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Fargues, Monique P.

Monterey, California. Naval Postgraduate School

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BOUNDS ON THE EXTREME GENERALIZED EIGENVALUES OF HERMITIAN PENCILS

by

Monique P. Fargues

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Known results based on the generalization of the Gershgorin theorem and norm inequalities are presented and compared to the proposed bounds. It is shown that the new bounds compare favorably with these known results; they are easier to compute, require less restrictions on the properties of the pencils studied, and they are in an average sense tighter than those obtained with the norm inequality bounds.

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December 1990

Monique P. Fargues

Electrical and Computer Engineering Department
Naval Postgraduate School
Monterey, CA 93943
on Research supported by the Naval Postgraduate School.

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Abstract

We present easily computable bounds on the extreme generalized eigenvalues of Hermitian pencils (R_{n+1}, B_{n+1}) with finite eigenvalues, and positive definite B_{n+1} matrices. The proposed bounds are derived in terms of the generalized eigenvalues of the subpencil of maximum dimension (R_n, B_n) contained in (R_{n+1}, B_{n+1}).

Known results based on the generalization of the Gershgorin theorem and norm inequalities are presented and compared to the proposed bounds. It is shown that the new bounds compare favorably with these known results; they are easier to compute, require less restrictions on the properties of the pencils studied, and they are in an average sense tighter than those obtained with the norm inequality bounds.

1.0 Introduction

The Hermitian (regular or generalized) eigenproblem occurs in a variety of applications in signal processing. It is commonly encountered in array processing [1,2,3], in spectral estimation [4], filtering [5], and other areas. Different bounds for the extreme eigenvalues of the regular Hermitian problems have been presented in the literature [6,7]. Some of them have then been extended to the generalized Hermitian eigenproblem by either backtransforming the generalized problem into the corresponding eigenproblem, or by generalizing the results originally derived for the regular eigenproblem [8]. Classical bounds derived using norm inequalities can be extended to the generalized positive definite eigenproblem by backtransforming the pencil (R,B) into a regular problem $(C^{-1}RC^{-*},I)$ where $B = CC^*$. However, such a transformation requires the Choleski decomposition of B. The generalization of the Gershgorin theorem proposed by Stewart [8] does not have such a restriction, but the tightness of the bounds depends strongly on the characteristics of the pencils under study.

Here we present new bounds for the extreme generalized eigenvalues based on an order-recursive eigenproblem decomposition. This work can be considered as an extension of the ideas of Slepian et al [9] and Dembo [10] who considered the regular eigenproblem. The original idea behind the following work is connected to the derivation of the order-recursive RITE [11] and C-RITE [12] algorithms. These algorithms take advantage of the interlacing property [6]:

$$\lambda_{1,n+1} \leq \lambda_{1,n} \leq \lambda_{2,n+1} \leq \cdots \leq \lambda_{n+1,n+1}$$

where $\lambda_{i,n+1}$ is associated with the $(n+1)^{th}$ dimensional pencil (R_{n+1}, B_{n+1}) and $\lambda_{i,n}$ is associated with the n-dimensional subpencil (R_n, B_n) contained in (R_{n+1}, B_{n+1}). This property allows us to define intervals in which $\lambda_{2,n+1}, \ldots, \lambda_{n,n+1}$ may be found via iterative search techniques [12,13]. However, the interlacing property does not provide an upper bound on the largest generalized eigenvalue or a lower bound on the smallest generalized eigenvalue. The proposed bounds on the extreme eigenvalues take advantage of the information available at the previous order (assumed to be known), and are easy to compute.

