+\mu) + Br^2(3+\mu) + \frac{C}{r^4}(1+\mu) - \frac{D}{r^2}(1-\mu)\right]$$

Again using

$A = - S/4$

$B = 0$

$C = -\frac{a^4}{4}S$

$D = \frac{a^2}{2}S$

$$v = -\frac{Sr}{2E}\left[(1+\mu) + \frac{a^4}{r^4}(1+\mu) + \frac{2a^2}{r^2}(1-\mu)\right]\sin 2\theta$$

Since there is no tangential displacement (v) due to the symmetrical loading by the constant component $\frac{1}{2}$ S of the normal forces, this is also the expression for the total tangential displacement, and as such, it agrees with the expression for v on page 14 of section II.

To determine σ_r and σ_θ in the ring and plate due to the normal forces ($\frac{1}{2}$ S cos 2θ) and the shearing forces (- $\frac{1}{2}$ S sin 2θ), the constants A, B, C, and D must be determined in both regions, i and O, as indicated in Fig. 8.

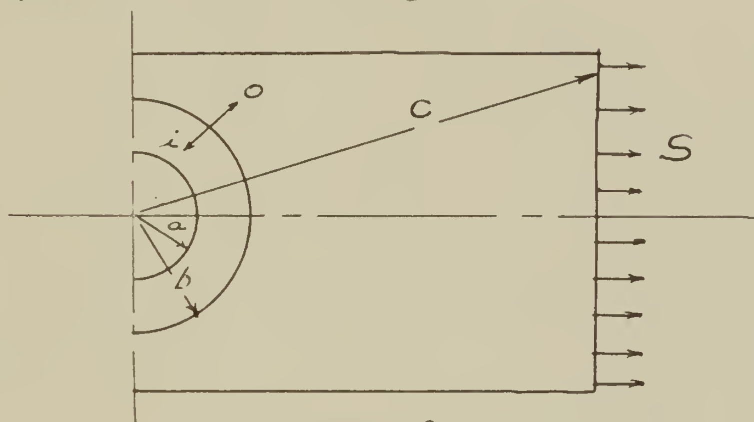


Fig. 8.

At the boundary b, a stress flow is assumed as in Fig. 9 (b) instead of the actual stress flow shown as it would be in Fig. 9 (a).

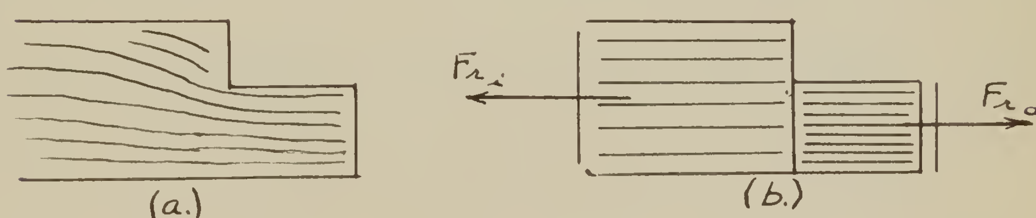


Fig. 9.

Since there will be eight unknowns, A_i, B_i, C_i, D_i, A_o, B_o, C_o, and D_o, there must be eight equations.

The boundary conditions for these eight equations are as follows:

At r = a

(1.) $\sigma_{r_i} = 0$

(2.) $\tau_{r\theta_i} = 0$

At r = C

(3.) $\sigma_{r_o} = \frac{1}{2} S \cos 2\theta$

(4.) $\tau_{r\theta_o} = -\frac{1}{2} S \sin 2\theta$

At r = b $(F_{r_i} = F_{r_o})$

(5.) $\sigma_{r_i} t_i = \sigma_{r_o} t_o$

(6.) $\tau_{r\theta_i} t_i = \tau_{r\theta_o} t_o$

(7.) $u_i = u_o$

(8.) $v_i = v_o$

The general equations to be used with these boundary conditions are:

$$\sigma_r = -\left(2A + \frac{6}{r^4} C + \frac{4}{r^2} D\right) \cos 2\theta$$

$$\tau_{r\theta} = \left(2A + 6r^2 B - \frac{6}{r^4} C - \frac{2}{r^2} D\right) \sin 2\theta$$

$$u = \frac{2r \cos 2\theta}{E}\left[-(1+\mu) A - 2r^2\mu B + \frac{1+\mu}{r^4} C + \frac{2}{r^2} D\right]$$

$$v = \frac{2r \sin 2\theta}{E}\left[(1+\mu) A + (3+\mu) r^2 B + \frac{1+\mu}{r^4} C - \frac{1-\mu}{r^2} D\right]$$

Using these formulas and the above boundary conditions, the following equations are set up for determination of the constants:

The subscripts are as indicated in Fig. 8 with relation to the radius b,

i - inside

0 - outside

These equations also assume that

$$E_i = E_o$$

It should be noted that these constants are not the same as those which are evaluated in section III.

(1.) $A_i + \frac{3}{a^4} C_i + \frac{2}{a^2} D_i = 0$

(2.) $A_i + 3a^2 B_i - \frac{3}{a^4} C_i - \frac{1}{a^2} D_i = 0$

(3.) $A_o + \frac{3}{c^4} C_o + \frac{2}{c^2} D_o = -\frac{5}{4}$

(4.) $A_o + 3c^2 B_o - \frac{3}{c^4} C_o - \frac{1}{c^2} D_o = -\frac{5}{4}$

(5.) $t_i A_i + \frac{3t_i}{b^4} C_i + \frac{2t_i}{b^2} D_i - t_o A_o - \frac{3t_o}{b^4} C_o - \frac{2t_o}{b^2} D_o = 0$

(6.) $t_i A_i + 3t_i b^2 B_i - \frac{3t_i}{b^4} C_i - \frac{t_i}{b^2} D_i$

$$-t_o A_o - 3t_o b^2 B_o + \frac{3t_o}{b^4} C_o + \frac{t_o}{b^2} D_o = 0$$

$$(7.)\ (1+\mu) A_i + 2b^2\mu\ B_i - \frac{1+\mu}{b^4} C_i - \frac{2}{b^2} D_i$$
$$- (1+\mu) A_o - 2b^2\mu\ B_o + \frac{1+\mu}{b^4} C_o + \frac{2}{b^2} D_o = 0$$

$$(8.)\ (1+\mu) A_i + (3+\mu) b^2 B_i + \frac{1+\mu}{b^4} C_i - \frac{1-\mu}{b^2} D_i$$
$$- (1+\mu) A_o - (3+\mu) b^2 B_o - \frac{1+\mu}{b^4} C_o + \frac{1-\mu}{b^2} D_o = 0$$

Solving these equations by the method of determinants for the constants, and imposing the condition of $c \rightarrow \infty$ gives the results listed below.

