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Torsion in an incomplete tore: an approximate solution for the stress distribution in a circular ring sector under uniform torsion using energy methods

Callaway, William Franklin

Monterey, California. Naval Postgraduate School

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TORSION IN AN INCOMPLETE TORUS

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TORSION IN AN INCOMPLETE TORSION

An approximate solution for the stress distribution in a circular ring sector under uniform torsion using energy methods

by

William Franklin Callaway
Lieutenant Commander, United States Navy

Submitted in partial fulfillment
of the requirements
for the degree of
MASTER OF SCIENCE
IN MECHANICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California
1952

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Monterey, California

June 1952

MEMORANDUM

The author desires to express his grateful appreciation for the
kindness and encouragement given by Professor Robert Gordon, U. S. Naval
Institute School, during the preparation of this work.

University, California

June 1952

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INTRODUCTION

The stress distribution in an incomplete tore loaded as shown in Fig. 1 is of particular interest since it very closely approximates that in heavy close-coiled helical springs under axial tension or compression. Necessarily the spring helix angle must be small, which is the case in a close-coiled spring. By a heavy spring is meant one whose ratio of mean diameter to cross-sectional diameter is such a value that the curvature of the section must be considered.

It should be noted that the stress distribution arising from the loading in Fig 1. is not pure torsion in the usual sense, but is a combination of torsion and direct shear. The problem therefore resolves itself into one of

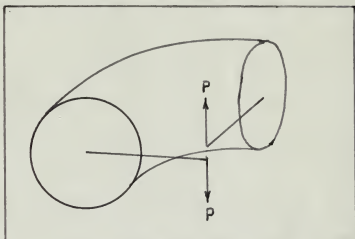


Fig. 1.

finding a single stress function which defines the true stress distribution in a cross-section of the circular ring sector.

Several solutions to the problem are in the literature, all of which by various means solve the differential equation arising from the conditions of compatibility. The first, by Michell (1) in 1899, used polynomial stress functions and obtained solutions for approximately circular cross-sections. Göhner (2) used successive approximations to approach an exact solution. Shepherd (3) used a method similar to both Göhner and Michell by finding a sequence of functions for approximately circular cross-sections and combining them linearly in such a manner that the sum was a solution.

Illustration

The correct distribution in the neighborhood of the origin is shown in Fig. 1. It is a probability density function which is very closely approximated by the heavy line-drawn curve. The light line-drawn curve is the case in which the force is zero. The heavy line-drawn curve is the case in which the force is non-zero. The heavy line-drawn curve is the case in which the force is non-zero.

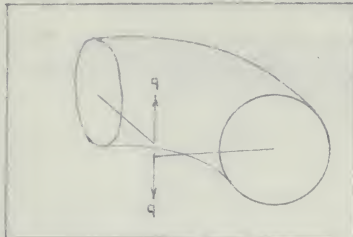


Fig. 1.

of the curve may be considered. It would be noted that the heavy line-drawn curve is the case in which the force is non-zero. The light line-drawn curve is the case in which the force is zero. The heavy line-drawn curve is the case in which the force is non-zero.

Using a single three degree function which defines the force distribution is a representation of the function for velocity.

Several solutions to the problem are in the literature, all of which by various means solve the differential equation arising from the conditions of separability. The first, by G. I. Taylor, is based on the use of three functions and special solutions for symmetrical circular motion. The second, by G. I. Taylor, is based on the use of three functions and special solutions for symmetrical circular motion. The third, by G. I. Taylor, is based on the use of three functions and special solutions for symmetrical circular motion.

Wahl (4) obtained a solution using curved bar theory and assuming a displacement of the center of rotation. Southwell (5) presented a formal solution for an arbitrary cross-section with a view towards a "relaxation" approach. Frieberger (6) has presented an exact solution for a circular cross-section by finding a stress function analogous to the ordinary torsion function and solving the problem in toroidal harmonics.

In this paper an approximate solution is obtained using the principle of least work. A stress function is found satisfying the equations of equilibrium and the boundary conditions and whose corresponding stresses make the strain energy a minimum. The solution of the differential equation of compatibility has therefore been replaced by the problem of minimizing the strain energy. In the energy method, the condition of minimum strain energy is equivalent to satisfying compatibility not in a point by point sense, but "on the average" throughout the body.

The purpose of this investigation has been to answer two questions in the author's mind. Namely, in view of the fact that nowhere was the author able to find the energy method used in the literature:

- (1) Can the problem be solved by this method, and how do the results compare with those of other solutions?
- (2) Does the problem particularly lend itself to solution by energy methods?

It was found that the problem is not adaptable to an exact solution by energy methods, but by making some approximations, excellent results are obtained that agree very closely with Frieberger's exact solution.

FORMULATION OF THE PROBLEM

We will consider a sector of a circular ring with mean radius of curvature \underline{R} and cross-sectional radius \underline{a} . A load \underline{P} is applied to one terminal cross-section as shown in Fig. 2, the other remaining fixed. Cylindrical coordinates are used, where the \underline{z} axis coincides with the toroidal axis, and the axis of the ring sector lies in the $\underline{r\theta}$ plane.

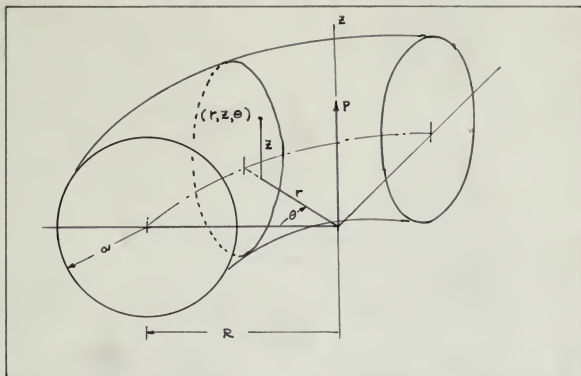


Fig. 2.

$\underline{\theta}$ increases positively as shown in the figure and \underline{r} increases outward from the toroidal axis. Later in the solution the coordinates will be transformed, but for the present purpose of establishing a stress function satisfying the equations of equilibrium, cylindrical coordinates are most convenient.

Assuming zero body forces, from THEORY OF ELASTICITY, Timoshenko & Goodier, Equations (170) the differential equations of equilibrium are

PROPERTIES OF THE PROBLEM

We will consider a vector of a function f and also some values of f for θ and $\theta + \Delta\theta$. The vector f is applied as one function in the θ direction, the other vector $f + \Delta f$ is applied in the $\theta + \Delta\theta$ direction, where the Δf is connected with the $\Delta\theta$ and the rate of the vector lies in the θ plane.

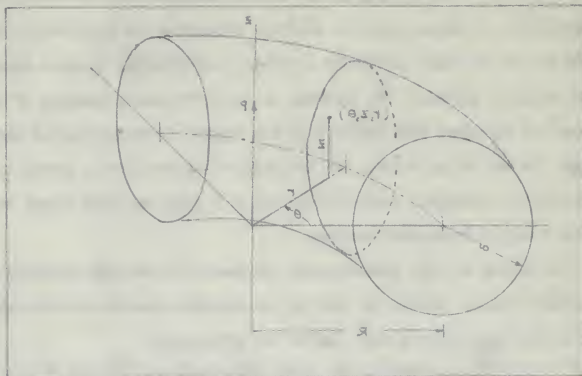


Fig. 5.

θ increases positively as shown in the figure and f increases along from the vertical axis. Later in the solution the coordinates will be given, but for the present purpose of establishing a clear mental picture the equations of unit vectors, spherical coordinates are not essential. Assuming zero body forces, from THOMAS OR BARNETT, *Thomson's & Hodder's* Dynamics (1907) the differential equations of equilibrium are

$$(1) \begin{cases} \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \\ \frac{\partial T_{rz}}{\partial r} + \frac{1}{r} \frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{T_{rz}}{r} = 0 \\ \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{2T_{r\theta}}{r} = 0 \end{cases}$$

Using the same assumptions made by Göhner in this case, namely that the only non-vanishing stresses are $T_{r\theta}$, $T_{z\theta}$ and that the stress distribution in any cross-section is independent of θ these reduce to

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{\partial T_{z\theta}}{\partial z} + \frac{2T_{r\theta}}{r} = 0$$

This may also be written

$$\left[\frac{\partial}{\partial r} (r^2 T_{r\theta}) + \frac{\partial}{\partial z} (r^2 T_{z\theta}) \right] = 0$$

A stress function ϕ satisfying the above is

$$GR^2 \frac{\partial \phi}{\partial z} = r^2 T_{r\theta} \quad GR^2 \frac{\partial \phi}{\partial r} = -r^2 T_{z\theta}$$

Where G is a constant (actually the modulus of rigidity).

Therefore the stresses may be expressed as

$$(2) \quad T_{r\theta} = \frac{GR^2}{r^2} \frac{\partial \phi}{\partial z} \quad \text{and} \quad T_{z\theta} = -\frac{GR^2}{r^2} \frac{\partial \phi}{\partial r}$$

At this point it is convenient to transform the cylindrical coordinates r, θ, z into toroidal coordinates ρ, ψ, θ (refer to Fig. 3).

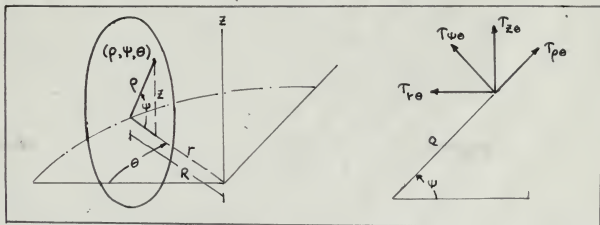


Fig. 3.

$$\left. \begin{aligned} 0 &= \frac{0.3}{4} - \frac{0.2}{4} + \frac{0.76}{4} + \frac{0.76}{4} + \frac{0.26}{4} \\ 0 &= \frac{1.2}{4} + \frac{0.2}{4} - \frac{0.76}{4} + \frac{0.76}{4} + \frac{0.76}{4} \\ 0 &= \frac{0.7}{4} + \frac{0.2}{4} - \frac{0.76}{4} + \frac{0.76}{4} \end{aligned} \right\} \text{--- (1)}$$

and the characteristic equation is $\det(A - \lambda I) = 0$.
 The eigenvalues are $\lambda_1 = 0.26$ and $\lambda_2 = 0.76$.
 The corresponding eigenvectors are $\vec{v}_1 = \begin{bmatrix} 0.76 \\ 0.26 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0.26 \\ 0.76 \end{bmatrix}$.

$$0 = \frac{0.76}{4} + \frac{0.26}{4} - \frac{0.76}{4} + \frac{0.76}{4}$$

Using the same method

$$0 = \left[\frac{0.26}{4} - \lambda \right] \frac{0.26}{4} + \left[\frac{0.76}{4} - \lambda \right] \frac{0.76}{4}$$

or $\frac{0.26}{4} (0.26 - \lambda) + \frac{0.76}{4} (0.76 - \lambda) = 0$

$$\frac{0.26}{4} (0.26 - \lambda) = -\frac{0.76}{4} (0.76 - \lambda)$$

or $0.26(0.26 - \lambda) = -0.76(0.76 - \lambda)$

or $0.26 \times 0.26 - 0.26\lambda = -0.76 \times 0.76 + 0.76\lambda$

$$\frac{0.26}{4} \times \frac{0.26}{4} - \frac{0.26}{4} \lambda = -\frac{0.76}{4} \times \frac{0.76}{4} + \frac{0.76}{4} \lambda \text{ --- (2)}$$

or $0.26 \times 0.26 - 0.26\lambda = -0.76 \times 0.76 + 0.76\lambda$

or $0.26 \times 0.26 + 0.76 \times 0.76 = 0.26\lambda + 0.76\lambda$



Fig. 1.1

If ϕ is a function of r and z , where

$$r = R - \rho \cos \psi$$

$$z = \rho \sin \psi$$

from Fig. 3.