2.0 Derivation of the new bounds

Let (R_{n+1}, B_{n+1}) be a (n + 1)-dimensional Hermitian pencil with finite eigenvalues. Let us assume that the generalized eigenvalues of (R_n, B_n) are known. The eigenvalues λ associated with the pencil satisfy the relation:

$$\det(R_{n+1} - \lambda B_{n+1}) = 0 \tag{1}$$

which can be expressed as:

$$\det\left(\begin{bmatrix} r_0 - \lambda b_0 & r^* - \lambda b^* \\ r - \lambda b & R_n - \lambda B_n \end{bmatrix}\right) = 0 \tag{2}$$

Therefore, using [6] the determinant of the extended pencil (R_{n+1},B_{n+1}) may be expressed as:

$$\begin{aligned} \mathrm{DET} &= \det\left[(R_n - \lambda B_n)\left(r_0 - \lambda b_0 - (s - \lambda q)^* (\Lambda_n - \lambda I)^{-1}(s - \lambda q)\right)\right] \\ &= \det[R_n - \lambda B_n]\det\left[r_0 - \lambda b_0 - (s - \lambda q)^* (\Lambda_n - \lambda I)^{-1}(s - \lambda q)\right] \end{aligned} \tag{3}$$

with $s = U_n^* r$, and $q = U_n^* b$, where $U_n = [u_1, \dots, u_n]$ is the B-orthonormalized eigenvector matrix associated with (R_n, B_n). The eigenvalue search function $h(\lambda)$ is defined as:

$$h(\lambda) = \frac{\mathrm{DET}}{\det[R_n - \lambda B_n]} \tag{4}$$

Expanding (4) leads to:

$$h(\lambda) = (r_0 - \lambda b_0) - \sum_{k=1}^{n} \frac{|\beta_k|^2}{\lambda_{k,n} - \lambda} \tag{5}$$

with $\beta_k = (s_k - \lambda q_k)$, where $s_k = u_k^* r$ and $q_k = u_k^* b$. The zeros of (5) are the generalized eigenvalues of the increased order pencil (R_{n+1}, B_{n+1}). The function $h(\lambda)$ is monotone decreasing between its poles, as shown in Appendix A. Note that similarly to the regular eigenproblem [9,14], $h(\lambda)$ fails to have (n + 1) real roots only when it has less than n distinct poles. This happens when (R_{n+1}, B_{n+1}) has multiple eigenvalues, or when $s_k = q_k = 0$ for some k. Slepian et al [6] indicated that three possible situations related to the multiple eigenvalue case can occur for the regular eigenproblem. These comments can be extended to the generalized problem. Let $\lambda_{m,n}$ be an eigenvalue of (R_n, B_n) with multiplicity k. If $(s_p, q_p) = (0,0)$ for $p = m, \ldots, m+k$, and $\lambda_{m,n}$ is not a root of $h(\lambda)$, then $\lambda_{m,n}$ is an eigenvalue of multiplicity k for the (n + 1) dimensional pencil. If $(s_p, q_p) = (0,0)$ for $p = m, \ldots, m+k$, and $\lambda_{m,n}$ is a root of $h(\lambda)$, then $\lambda_{m,n}$ is an eigenvalue of multiplicity $k+1$ for the $(n+1)^{th}$ dimensional pencil. Finally, if $(s_p, q_p) \neq (0,0)$ for some p where $m \leq p \leq m+k$, then $\lambda_{m,n}$ is an eigenvalue of multiplicity $k-1$ for (R_{n+1}, B_{n+1}).

The idea now is to find a lower bound on $\lambda_{1,n+1}$ and a higher bound on $\lambda_{n+1,n+1}$ by approximating the rational portion of the eigenvalue search function. To that end we note that:

$$\sum_{k=1}^{n} \frac{|\beta_k|^2}{\lambda_{k,n} - \lambda} \leq G_{\min}(\lambda) \triangleq \sum_{k=1}^{n} \frac{|\beta_k|^2}{\lambda_{1,n} - \lambda} \tag{6a}$$

and

$$\sum_{k=1}^{n} \frac{|\beta_k|^2}{\lambda_{k,n} - \lambda} \geq G_{\max}(\lambda) \triangleq \sum_{k=1}^{n} \frac{|\beta_k|^2}{\lambda_{n,n} - \lambda} \tag{6b}$$

Thus from (6) we get

$$h(\lambda) \geq h_{\min}(\lambda) \triangleq r_0 - \lambda b_0 - G_{\min}(\lambda) \quad \text{for} \quad (-\infty, \lambda_{1,n+1}) \tag{7a}$$

$$h(\lambda) \leq h_{\max}(\lambda) \triangleq r_0 - \lambda b_0 - G_{\max}(\lambda) \quad \text{for} \quad (\lambda_{n,n}, \infty) \tag{7b}$$

