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COMPARISON OF A DETERMINISTIC AND A STOCHASTIC

FORMULATION FOR THE OPTIMAL CONTROL OF A LANCHESTER-

TYPE ATTRITION PROCESS

by

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and

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April 1977

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

The structure of the optimal fire distribution policy obtained using a deterministic combat attrition model is compared with that for a stochastic one. The same optimal control problem for a homogeneous force in Lanchester combat against heterogeneous forces is studied using two different models for the combat dynamics (the usual deterministic Lanchester-type differential equation formulation and a continuous parameter Markov chain with stationary transition probabilities). Both versions are solved using modern optimal control theory

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(the maximum principle (including the theory of state variable inequality constraints) for the deterministic control problem and the formalism of dynamic programming for the stochastic control problem). Numerical results have been generated using a digital computer and are compared.

1. Introduction.

In today's complex world an understanding of the impact of modelling assumptions upon optimum military strategies derived from mathematical models is essential for the determining of optimal solutions to complex problems of international significance. In this paper we continue the study of one of the authors on the effects of various modelling assumptions on the structure of optimal tactical allocation policies by systematically contrasting the solutions for a sequence of idealized models. These combat models are too simple to be taken literally but should be interpreted as indicating general principles to serve as hypotheses for subsequent higher resolution studies of real world problems via computer simulation or field experimentation.

In previous papers [34], [35], [36], [37], [38] one of us has studied the optimal control of deterministic Lanchester attrition processes. A major result of this previous research was that optimal tactical allocation policies are quite sensitive to the precise nature of the combat model adopted, even as to whether the tactical scenario lasts for a specified period of time or terminates only when a predetermined "break-point" has been reached. We have shown [36] that whether or not concentration of all fire on a single enemy target type is always the optimal fire distribution policy depends on whether, for example, enemy target types undergo a "square-law" or "linear-law" attrition process (see also [38]). In the paper at hand, we examine the effects on the structure of the optimal fire distribution policy of whether combat attrition is modelled as a deterministic or a stochastic process. Although there has been a continuing discussion among military operations analysts about the relative

merits of deterministic versus stochastic combat attrition models (in particular, see [4], [9]), there apparently has been no systematic attempt to contrast optimal military strategies derived from such different modelling approaches.

In order to keep the impact of modelling assumptions on optimal strategies in sharp focus and also for reasons of mathematical tractability, we consider a simple fire distribution problem for a homogeneous Y force in Lanchester combat against heterogeneous X forces composed of two types of weapon systems. Our research approach is to study the same scenario (prescribed duration battle) using a deterministic combat attrition model and also a stochastic one and then to compare the corresponding optimal fire distribution policies.

The solution to the deterministic problem is obtained using modern optimal control theory (see [8], [27]). As discussed in [37] and [41], the non-negativity restrictions on the force levels are state variable inequality constraints (henceforth abbreviated as SVIC's) and require special treatment (appropriate modification of the usual maximum principle[||]) when active (see Chapter 6 of [27], [40]). In this paper we shall treat SVIC's by the method of Speyer and Bryson [32] (see also [19], [24]) of adjoining an SVIC directly to the return functional with a (Lagrange) multiplier (see [41]). Unlike the corresponding terminal control problem studied in [34], however, this "solution" requires several computer assisted computations for implementation.

The solution to the stochastic problem is obtained using the well-known dynamic programming approach to optimal stochastic control [13], [21],

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[||] In this paper we employ an equivalent statement of the Pontryagin maximum principle [27] commonly used by engineers in the United States. There is a minor sign difference (see p. 108 of [8]) between these versions.

[12]. The basic equations of optimality (the fundamental functional equation for the optimal expected-value function (see [12])) are developed. We derive analytic solutions to these equations for very small numbers of combatants and thus obtain the optimal closed-loop control. As is the case for the Lanchester stochastic process (see [9], [20]), a general solution for arbitrary numbers of combatants has not been obtained for the fundamental functional equation (actually a system of differential-difference equations), although solutions for specific (small) numbers of combatants are readily obtained. Therefore, we have used finite-difference methods to generate a numerical approximate solution.