Then

$$\frac{\partial \phi}{\partial \rho} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial \rho} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \rho}$$

$$\frac{\partial \phi}{\partial \psi} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial \psi} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \psi}$$

where

$$\frac{\partial r}{\partial \rho} = -\cos \psi$$

$$\frac{\partial r}{\partial \psi} = \rho \sin \psi$$

$$\frac{\partial z}{\partial \rho} = \sin \psi$$

$$\frac{\partial z}{\partial \psi} = \rho \cos \psi$$

Substituting

$$\frac{\partial \phi}{\partial \rho} = \frac{\partial \phi}{\partial r} (-\cos \psi) + \frac{\partial \phi}{\partial z} (\sin \psi)$$

$$\frac{\partial \phi}{\partial \psi} = \frac{\partial \phi}{\partial r} (\rho \sin \psi) + \frac{\partial \phi}{\partial z} (\rho \cos \psi)$$

Solving for $\frac{\partial \phi}{\partial r}$ and $\frac{\partial \phi}{\partial z}$

$$(3) \begin{cases} \frac{\partial \phi}{\partial r} = \left(\frac{\sin \psi}{\rho} \right) \frac{\partial \phi}{\partial \psi} - (\cos \psi) \frac{\partial \phi}{\partial \rho} \\ \frac{\partial \phi}{\partial z} = \left(\frac{\cos \psi}{\rho} \right) \frac{\partial \phi}{\partial \psi} + (\sin \psi) \frac{\partial \phi}{\partial \rho} \end{cases}$$

In a plane cross-section determined by θ a constant

$$(4) \begin{cases} T_{\rho\theta} = -T_{r\theta} \cos \psi + T_{z\theta} \sin \psi \\ T_{\psi\theta} = T_{r\theta} \sin \psi + T_{z\theta} \cos \psi \end{cases}$$

Using Equations (2), (3) and (4) the following result is obtained.

$$T_{\rho\theta} = \frac{GR^2}{(R-\rho \cos \psi)^2} \left[\frac{\cos \psi}{\rho} \frac{\partial \phi}{\partial \psi} + \sin \psi \frac{\partial \phi}{\partial \rho} \right] - \frac{GR^2 \sin \psi}{(R-\rho \cos \psi)^2} \left[\frac{\sin \psi}{\rho} \frac{\partial \phi}{\partial \psi} - \cos \psi \frac{\partial \phi}{\partial \rho} \right]$$

$$T_{\psi\theta} = \frac{GR^2}{(R-\rho \cos \psi)^2} \left[\frac{\cos \psi}{\rho} \frac{\partial \phi}{\partial \psi} + \sin \psi \frac{\partial \phi}{\partial \rho} \right] - \frac{GR^2 \cos \psi}{(R-\rho \cos \psi)^2} \left[\frac{\sin \psi}{\rho} \frac{\partial \phi}{\partial \psi} - \cos \psi \frac{\partial \phi}{\partial \rho} \right]$$

Reducing

$$(5) \quad T_{\rho\theta} = \frac{GR^2}{(R-\rho \cos \psi)^2} \frac{1}{\rho} \frac{\partial \phi}{\partial \psi}$$

$$\text{and} \quad T_{\psi\theta} = \frac{GR^2}{(R-\rho \cos \psi)^2} \frac{\partial \phi}{\partial \rho}$$

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$$\psi_{200} - \psi_{112} = \frac{76}{96}$$

$$\psi_{200} = \frac{76}{96} + \psi_{112}$$

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$$\psi_{112} = \frac{76}{96}$$

$$\psi_{200} = \frac{76}{96} + \psi_{112}$$

$$\frac{76}{96} + \frac{76}{96} = \frac{152}{96}$$

$$\frac{76}{96} + \frac{76}{96} = \frac{152}{96}$$

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$$(\psi_{112}) \frac{76}{96} + (\psi_{200}) \frac{76}{96} = \frac{76}{96}$$

$$(\psi_{200}) \frac{76}{96} + (\psi_{112}) \frac{76}{96} = \frac{76}{96}$$

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$$\left. \begin{aligned} \frac{76}{96} (\psi_{200}) - \frac{76}{96} (\psi_{112}) &= \frac{76}{96} \\ \frac{76}{96} (\psi_{112}) + \frac{76}{96} (\psi_{200}) &= \frac{76}{96} \end{aligned} \right\} \text{--- (1)}$$

1971 1971 1971 1971 1971 1971

$$\left. \begin{aligned} \psi_{112} e^{\frac{76}{96}} + \psi_{200} e^{\frac{76}{96}} &= e^{\frac{76}{96}} \\ \psi_{200} e^{\frac{76}{96}} + \psi_{112} e^{\frac{76}{96}} &= e^{\frac{76}{96}} \end{aligned} \right\} \text{--- (2)}$$

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$$\left[\frac{76}{96} \psi_{200} - \frac{76}{96} \psi_{112} \right] \frac{76}{96} - \left[\frac{76}{96} \psi_{112} + \frac{76}{96} \psi_{200} \right] \frac{76}{96} = \frac{76}{96}$$

$$\left[\frac{76}{96} \psi_{200} - \frac{76}{96} \psi_{112} \right] \frac{76}{96} - \left[\frac{76}{96} \psi_{112} + \frac{76}{96} \psi_{200} \right] \frac{76}{96} = \frac{76}{96}$$

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$$\frac{76}{96} \frac{76}{96} = \frac{76}{96} \frac{76}{96}$$

$$\frac{76}{96} \frac{76}{96} = \frac{76}{96} \frac{76}{96}$$

The latter expressions relate the stress function and the stresses in the new system of coordinates.

It follows that since the shear stress $\tau_{\rho\theta}$ is normal to the boundary, it must vanish everywhere on the boundary. This is true because the surface of the body is free from any external forces. Using this condition with Equation (5), it is apparent that $\frac{\partial\phi}{\partial n} = 0$ and ϕ must be constant on the boundary.

The circular ring sector we are considering is a singly connected body, hence the constant may be chosen arbitrarily. Therefore the boundary condition is taken as $\phi = 0$ everywhere on the boundary.

The only action on a cross-section is a force \underline{P} directed along the toroidal axis. This may be resolved into a force and a couple as shown in Fig. 4.

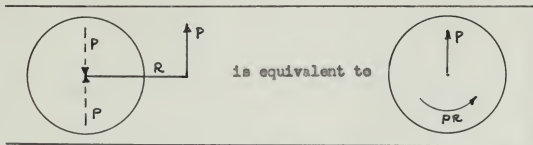


Fig. 4.

It is now seen that the two conditions of static equilibrium to be satisfied are that the resultant stress on a cross-section produce a force \underline{P} directed along the \underline{z} axis and a moment about the center \underline{PR} . These requirements may be written

$$(6) \left\{ \begin{array}{l} P = \int_0^a \int_0^{2\pi} (\tau_{\rho\theta} \sin\psi + T_{\psi\theta} \cos\psi) \rho d\rho d\psi \\ PR = \int_0^a \int_0^{2\pi} T_{\psi\theta} \rho^2 d\rho d\psi \end{array} \right.$$

The lateral equilibrium is maintained by the forces and the stresses in the new system of coordinates.

It follows that since the shear stress τ_{θ} is normal to the boundary, it must induce a moment on the boundary. This is true because the direction of the body is free from any external forces. Being this condition with Equation (2), it is apparent that $\frac{\partial \phi}{\partial r} = 0$ and $\phi = 0$ must be constant on the boundary.

The classical ring section we are considering is a simply connected body, hence the boundary can be chosen arbitrarily. Therefore the boundary condition is taken as $\phi = 0$ everywhere on the boundary.

The ring section in a cross-section is a force P directed along the vertical axis. This may be resolved into a force and a couple as shown in

Fig. 4.

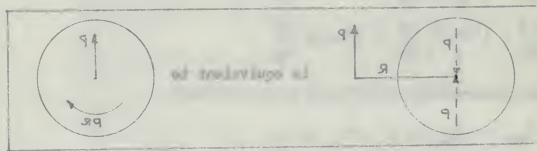


Fig. 4.

It is now seen that the two conditions of rigid equilibrium can be satisfied and that the boundary stress on a cross-section involves a force P directed along the y axis and a moment about the origin O . These requirements may be written

$$\left. \begin{aligned} P &= \int_0^{2\pi} \int_0^a (\tau_{\theta} \cos \psi + \tau_{\theta} \sin \psi) r \, d\psi \, dr \\ PR &= \int_0^{2\pi} \int_0^a \tau_{\theta} r^2 \, d\psi \, dr \end{aligned} \right\} \quad (5)$$

The strain energy per unit angle θ is

$$(7) \text{---} \quad U = \frac{1}{2G} \int_0^a \int_0^{2\pi} (\tau_{\rho\theta}^2 + \tau_{\psi\theta}^2) (R - \rho \cos \psi) \rho \, d\rho \, d\psi$$

The method of solution will now be to take the stress function in the form

$$\phi = \sum_{i=0}^N \alpha_i \phi_i \quad , \quad \text{where } \phi_i \text{ are suitably selected functions of } \rho \text{ and}$$

ψ , each of which satisfies the boundary condition $\phi_i = 0$ when $\rho = a$.

The coefficients α_i are constants which are evaluated from the minimum condition of strain energy.