As shown in Appendix A, the function $h_{\min}(\lambda)$ is monotone decreasing in $(-\infty, \lambda_{1,n})$, and $h_{\max}(\lambda)$ is monotone decreasing in $(\lambda_{n,n}, \infty)$. Thus, $h_{\min}(\lambda)$ has a root $\lambda_{\min}$ in the interval $(-\infty, \lambda_{1,n})$ such that $\lambda_{\min} \leq \lambda_{1,n+1}$, as illustrated in Figure 1. Similarly, $h_{\max}(\lambda)$ has a root $\lambda_{\max}$ in the interval $(\lambda_{n,n}, \infty)$ such that $\lambda_{\max} \geq \lambda_{n+1,n+1}$. The roots $\lambda_{\min}$ and $\lambda_{\max}$ can easily be computed by solving for the roots of the second order polynomials:

$$\begin{aligned} h_\eta(\lambda) &= (r_0 - \lambda b_0)(\eta - \lambda) - |s - \lambda q|^2 \\ &= \lambda^2\left(m_0 - |q|^2\right) + \lambda\left(2\text{real}(s^* q) - r_0 - m_0\eta\right) + r_0\eta - |s|^2 = 0 \end{aligned} \tag{8}$$

for $\eta = \lambda_{1,n}$ or $\eta = \lambda_{n,n}$.

Note that for the regular eigenproblem, where $B = I$, (8) becomes:

$$\lambda^2 m_0 + \lambda(-r_0 - m_0\eta) + r_0\eta - |s|^2 = 0 \tag{9}$$

which is the same expression as the one obtained by Slepian et al [9] and by Dembo [10].

3.0 Comparisons with known bounds

This section first reviews two types of known bounds on the extreme generalized eigenvalues of pencils, and next presents some comparisons of the proposed bounds with classical results based on norm inequalities.

The generalized Gershgorin theorem

Stewart [8] derived a generalization of the Gershgorin theorem and showed that the generalized eigenvalues of $Rx = \lambda Bx$ lie in the union of the neighborhoods $\tilde{G}_i$ defined as:

$$\tilde{G}_i = \left\{ \lambda : \chi\left(\frac{r_{ii}}{b_{ii}}, \lambda\right) \leq \rho_i \right\}$$

where

$$\rho_i \stackrel{\Delta}{=} \frac{\| r_{ii}\beta_i^* - b_{ii}\alpha_i^* \|_1}{\sqrt{|r_{ii}|^2 + |b_{ii}|^2}\,\sqrt{r'^2_{ii} + b'^2_{ii}}} \tag{10}$$

with

$$\begin{aligned} \alpha_i^* &= (r_{i1}, \ldots, r_{i,i-1}, r_{i,i+1}, \ldots, r_{i,n}),\ \beta_i^* = (b_{i1}, \ldots, b_{i,i-1}, b_{i,i+1}, \ldots, b_{i,n}) \\ r'_{ii} &= \max\left\{0,\ |r_{ii}| - \|\alpha_i\|_1\right\} \\ b'_{ii} &= \max\left\{0,\ |b_{ii}| - \|\beta_i\|_1\right\} \end{aligned} \tag{11}$$

$\chi(\lambda, \lambda')$ is the chordal distance between λ and λ'. It is defined as the length of the chord joining the points a and b located on the Riemann sphere [15,16], as shown in Figure 2.

Several comments can be made here:

1. A tighter bound may be obtained by replacing in ρ_i norm 1 with norm 2, as shown in Appendix B.

2. A finite value for ρ_i is obtained only when $(r'_{ii}, b'_{ii}) = (0,0)$, or when $(r_{ii}, b_{ii}) = (0,0)$. Note that $(r'_{ii}, b'_{ii}) \neq (0,0)$, requires that at least one of the matrices of the pencil (R,B) to be diagonally dominant[1] . Thus, ρ_i may be infinite when at least one of the matrices of the pencil studied (R,B) is not diagonally dominant and no restrictions on the diagonal elements of the pencil are made. This indicates that the bounds obtained via the Generalized Gershgorin (G.G.) theorem are not insured to be finite in all situations where the pencil has finite eigenvalues.