Because of the length of the terms, the numerators are listed separately, as is the denominator. They are not combined to give the complete constants as it is more convenient to work them out separately for a specific case.

The <u>numerators</u> of the constants are:

$$\text{Num } A_i = -\frac{S}{4}\left\{ \frac{36}{a^6 b^4}\left[t_o^2 (3-\mu) + t_i t_o (1+\mu)\right] - \frac{108}{a^2 b^8}\left[t_o^2 (1+\mu) - t_i t_o (1+\mu)\right] + \frac{144}{b^{10}}\left[t_o^2 (1+\mu) - t_i t_o (1+\mu)\right] \right\}$$

$$\text{Num } B_i = -\frac{S}{4}\left\{ -\frac{72}{a^4 b^8}\left[t_i t_o (1+\mu) - t_o^2 (1+\mu)\right] + \frac{72}{a^2 b^{10}}\left[t_i t_o (1+\mu) - t_o^2 (1+\mu)\right] \right\}$$

$$\text{Num } C_i = -\frac{S}{4}\left\{\frac{36}{a^2b^4}\left[t_i t_o(1+\mu)+t_o^2(3-\mu)\right] + \frac{36a^2}{b^8}\left[t_o^2(1+\mu) - t_i t_o(1+\mu)\right]\right\}$$

$$\text{Num } D_i = -\frac{S}{4}\left\{-\frac{72}{a^4b^4}\left[t_i t_o(1+\mu)+t_o^2(3-\mu)\right] + \frac{72a^2}{b^{10}}\left[t_i t_o(1+\mu) - t_o^2(1+\mu)\right]\right\}$$

$$\text{Num } A_o = -\frac{S}{4}\left\{\text{Denominator}\right\}$$

$$\text{Num } B_o = 0$$

$$\begin{aligned}\text{Num } C_o = -\frac{S}{4}\Big\{ & -\frac{9}{a^6}\left[2t_i t_o(1-\mu^2) + t_i^2(1+\mu)^2 - t_o^2(3-\mu)(1+\mu)\right] \\ & -\frac{36}{a^4b^2}\left[2t_i t_o(2+\mu+\mu^2) - t_i^2(1+\mu)^2 - t_o^2(3+\mu^2)\right] \\ & +\frac{18}{a^2b^4}\left[2t_i t_o(7+6\mu+3\mu^2) - 3t_i^2(1+\mu)^2 - 3t_o^2(1+\mu)^2\right] \\ & -\frac{36}{b^6}\left[2t_i t_o(1+\mu)^2 - t_i^2(1+\mu)^2 - t_o^2(1+\mu)^2\right] \\ & -\frac{9a^2}{b^8}\left[2t_i t_o(1-\mu^2) + t_i^2(1+\mu)^2 - t_o^2(3-\mu)(1+\mu)\right]\Big\}\end{aligned}$$

$$\begin{aligned}\text{Num } D_o = -\frac{S}{4}\Big\{ & \frac{18}{a^6b^2}\left[2t_i t_o(1-\mu^2) + t_i^2(1+\mu)^2 - t_o^2(3-\mu)(1+\mu)\right] \\ & +\frac{72}{a^4b^4}\left[2t_i t_o\mu(1+\mu) - t_i^2(1+\mu)^2 - t_o^2(3+\mu^2)\right] \\ & -\frac{108}{a^2b^6}\left[2t_i t_o(1+\mu)^2 - t_i^2(1+\mu)^2 - t_o^2(1+\mu)^2\right] \\ & +\frac{72}{b^8}\left[2t_i t_o(1+\mu)^2 - t_i^2(1+\mu)^2 - t_o^2(1+\mu)^2\right] \\ & +\frac{18a^2}{b^{10}}\left[2t_i t_o(1-\mu^2) + t_i^2(1+\mu)^2 - t_o^2(3-\mu)(1+\mu)\right]\Big\}\end{aligned}$$

$$\text{Den} = \frac{9}{a^6 b^4}\left[2t_i t_o(5-2\mu+\mu^2) + t_i^2(3-\mu)(1+\mu) + t_o^2(3-\mu)(1+\mu)\right]$$

$$+\frac{36}{a^4 b^6}\left[2t_i t_o \mu(1-\mu) - t_i^2(3-\mu)(1+\mu) + t_o^2(3+\mu^2)\right]$$

$$-\frac{54}{a^2 b^8}\left[2t_i t_o(1-\mu^2) - t_i^2(3-\mu)(1+\mu) + t_o^2(1+\mu)^2\right]$$

$$+\frac{36}{b^{10}}\left[2t_i t_o(1-\mu^2) - t_i^2(3-\mu)(1+\mu) + t_o^2(1+\mu)^2\right]$$

$$+\frac{9a^2}{b^{12}}\left[(t_i - t_o)^2(3-\mu)(1+\mu)\right]$$

For the stresses and displacement due to the normal forces ($\frac{1}{2}$ S cos 2θ) and the shearing forces ($-\frac{1}{2}$ S sin 2θ), the above constants are to be placed in the equations as listed below, using either the subscripts i or o as appropriate.

$$\sigma_r = -\left(2A + \frac{6}{r^4}C + \frac{4}{r^2}D\right)\cos 2\theta$$

$$\sigma_\theta = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = \left(2A + 6r^2B - \frac{6}{r^4}C - \frac{2}{r^2}D\right)\sin 2\theta$$

$$u = \frac{2r\cos 2\theta}{E}\left[-(1+\mu)A - 2r^2\mu B + \frac{1+\mu}{r^4}C + \frac{2}{r^2}D\right]$$

$$v = \frac{2r\sin 2\theta}{E}\left[(1+\mu)A + (3+\mu)r^2B + \frac{1+\mu}{r^4}C - \frac{1-\mu}{r^2}D\right]$$

THEORETICAL ANALYSIS

V. SUMMARY OF THEORETICAL ANALYSIS

The radial and tangential stresses due to both the constant component ($\frac{1}{2}$ S) of the normal forces and the normal forces ($\frac{1}{2}$ S cos 2θ) together with the shearing forces ($-\frac{1}{2}$ S sin 2θ) must be added together to give the total radial and tangential stresses in the ring and plate.