The body of this paper is organized in the following fashion. First, we review a few relevant facts about the Lanchester stochastic process. Then we state the two optimal control problems that this paper compares. The method of solving the deterministic problem is outlined. The basic equations of optimality for the stochastic control problem are developed, and obtaining an analytic solution to these equations is discussed. The use of finite difference methods for generating a numerical solution is described. Then we compare results obtained from the two models and discuss these results. The implications of these results for defense planners and military operations analysts are pointed out.

2. The Lanchester Stochastic Process.

In 1914 in the British journal Engineering F. W. Lanchester [23] postulated that under the condition of "modern warfare" combat between two homogeneous forces could be described by the equations[||]

[||] See [45] for a discussion of the assumptions inherent in (1). A further discussion of Lanchester-type equations of warfare can be found in [39]. Further references on deterministic Lanchester formulations can be found there [39] or in [11].

$$\frac{dx}{dt} = -ay,$$

$$\frac{dy}{dt} = -bx, \tag{1}$$

where a,b are commonly referred to as the Lanchester attrition-rate

coefficients and x(t),y(t) are force levels. During World War II,

B. Koopman suggested a reformulation of such a model in stochastic form

[25]. Subsequent work on stochastic models of combat attrition has been

by R. Snow [31], R. Brown [6], [7], G. Weiss [44], D. Smith [30], and

G. Clark [9]. The stochastic process corresponding to a model like (1)

has been called the Lanchester stochastic process by B. Koopman [20].

Before considering the optimal stochastic control problem, it seems

appropriate for us to review a few results for the Lanchester stochastic

process. Consider combat between a homogeneous X force and a homogeneous

Y force. Let us model this combat as a continuous parameter Markov chain

with stationary transition probabilities (see pp. 188-189 of [26] for a

further discussion of terminology). Let M(t) denote the (integer)

number of X combatants "alive" at time t after the battle begins, and

let N(t) denote the number of Y combatants.[||] We denote the state proba-

bility by P(t,m,n), and thus

$$P(t,m,n) = \text{Prob}[M(t)=m, N(t)=n].$$

Making standard assumptions (see [5]), we find that the state probabilities

satisfy the following system of differential-difference equations

for $1 \le m \le m_0$ and $1 \le n \le n_0$[¶]

[||]Random variables are denoted by capital letters, while their realizations
are denoted by the corresponding lower case letters.

[¶]We adopt the convention that $P(t,m,n) = 0$ for either $m > m_0$ or $n > n_0$.

$$\frac{dP}{dt}(t,m,n) = P(t,m+1,n)A(m+1,n) + P(t,m,n+1)B(m,n+1)$$

$$-\{A(m,n) + B(m,n)\}P(t,m,n), \quad (2)$$

where m_0 (n_0) is the number of X (Y) combatants at the beginning of battle at $t = 0$, i.e. $M(t=0) = m_0$ with probability one; $A(m,n)$ is the rate of attrition of the X forces with $A(0,n) = 0$; and $B(m,n)$ is the rate of attrition of the Y forces with $B(m,0) = 0$. In other words, we have

$$\text{Prob}\begin{bmatrix} \text{one X casualty in time} \\ \text{interval from } t \text{ to } t + \Delta t \end{bmatrix} = A(m,n)\Delta t.$$

(Moreover, $P(t,m,n)$ is, more precisely, the transition probability

$$P(t,m,n) = P(t,m,n;t=0,m_0,n_0) = \text{Prob}\begin{bmatrix} M(t)=m & M(t=0)=m_0 \\ N(t)=n & N(t=0)=n_0 \end{bmatrix}.)$$

Of course, the state space is discrete, i.e. $m = 0,1,\ldots,m_0$ and $n = 0,1,\ldots,n_0$. At state space boundaries, i.e. $m = 0$ or $n = 0$, equation (2) takes the form

$$\frac{dP}{dt}(t,m,0) = P(t,m+1,0)A(m+1,0) + P(t,m,1)B(m,1)$$

$$- P(t,m,0)A(m,0),$$

$$\frac{dP}{dt}(t,0,n) = P(t,0,n+1)B(0,n+1) + P(t,1,n)A(1,n)$$

$$- P(t,0,n)B(0,n),$$

$$\frac{dP}{dt}(t,0,0) = P(t,1,0)A(1,0) + P(t,0,1)B(0,1). \quad (3)$$

Initial conditions for (2) and (3) are

$$P(t=0,m,n) = \begin{cases} 1 & \text{for } m = m_0, \ n = n_0, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Let us adopt the following terminology for the attrition rates (and hence the process itself). We say that we have a

    (a) <u>linear-law attrition process</u> when

$$A(m,n) = amn,$$

$$B(m,n) = bmn, \tag{5}$$

and   (b) <u>square-law attrition process</u> when

$$A(m,n) = \beta m + an,$$

$$B(m,n) = bm + \alpha n, \tag{6}$$

where α, β may be referred to as operational loss rates.