FIRST APPROXIMATION

For a first approximation we shall take a function ϕ , satisfying the boundary condition that it vanish everywhere on the boundary, in the form $\phi = (\rho^2 - a^2)(\alpha_0 + \frac{\alpha_1 \rho}{R} \cos \psi)$. The reasons for this particular choice are discussed in Appendix A. Taking the partial derivatives of ϕ with respect to the two variables ρ and ψ

$$\frac{\partial \phi}{\partial \rho} = 2\rho\alpha_0 + \frac{\alpha_1(\rho^2 - a^2)}{R} \cos \psi \quad \text{and} \quad \frac{\partial \phi}{\partial \psi} = -\frac{\alpha_1(\rho^2 - a^2)}{R} \sin \psi$$

Substituting in Equations (5), the following expressions are obtained for $T_{\rho\theta}$ and $T_{\psi\theta}$.

$$(8) \text{-----} T_{\rho\theta} = \frac{GR^2}{(R - \rho \cos \psi)^2} \left[\frac{\alpha_1(\rho^2 - a^2)}{R} \sin \psi \right] \quad \text{and} \quad T_{\psi\theta} = \frac{GR^2}{(R - \rho \cos \psi)^2} \left[2\rho\alpha_0 + \frac{\alpha_1(\rho^2 - a^2)}{R} \cos \psi \right]$$

The appearance of the term $\frac{1}{(R - \rho \cos \psi)^2}$ in the stress equations makes the integration required in (6) and (7) very complicated and the results largely unmanageable in the evaluation of the unknown coefficients in ϕ . (See Appendix B). This is particularly true when additional terms are used in ϕ for a higher order of approximation, and in the evaluation of the strain energy where the stresses appear as squared terms.

Since $\frac{\rho}{R}$ is always less than unity, we may write

$$\frac{R^2}{(R - \rho \cos \psi)^2} = \frac{1}{\left(1 - \frac{\rho}{R} \cos \psi\right)^2} = 1 + 2\left(\frac{\rho}{R}\right) \cos \psi + 3\left(\frac{\rho}{R}\right)^2 \cos^2 \psi + \dots$$

Utilizing this expansion, the exact stress expressions (5) may be approximated as follows

$$T_{\rho\theta} = -\frac{G}{\rho} \left[\left(1 + 2\frac{\rho}{R} \cos \psi\right) \frac{\partial \phi_0}{\partial \psi} \alpha_0 + \frac{\partial \phi_1}{\partial \psi} \alpha_1 \right]$$

$$T_{\psi\theta} = G \left[\left(1 + 2\frac{\rho}{R} \cos \psi\right) \frac{\partial \phi_0}{\partial \rho} \alpha_0 + \frac{\partial \phi_1}{\partial \rho} \alpha_1 \right]$$

This particular form of approximation accomplishes the desired result

BOUNDARY CONDITIONS

For a finite approximation we shall take a function ϕ , satisfying the boundary conditions that in every neighborhood of the boundary, in the sense of the mean value theorem, the function ϕ is harmonic. Let us take the partial derivatives of ϕ with respect to the two variables ψ and ρ

$$\frac{\partial \phi}{\partial \rho} = \frac{\partial \phi}{\partial \rho} \cos \psi + \frac{\partial \phi}{\partial \psi} \sin \psi \quad \text{and} \quad \frac{\partial \phi}{\partial \psi} = \frac{\partial \phi}{\partial \psi} \cos \psi - \frac{\partial \phi}{\partial \rho} \sin \psi$$

Substituting in conditions (1), the following expressions are

obtained for T_{ρ} and T_{ψ}

$$(8) \quad T_{\rho} = \frac{\partial \phi}{\partial \rho} \cos \psi + \frac{\partial \phi}{\partial \psi} \sin \psi \quad \text{and} \quad T_{\psi} = \frac{\partial \phi}{\partial \psi} \cos \psi - \frac{\partial \phi}{\partial \rho} \sin \psi$$

The appearance of the term $\frac{1}{(R-\rho \cos \psi)^2}$ in the stress equations

makes the integration required in (6) and (7) very complicated and the results largely meaningless in the evaluation of the unknown coefficients in ϕ . (See Appendix B). This is particularly true when additional terms are used in ϕ for a higher order of approximation, and in the evaluation of the strain energy where the stresses appear as squared terms.

Since $\frac{\rho}{R}$ is always less than unity, we may write

$$\frac{1}{(1 - \frac{\rho}{R} \cos \psi)^2} = 1 + 2 \left(\frac{\rho}{R}\right) \cos \psi + 3 \left(\frac{\rho}{R}\right)^2 \cos^2 \psi + \dots$$

Utilizing this expansion, the exact stress equations (2) may be

approximated as follows

$$T_{\rho} = \frac{\partial \phi}{\partial \rho} \left[1 + 2 \frac{\rho}{R} \cos \psi \right] + \frac{\partial \phi}{\partial \psi} \left[\sin \psi + 2 \frac{\rho}{R} \cos \psi \sin \psi \right]$$

$$T_{\psi} = \frac{\partial \phi}{\partial \psi} \left[\cos \psi - 2 \frac{\rho}{R} \sin \psi \cos \psi \right] - \frac{\partial \phi}{\partial \rho} \left[\sin \psi - 2 \frac{\rho}{R} \cos \psi \sin \psi \right]$$

This particular form of approximation simplifies the resulting results

of limiting the highest power to which the ratios $\frac{a}{R}$ and $\frac{\rho}{R}$ appear in the stress equations.

Since $\phi_0 = (\rho^2 - a^2)$ and $\phi_1 = \frac{\rho(\rho^2 - a^2)}{R} \cos \psi$

The partial derivatives are

$$\frac{\partial \phi_0}{\partial \rho} = 2\rho \quad \frac{\partial \phi_1}{\partial \rho} = \frac{(3\rho^2 - a^2)}{R} \cos \psi$$

$$\frac{\partial \phi_0}{\partial \psi} = 0 \quad \frac{\partial \phi_1}{\partial \psi} = -\frac{\rho(\rho^2 - a^2)}{R} \sin \psi$$

Substituting, we arrive at the following approximate expressions for the stresses.

$$(9) \left\{ \begin{array}{l} T_{\rho\theta} = G \left[\frac{\alpha_1 (\rho^2 - a^2)}{R} \sin \psi \right] \\ T_{\psi\theta} = G \left[2\rho\alpha_0 + \frac{(4\rho^2\alpha_0 + 3\rho^2\alpha_1 - a^2\alpha_1)}{R} \cos \psi \right] \end{array} \right.$$

Substituting these values of $T_{\rho\theta}$ and $T_{\psi\theta}$ in the first of Equations (6), and integrating we obtain

$$P = \frac{G\pi a^4}{R} \alpha_0 \quad \therefore \alpha_0 = \frac{PR}{G\pi a^4}$$

The same result is obtained from the second condition of Equations (6).

It follows that α_0 is fixed by the requirements of static equilibrium and α_1 may now be determined by the condition of minimum strain energy

that $\frac{\partial U}{\partial \alpha_1} = 0$.

From Equation (7)

$$\frac{\partial U}{\partial \alpha_1} = \frac{1}{G} \int_0^a \int_0^{2\pi} (T_{\rho\theta} \frac{\partial T_{\rho\theta}}{\partial \alpha_1} + T_{\psi\theta} \frac{\partial T_{\psi\theta}}{\partial \alpha_1}) (R - \rho \cos \psi) \rho d\rho d\psi$$

Substituting the stresses from Equation (9) and integrating

$$\frac{\partial U}{\partial \alpha_1} = \frac{R\pi}{G} \left[\left(\frac{1}{2} \frac{a^4}{R^2} \right) \alpha_0 + \left(\frac{2}{3} \frac{a^4}{R^2} \right) \alpha_1 \right]$$

Setting $\frac{\partial U}{\partial \alpha_1} = 0$ and solving for α_1 ,

of initial and final wave functions in the interval $\frac{a}{R}$ and $\frac{b}{R}$ are given by

$$\psi_0 = \phi \quad \text{and} \quad \psi_1 = \phi \cos \frac{(b-a)q}{R}$$

The initial derivatives are

$$\begin{aligned} \psi_0' &= \frac{6\phi}{9b} & \psi_1' &= \frac{6\phi}{9b} \\ \psi_0' &= \frac{6\phi}{9a} & \psi_1' &= \frac{6\phi}{9a} \cos \frac{(b-a)q}{R} - \frac{6\phi}{9R} \sin \frac{(b-a)q}{R} \end{aligned}$$

Integrating, we write as the following approximate expressions for

$$\left. \begin{aligned} \int \psi_0^2 dx &= \int_a^b \phi^2 dx \\ \int \psi_1^2 dx &= \int_a^b \phi^2 \cos^2 \frac{(b-a)q}{R} dx \end{aligned} \right\} \text{the energies}$$

Substituting these values of $\int \psi^2 dx$ in the three equations (5)

and integrating we obtain

$$E = \frac{\hbar^2 \pi^2 a^2}{2mR^2} \quad \therefore \quad \alpha = \frac{\pi R}{2a}$$

The same result is obtained from the second condition of equation (5).

It follows that α is fixed by the requirement of finite normalization

and α may now be determined by the condition of stationarity energy

$$\frac{\partial E}{\partial \alpha} = 0$$

From equation (7)

$$\frac{\partial E}{\partial \alpha} = \frac{\hbar^2 \pi^2 a^2}{2mR^2} \left(\frac{1}{\alpha} + \frac{1}{\alpha^3} \right) \int_a^b \psi^2 dx - \frac{\hbar^2 \pi^2 a^2}{2mR^2} \int_a^b \psi^2 dx$$

Calculating the integrals from equation (7) and integrating

$$\left[\frac{1}{\alpha} + \frac{1}{\alpha^3} \right] \int_a^b \psi^2 dx - \int_a^b \psi^2 dx = 0$$

and solving for α

Using these results in Equations (9) we arrive at the expressions for the first approximation of the stress distribution in a cross-section of the incomplete tore

$$(10) \left\{ \begin{array}{l} T_{\rho\theta} = - \frac{PR}{\pi a^4} \left[\frac{3}{4} \frac{(\rho^2 - a^2)}{R} \sin \psi \right] \\ T_{\psi\theta} = \frac{PR}{\pi a^4} \left[2\rho + \frac{(\rho^2 + 3a^2)}{R} \cos \psi \right] \end{array} \right.$$

At the point of maximum stress where $\rho = a$ and $\psi = 0$ the above reduce to

$$(11) \left\{ \begin{array}{l} T_{\rho\theta} = 0 \\ [T_{\psi\theta}]_{\max} = \frac{2PR}{\pi a^3} \left[1 + \frac{5}{4} \left(\frac{a}{R} \right) \right] \end{array} \right.$$

It is interesting to note at this point that for this particular solution, one of the unknown coefficients in ϕ is determined directly from the requirements of static equilibrium, and the other directly from the minimum strain energy condition without constraint arising from static equilibrium.