The radial and tangential stresses due to the constant component ($\frac{1}{2}$ S), together with the proper constants are:

$$\sigma_{r_i} = E\left[(1+\mu)A_i - (1-\mu)\frac{B_i}{r^2}\right]; \quad \sigma_{\theta_i} = E\left[(1+\mu)A_i + (1-\mu)\frac{B_i}{r^2}\right]$$

$$\sigma_{r_o} = E\left[(1+\mu)A_o - (1-\mu)\frac{B_o}{r^2}\right]; \quad \sigma_{\theta_o} = E\left[(1+\mu)A_o + (1-\mu)\frac{B_o}{r^2}\right]$$

$$A_i = -\frac{b^2 t_o S}{E(1+\mu)}\left[\frac{1}{a^2(1+\mu)(t_i - t_o) - b^2 t_i(1+\mu) - b^2 t_o(1-\mu)}\right]$$

$$B_i = -\frac{a^2 b^2 t_o S}{E(1-\mu)}\left[\frac{1}{a^2(1+\mu)(t_i - t_o) - b^2 t_i(1+\mu) - b^2 t_o(1-\mu)}\right]$$

$$A_o = \frac{S}{2E(1+\mu)}$$

$$B_o = \frac{b^2 S}{2E(1-\mu)}\left[\frac{b^2(1-\mu)(t_i - t_o) - a^2 t_i(1-\mu) - a^2 t_o(1+\mu)}{a^2(1+\mu)(t_i - t_o) - b^2 t_i(1+\mu) - b^2 t_o(1-\mu)}\right]$$

The radial and tangential stresses due to the normal forces ($\frac{1}{2}$ S cos 2θ) and the shearing forces ($-\frac{1}{2}$ S sin 2θ), together with the numerators and denominator of the constants are:

$$\sigma_{r_i} = -\left(2A_i + \frac{6}{r^4} C_i + \frac{4}{r^2} D_i\right) \cos 2\theta$$

$$\sigma_{r_o} = -\left(2A_o + \frac{6}{r^4} C_o + \frac{4}{r^2} D_o\right) \cos 2\theta$$

$$\sigma_{\theta_i} = \left(2A_i + 12 B_i r^2 + \frac{6}{r^4} C_i\right) \cos 2\theta$$

$$\sigma_{\theta_o} = \left(2A_o + 12 B_o r^2 + \frac{6}{r^4} C_o\right) \cos 2\theta$$

$$\text{Num } A_i = -\frac{S}{4}\left\{\frac{36}{a^6 b^4}\left[t_o^2(3-\mu) + t_i t_o(1+\mu)\right] - \frac{108}{a^2 b^8}\left[t_o^2(1+\mu) - t_i t_o(1+\mu)\right] + \frac{144}{b^{10}}\left[t_o^2(1+\mu) - t_i t_o(1+\mu)\right]\right\}$$

$$\text{Num } B_i = -\frac{S}{4}\left\{-\frac{72}{a^4 b^8}\left[t_i t_o(1+\mu) - t_o^2(1+\mu)\right] + \frac{72}{a^2 b^{10}}\left[t_i t_o(1+\mu) - t_o^2(1+\mu)\right]\right\}$$

$$\text{Num } C_i = -\frac{S}{4}\left\{\frac{36}{a^2 b^4}\left[t_i t_o(1+\mu) + t_o^2(3-\mu)\right] + \frac{36a^2}{b^8}\left[t_o^2(1+\mu) - t_i t_o(1+\mu)\right]\right\}$$

$$\text{Num } D_i = -\frac{S}{4}\left\{-\frac{72}{a^4 b^4}\left[t_i t_o(1+\mu) + t_o^2(3-\mu)\right] + \frac{72a^2}{b^{10}}\left[t_i t_o(1+\mu) - t_o^2(1+\mu)\right]\right\}$$

$$\text{Num } A_o = -\frac{S}{4}\left\{\text{Denominator}\right\}$$

$$\text{Num } B_o = 0$$

$$\text{Num } C_0 = -\frac{5}{4}\left\{-\frac{9}{a^6}\left[2\,t_i t_0\,(1-\mu^2) + t_i^{\,2}(1+\mu)^2 - t_0^{\,2}(3-\mu)(1+\mu)\right]\right.$$

$$-\frac{36}{a^4 b^2}\left[2\,t_i t_0\,(2+\mu+\mu^2) - t_i^{\,2}(1+\mu)^2 - t_0^{\,2}(3+\mu^2)\right]$$

$$+\frac{18}{a^2 b^4}\left[2\,t_i t_0\,(7+6\mu+3\mu^2) - 3\,t_i^{\,2}(1+\mu)^2 - 3\,t_0^{\,2}(1+\mu)^2\right]$$

$$-\frac{36}{b^6}\left[2\,t_i t_0\,(1+\mu)^2 - t_i^{\,2}(1+\mu)^2 - t_0^{\,2}(1+\mu)^2\right]$$

$$\left.-\frac{9a^2}{b^8}\left[2\,t_i t_0\,(1-\mu^2) + t_i^{\,2}(1+\mu)^2 - t_0^{\,2}(3-\mu)(1+\mu)\right]\right\}$$

$$\text{Num } D_0 = -\frac{5}{4}\left\{\frac{18}{a^6 b^2}\left[2\,t_i t_0\,(1-\mu^2) + t_i^{\,2}(1+\mu)^2 - t_0^{\,2}(3-\mu)(1+\mu)\right]\right.$$

$$+\frac{72}{a^4 b^4}\left[2\,t_i t_0\,\mu(1+\mu) - t_i^{\,2}(1+\mu)^2 - t_0^{\,2}(3+\mu^2)\right]$$

$$-\frac{108}{a^2 b^6}\left[2\,t_i t_0\,(1+\mu)^2 - t_i^{\,2}(1+\mu)^2 - t_0^{\,2}(1+\mu)^2\right]$$

$$+\frac{72}{b^8}\left[2\,t_i t_0\,(1+\mu)^2 - t_i^{\,2}(1+\mu)^2 - t_0^{\,2}(1+\mu)^2\right]$$

$$\left.+\frac{18a^2}{b^{10}}\left[2\,t_i t_0\,(1-\mu^2) + t_i^{\,2}(1+\mu)^2 - t_0^{\,2}(3-\mu)(1+\mu)\right]\right\}$$

$$
\begin{aligned}
\text{Den} = {} & \frac{9}{a^6 b^4}\left[2 t_i t_o (5 - 2\mu + \mu^2) + t_i^2 (3-\mu)(1+\mu) + t_o^2 (3-\mu)(1+\mu)\right] \\
& + \frac{36}{a^4 b^6}\left[2 t_i t_o \mu (1-\mu) - t_i^2 (3-\mu)(1+\mu) + t_o^2 (3+\mu^2)\right] \\
& - \frac{54}{a^2 b^8}\left[2 t_i t_o (1-\mu^2) - t_i^2 (3-\mu)(1+\mu) + t_o^2 (1+\mu)^2\right] \\
& + \frac{36}{b^{10}}\left[2 t_i t_o (1-\mu^2) - t_i^2 (3-\mu)(1+\mu) + t_o^2 (1+\mu)^2\right] \\
& + \frac{9a^2}{b^{12}}\left[(t_i - t_o)^2 (3-\mu)(1+\mu)\right]
\end{aligned}
$$