Although it is well-known that (2) through (4) yield an exponential solution (the Chapman-Kolmogorov equation expresses the semi-group property of the state probabilities (see [20])) when $A(m,n)$ and $B(m,n)$ have been specified (for example, by (6)), general solutions which apply for all values of m_0 and n_0 have only been obtained to this system only in a few special cases. In the special case when $a + \alpha = b + \beta$, Isbell and Marlow [18] developed a general solution to (2) through (4) for a square-law stochastic attrition process. Recently, Clark (see pp. 102-104 of [9]) developed the general solution to the linear-law stochastic attrition process (i.e. $A(m,n)$ and $B(m,n)$ are given by (5)).

One reason why we have reviewed this material is to now point out to the reader that a general solution to (2) through (4) only exists for a linear-law attrition process and is very complex (see pp. 102-104 of [9]). In considering the optimal control of the Lanchester stochastic (square-law) process, we will encounter a similar system of equations for the optimal expected-value function. Keeping in mind that a general solution has not been obtained to the corresponding equations (2) through (4) for the state

probabilities of the square-law stochastic attrition process, the reader will not be surprised to learn that we have not developed an analytic solution for the general case of these equations.

Additionally, using the above noted solutions for the Lanchester stochastic process, Clark (following results in [25] and qualitative results in [31]) made comparisons [9] (see also Chapter 11 of [4]) of the average force levels in the stochastic process (denoted as $\bar{m}(t)$ and $\bar{n}(t)$) and the corresponding force levels $x(t)$ and $y(t)$ in the deterministic formulation (such as (1)). Unlike the corresponding situation for the Yule-Ferry linear birth process (see pp. 77-78 of [3] or pp. 156-159 of [10]), there is a bias (due to "boundary effects") in the dynamical behavior of $x(t)$ and $y(t)$ as compared with $\bar{m}(t)$ and $\bar{n}(t)$ for the same values of a and b. It turns out that $\bar{m}(t)$ lies above $x(t)$, and the amount of separation grows over time.

The above is a major result of Clark's careful investigation in which several numerical examples are given to prove such points. He concludes that (see p. 11-19 of [4]) "the deterministic model would have difficulty approximating a stochastic simulation" with respect to the time history of force levels. Clark's solution to the stochastic linear-law process was important in making such a comparison. This fact that the average of the Lanchester stochastic process does not behave identically to the corresponding force levels $x(t)$ and $y(t)$ computed according to the corresponding deterministic model has motivated the paper at hand.

3. The Optimal Control Problems.

In this section we state the two optimal control problems that are considered in this paper. The deterministic optimal control problem considered is

$$\underset{\phi_D(t)}{\text{maximize}}\{ry(t_f) - px_1(t_f) - qx_2(t_f)\} \quad \text{with} \quad t_{max} \quad \text{specified},$$

subject to:

$$\frac{dx_1}{dt} = -\phi_D a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi_D)a_2 y,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2, \tag{7}$$

$$x_1, x_2, y \geq 0, \qquad 0 \leq \phi_D \leq 1, \qquad \text{and} \quad t_f \leq t_{max},$$

with initial conditions

$$x_1(t=0) = x_1^o, \qquad x_2(t=0) = x_2^o, \qquad y(t=0) = y_0,$$

where all symbols are explained in the Appendix. In this problem x_1, x_2, and y are called state variables, while ϕ_D is called a control (or decision) variable. A constraint such as $x_1 \geq 0$ is called a state variable inequality constraint (SVIC) and requires special treatment (see below).