3. The chordal distance $\chi(\lambda, \lambda')$ has a maximum value of 1 [16]. Thus, a finite Gershgorin neighborhood is obtained only when $\rho_i < 1$. The value of the parameter ρ_i defined in (10) depends on the pencil (R,B), it can have values larger than 1 even when the pencil eigenvalues are finite. In such a case, the regions $\tilde{G}_i$ containing the eigenvalues cover the whole space, and no new information is gained by applying the G.G. theorem.

The above comments indicate that, when dealing with pencils with finite eigenvalues, additional information can be gained from the G. G. only when $\rho_i < 1$ for all i. This further restricts the usefulness of the G.G. theorem. By comparison, the proposed bounds are limited only to Hermitian pencils (R,B) with positive definite B matrices. Furthermore, the bounds are insured to be finite when the original pencil has finite eigenvalues to start with. Therefore, the G.G. bounds will not be used in the following statistical comparisons because they require too many restrictions on the pencils studied in order to bring additional information.

Norm Inequality bounds

Bounds on the extreme eigenvalues of the regular eigenproblem $Ax = \lambda x$ based on norm inequalities have been proposed [6,7]. Recall that for such bounds, we have:

---

[1] The matrix R is diagonally dominant if $|r_{ii}| > \sum_{j \neq i} |r_{ij}|$ for all i.

$$
\begin{aligned}
&\lambda_{n+1,n+1} \leq \|A\|_F \\
&\lambda_{n+1,n+1} \leq \|A\|_1 \\
&\lambda_{1,n+1} \geq \frac{1}{\sqrt{(n+1)}\,\|A^{-1}\|_\infty}
\end{aligned}
\tag{12}
$$

The above inequalities can be extended to the generalized eigenproblem by backtransforming the pencil (R,B) into $(C^{-1}RC^{-*},I)$ when $B = CC^*$ is positive definite. It was not possible to perform an analytical comparison of the new bounds with the norm inequality bounds. As a result, the behavior of the bounds was studied statistically using simulations.

The errors between computed bounds and the true eigenvalue are defined as:

$$
\begin{aligned}
&err_{\min_{est}} = \frac{\lambda_{1,n+1} - \lambda_{\min}}{|\lambda_{1,n+1}|} \qquad && err_{\min_{inf}} = \frac{\lambda_{1,n+1} - \lambda_{norm_{inf}}}{|\lambda_{1,n+1}|} \\
&\mathrm{err}_{\max_{est}} = \frac{\lambda_{\max} - \lambda_{n+1,n+1}}{|\lambda_{n+1,n+1}|} \qquad && \mathrm{err}_{\max_1} = \frac{\lambda_{norm_1} - \lambda_{n+1,n+1}}{|\lambda_{n+1,n+1}|} \\
&\mathrm{err}_{\max_F} = \frac{\lambda_{norm_F} - \lambda_{n+1,n+1}}{|\lambda_{n+1,n+1}|}
\end{aligned}
\tag{13}
$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the eigenvalue bounds proposed using the order-recursive technique. λ_{norm_F} and λ_{norm_1} are the upper bounds respectively obtained using norm F and norm 1, as defined in (12). $\lambda_{norm_{inf}}$ is the lower bound obtained using norm infinity, as defined in (12).

We considered pencils in which the elements are randomly generated from a uniform distribution. Note that bounds derived using matrix norm inequalities are only valid with positive definite pencils. Thus, in order to compare the proposed bounds with the matrix norm inequalities bounds, the eigenvalues of (R,B) are shifted to insure that the pencils under study are positive definite. Table 1 presents the means and standard deviations obtained for the error measures defined in (13). 3000 randomly generated positive definite pencils were used to generate the results in each case. This table shows that the proposed bounds are tighter than the norm inequality bounds in an average sense only, i.e., the relative tightness of the bounds around the true eigenvalue depends upon the pencil under consideration much more than the norm inequality based norms. Furthermore, the results indicate that the larger the eigenvalues are, the better the performance of the proposed $\lambda_{\min}$

is. Note that $\lambda_{\min}$ is not bounded by 0, as is $\lambda_{perm_{inf}}$, and can be negative. Therefore, the likelihood of $\lambda_{\min} < 0$ increases when the true eigenvalues are close to 0 to start with.