It is not practicable to combine the above numerators and denominator because of their length. At this point, it is best to substitute actual values for a specific case when using the above expressions to determine the radial and tangential stresses.

APPLICATION OF
THEORETICAL ANALYSIS TO TEST SPECIMEN

The results of the theoretical analysis are now applied to a flat plate under a tension load, with a circular cutout symmetrically placed and reinforced with a circular ring.

The dimensions of the test specimen are taken from a structure which was tested experimentally for verification of this analysis. The dimensions and loading are:

$a = 2.5$ inches

$b = 3.5$ inches

$t_o = 0.040$ inches

$t_i = 0.120$ inches

$\mu = 0.3$

$E = 10.3 \times 10^6$ lbs./sq.in.

See the section TEST DATA AND EXPERIMENTAL RESULTS for details of the test specimen.

Applying the equations of Part V of the theoretical analysis:

For the constant component ($\frac{1}{2}$ S),

$$\sigma_{r_i} = S\left(0.305 - \frac{1.91}{r^2}\right)$$

$$\sigma_{\theta_i} = S\left(0.305 + \frac{1.91}{r^2}\right)$$

$$\sigma_{r_o} = S\left(0.5 - \frac{0.626}{r^2}\right)$$

$$\sigma_{\theta_o} = S\left(0.5 + \frac{0.626}{r^2}\right)$$

For $\frac{1}{2}$ S cos 2θ component of normal forces and - $\frac{1}{2}$ S sin 2θ shearing forces,

$$\sigma_{r_i} = S\left(0.444 + \frac{42.6}{r^4} - \frac{9.6}{r^2}\right)\cos 2\theta$$

$$\sigma_{\theta_i} = S\left(-0.444 + 0.03912\,r^2 - \frac{42.6}{r^4}\right)\cos 2\theta$$

$$\sigma_{r_o} = S\left(0.5 - \frac{32.64}{r^4} - \frac{5.472}{r^2}\right)\cos 2\theta$$

$$\sigma_{\theta_o} = S\left(-0.5 + \frac{32.64}{r^4}\right)\cos 2\theta$$

Combining the above expressions to give the equations for radial and tangential stresses due to the total loading:

$$\sigma_{r_i} = S\left\{0.305 - \frac{1.91}{r^2} + \left(0.444 + \frac{42.6}{r^4} - \frac{9.6}{r^2}\right)\cos 2\theta\right\}$$

$$\sigma_{r_o} = S\left\{0.5 - \frac{0.626}{r^2} + \left(0.5 - \frac{32.64}{r^4} - \frac{5.472}{r^2}\right)\cos 2\theta\right\}$$

$$\sigma_{\theta_i} = S\left\{0.305 + \frac{1.91}{r^2} + \left(-0.444 + 0.03912\,r^2 - \frac{42.6}{r^4}\right)\cos 2\theta\right\}$$

$$\sigma_{\theta_o} = S\left\{0.5 + \frac{0.626}{r^2} + \left(-0.5 + \frac{32.64}{r^4}\right)\cos 2\theta\right\}$$

In the experimental testing, the data was taken at

$$\theta = 90^\circ$$

$$\cos 2\theta = -1.$$

For this location, the above equations reduce to:

$$\sigma_{r_i} = S\left(-0.139 + \frac{7.69}{r^2} - \frac{42.6}{r^4}\right)$$

$$\sigma_{r_o} = S\left(\frac{4.846}{r^2} + \frac{32.64}{r^4}\right)$$

$$\sigma_{\theta_i} = S\left(0.749 + \frac{1.91}{r^2} - 0.03912 r^2 + \frac{42.6}{r^4}\right)$$

$$\sigma_{\theta_o} = S\left(1.0 + \frac{0.626}{r^2} - \frac{32.64}{r^4}\right)$$

Since the experimental data taken was for ϵ_θ at $\theta = 90^\circ$, and for r = 2.6, 3.0, 3.4, and 3.6, the above equations were used to obtain the following table.

TABLE I

r	σ_{θ_i}	σ_{θ_o}
2.5	S(1.900)	
2.6	S(1.701)	
3.0	S(1.135)	
3.4	S(0.781)	
3.5	S(0.710)	S(0.833)
3.6		S(0.854)

Since the cross sectional area of the test specimen at the point where the loading is applied is 1.2 square inches,

$$S = \frac{1000}{1.2} = 833 \frac{lbs.}{sq.in.} \text{ for each}$$

P = 1000 lbs. which is applied to the structure.

See Fig. 12, page 60 for a plot of the theoretical data as determined above.

TEST DATA AND EXPERIMENTAL RESULTS FROM TEST SPECIMEN

The details of the test specimen are shown in Fig. 10, page 55. The reinforcing rings are fastened to both sides of the flat plate by hammered rivets, spaced as shown. Past experience has shown that the reinforcement acts very much as an integral part of the sheet when it is reinforced with two concentric rows of rivets.

Figure 11 contains photographs of the test section and test equipment, made while the tests were being made.

Table II is made up of the data as taken during the experiment and it is used to make the plot of experimental data as shown in Fig. 12, page 60.

Strains were taken by use of the SR-4 strain indicator together with a multiple switch box for ease of taking the readings. The gage factor setting on the strain indicator was 1.77 and the gage factor of the strain gages was 1.68. This necessitated that all strain measurements be multiplied by the factor of $\frac{1.77}{1.68}$.