From these results in Section (I) we derive the expression for the first approximation of the stress distribution in a semi-infinite body of the homogeneous type

$$\left. \begin{aligned} T_{\theta} &= - \frac{PR}{\pi a^2} \left[\frac{3}{4} \frac{q^2 - a^2}{R} \sin \psi \right] \\ T_{\psi} &= \frac{PR}{\pi a^2} \left[\frac{1}{2} q + \frac{(T_0^2 + 3a^2)}{R} \cos \psi \right] \end{aligned} \right\} \text{(10)}$$

At the point of contact where $\psi = \alpha$ and $\psi = 0$ the above

$$\left. \begin{aligned} T_{\theta} &= 0 \\ T_{\psi} &= \left[\frac{PR}{\pi a^2} \left(1 + \frac{3}{4} \left(\frac{a}{R} \right)^2 \right) \right] \end{aligned} \right\} \text{(11)}$$

It is interesting to note at this point that for this particular solution, one of the maximum compressions in ϕ is obtained directly from the positive mode of stress equation, and the other directly from the strain strain energy condition which governs the stress equilibrium.

SECOND APPROXIMATION

A closer approximation to the true stress conditions will result if higher order terms of a suitable nature are used in the stress function. We shall now take ϕ as

$$\phi = (\rho^2 - a^2) \left(\alpha_0 + \frac{\alpha_1 \rho}{R} \cos \psi + \frac{\alpha_2 \rho^2}{R^2} \cos^2 \psi + \frac{\alpha_3 \rho^2}{R^2} \sin^2 \psi + \frac{\alpha_4 a^2}{R^2} \right)$$

Reasons for this particular choice of functions are discussed in Appendix A.

Again employing the binomial expansion of $\frac{1}{(R - \rho \cos \psi)^2}$ we write approximate expressions for $\tau_{\rho\theta}$ and $\tau_{\psi\theta}$.

$$\tau_{\rho\theta} = -\frac{G}{\rho} \left[\left(1 + 2\frac{\rho}{R} \cos \psi + 3\frac{\rho^2}{R^2} \cos^2 \psi \right) \frac{\partial \phi}{\partial \psi} \alpha_0 + \left(1 + 2\frac{\rho}{R} \cos \psi \right) \frac{\partial \phi}{\partial \psi} \alpha_1 + \frac{\partial \phi}{\partial \psi} \alpha_2 + \frac{\partial \phi}{\partial \psi} \alpha_3 + \frac{\partial \phi}{\partial \psi} \alpha_4 \right]$$

$$\tau_{\psi\theta} = G \left[\left(1 + 2\frac{\rho}{R} \cos \psi + 3\frac{\rho^2}{R^2} \cos^2 \psi \right) \frac{\partial \phi}{\partial \rho} \alpha_0 + \left(1 + 2\frac{\rho}{R} \cos \psi \right) \frac{\partial \phi}{\partial \rho} \alpha_1 + \frac{\partial \phi}{\partial \rho} \alpha_2 + \frac{\partial \phi}{\partial \rho} \alpha_3 + \frac{\partial \phi}{\partial \rho} \alpha_4 \right]$$

Where

$$\phi_0 = (\rho^2 - a^2) \quad \phi_2 = \frac{\rho^2(\rho^2 - a^2)}{R^2} \cos^2 \psi \quad \phi_4 = \frac{a^2(\rho^2 - a^2)}{R^2}$$

$$\phi_1 = \frac{\rho(\rho^2 - a^2)}{R} \cos \psi \quad \phi_3 = \frac{\rho^2(\rho^2 - a^2)}{R^2} \sin^2 \psi$$

This is an extension of the device used before to limit the highest power to which the ratios $\frac{a}{R}$ and $\frac{\rho}{R}$ appear in each term of the stress equations. Since $\frac{a}{R}$ and $\frac{\rho}{R}$ occur in a like manner in ϕ_2 , ϕ_3 and ϕ_4 , these latter terms are grouped together and treated in similar fashion when introduced into the approximate expressions for the stresses.

Taking the partial derivatives, substituting and rearranging the terms for convenient integration, the following approximate expressions for $\tau_{\rho\theta}$ and $\tau_{\psi\theta}$ are obtained.

$$(12) \left\{ \begin{aligned} \tau_{\rho\theta} &= G \left[\frac{\alpha_1(\rho^2 - a^2)}{R} \sin \psi + \frac{2\rho(\alpha_1 + \alpha_2 - \alpha_3)(\rho^2 - a^2)}{R^2} \sin \psi \cos \psi \right] \\ \tau_{\psi\theta} &= G \left\{ \left[\frac{6\rho^3 \alpha_0}{R^2} + \frac{2\rho \alpha_1(3\rho^2 - a^2)}{R^2} + \frac{2\rho \alpha_2(2\rho^2 - a^2)}{R^2} \right] \cos^2 \psi + \left[\frac{4\rho^2 \alpha_0}{R} + \frac{\alpha_1(3\rho^2 - a^2)}{R} \right] \cos \psi + 2\rho \alpha_0 + \frac{2\rho \alpha_3(2\rho^2 - a^2)}{R^2} \sin^2 \psi + \frac{2a^2 \alpha_4 \rho}{R^2} \right\} \end{aligned} \right.$$

THE SOLUTION

It is assumed that the wave function ψ is of the form $\psi = \phi e^{i\theta}$. The function ϕ is assumed to be a function of r only, and θ is assumed to be a function of t only. The wave function ψ is assumed to be a function of r and t only.

$$\phi \left(\frac{d^2 \psi}{dt^2} + \psi \left(\frac{d^2 \theta}{dt^2} + \frac{d^2 \psi}{dr^2} + \frac{2}{r} \frac{d\psi}{dr} \right) \right) = 0$$

Because the two terms in brackets are assumed to be equal to zero, the function ϕ is assumed to be a function of r only, and θ is assumed to be a function of t only. The function ϕ is assumed to be a function of r only, and θ is assumed to be a function of t only.

$$\left[\frac{d^2 \psi}{dt^2} + \psi \left(\frac{d^2 \theta}{dt^2} + \frac{d^2 \psi}{dr^2} + \frac{2}{r} \frac{d\psi}{dr} \right) \right] = 0$$

$$\phi \left(\frac{d^2 \psi}{dt^2} + \psi \left(\frac{d^2 \theta}{dt^2} + \frac{d^2 \psi}{dr^2} + \frac{2}{r} \frac{d\psi}{dr} \right) \right) = 0$$

This is an equation of the wave function ψ in the r direction. The function ϕ is assumed to be a function of r only, and θ is assumed to be a function of t only. The function ϕ is assumed to be a function of r only, and θ is assumed to be a function of t only.

Using the method of separation of variables, the wave function ψ is assumed to be a function of r and t only. The function ϕ is assumed to be a function of r only, and θ is assumed to be a function of t only.

$$\left[\frac{d^2 \psi}{dt^2} + \psi \left(\frac{d^2 \theta}{dt^2} + \frac{d^2 \psi}{dr^2} + \frac{2}{r} \frac{d\psi}{dr} \right) \right] = 0$$

From the first of the static equilibrium conditions in (6) (that the resultant stress must produce a force \underline{P} in the \underline{Z} direction) it is again found that $\alpha_0 = \frac{PR}{G\pi a^4}$.

The second static equilibrium condition (that the resultant stress must produce a moment about the center equal to \underline{PR}) gives the following result.

$$\frac{PR}{G\pi a^4} = \left[\left(1 + \frac{a^2}{R^2}\right)\alpha_0 + \left(\frac{1}{2} \frac{a^2}{R^2}\right)\alpha_1 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\alpha_2 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\alpha_3 + \left(\frac{a^2}{R^2}\right)\alpha_4 \right]$$

However, since $\alpha_0 = \frac{PR}{G\pi a^4}$

$$\left(\frac{1}{2} \frac{a^2}{R^2}\right)\alpha_1 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\alpha_2 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\alpha_3 + \left(\frac{a^2}{R^2}\right)\alpha_4 + \left(\frac{a^2}{R^2}\right)\left(\frac{PR}{G\pi a^4}\right) = 0$$

Since $\alpha_1, \alpha_2, \alpha_3$ and α_4 will ultimately all contain the factor $\frac{PR}{G\pi a^4}$ some simplification of the algebra will be afforded if we make the following substitutions

$$\beta_n = \alpha_n \left(\frac{G\pi a^4}{PR}\right) \quad \text{where } n = 1, 2, 3, 4$$

Finally the constraining function derived from the conditions of static equilibrium to be used in minimizing the strain energy is

$$(13) \text{-----} \left(\frac{1}{2} \frac{a^2}{R^2}\right)\beta_1 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\beta_2 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\beta_3 + \left(\frac{a^2}{R^2}\right)\beta_4 + \frac{a^2}{R^2} = 0$$

The work involved in obtaining the partial derivatives of the strain energy with respect to the unknown coefficients $\alpha_1, \alpha_2, \alpha_3$ and α_4 will be simplified by differentiating under the integral sign.