Table 1. Bound error measures

average min & max eigenv.	$err_{\min}$	$err_{\min_{inf}}$	$err_{\max}$	$err_{\max_1}$	$err_{\max_F}$
	mean (standard deviation)				
28.677 3079	0.5381 (2.1267)	0.7089 (0.0225)	0.2813 (0.4057)	0.3282 (0.1194)	0.5016 (0.3109)
219 10978	0.5730 (1.4311)	0.7913 (0.0156)	0.4356 (0.6020)	0.5005 (0.1383)	0.5730 (0.3724)
8407 3.610^5	0.1631 (0.0771)	0.7884 (0.0166)	0.4356 (0.6450)	0.4771 (0.1486)	0.6485 (0.4410)

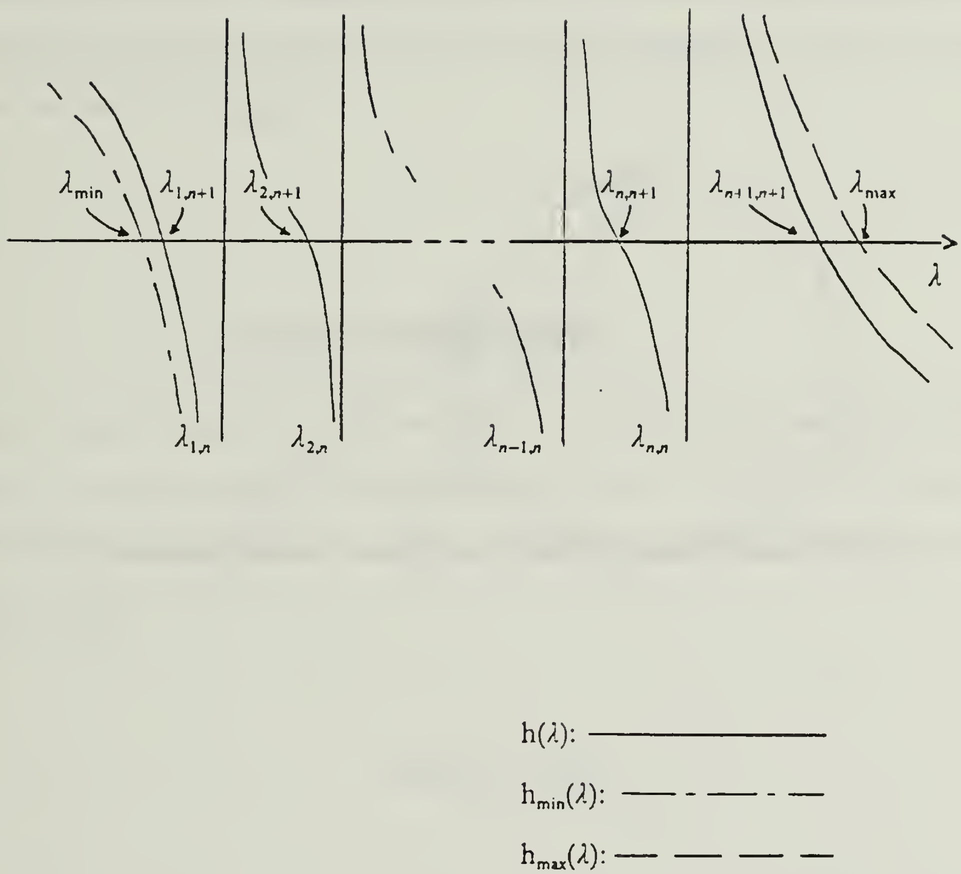
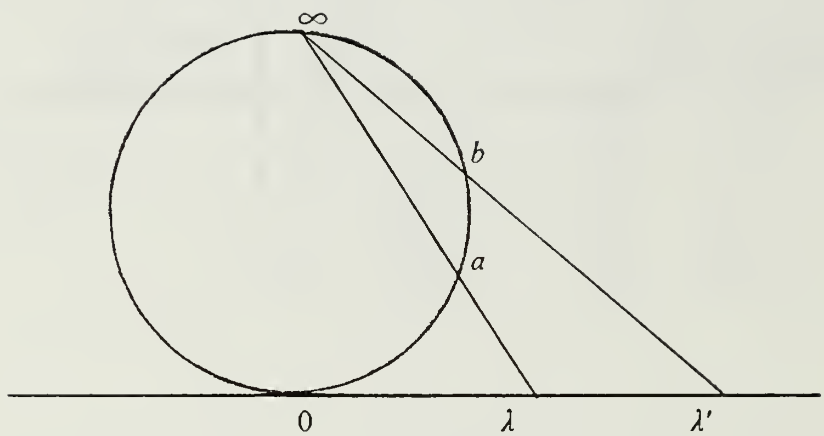