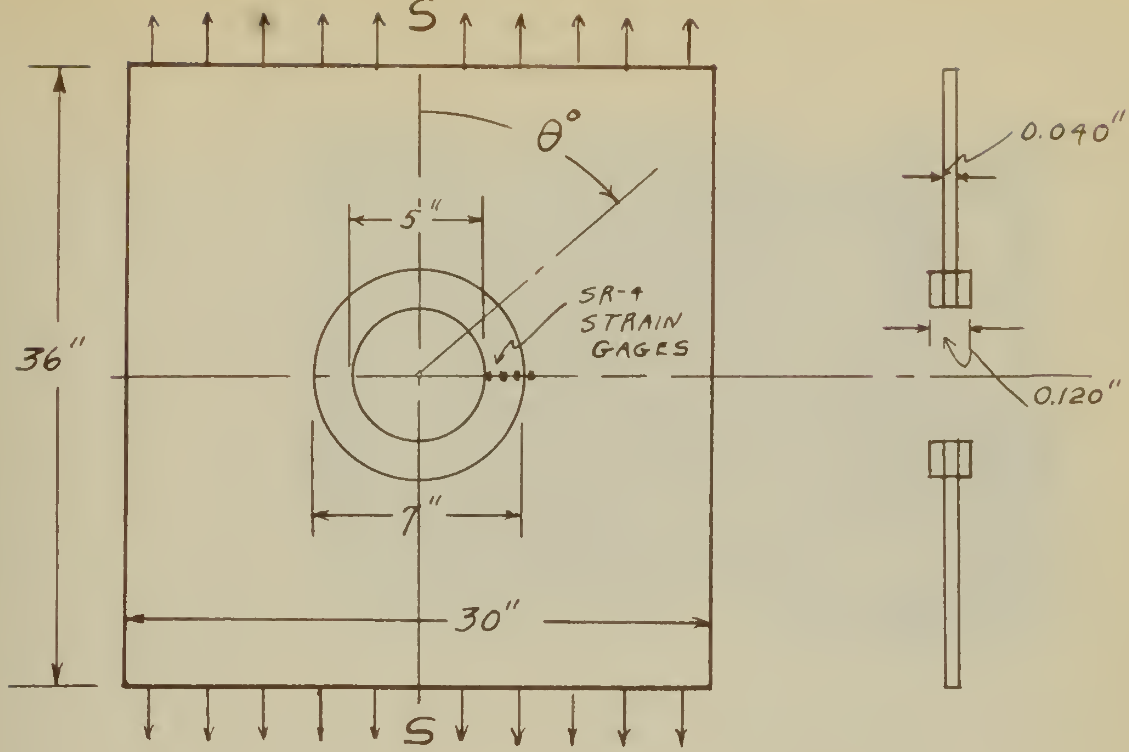
In the determination of σ_θ it was assumed that

$\sigma_r = 0$, an assumption which is obviously quite accurate. This allows a calculation of σ_θ by the simple relationship:

$$\sigma_\theta = E \epsilon_\theta$$

where ϵ_θ is the strain as measured by the strain gages on the test specimen.

DETAILS OF TEST SPECIMEN



Ring and Plate - 24ST Aluminum Alloy.

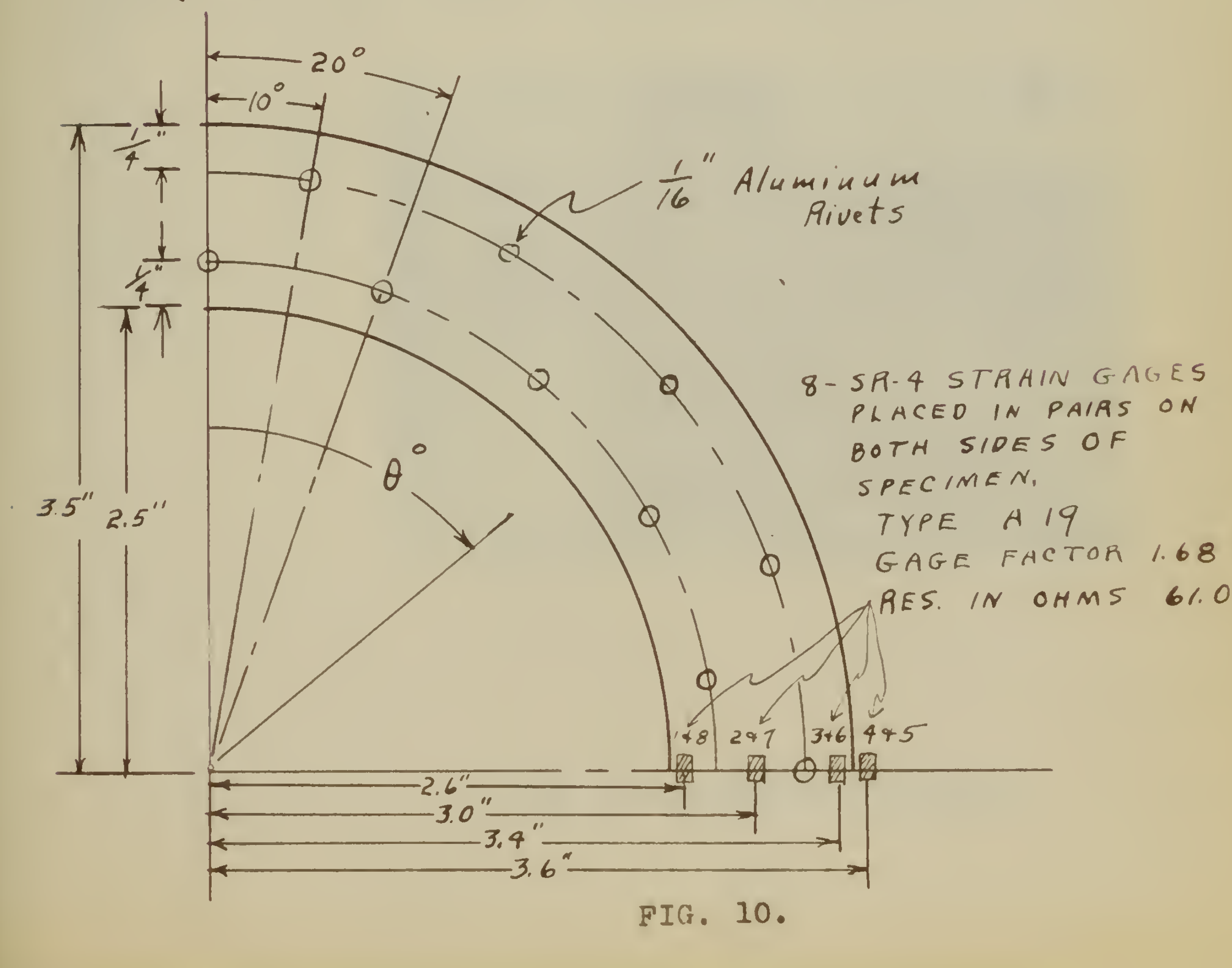
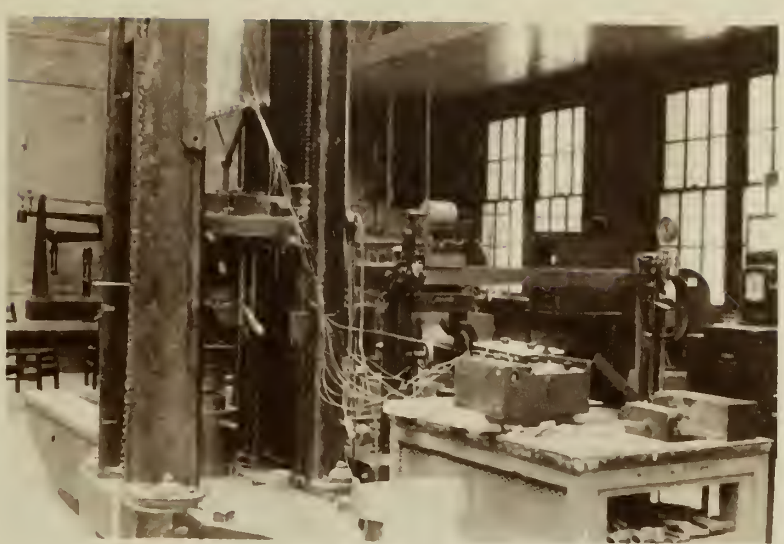