Therefore

$$\frac{\partial U}{\partial \alpha_n} = \frac{R}{G} \int_0^a \int_0^{2\pi} \left(T_{\rho\theta} \frac{\partial T_{\rho\theta}}{\partial \alpha_n} + T_{\psi\theta} \frac{\partial T_{\psi\theta}}{\partial \alpha_n} \right) \left(1 - \frac{\rho}{R} \cos \psi \right) \rho \, d\rho \, d\psi$$

$\frac{\partial U}{\partial \alpha_0}$ is not required since α_0 has already been evaluated from the

From the form of the general solution in (1) it is seen that the coefficients must be chosen so that the function is periodic in x with period 2π .

$$\frac{pR}{2\pi a} \alpha_0 = \alpha_0$$

The second order condition (and the condition of periodicity) must be satisfied at the center point of the interval $0 < x < 2\pi$.

$$\left[\frac{pR}{2\pi a} \alpha_0 + \left(\frac{1}{2} \frac{pR}{a} \right) \alpha_1 + \left(\frac{1}{6} \frac{pR}{a} \right) \alpha_2 + \left(\frac{1}{24} \frac{pR}{a} \right) \alpha_3 + \left(\frac{1}{120} \frac{pR}{a} \right) \alpha_4 \right] = 0$$

$$\frac{pR}{2\pi a} \alpha_0 = \alpha_0$$

$$\left(\frac{1}{2} \frac{pR}{a} \right) \alpha_1 + \left(\frac{1}{6} \frac{pR}{a} \right) \alpha_2 + \left(\frac{1}{24} \frac{pR}{a} \right) \alpha_3 + \left(\frac{1}{120} \frac{pR}{a} \right) \alpha_4 = 0$$

Since $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are all arbitrary constants the first

condition is satisfied if the second is satisfied. It is seen that

$$\left(\frac{pR}{2\pi a} \right) \alpha_0 = \alpha_0$$

$$\alpha_0 = 1.2333 \alpha_1$$

Finally the condition of periodicity must be satisfied at $x = 2\pi$. This is done by substituting $x = 2\pi$ in the general solution and requiring that the function be periodic.

$$\left(\frac{1}{2} \frac{pR}{a} \right) \alpha_1 + \left(\frac{1}{6} \frac{pR}{a} \right) \alpha_2 + \left(\frac{1}{24} \frac{pR}{a} \right) \alpha_3 + \left(\frac{1}{120} \frac{pR}{a} \right) \alpha_4 = 0 \quad (1)$$

The next step is to obtain the partial derivatives of the strain energy with respect to the unknown coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and to set them equal to zero. This is done by differentiating under the integral sign.

$$\frac{\partial U}{\partial \alpha_1} = \frac{pR}{2\pi a} \int_0^{2\pi} \left(\frac{1}{2} \frac{pR}{a} \right) \alpha_1 \cos \psi \, d\psi + \frac{pR}{2\pi a} \int_0^{2\pi} \left(\frac{1}{6} \frac{pR}{a} \right) \alpha_2 \cos \psi \, d\psi + \frac{pR}{2\pi a} \int_0^{2\pi} \left(\frac{1}{24} \frac{pR}{a} \right) \alpha_3 \cos \psi \, d\psi + \frac{pR}{2\pi a} \int_0^{2\pi} \left(\frac{1}{120} \frac{pR}{a} \right) \alpha_4 \cos \psi \, d\psi$$

It is noted that α_0 has already been evaluated from the

conditions of static equilibrium. Since $\frac{\partial U}{\partial \alpha_n} = (\text{Constant}) \frac{\partial U}{\partial \beta_n}$, it is convenient here to take $\frac{\partial U}{\partial \beta_n}$. After substituting for $\tau_{\rho\theta}$ and $\tau_{\psi\theta}$ from Equations (12) and performing the required integration, the following expressions are obtained.

$$(14) \left\{ \begin{aligned} \frac{\partial U}{\partial \beta_1} &= \frac{\pi a^4}{GR} \left[\left(\frac{3}{2} - \frac{5}{16} \frac{a^2}{R^2} \right) + \frac{2}{3} \beta_1 + \frac{13}{48} \frac{a^2}{R^2} \beta_2 + \frac{1}{16} \frac{a^2}{R^2} \beta_3 + \frac{1}{2} \frac{a^2}{R^2} \beta_4 \right] \\ \frac{\partial U}{\partial \beta_2} &= \frac{\pi a^4}{GR} \left[\left(\frac{1}{3} - \frac{1}{4} \frac{a^2}{R^2} \right) + \frac{13}{16} \frac{a^2}{R^2} \beta_1 + \frac{7}{24} \frac{a^2}{R^2} \beta_2 + \frac{1}{24} \frac{a^2}{R^2} \beta_3 + \frac{1}{3} \frac{a^2}{R^2} \beta_4 \right] \\ \frac{\partial U}{\partial \beta_3} &= \frac{\pi a^4}{GR} \left[\left(\frac{1}{3} + \frac{1}{12} \frac{a^2}{R^2} \right) + \frac{1}{16} \frac{a^2}{R^2} \beta_1 + \frac{1}{24} \frac{a^2}{R^2} \beta_2 + \frac{7}{24} \frac{a^2}{R^2} \beta_3 + \frac{1}{3} \frac{a^2}{R^2} \beta_4 \right] \\ \frac{\partial U}{\partial \beta_4} &= \frac{\pi a^4}{GR} \left[\left(2 + \frac{2}{3} \frac{a^2}{R^2} \right) + \frac{1}{2} \frac{a^2}{R^2} \beta_1 + \frac{1}{3} \frac{a^2}{R^2} \beta_2 + \frac{1}{3} \frac{a^2}{R^2} \beta_3 + 2 \frac{a^2}{R^2} \beta_4 \right] \end{aligned} \right.$$

To minimize the strain energy and evaluate the unknown coefficients β_1 , β_2 , β_3 and β_4 , the method of Lagrangian multipliers will be used with the constraining function (13) established by the requirements of static equilibrium. The constant $\frac{\pi a^4}{GR}$ appearing in the partial derivatives of the strain energy will be incorporated in the multiplier. Letting λ be a Lagrangian multiplier, and $f(\beta_1, \beta_2, \beta_3, \beta_4) = 0$ the constraining function, we may write

$$\frac{\partial U}{\partial \beta_n} + \lambda \frac{\partial f}{\partial \beta_n} = 0$$

$$f(\beta_1, \beta_2, \beta_3, \beta_4) = 0$$

where $n = 1, 2, 3, 4$

Using (13) and (14) in the above, and the fact that

$$\frac{\partial f}{\partial \beta_1} = \frac{1}{2} \frac{a^2}{R^2} \quad \frac{\partial f}{\partial \beta_2} = \frac{1}{6} \frac{a^2}{R^2} \quad \frac{\partial f}{\partial \beta_3} = \frac{1}{6} \frac{a^2}{R^2} \quad \frac{\partial f}{\partial \beta_4} = \frac{a^2}{R^2}$$

$\frac{\partial U}{\partial \epsilon}$ (strain) = $\frac{\partial U}{\partial \epsilon}$ (stress) ϵ
 is converted into $\frac{\partial U}{\partial \epsilon}$ (stress) ϵ
 the strain energy (15) and performing the partial differentiation
 the following equations are obtained.

$$\left. \begin{aligned}
 \frac{\partial U}{\partial \epsilon_1} &= \left[\frac{\pi}{2} \frac{\partial U}{\partial \epsilon} - \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} \right] + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} \\
 \frac{\partial U}{\partial \epsilon_2} &= \left[\frac{\pi}{2} \frac{\partial U}{\partial \epsilon} - \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} \right] + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} \\
 \frac{\partial U}{\partial \epsilon_3} &= \left[\frac{\pi}{2} \frac{\partial U}{\partial \epsilon} - \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} \right] + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} \\
 \frac{\partial U}{\partial \epsilon_4} &= \left[\frac{\pi}{2} \frac{\partial U}{\partial \epsilon} - \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} \right] + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon} + \frac{\pi}{2} \frac{\partial U}{\partial \epsilon}
 \end{aligned} \right\} (1)$$

To minimize the strain energy we evaluate the strain energy
 by the method of Lagrange multipliers. Let λ be
 used with the constraint function (1) in the method of
 static equilibrium. The constant $\frac{\pi}{2}$ appearing in the partial derivatives
 of the strain energy will be incorporated in the multiplier. Let λ be
 a Lagrange multiplier, and $\phi = \frac{\partial U}{\partial \epsilon} - \lambda \left(\frac{\partial U}{\partial \epsilon} - \frac{\partial U}{\partial \epsilon} \right)$ be the
 we set

$$\phi = \frac{\partial U}{\partial \epsilon} - \lambda \left(\frac{\partial U}{\partial \epsilon} - \frac{\partial U}{\partial \epsilon} \right) = 0$$

Using (1) and (2) in the above, and the first law

$$\frac{\partial \phi}{\partial \epsilon_1} = \frac{\partial U}{\partial \epsilon_1} - \lambda \left(\frac{\partial U}{\partial \epsilon_1} - \frac{\partial U}{\partial \epsilon_1} \right) = 0$$

we arrive at the following set of equations, the solution of which will evaluate $\beta_1, \beta_2, \beta_3, \beta_4$ and determine the stress function.

$$(15) \left\{ \begin{aligned} \left(\frac{3}{2} - \frac{5}{16} \frac{a^2}{R^2} \right) + \frac{2}{3} \beta_1 + \frac{13}{48} \frac{a^2}{R^2} \beta_2 + \frac{1}{16} \frac{a^2}{R^2} \beta_3 + \frac{1}{2} \frac{a^2}{R^2} \beta_4 &= -\frac{\lambda' a^2}{2 R^2} \\ \left(\frac{1}{3} - \frac{1}{4} \frac{a^2}{R^2} \right) + \frac{13}{16} \frac{a^2}{R^2} \beta_1 + \frac{7}{24} \frac{a^2}{R^2} \beta_2 + \frac{1}{24} \frac{a^2}{R^2} \beta_3 + \frac{1}{3} \frac{a^2}{R^2} \beta_4 &= -\frac{\lambda' a^2}{6 R^2} \\ \left(\frac{1}{3} + \frac{1}{12} \frac{a^2}{R^2} \right) + \frac{1}{16} \frac{a^2}{R^2} \beta_1 + \frac{1}{24} \frac{a^2}{R^2} \beta_2 + \frac{7}{24} \frac{a^2}{R^2} \beta_3 + \frac{1}{3} \frac{a^2}{R^2} \beta_4 &= -\frac{\lambda' a^2}{6 R^2} \\ \left(2 + \frac{2}{3} \frac{a^2}{R^2} \right) + \frac{1}{2} \frac{a^2}{R^2} \beta_1 + \frac{1}{3} \frac{a^2}{R^2} \beta_2 + \frac{1}{3} \frac{a^2}{R^2} \beta_3 + 2 \frac{a^2}{R^2} \beta_4 &= -\frac{\lambda' a^2}{R^2} \\ \frac{a^2}{R^2} + \frac{1}{2} \frac{a^2}{R^2} \beta_1 + \frac{1}{6} \frac{a^2}{R^2} \beta_2 + \frac{1}{6} \frac{a^2}{R^2} \beta_3 + \frac{a^2}{R^2} \beta_4 &= 0 \end{aligned} \right.$$

where $\lambda' = \left(\frac{GR}{\pi a_0} \right) \lambda$

Solving for $\beta_1, \beta_2, \beta_3$ and β_4

$$\beta_1 = -\frac{3}{4} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right]$$

$$\beta_2 = \frac{57}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] - \frac{56}{96}$$

$$\beta_3 = \frac{8}{96} - \frac{3}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right]$$

$$\beta_4 = -\frac{88}{96} + \frac{27}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right]$$

The corresponding values of α are

$$\alpha_1 = -\frac{3}{4} \frac{PR}{G\pi a^4} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right]$$

$$\alpha_2 = \frac{PR}{G\pi a^4} \left\{ \frac{57}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] - \frac{56}{96} \right\}$$

$$\alpha_3 = \frac{PR}{G\pi a^4} \left\{ \frac{8}{96} - \frac{3}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] \right\}$$

$$\alpha_4 = \frac{PR}{G\pi a^4} \left\{ -\frac{88}{96} + \frac{27}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] \right\}$$