Figure 1. Eigenvalue search function $h(\lambda)$, and bound search functions $h_{min}(\lambda)$, and $h_{max}(\lambda)$



$$\chi(\lambda, \lambda') = \mathrm{d}(a,b)$$

Figure 2. Definition of the Chordal distance: from Parlett [15]

4.0 Conclusions

This report presents new bounds on the extreme eigenvalues of Hermitian pencils (R,B) with finite eigenvalues, when B is positive definite. The bounds are based on an order-recursive eigendecomposition of the pencil. Simulations indicate that the proposed bounds depend more strongly on the pencil considered than those derived using norm inequalities. However, they are not as restricted and are easy to compute.

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Appendix A. Monotone behavior of the eigenvalue search function, and the bound search functions

This appendix shows that the generalized eigenvalue search function $h(\lambda)$, and the bound search functions $h_{min}(\lambda)$, and $h_{max}(\lambda)$ are monotone decreasing between their poles.

Proof:

Recall from (5) that the eigenvalue search function is defined as:

$$h(\lambda) = r_0 - \lambda b_0 - \sum_{k=1}^{n} \frac{|s_k - \lambda q_k|^2}{\lambda_{k,n} - \lambda} \tag{A.1}$$

Consider the following matrix equation:

$$\begin{bmatrix} r_0 - \lambda b_0 & s^* - \lambda q^* \\ s - \lambda q & \Lambda - \lambda I \end{bmatrix} \begin{bmatrix} C \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ O \end{bmatrix} \tag{A.2}$$

with $\Lambda = \mathrm{diag}(\lambda_{1,n}, \dots, \lambda_{n,n})$. Solving equation (A.2) for C leads to:

$$\begin{aligned} C(\lambda) &= \left[r_0 - \lambda b_0 - (s - \lambda q)^*(\Lambda - \lambda I)^{-1}(s - \lambda q)\right]^{-1} \\ &= [h(\lambda)]^{-1} \end{aligned} \tag{A.3}$$

$C(\lambda)$ may be rewritten as:

$$C(\lambda) = e_1^*\left(\hat{R} - \lambda \hat{B}\right)^{-1} e_1 \tag{A.4}$$

with

$$\hat{R} = \begin{bmatrix} r_0 & s^* \\ s & \Lambda \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} b_0 & q^* \\ q & I \end{bmatrix} \qquad e_1 = [1, 0, \dots, 0]^t$$

so that from (A.4) we get:

$$C'(\lambda) = e_1^* \left(\hat{R} - \lambda \hat{B} \right)^{-1} \hat{B} \left(\hat{R} - \lambda \hat{B} \right)^{-1} e_1 > 0 \tag{A.5}$$

Using (A.3) and (A.5), $h'(\lambda)$ becomes:

$$h'(\lambda) = \frac{d}{d\lambda} \left[\frac{1}{C(\lambda)} \right] = \frac{-C'(\lambda)}{C(\lambda)^2} < 0 \tag{A.6}$$

Therefore, $h(\lambda)$ is monotone decreasing between its poles.