FIG. 10.



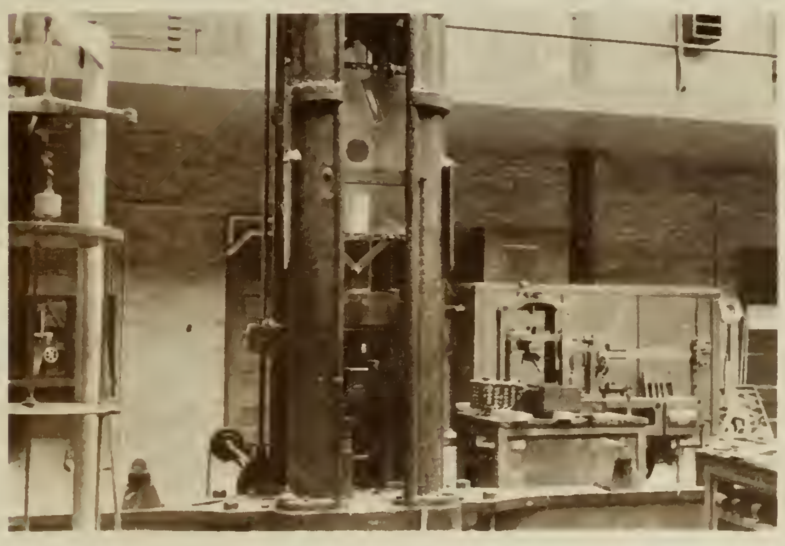


FIG. 11. Experimental Test Set-up

TABLE II

EXPERIMENTAL DATA

Gage	Strain indicator reading for P equal to: 0	2,000	4,000	6,000	8,000	10,000	12,000	14,000
1	13,568	13,902	14,175	14,430	14,675	14,905	15,120	15,290
2	14,482	14,648	14,742	14,835	14,915	14,996	15,052	15,070
3	9,030	9,140	9,193	9,233	9,265	9,287	9,300	9,245
4	12,975	13,165	13,305	13,438	13,570	13,703	13,836	14,020
5	15,220	15,288	15,420	15,560	15,712	15,868	16,010	16,210
6	10,012	9,950	9,975	10,023	10,083	10,130	10,130	10,015
7	11,830	11,860	11,952	12,050	12,168	12,285	12,402	12,548
8	9,158	9,250	9,434	9,620	9,830	10,038	10,255	10,535

Gage	Strain (inches per inch) for P equal to: 2,000	4,000	6,000	8,000	10,000	12,000	14,000
1	351	637	905	1,162	1,405	1,630	1,810
2	174	273	371	455	540	598	618
3	115	172	214	247	270	284	226
4	190	330	463	595	728	861	1,045
5	68	200	340	492	648	790	990
6	-65	-39	12	76	124	124	3
7	32	128	231	355	478	601	755
8	97	290	485	705	924	1,155	1,448

Gage	Average Strain for P equal to: Radius	2,000	4,000	6,000	8,000	10,000	12,000	14,000
1 and 8	2.6"	224	464	695	934	1,165	1,393	1,629
2 and 7	3.0"	103	201	301	405	509	600	687
3 and 6	3.4"	24	67	113	162	198	204	115
4 and 5	3.6"	129	265	402	544	688	826	1,018

Gage	σ_θ for P equal to: Radius	2,000	4,000	6,000	8,000	10,000	12,000	14,000
1 and 8	2.6"	2,310	4,780	7,160	9,620	12,000	14,340	16,800
2 and 7	3.0"	1,060	2,070	3,100	4,160	5,240	6,180	7,080
3 and 6	3.4"	260	690	1,160	1,670	2,030	2,100	1,190
4 and 5	3.6"	1,330	2,730	4,130	5,600	7,090	8,510	10,480