The battle lasts for $0 \leq t \leq t_{max}$ unless, of course, one side or the other is annihilated before t_{max}. To be more precise, the battle terminates under one of the three following circumstances:

(1) $x_1(t_f) = x_2(t_f) = 0$ and $t_f \leq t_{max}$,

(2) $y(t_f) = 0$ and $t_f \leq t_{max}$,

(3) $t_f = t_{max}$,

where t_f denotes the time at which the battle ends. Upon further analysis, it has been convenient to consider that there are eight "terminal states," or "target sets." These are shown in Table I. The reader should note that for S_4 through S_8 the battle ends by the system (as described by the three state variables x_1, x_2, and y) being driven to a prescribed terminal state. For these terminal states, t_f is undetermined when $t_f < t_{max}$, since it is then determined by entry to the terminal state, and this depends upon the control used. For these cases a well-known transversality condition must hold. The above problem (7) is called a _prescribed duration battle_, since the battle lasts for a maximum duration of t_{max}, i.e. $t_f \leq t_{max}$.

The corresponding _stochastic optimal control_ problem considered is

$$\underset{\phi_S}{\text{maximize}} \ E[rN(t_f) - pM_1(t_f) - qM_2(t_f)] \ \text{with} \ t_f \ \text{specified},$$

subject to: casualties occur randomly as a continuous parameter Markov chain with stationary transition probabilities corresponding to the deterministic process (7), (8)

$$M_1, M_2, N \geq 0 \ \text{and} \ 0 \leq \phi_S \leq 1,$$

where the random variables $M_1(t)$, $M_2(t)$, $N(t)$ are force levels (integers), $E[\cdot]$ denotes mathematical expectation, and all other symbols are explained in the Appendix. In (8) $\phi_S = \phi_S(t, m_1, m_2, n)$ denotes a closed-loop control (see [16]). For the deterministic problem (7) we have not been precise about this point, since it is well-known that open-loop control (e.g. $\phi_D = \phi_D \ (t; x_1^o, x_2^o, y_0)$) and closed-loop control (e.g. $\phi_D = k(t, x_1, x_2, y)$) are equivalent and yield identical results in trajectory and payoff [16]. For stochastic control problems this equivalence is, of course, not true (see [12]).

Table I. Definition of Terminal States for Deterministic Optimal Control Problem (Prescribed Duration Battle).

S_1: $x_1(t_f) > 0$, $x_2(t_f) > 0$, $y(t_f) > 0$, $t_f = t_{max}$

S_2: $x_1(t_f) = x_1(t_1) = 0$, $x_2(t_f) > 0$, $y(t_f) > 0$, $t_f = t_{max}$

where $t_1 < t_f$

S_3: $x_1(t_f) = x_1(t_3) > 0$, $x_2(t_f) = 0$, $y(t_f) > 0$, $t_f = t_{max}$

where $t_3 < t_f$

S_4: $x_1(t_f) > 0$, $x_2(t_f) > 0$, $y(t_f) = 0$, $t_f \leq t_{max}$

S_5: $x_1(t_f) = x_1(t_1) = 0$, $x_2(t_f) > 0$, $y(t_f) = 0$, $t_f \leq t_{max}$

where $t_1 < t_f$

S_6: $x_1(t_f) = x_1(t_2) > 0$, $x_2(t_f) = 0$, $y(t_f) = 0$, $t_f \leq t_{max}$

where $t_2 < t_f$

S_7: $x_1(t_f) = x_1(t_1) = 0$, $x_2(t_f) = 0$, $y(t_f) > 0$, $t_f \leq t_{max}$

where $t_1 < t_f$

S_8: $x_1(t_f) = 0$, $x_2(t_f) = x_2(t_4) = 0$, $y(t_f) > 0$, $t_f \leq t_{max}$

where $t_4 < t_f$

4.   Determination of an Optimal Policy for Deterministic Problem.

In this section we outline how an optimal policy (expressed as a closed-loop control) may be determined for (7). In order to keep the length of the paper at hand within reasonable limits we will only be able to highlight the main points. Details which are available elsewhere in the open literature will be omitted. In order to contain the length of this paper the entire "solution" will not be given here.$^{||}$

4.1.   Outline of Solution Procedure.

Before giving our solution algorithm, it seems appropriate to define some terms. We have then

Definition 1:   By an extremal path we mean a path on which the necessary

conditions of optimality are everywhere satisfied (we use

the work everywhere, since we take the class of admissible

controls to be the space of piecewise-continuous functions).

Definition 2:   By an extremal control we mean the control used in order

that the system follow an extremal path.