... ..

$$\left. \begin{aligned}
 & \left(\frac{1}{5} - \frac{1}{15} \frac{q_1^5}{q_1^5} + \frac{1}{15} \frac{q_2^5}{q_1^5} + \frac{1}{15} \frac{q_3^5}{q_1^5} + \frac{1}{15} \frac{q_4^5}{q_1^5} + \frac{1}{15} \frac{q_5^5}{q_1^5} \right) \frac{1}{q_1^5} \\
 & \left(\frac{1}{3} - \frac{1}{4} \frac{q_1^5}{q_1^5} + \frac{1}{10} \frac{q_2^5}{q_1^5} + \frac{1}{10} \frac{q_3^5}{q_1^5} + \frac{1}{10} \frac{q_4^5}{q_1^5} + \frac{1}{10} \frac{q_5^5}{q_1^5} \right) \frac{1}{q_1^5} \\
 & \left(\frac{1}{3} + \frac{1}{4} \frac{q_1^5}{q_1^5} + \frac{1}{10} \frac{q_2^5}{q_1^5} + \frac{1}{10} \frac{q_3^5}{q_1^5} + \frac{1}{10} \frac{q_4^5}{q_1^5} + \frac{1}{10} \frac{q_5^5}{q_1^5} \right) \frac{1}{q_1^5} \\
 & \left(\frac{1}{5} + \frac{1}{5} \frac{q_1^5}{q_1^5} + \frac{1}{5} \frac{q_2^5}{q_1^5} + \frac{1}{5} \frac{q_3^5}{q_1^5} + \frac{1}{5} \frac{q_4^5}{q_1^5} + \frac{1}{5} \frac{q_5^5}{q_1^5} \right) \frac{1}{q_1^5} = 0
 \end{aligned} \right\} (12)$$

$$\chi_1 = \left(\frac{q_1^5}{q_1^5} \right) \chi$$

... ..

$$\left[\begin{array}{c} \frac{21}{21} \\ 1 - \frac{31}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{43}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{55}{15} \frac{q_1^5}{q_1^5} \end{array} \right] \frac{1}{21} = \left[\begin{array}{c} \frac{31}{15} \\ 1 - \frac{43}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{55}{15} \frac{q_1^5}{q_1^5} \end{array} \right] \frac{1}{21}$$

$$\left[\begin{array}{c} \frac{8}{8} \\ 1 - \frac{31}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{43}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{55}{15} \frac{q_1^5}{q_1^5} \end{array} \right] \frac{1}{8} = \left[\begin{array}{c} \frac{31}{15} \\ 1 - \frac{43}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{55}{15} \frac{q_1^5}{q_1^5} \end{array} \right] \frac{1}{8}$$

The corresponding values of α are

$$\alpha_1 = \frac{3}{4} \frac{q_1^5}{q_1^5} \left[\begin{array}{c} 1 - \frac{43}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{55}{15} \frac{q_1^5}{q_1^5} \end{array} \right] \frac{1}{4} \frac{q_1^5}{q_1^5} = \frac{q_1^5}{q_1^5} \frac{q_1^5}{q_1^5} \left[\begin{array}{c} \frac{21}{21} \\ 1 - \frac{31}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{43}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{55}{15} \frac{q_1^5}{q_1^5} \end{array} \right] \frac{1}{21}$$

$$\alpha_2 = \frac{q_1^5}{q_1^5} \frac{q_1^5}{q_1^5} \left[\begin{array}{c} \frac{8}{8} \\ 1 - \frac{31}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{43}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{55}{15} \frac{q_1^5}{q_1^5} \end{array} \right] \frac{1}{8} = \frac{q_1^5}{q_1^5} \frac{q_1^5}{q_1^5} \left[\begin{array}{c} \frac{31}{15} \\ 1 - \frac{43}{15} \frac{q_1^5}{q_1^5} \\ 1 - \frac{55}{15} \frac{q_1^5}{q_1^5} \end{array} \right] \frac{1}{8}$$

Using these results in Equations (12), we may now write expressions representing a second approximation of the stress distribution in a cross-section of the incomplete tore.

$$(16) \left\{ \begin{aligned} \tau_{\rho\theta} &= -\frac{PR}{\pi a^3} \left[\frac{3}{4} \delta (\rho^2 - a^2) \frac{\sin \psi}{R} + \left(\frac{2}{3} + \frac{1}{8} \delta \right) \frac{\sin 2\psi}{R^2} \right] \text{ where } \delta = \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] \\ \tau_{\psi\theta} &= \frac{PR}{\pi a^3} \left\{ \left[\left(\frac{10}{3} - 2\delta \right) \rho^3 + \left(\frac{4}{3} + \frac{1}{4} \delta \right) \rho a^2 \right] \frac{\cos^2 \psi}{R^2} + \left[\left(4 - \frac{9}{4} \delta \right) \rho^2 + \left(\frac{3}{4} \delta \right) a^2 \right] \frac{\cos \psi}{R} + \left[\left(\frac{1}{3} - \frac{1}{8} \delta \right) \rho^3 - \left(2 - \frac{5}{8} \delta \right) \rho a^2 \right] \frac{1}{R^2} + 2\rho \right\} \end{aligned} \right.$$

At the point of maximum stress, where $\rho = a$ and $\psi = 0$, the above reduce to

$$(17) \left\{ \begin{aligned} \tau_{\rho\theta} &= 0 \\ \left[\tau_{\psi\theta} \right]_{\max.} &= \frac{2PR}{\pi a^3} \left[1 + \left(2 - \frac{3}{4} \delta \right) \frac{a}{R} + \left(\frac{3}{2} - \frac{5}{8} \delta \right) \frac{a^2}{R^2} \right] \end{aligned} \right.$$

As is apparent from the foregoing development, further approximations utilizing additional terms in the stress function will result in extremely long and tedious calculations. This in itself is a limitation of this method. Therefore at this point, assuming the solution to be a rapidly converging one, we will stop and introduce actual values of the ratio of cross-sectional radius to the mean radius of curvature of the tore in order to compare results with other solutions.

Being these results in Equation (11), we may now write the expression corresponding to a second approximation of the wave function in a similar manner as the first approximation.

$$\left. \begin{aligned} \psi_0 &= -\frac{PR}{\pi\alpha^2} \left[\frac{3}{4} (p^2 - \alpha^2) \left(\frac{2i\alpha^2}{R} + \frac{1}{2} + \frac{1}{8} \right) + \frac{2i\alpha^2}{R} \psi \right] \\ \psi_1 &= \frac{PR}{\pi\alpha^2} \left[\left(\frac{10}{3} - 2\alpha^2 \right) p^3 + \left(\frac{1}{3} + \frac{1}{4} \right) \alpha^2 \left(\frac{2i\alpha^2}{R} + \frac{1}{2} + \frac{1}{8} \right) \left[4 - \frac{1}{2} \alpha^2 \right] p^2 + \left(\frac{3}{2} \alpha^2 \right) \alpha^2 \left[\frac{2i\alpha^2}{R} + \frac{1}{2} + \frac{1}{8} \right] \cos \psi + \left[\frac{1}{2} - \frac{1}{8} \alpha^2 \right] p^3 - \left(\frac{2}{8} \alpha^2 \right) \alpha^2 \left[\frac{2i\alpha^2}{R} + \frac{1}{2} + \frac{1}{8} \right] \right] \end{aligned} \right\} (12)$$

In the limit of infinite waves, where $\psi = 0$, the above

$$\left. \begin{aligned} \psi_0 &= 0 \\ \psi_1 &= \frac{PR}{\pi\alpha^2} \left[1 + \left(2 - \frac{3}{4} \alpha^2 \right) \frac{1}{R} + \left(\frac{3}{2} - \frac{1}{8} \alpha^2 \right) \frac{2i\alpha^2}{R} \right] \end{aligned} \right\} (13)$$

as is apparent from the foregoing development, further approximations utilizing additional terms in the wave function will result in extremely long and tedious calculations. This in itself is a justification of this method. Therefore at this point, assuming the solution to be a rapidly converging one, we will stop and introduce actual values of the ratio of cross-sectional areas to the mean radius of curvature of the face in order to compare results with other solutions.

RESULTS

The distribution of shearing stress on a horizontal diameter is shown in Fig. 5 and the circumferential stress distribution in Fig. 6. In both cases a ratio of R/a equal to 4 is used since this realizes the worst condition (i.e. greatest curvature for a given cross-section) of any practical significance.

The quantity \underline{S} appearing as the ordinate in both curves is a dimensionless quantity and is equal to $\frac{\tau_{\psi\theta} \pi a^2}{P}$, since $\tau_{\theta e}$ vanishes on both a horizontal diameter and the periphery. A stress distribution representing pure torsion of a straight circular shaft is shown by a dotted line in both figures. It is seen that the maximum stress actually existing in the core is considerably greater than that derived from ordinary torsion theory. Both curves are in good agreement with similar ones derived from the exact solution by Frieberger. Points from Frieberger's curves appear as the small circles in Fig. 5 and 6.

In comparing the results of this solution with others, namely Göhner, Wahl, and Frieberger, the point of maximum stress will be used as a reference with different values of R/a . Table 1 gives values of \underline{K} in the expression

$$[\tau_{\psi\theta}]_{\max} = \frac{2PR}{\pi a^3} [K] \quad \text{for the several solutions.}$$

Table 1.

R/a	Exact	This Solution		Other Approx. Solutions	
	Frieberger	1st Approx.	2nd Approx.	Göhner	Wahl
4	1.376	1.313	1.371	1.372	1.400
5	1.293	1.250	1.287	1.295	1.310
6	1.237	1.209	1.234	1.239	1.252
8	1.171	1.156	1.171	1.172	1.184
10	1.136	1.125	1.134	1.135	1.145

TABLE I

The distribution of bending stress in a rectangular beam is shown in Fig. 1 and the corresponding stress distribution in Fig. 2. In both cases a value of σ is given at the neutral axis which is the same as the greatest maximum (or a given stress-value) of any particular situation.

The quantity $\frac{1}{q}$ depending on the modulus in both cases is a dimensionless quantity and is equal to $\frac{1}{q} = \frac{1}{\sigma} \frac{d\sigma}{dy}$; also $\frac{1}{q} = \frac{1}{\sigma} \frac{d\sigma}{dy}$. A stress distribution corresponding to a certain distance y is shown by a dashed line in both figures. It is seen that the maximum stress normally existing in the beam is considerably greater than that derived from ordinary bending theory. With stresses up to yield stresses and also similar and higher than the yield stresses by stretching, strains from stretching curves appear as the same as in Fig. 1 and 2. In computing the results of this relation with stress, namely σ versus y , and stretching, the point of maximum stress will be used as a reference with different values of σ . Table I gives values of $\frac{1}{q}$ in the Appendix.