COMPARISON AND DISCUSSION OF TEST DATA AND THEORY

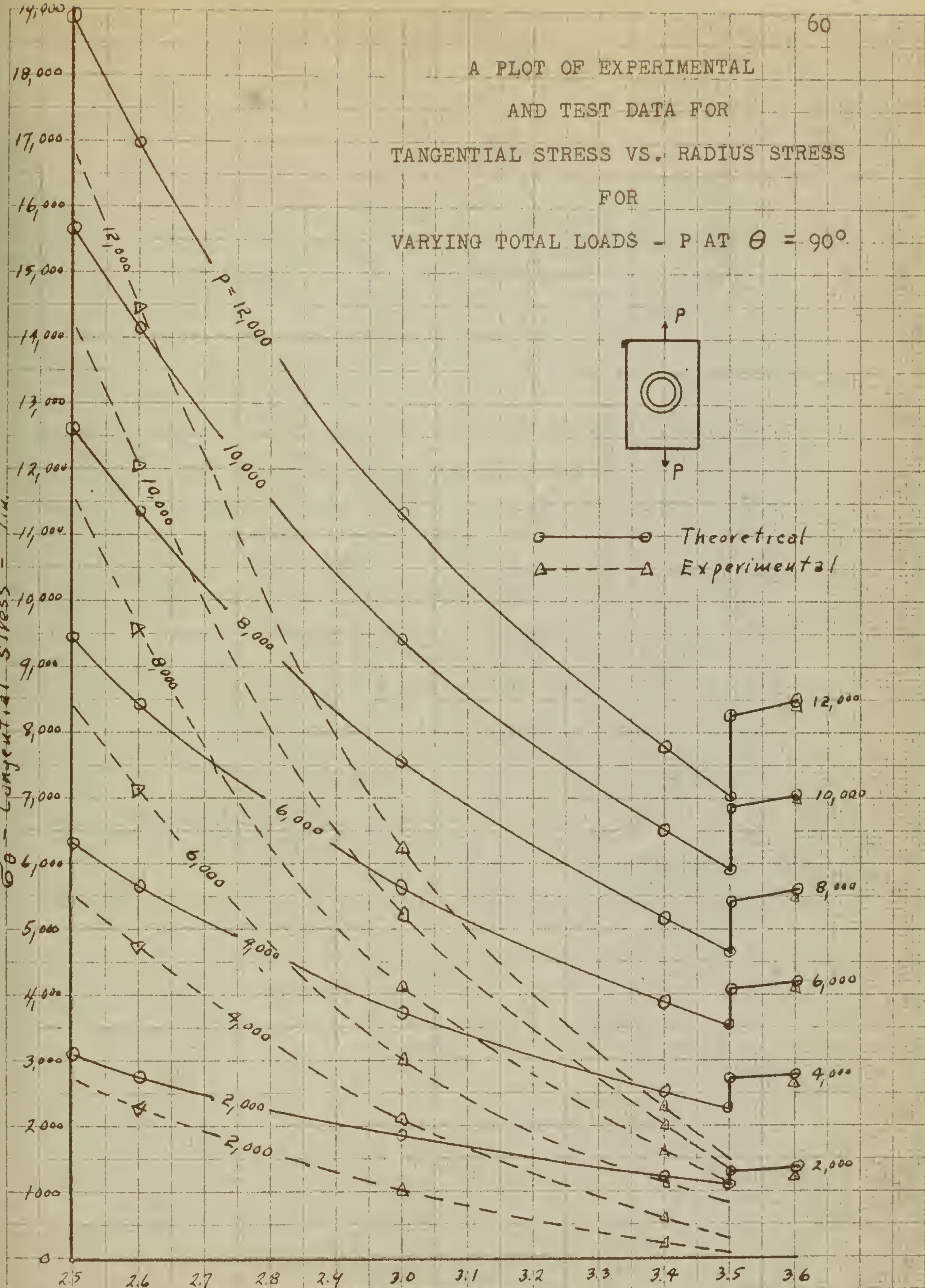
A plot of experimental and test data for tangential stress versus radius for varying total loads is shown in Fig. 12. In general, the agreement is excellent, especially at the inner radius where the stress concentration is critical. At this point, the per cent variance of the theoretical tangential stress from the experimental tangential stress at the varying total loads is:

At P = 12,000 lbs., 10.5%
P = 10,000 lbs., 9.7%
P = 8,000 lbs., 8.7%
P = 6,000 lbs., 11.8%
P = 4,000 lbs., 14.3%
P = 2,000 lbs., 10.7%

Observation of the curves of Fig. 12, indicates that the original sheet between the rings is under a slightly higher stress than the rings, perhaps on the order of ten per cent, since the experimental stresses (taken on the rings) are somewhat lower than the theory indicates. This is probably due to the fact that the rings are not acting in complete accord with the assumption that they are integral

with the sheet. It is likely that an increased number of rivets would result in better agreement.

Non-agreement between the theoretical data and the experimental data is quite pronounced in the ring area towards the outer edge. This is probably due to the fact that the ring does not act completely as an integral part of the sheet, and if a greater number of rivets were used, or if the outer row of rivets were placed nearer the outer boundary of the ring, this part of the ring would tend to carry more of the load and give better agreement with the theory. The assumption of an abrupt change of stress flow at the outer boundary of the ring as explained on page 39 is also responsible for making the theoretical values high at that point. However, the stresses at the outer edge of the ring are not critical, so that the variance in this area is inconsequential.



A PLOT OF EXPERIMENTAL
AND TEST DATA FOR
TANGENTIAL STRESS VS. RADIUS STRESS
FOR
VARYING TOTAL LOADS - P AT θ = 90°
P
P
Theoretical
Experimental
σ_θ - tangential stress - #/in.²
r - radius in inches
19,000
18,000
17,000
16,000
15,000
14,000
13,000
12,000
11,000
10,000
9,000
8,000
7,000
6,000
5,000
4,000
3,000
2,000
1000
0
2.5
2.6
2.7
2.8
2.9
3.0
3.1
3.2
3.3
3.4
3.5
3.6
P = 12,000
10,000
8,000
6,000
4,000
2,000
12,000
10,000
8,000
6,000
4,000
2,000
12,000
10,000
8,000
6,000
4,000
2,000

FIG. 12

CONCLUSIONS

As discussed in the previous section, the comparison of the theoretical computations and the experimental data for the tangential stress showed excellent agreement, and indicates that this method might be used to solve problems of reinforcement design in the case of a flat plate in tension with a circular cutout. Sets of curves could be drawn up for various thicknesses of plate and ring and also for varying widths of the ring. It would also be useful to apply this type of analysis to investigate the possible optimum widths of rings and thicknesses of rings for reducing the stress concentration by a predetermined amount.

A study of the effect of rivet placement on the stress distribution across the ring is suggested as a worthwhile topic for experimental research in this field.

APPENDIX A

REFERENCES

(a.) Timoshenko: "Theory of Elasticity", 1934, First Edition, Eighth Impression, pp. 52-79, McGraw-Hill Book Company, Inc. New York and London.

(b.) Timoshenko: "Strength of Materials", Part II, Advanced Theory and Problems, 1930, pp. 454-457, D. Van Nostrand Company, Inc. New York.

APPENDIX B

BIBLIOGRAPHY

1. Kuhn, Paul and Moggio, Edwin M.: Stresses Around Rectangular Cutouts in Skin-Stringer Panels under Axial Loads. NACA ARR June, 1942.