Definition 3:   By the domain of controllability for extremals to a given

terminal state we mean that subset of the initial state

space from which extremals lead to the terminal state.

Definition 4:   By the synthesis of an extremal control we mean using the

basic necessary conditions of optimality to explicitly

determine the time history of an extremal control from

initial to terminal time as a function of initial conditions.

$^{||}$Complete results in a form suitable for numerical determination are to be found in Appendix G of [43]. The "solution" occupies twenty pages in [43], and this should explain why for the purposes of the paper at hand only representative results are given.

Our solution algorithm then is as follows:[‖]

(a)   an extremal control law is developed from the maximum principle

      (which must be modified when the trajectory lies on the boundary of

      the state space); for Lanchester "square-law" attrition structures

      the extremal control law in many cases depends only on relationships

      between dual variables (marginal returns from destroying targets),

(b)   for each terminal state an extremal control is synthesized by com-

      bining a backwards integration of the adjoint system of differential

      equations with the extremal control law and corner conditions,

(c)   for each terminal state the domain of controllability for extremals

      is determined by forwards integration of the state equations using

      the synthesized extremal control from (b),

(d)   the solution is determined at this point for regions of the initial

      state space which are covered by only (part of) the domain of con-

      trollability for extremals to one terminal state; one must also verify

      that the entire initial state space has been accounted for, since

      otherwise one may have overlooked some type of "singular" surface,

(e)   if domains of controllability overlap so that for a point of the

      initial state space contained in their intersection there is more

      than one extremal leading to the terminal surface, then one computes

      the return (or payoff) associated with each extremal; the optimal

      trajectory is selected from the extremals by comparing these values.

      The above solution algorithm is a refinement of the one presented

in [34]. Let us make a few remarks about the application of this procedure

to the prescribed duration battle 7) For this problem we may think of

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[‖] For this approach to work it is essential that an optimal policy exist for
(7). This has previously been established in [37], [41]. In this case
one of the extremals must be an optimal trajectory.

time as being an additional state variable. On the other hand, for the
Isbell-Marlow terminal control problem [34] time may be considered as being
a parameter and consequently was eliminated for the determinations of step
(c) above. In other words, for the Isbell-Marlow problem a domain of
controllability was determined by inequalities involving the three state
variables; for the prescribed duration battle (7) such a determination
involves the four variables t_{max}, x_1^o, x_2^o, and y_0.

For the prescribed duration battle we have not been able in all
cases to develop analytic expressions at step (c) in the above algorithm
as we did for the terminal control problem studied in [34]. Consequently,
we could not analytically accomplish steps (d) and (e) for the problem at
hand. We have, however, used computational methods to determine the optimal
control. We have expressed our "solution" (partially presented in the next
section) so that given a point $P^o = (x_1^o, x_2^o, y_0)$ in the initial state space
and t_{max}, one can determine which terminal states are reached by extremals.
Thus, we can determine to which domains of controllability P^o belongs.
Then, using the extremal control, we can numerically compute the return
(or payoff) associated with each extremal and select the optimal policy
from among a finite number of possibilities. A computer program was written
in FORTRAN to do the above and computations performed on an IBM 360 computer.

4.2. Summary of Solution.

We have applied the solution procedure of Section 4.1 to develop
a "solution" in the sense discussed there. Without loss of generality we
assume that $a_1 b_1 > a_2 b_2$, i.e. $R > 1$. There are two cases to be considered

$$(1) \quad \delta \geq 1,$$

and $\quad (2) \quad 0 \leq \delta < 1,$

where $\delta = a_1 p / (a_2 q)$.

For Case (1): $\delta \geq 1$, the domains of controllability do not overlap each other, and hence extremals extremals are unique. The extremal control is thus the optimal control. The optimal policy, moreover, may be expressed in a particularly simple form: always concentrate all fire on X_1 while $x_1 > 0$. Further details on domains of controllability and "event" times are to be found in Table II of [43].

For Case (2): $0 \leq \delta < 1$, some domains of controllability overlap each other, and hence extremals are not unique (in the sense that from a point in the initial state space the system may be steered along any one of several extremals to various end states of battle). (See [41] for a discussion of a similar case.) Thus, considerations "in the large" (i.e. step (e) of the above solution procedure) are required to determine the optimal policy. Unfortunately, explicit analytic expressions are not readily obtainable as they were for the Isbell-Marlow terminal control problem [34]. However, as discussed in Section 4.1 above, one can use the information presented in Tables III of [43] (which is fifteen pages long) to numerically determine an optimal fire distribution policy for any specific set of model input parameter values. A representative sample of this information is given in Table II.