Next, note that $h_{min}(\lambda)$ and $h_{max}(\lambda)$ are functions similar to $h(\lambda)$ in which Λ has been respectively replaced with $\mathrm{diag}(\lambda_{1,n}, \dots, \lambda_{1,n})$ and $\mathrm{diag}(\lambda_{n,n}, \dots, \lambda_{n,n})$. Thus, $h_{min}(\lambda)$ and $h_{max}(\lambda)$ are monotone decreasing between their respective multiple poles $\lambda_{1,n}$ and $\lambda_{n,n}$.

■

Appendix B. Generalized Gershgorin bounds

This appendix first reviews the generalized Gershgorin neighborhoods proposed earlier by Stewart [8]. Next, it shows that further tightness of the eigenvalue bounds may be obtained by replacing the norm 1 measure used by Stewart with the norm 2 measure.

Generalized eigenvalues and the Chordal distance

Some insight into the properties of the pencil (R,B) can be gained by looking at the Generalized Schur (G.S.) decomposition [7]. Recall that the G.S. decomposition leads to the following result:

Theorem: If R and B are in $\mathbb{C}^{n \times n}$, then there exists unitary Q and Z such that $Q^{*}AZ = T$ and $Q^{*}BZ = S$ are upper triangular. If for some k, t_{kk} and s_{kk} are both zero, then $\lambda(R,B) = \mathbb{C}$. Otherwise,

$$\lambda(R,B) = \left\{ t_{ii}/s_{ii} \mid s_{ii} \neq 0 \right\} \tag{B.1}$$

Equation (B.1) shows that $\lambda(R,B)$ may be very sensitive to small changes if s_{ii} is small. However, Stewart [7] noted that the reciprocal s_{ii}/t_{ii} may be a well behaved (i.e., not sensitive to small changes of its parameters) eigenvalue of the pencil (B,R), and pointed out that it may be better to treat the eigenvalues as pairs (t_{ii}, s_{ii}) than as quotients. As a consequence, Stewart [8] identified the eigenvalues $\lambda = t/s$ of pencils with the point in the projective complex line defined as:

$$[t,s] = \{(t,s) \neq (0,0) : t/s = \lambda\}$$

The Chordal metric[2] χ is used to measure the eigenvalue separation. It is expressed as:

$$\chi([s,t]\,,[s',t']) \triangleq \frac{|st' - s't|}{\sqrt{|s|^2 + |t|^2}\,\sqrt{|s'|^2 + |t'|^2}} \tag{B.2}$$

---

[2] This metric results from the introduction of the extended complex plane (complex plane + infinity) in complex analysis [16]. The Riemann sphere is chosen to represent the extended complex plane which is not easy to visualize directly. The correspondence between the two geometric representations is then set up with the aid of a stereographic projection [16].

For $\lambda = s/t$ and $\lambda' = s'/t'$ the chordal distance can be expressed as:

$$\chi(\lambda, \lambda') = \frac{|\lambda - \lambda'|}{\sqrt{1 + \lambda^2}\sqrt{1 + \lambda'^2}} \tag{B.3}$$

The distance $\chi(\lambda, \lambda')$ is the length of the chord joining a and b, as shown in Figure 2.

Some useful properties of the Chordal metric [16]

1. The chordal metric is invariant under reciprocation; i.e., $\chi(\lambda, \lambda') = \chi(1/\lambda, 1/\lambda')$.

2. The chordal metric is bounded; i.e., $0 \le \chi(\lambda, \lambda') \le 1$.

3. $\chi(\lambda_1, \lambda_2) \le \chi(\lambda_1, \lambda_3) + \chi(\lambda_3, \lambda_2)$.

4. $\chi(\lambda, \lambda') = \chi(\lambda', \lambda)$.