For the general equation
$$\left[\frac{1}{q} \right] = \frac{\sigma}{E} \left[\frac{1}{\sigma} \frac{d\sigma}{dy} \right]$$

TABLE I

n	Stress distribution		Stress distribution	
	at neutral axis	at surface	at surface	at neutral axis
1	1.170	1.171	1.171	1.170
2	1.192	1.193	1.193	1.192
4	1.237	1.239	1.239	1.237
6	1.271	1.274	1.274	1.271
10	1.316	1.321	1.321	1.316

Table 1. indicates that the energy method applied to this problem produces results which compare favorably with other solutions. It also appears that the solution converges rapidly, since only five terms were used in the stress function.

The following table shows the results of the experiments conducted on the effect of the concentration of the solution on the rate of reaction. The rate of reaction was measured by the volume of gas evolved per unit time.

The results show that the rate of reaction increases with the concentration of the solution. This is because a higher concentration of the solution means there are more particles available to react, leading to a higher frequency of collisions and thus a faster rate of reaction.

Table 1: Effect of concentration on the rate of reaction.

Concentration (M)	Time (s)	Volume of gas (cm ³)	Rate of reaction (cm ³ /s)
0.1	100	10	0.1
0.2	50	10	0.2
0.3	33	10	0.3
0.4	25	10	0.4
0.5	20	10	0.5

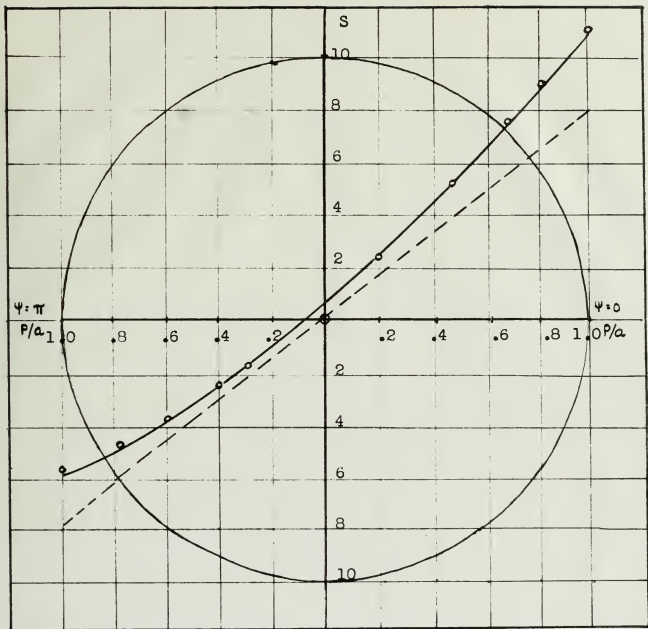


Fig. 5.

Distribution of shearing stress on a horizontal diameter for $a/R=1/4$. ($\psi=0, \pi$). $S=(\tau_{\psi 0})(\pi a^2)/P$. Frieberger's points are indicated by small circles.

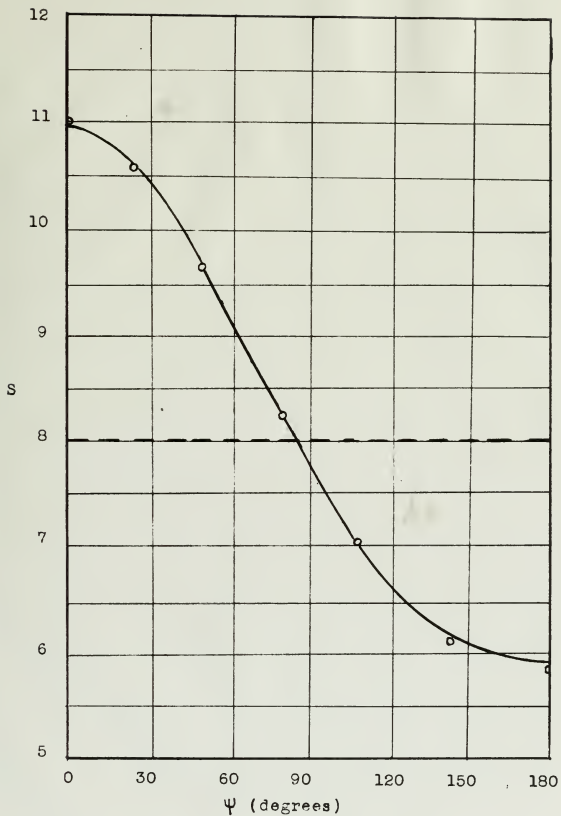


Fig. 6.

Circumferential stress distribution for $a/R=1/4$. $S=(\tau_{\psi\theta})(\pi a^2)/P$. Frieberger's points are indicated by small circles.

CONCLUSIONS

In discussing any conclusions from this investigation, it would be appropriate to recall the two questions that prompted it.

- (1) Can the problem be solved by this method, and how do the results compare with those of other solutions?
- (2) Does the problem particularly lend itself to solution by energy methods?

First, the method will work and acceptable results are obtained with a relatively few terms in the stress function. This is in itself worthy of note, since it allows a very complex problem to be attacked by the more elementary methods of mathematics.

However, in reference to the second question, there are limitations both inherent in the energy method and peculiar to this particular application, that strongly indicate the problem is not especially adapted to a solution by energy methods.

The energy method, except in unusual circumstances, does not provide an exact solution. Consequently, in the absence of an exact solution, there is no real basis for judging the results. The fact also that the energy method requires minimizing an integral, which is done only with extreme difficulty with any number of terms in the stress function, is a limitation to its adaptability.

In conclusion then, it may be said that this solution has the value of arriving at very good results using a relatively uncomplicated stress function of only five terms.

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methods of mathematics.

However, in reference to the second question, there are limitations both inherent in the energy method and peculiar to this particular application, that strongly indicate the problem is not especially adapted to a solution by energy methods.

The energy method, except in unusual circumstances, does not provide an exact solution. Consequently, in the absence of an exact solution, there is no real basis for judging the results. The fact also that the energy method requires

maintaining an integral, which is done only with extreme difficulty when the number of terms in the stress function, is a limitation as to applicability.

It concludes that, if not to solve the problem has the value of results is very good results using a relatively uncomplicated stress function of only five terms.

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APPENDIX A

According to the principle of least work which is used in this solution, an exact stress function would require selecting from all functions that satisfy the boundary condition those which minimize the strain energy.

Since in general this procedure is too difficult, a limited number N of suitable functions was selected to determine an approximate stress function.

In choosing functions of ρ and ψ in $\phi = \sum_{i=0}^N \alpha_i \phi_i$, the first consideration was the boundary condition $\phi_i = 0$ when $\rho = a$. This condition was satisfied by taking each ϕ_i to contain the factor $(\rho^2 - a^2)$.

Then
$$\phi = (\rho^2 - a^2) \sum_{i=0}^N \alpha_i f_i(\rho, \psi)$$

With rectangular coordinates (ξ, η) in mind, where $\xi = \rho \cos \psi$ and $\eta = \rho \sin \psi$, the next logical step was to express $\sum f_i(\xi, \eta)$ in a power series. The first six terms of such a series were considered, namely those involving

$$1, \xi, \eta, \xi^2, \eta^2, \xi\eta$$

Since a horizontal diameter ($\eta = 0$) on a plane cross-section (θ is a constant) is an axis of symmetry for the ϕ surface, ϕ must be even in η and not contain terms involving odd powers of η . In general ξ will appear to all powers since $(\xi = 0)$ is not axis of symmetry. Therefore the remaining terms expressed as functions of ρ and ψ are

$$1, \rho \cos \psi, \rho^2 \cos^2 \psi, \rho^2 \sin^2 \psi$$

The term $\frac{\rho^2}{R^2}$ in ϕ_4 while not consistent with this line of reasoning, appeared as a result of the binomial expansion used in approximating the stresses. It was extracted from G6hners solution where a like approximation was used.

APPENDIX I

According to the principle of least work which is used in this solution, an exact stress function would require selecting from all functions that satisfy the boundary condition those which minimize the strain energy.

Since in general the stress function is too difficult to find, a limited number of suitable functions was selected for derivation an approximate stress function.

in choosing functions of ρ and ψ in the form

$$\phi = \sum_{j=0}^n a_j \rho^j \psi^j$$

the boundary condition was the boundary condition

to contain the factor ϕ to contain the factor $(\rho^2 - a^2)$

$$\phi = \sum_{j=0}^n a_j (\rho^2 - a^2)^j \psi^j$$

With rectangular coordinates (r, θ) in which $\rho = r \cos \theta$ and $\psi = r \sin \theta$, the next logical step was to express

$$\phi = \sum_{j=0}^n a_j (r^2 - a^2)^j \cos^j \theta \sin^j \theta$$

series. The first six terms of such a series were determined, namely those

involving

$$1, \cos^2 \theta, \sin^2 \theta, \cos^4 \theta, \sin^4 \theta, \cos^2 \theta \sin^2 \theta$$

Since a horizontal diameter $(\theta = 0)$ or a plane cross-section $(\theta = \pi/2)$ is an axis of symmetry for the ϕ surface, ϕ must be even in θ and

the terms involving odd powers of θ in general will appear in all

powers since $(\theta = 0)$ is not axis of symmetry. Therefore the remaining terms

expressed as functions of ρ and ψ are

$$\phi = a_0 + a_1 \rho^2 + a_2 \psi^2 + a_3 \rho^4 + a_4 \psi^4 + a_5 \rho^2 \psi^2$$

The form $\phi = \sum_{j=0}^n a_j \rho^j \psi^j$ was not considered but this line of reasoning

appeared as a result of the binomial expansion used in representing the stress.

It was expected that there would be a line of symmetry and hence

Consequently the stress function was taken in the form

$$\phi = (\rho^2 - a^2) \left[\alpha_0 + \alpha_1 \left(\frac{\rho}{R} \right) \cos \psi + \alpha_2 \left(\frac{\rho}{R} \right)^2 \cos^2 \psi + \alpha_3 \left(\frac{\rho}{R} \right)^2 \sin^2 \psi + \alpha_4 \frac{a^2}{R^2} \right]$$

Here ρ was replaced by $\frac{\rho}{R}$ so that α_i would in all cases be the product of a dimensionless number and the factor $\frac{PR}{G\pi a^4}$.

In the first approximation the first two terms were used and in the second all five were introduced in ϕ .