2. Kuhn, Paul; Duberg, John E. and Diskin, Simon H.: Stresses around Rectangular Cutouts in Skin-Stringer Panels under Axial Loads - II. NACA ARR 3J02, October, 1943.

3. Farb, Daniel: Experimental Investigation of the Stress Distribution around Reinforced Circular Cutouts in Skin-Stringer Panels under Axial Loads. NACA TN 1241, March, 1947.

4. Kuhn, Paul; Rafel, Norman and Griffith, George E.: Stresses around Rectangular Cutouts with Reinforced Coaming Stringers. NACA TN 1176, January, 1947.

5. Dumont, C.: Stress Concentration around an Open Circular Hole in a Plate Subjected to Binding Normal to the Plane of the Plate. TN 740, December, 1939.

6. Podorazhny, A.A.: Investigation of the Behavior of Thin-Walled Panels with Cutouts. TM 1094, September, 1946.

7. Kuhn, Paul and Moggio, Edwin M.: Stresses around Large Cutouts in Torsion Boxes. TN 1066, May, 1946.

8. Moore, R.L.: Observations on the Behavior of Some Non-Circular Aluminum Alloy Sections Loaded to Failure in Torsion. TN 1097, February, 1947.

9. Ruffner, Benjamin F. and Schmidt; Calvin L.: Stresses at Cutouts in Shear Resistant Webs as Determined by the Photoelastic Method. NACA TN 984, October, 1945.

10. Kuhn, Paul: Skin Stresses around Inspection Cutouts.
NACA ARR, December, 1941.

11. Kuhn, Paul: The Strength and Stiffness of Shear Webs with and without Lightening Holes.
NACA ARR, June, 1942.

12. Kuhn, Paul: The Strength and Stiffness of Shear Webs with Round Lightening Holes Having 45° Flanges.
NACA ARR, December, 1942.

13. Kuhn, Paul and Levin, L. Ross: Tests of 10-inch 24ST Aluminum Alloy Shear Panels with $1\frac{1}{2}$ inch Holes.
NACA RB, 3F29, June, 1943.

14. Kuhn, Paul and Levin, L. Ross: Tests of 10-inch 24ST Aluminum Alloy Shear Panels with $1\frac{1}{2}$ inch Holes.
II Panels Having Holes with Notched Edges.
NACA RB L4D01, April, 1944.

15. Moggio, Edwin M. and Brilmyer, Harold G.: A Method for Estimation of Maximum Stresses around a Small Rectangular Cutout in a Sheet--Stringer Panel in Shear.
NACA ARR L4D27, April, 1944.

16. Hoff, N.J. & Kempner, Joseph: Numerical Procedures for the Calculation of the Stresses in Monocoques.
II Diffusion of Tensile Stringer Loads in Reinforced Flat Panels with Cutouts.
NACA TN 950, November, 1944.

17. Kuhn, Paul and Diskin, Simon H.: On the Shear Strength of Skin-Stiffener Panels with Inspection Cutouts.
ARR L5C01a, March, 1945.

18. Hill, H.N. and Barker, R.S.: Effect of Open Circular Holes on Tensile Strength and Elongation of Sheet Specimens of some Aluminum Alloys.
TN 1974, October, 1949.

19. Reissner, H. and Morduchow, M.: Reinforced Circular Cutouts in Plane Sheets.
TN 1852 April, 1949.

20. Hoff, Boley and Klein: Stresses in and General Instability of Monocoque Cylinders with Cutouts:
I. Experimental Investigation of Cylinders with a Symmetric Cutout Subjected to Pure Bending.
NACA TN 1013, 1946.

II. Calculation of the Stresses in Cylinders with Symmetric Cutout.
NACA TN 1014, 1946.

III. Calculation of the Buckling Load of Cylinders with Symmetric Cutout Subjected to Pure Bending.
NACA TN 1263, 1947.

IV. Pure Bending Tests of Cylinders with Side Cutout.
NACA TN 1264, 1948.

V. Calculation of the Stresses in Cylinders with Side Cutout.
NACA TN 1435, 1948.

VI. Calculation of the Buckling Load of Cylinders with Side Cutout Subjected to Pure Bending.
NACA TN 1436, March, 1948.

21. Journal of the Aeronautical Sciences
(a.) Vol. 15, pp. 171, 1948.
Effects of Cutouts in Semi-monocoque Structures by P. Cicala.
(not assuming a rigid frame)

(b.) Vol. 12, pp. 47, 1945.
Graphical Solution for Strains and Stresses from Strain
Rosette Data by Continu. pp. 235 Stress in a ring loaded Normal to its Plane by E.R. Walters.

(c.) Vol. 6, pp. 460, 1938-39.
Analysis of Circular Rings for Monocoque Fuselages by J.A. Wise.

(d.) Vol. 7, pp. 438, 1939-40.
Circles of Strain by J.A. Wise.

22. Lundquist, E.E. & Burke, W.F.: General Equations for Stress Analysis of Rings. NACA Tech. Report No. 509, 1934.

23. Niles, & Newell, Airplane Structures, Vol. II. Second Edition, pp. 265-281. Circular Rings.

APPENDIX C

SYMBOLS

(1.) a - internal radius of reinforcing ring - radius of cutout.

(2.) b - external radius of reinforcing ring.

(3.) E - modulus of elasticity.

(4.) ϵ_r - radial strain.

(5.) ϵ_θ - tangential strain.

(6.) G - modulus of shear.

(7.) $\gamma_{r\theta}$ - shearing strain.

(8.) P - total load on plate - lbs.

(9.) r - radius in inches.

(10.) ϕ - stress function.

(11.) S - loading on plate - lbs. per sq. in.

(12.) σ_{r_i} - radial stress in ring - lbs. per sq. in.

(13.) σ_{r_o} - radial stress outside ring - lbs. per sq. in.

(14.) σ_{θ_i} - tangential stress inside ring - lbs. per sq. in.

(15.) σ_{θ_o} - tangential stress outside ring - lbs. per sq. in.

(16.) t_i - thickness of ring and sheet.

(17.) t_o - thickness of sheet.

(18.) $\tau_{r\theta}$ - shearing stress - lbs. per sq. in.

(19.) θ - angle measured from direction of load (see Fig. 1.)

(20.) u - radial displacement.

(21.) μ - Poisson's Ratio.

(22.) v - tangential displacement.