Generalized Gershgorin bounds

Stewart [8, th. 2.1] showed that the generalized eigenvalues λ of the pencil (R,B) lie in the union of the regions G_i defined by:

$$G_i = \left\{\left[r_{ii} + \alpha_i^{*}\tilde{x},\, b_{ii} + \beta_i^{*}\tilde{x}\right] : \|\tilde{x}\|_\infty \le 1\right\}, \quad (i = 1, \dots, n) \tag{B.4}$$

where $\alpha_i^* = (r_{i,1}, \dots, r_{i,i-1}, r_{i,i+1}, \dots, r_{i,n})$, $\beta_i^* = (b_{i,1}, \dots, b_{i,i-1}, b_{i,i+1}, \dots, b_{i,n})$, and $\tilde{x}$ is formed from the eigenvalue x by deleting its i^{th} component.

The sets G_i are not easy to work with. Thus, they are replaced with the following neighborhoods defined in terms of the chordal metric χ. This leads to:

$$\chi\left([r_{ii}, b_{ii}],\left[r_{ii} + \alpha_i^{*}\tilde{x},\, b_{ii} + \beta_i^{*}\tilde{x}\right]\right) = \frac{|r_{ii}\beta_i^{*} - b_{ii}\alpha_i^{*}\tilde{x}|}{\sqrt{|r_{ii} + \alpha_i^{*}\tilde{x}|^2 + |b_{ii} + \beta_i^{*}\tilde{x}|^2}} \tag{B.5}$$

Next, the sets $\tilde{G}_i$ are defined as follows:

$$\tilde{G}_i = \{\lambda : \chi(r_{ii}/b_{ii}, \lambda) \le \rho_i\} \tag{B.6}$$

where ρ_i is an upper bound on the chordal distance defined in (B.5). These regions $\tilde{G}_i$ contain the eigenvalues of the pencil (R,B); they are called the Gershgorin regions. Stewart [8] showed that the bound ρ_i introduced in (B.6) can be defined by:

$$\rho_i \stackrel{\Delta}{=} \frac{\|r_{ii}\beta_i^* - b_{ii}\alpha_i^*\|_1}{\sqrt{|r_{ii}|^2 + |b_{ii}|^2}\,\sqrt{r'^2_{ii} + b'^2_{ii}}} \tag{B.7}$$

where

$$r'_{ii} = \max\left\{0,\ |r_{ii}| - \|\alpha_i\|_1\right\}$$
$$b'_{ii} = \max\left\{0,\ |b_{ii}| - \|\beta_i\|_1\right\}$$

Modified Gershgorin bounds

A bound γ_i on the chordal distance tighter than the one proposed by Stewart with (B.7) can be derived by using the following vector norm inequality:

$$\|x\|_2 \leq \|x\|_1 \tag{B.8}$$

Using (B.8) in (B.5) leads to:

$$|(r_{ii}\beta_i^* - b_{ii}\alpha_i^*)\tilde{x}| \leq \|r_{ii}\beta_i^* - b_{ii}\alpha_i^*\|_2\|\tilde{x}\|_2 \tag{B.9}$$

and

$$|r_{ii} + \alpha_i^*\tilde{x}| \geq ||r_{ii}| - |\alpha_i^*\tilde{x}|| \tag{B.10}$$

Note that

$$|r_{ii}| - |\alpha_i^*\tilde{x}| \geq |r_{ii}| - \|\alpha_i^*\|_2\|\tilde{x}\|_2 \tag{B.11}$$

Thus, (B.8), (B.9), (B.10), and (B.11) lead to:

$$\chi(r_{ii}/b_{ii}, \lambda) \leq \gamma_i \stackrel{\Delta}{=} \frac{\|r_{ii}\beta_i^* - b_{ii}\alpha_i^*\|_2}{\sqrt{|r_{ii}|^2 + |b_{ii}|^2}\,\sqrt{|r'_{ii}|^2 + |b'_{ii}|^2}} \leq \rho_i \tag{B.12}$$

where

$$r'_{ii} = \max\left\{0,\ |r_{ii}| - \|\alpha_i\|_2\right\}$$
$$b'_{ii} = \max\left\{0,\ |b_{ii}| - \|\beta_i\|_2\right\}$$

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Note that the G.G. bound γ_i is tighter than ρ_i but similar comments to those made on ρ_i apply.

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