Consequently the stress function was taken in the form

$$\phi = (a_0 + a_1 \frac{r}{R} + a_2 \left(\frac{r}{R}\right)^2) \cos \psi + a_3 \left(\frac{r}{R}\right)^3 \cos^2 \psi + a_4 \left(\frac{r}{R}\right)^4 \cos^3 \psi + a_5 \left(\frac{r}{R}\right)^5 \cos^4 \psi$$

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the product of a dimensionless number and the factor $\frac{PR}{Gh\alpha^2}$.

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all five were introduced in ϕ .

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$$\phi = (a_0 + a_1 \frac{r}{R} + a_2 \left(\frac{r}{R}\right)^2) \cos \psi + a_3 \left(\frac{r}{R}\right)^3 \cos^2 \psi + a_4 \left(\frac{r}{R}\right)^4 \cos^3 \psi + a_5 \left(\frac{r}{R}\right)^5 \cos^4 \psi$$

the stress function was taken in the form

$$\phi = (a_0 + a_1 \frac{r}{R} + a_2 \left(\frac{r}{R}\right)^2) \cos \psi + a_3 \left(\frac{r}{R}\right)^3 \cos^2 \psi + a_4 \left(\frac{r}{R}\right)^4 \cos^3 \psi + a_5 \left(\frac{r}{R}\right)^5 \cos^4 \psi$$

the stress function was taken in the form

$$\phi = (a_0 + a_1 \frac{r}{R} + a_2 \left(\frac{r}{R}\right)^2) \cos \psi + a_3 \left(\frac{r}{R}\right)^3 \cos^2 \psi + a_4 \left(\frac{r}{R}\right)^4 \cos^3 \psi + a_5 \left(\frac{r}{R}\right)^5 \cos^4 \psi$$

APPENDIX B

The appearance of the term $\frac{1}{(R - \rho \cos \psi)^2}$ in the exact equations relating the stresses and the stress function gives rise to the occurrence of integrals of the type $\int_0^a \int_0^{2\pi} \frac{\rho^m \sin^n \psi \cos^q \psi \, \rho \, d\rho \, d\psi}{(R - \rho \cos \psi)^2}$ and $\int_0^a \int_0^{2\pi} \frac{\rho^m \sin^n \psi \cos^q \psi \, \rho \, d\rho \, d\psi}{(R - \rho \cos \psi)^3}$ in evaluating the strain energy and in consideration of the conditions of static equilibrium.

Taking the simplest form of the first type, where $m=1$, $n=0$ and $q=0$ we have
$$\int_0^a \int_0^{2\pi} \frac{\rho \, d\rho \, d\psi}{(R - \rho \cos \psi)^2}$$

Integrating first with respect to ψ and setting $R = c$ and $-\rho = b$

$$\int \frac{d\psi}{(c + b \cos \psi)^2}$$

Letting

$$P = \frac{\sin \psi}{(c + b \cos \psi)^2}$$

Then

$$\begin{aligned} \frac{dP}{d\psi} &= \frac{\cos \psi (c + b \cos \psi) + b(1 - \cos^2 \psi)}{(c + b \cos \psi)^2} = \frac{b + c \cos \psi}{(c + b \cos \psi)^2} \\ &= \frac{b - \frac{c^2}{b} + \frac{c}{b}(c + b \cos \psi)}{(c + b \cos \psi)^2} = \frac{c}{b} \left(\frac{1}{c + b \cos \psi} \right) - \frac{c^2 - b^2}{b} \left[\frac{1}{(c + b \cos \psi)^2} \right] \end{aligned}$$

Multiplying by $d\psi$ and integrating

$$\begin{aligned} \int \frac{dP}{d\psi} d\psi = P &= \frac{\sin \psi}{c + b \cos \psi} = \frac{c}{b} \int \frac{d\psi}{c + b \cos \psi} - \frac{c^2 - b^2}{b} \int \frac{d\psi}{(c + b \cos \psi)^2} \\ \int \frac{d\psi}{c + b \cos \psi} &= -\frac{b}{c^2 - b^2} \left(\frac{\sin \psi}{c + b \cos \psi} \right) + \frac{c}{c^2 - b^2} \int \frac{d\psi}{c + b \cos \psi} \\ &= \frac{1}{\sqrt{c^2 - b^2}} \cos^{-1} \left(\frac{b + c \cos \psi}{c + b \cos \psi} \right) \quad \text{where } c^2 > b^2 \end{aligned}$$

APPENDIX 2

The appearance of the term $\frac{1}{(R - \rho \cos \psi)^2}$ in the stress function requires the stresses and the stress function gives rise to the occurrence of integrals of the type $\int_0^\pi \frac{\rho^{2n} \cos^{2n} \psi \, d\psi}{(R - \rho \cos \psi)^2}$ and $\int_0^\pi \frac{\rho^{2n} \sin^{2n} \psi \, d\psi}{(R - \rho \cos \psi)^2}$ in evaluating the strain energy and in consideration of the condition of static equilibrium.

Taking the simplest form of the first type, where $m=1$, $n=0$ and $\delta=0$

we have
$$\int_0^\pi \frac{\rho \, d\psi}{(R - \rho \cos \psi)^2}$$

Integrating first with respect to ψ and setting $R = c$ and $-\rho = -d$

$$\int \frac{d\psi}{(c + d \cos \psi)^2}$$

Letting
$$\frac{2 \sin \psi}{c + d \cos \psi} = \eta$$

Then

$$\frac{d\psi}{(c + d \cos \psi)^2} = \frac{(\psi \cos \psi (c + d \cos \psi) + d(1 - \cos^2 \psi))}{(c + d \cos \psi)^2} = \frac{d + c \cos \psi}{(c + d \cos \psi)^2}$$

$$= \frac{d}{(c + d \cos \psi)^2} + \frac{c}{d} \left(\frac{1}{c + d \cos \psi} - \frac{1}{d} \right) = \frac{d}{(c + d \cos \psi)^2} + \frac{c}{d} \left(\frac{1}{c + d \cos \psi} - \frac{1}{d} \right)$$

Multiplying by $\rho \psi$ and integrating

$$\int \frac{\rho \psi \, d\psi}{(c + d \cos \psi)^2} = \int \frac{\rho \psi \, d\psi}{c + d \cos \psi} + \frac{c}{d} \int \frac{\rho \psi \, d\psi}{c + d \cos \psi} - \frac{c}{d} \int \rho \psi \, d\psi$$

$$= \int \frac{\rho \psi \, d\psi}{c + d \cos \psi} + \frac{c}{d} \int \frac{\rho \psi \, d\psi}{c + d \cos \psi} - \frac{c}{d} \int \rho \psi \, d\psi$$

$$= \frac{1}{\sqrt{c^2 - d^2}} \cos^{-1} \left(\frac{d + c \cos \psi}{c + d \cos \psi} \right) + \frac{c}{d} \int \frac{\rho \psi \, d\psi}{c + d \cos \psi} - \frac{c}{d} \int \rho \psi \, d\psi$$

Therefore

$$\int \frac{d\psi}{(c + b \cos \psi)^2} = -\frac{b}{c^2 - b^2} \left(\frac{\sin \psi}{c + b \cos \psi} \right) + \frac{c}{(c^2 - b^2)^{3/2}} \cos^{-1} \left[\frac{b + c \cos \psi}{c + b \cos \psi} \right]$$

Introducing the limits 0 and 2π , this reduces to

$$\frac{2\pi c}{(c^2 - b^2)^{3/2}} \quad \text{where} \quad \begin{array}{l} c = R \\ b = -\rho \end{array}$$

Therefore

$$\begin{aligned} \int_0^a \int_0^{2\pi} \frac{\rho d\rho d\psi}{(R - \rho \cos \psi)^2} &= 2\pi R \int_0^a \frac{\rho d\rho}{(R^2 - \rho^2)^{3/2}} = 2\pi R \left[\frac{1}{(R^2 - \rho^2)^{1/2}} \right]_0^a \\ &= 2\pi \left[\frac{R}{(R^2 - a^2)^{1/2}} - 1 \right] \end{aligned}$$

The other more complicated forms where $n \neq 0$ and $g \neq 0$ are integrable in finite terms by similar reduction methods, but it is apparent that the work becomes excessively involved. Also the results in the form just developed are not readily usable in evaluating the unknown coefficients in the stress function.

In view of the foregoing, despite the fact that it was not actually necessary, it was expedient to approximate the stresses in such a manner that the integration was simplified and the results put in a usable form.

This device of approximating the stress equations compromised the requirement that the stresses satisfy the equations of equilibrium. However, it appears that, since the stresses do satisfy the conditions of minimum strain energy and static equilibrium, and give satisfactory results, the compromise may be tolerated.

$$\left[\frac{\psi + b \cos \psi}{c + d \cos \psi} \right]^{1/n} \cos \psi \sqrt{c^2 - d^2} + \left(\frac{\psi + b \cos \psi}{c + d \cos \psi} \right)^{1/n} \frac{d}{c^2 - d^2} = \frac{\psi + b \cos \psi}{c + d \cos \psi} \sqrt{c^2 - d^2}$$

Integrating the first 0 and 2π , this becomes

$$\frac{2\pi c}{(c^2 - d^2)^{1/2}} = \frac{2\pi R}{(R^2 - b^2)^{1/2}}$$

$c = R$
 $d = -b$

$$\int_0^{2\pi} \left[\frac{\psi + b \cos \psi}{c + d \cos \psi} \right]^{1/n} \cos \psi \sqrt{c^2 - d^2} d\psi = \int_0^{2\pi} \left[\frac{\psi + b \cos \psi}{c + d \cos \psi} \right]^{1/n} \frac{d}{c^2 - d^2} d\psi = \int_0^{2\pi} \left[\frac{\psi + b \cos \psi}{c + d \cos \psi} \right]^{1/n} \sqrt{c^2 - d^2} d\psi$$

$$\left[\frac{R}{(R^2 - b^2)^{1/2}} \right]^{1/n} = \left[\frac{R}{(R^2 - b^2)^{1/2}} \right]^{1/n}$$

The other two equations have where $\psi = 0$ and $\psi = 2\pi$ the integrals

in finite form by similar reduction methods, but it is apparent that the work

becomes excessively involved. Also the results in the first two integrals are

not readily usable in evaluating the unknown coefficients in the stress function,

in view of the fact that, besides the fact that it is not readily integrable,

it was essential to approximate the integrals in such a way as to have the integrals

we simplify and the results are in a simple form.

This device of approximating the stress equations by means of the boundary

and the stresses satisfy the conditions of equilibrium. However, it appears that

since the stresses do satisfy the conditions of equilibrium, the stress function, and since

equilibrium, the stress function, and the boundary conditions are satisfied.

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