In Case (2) the optimal fire distribution policy cannot be expressed in the very simple form as in the first case. When Y wins in time less than t_{max} (S_7 for which the optimal policy is determined), the optimal fire distribution policy is precisely the same as when $\delta \geq 1$. However, for all other cases (i.e. terminal states S_1 through S_6) the extremal policy is to finish the prescribed duration battle by firing at X_2, regardless of whether or not X_1 has been annihilated. This differs from that when $\delta \geq 1$. Thus, we see that force levels affect the optimal fire distribution policy.

Table II. A Representative Part of the Solution

to the Prescribed Duration Battle for

$0 \leq \delta < 1$.

(Nonrestrictive assumption: $R > 1$, i.e. $a_1 b_1 > a_2 b_2$)

S_5: $x_1(t_f) = x_1(t_1) = 0$, $\qquad x_2(t_f) > 0$, $\qquad y(t_f) = 0$, $\qquad t_f \leq t_{max}$

Extremal Control: $\phi_D^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_1 \text{ where } x_1(t_1) = 0 \\ \\ 0 & \text{for } t_1 < t \leq t_f \end{cases}$

Domain of Controllability: $a_1 b_1 y_0^2 > s^2 - (b_2 x_2^o)^2$

$\qquad\qquad\qquad\qquad\qquad a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2^o)^2$

$$t_f = t_1 + \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1}\left\{ \frac{\sqrt{a_1 b_1 y_0^2 - s^2 + (b_2 x_2^o)^2}}{b_2 x_2^o \sqrt{R}} \right\} \leq t_{max}$$

where $t_1 = t_1(S_5) = t_1(S_2)$ is given by

(1) for $a_1 b_1 y_0^2 > s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ell n\left\{ \frac{\sqrt{a_1 b_1 y_0^2 - s^2 + (b_2 x_2^o)^2} - b_2 x_2^o}{\sqrt{a_1 b_1}\, y_0 - s} \right\}$$

(2) for $a_1 b_1 y_0^2 < s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ell n\left\{ \frac{b_2 x_2^o - \sqrt{a_1 b_1 y_0^2 - s^2 + (b_2 x_2^o)^2}}{s - \sqrt{a_1 b_1}\, y_0} \right\}$$

(3) for $a_1 b_1 y_0^2 = s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ell n\left\{ \frac{s}{b_2 x_2^o} \right\}$$

NOTE: for $0 \leq \delta < R - \sqrt{R(R-1)}$ optimal paths also satisfy (equality

yielding a _dispersal surface_)

for $0 \leq x_1^o < (b_2 x_2^o)/(k b_1)$

$$a_1 b_1 y_0^2 \geq R s^2 - \frac{R}{z^2}\left\{ b_1 x^o \cdot \frac{[z^2(R-1) + R]}{2R} + b_2 x_2^o \right\}^2,$$

where k is given by $k = \{z^2 - R(z-1)^2\}/(2R)$.

4.3. Development of Basic Necessary Conditions of Optimality.

We will use Speyer and Bryson's approach [32][||] of adjoining the state variable constraint directly to the criterion functional with a Lagrange multiplier. The Hamiltonian is given by (see also [19])

$$H(t,\underset{\sim}{x},\underset{\sim}{p},\phi_D) = -p_1\phi_D a_1 y - p_2(1-\phi_D)a_2 y - p_3(b_1 x_1 + b_2 x_2) + \eta_1(t)x_1 + \eta_2(t)x_2, \quad (9)$$

where

$$\eta_i(t) \begin{cases} = 0 & \text{for } x_i > 0, \\ \\ \geq 0 & \text{for } x_i = 0. \end{cases}$$

The adjoint system of differential equations for the dual variables is

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x_1}(t,\underset{\sim}{x},\underset{\sim}{p},\phi_D^*) = b_1 p_3 - \eta_1(t), \quad (10)$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial x_2}(t,\underset{\sim}{x},\underset{\sim}{p},\phi_D^*) = b_2 p_3 - \eta_2(t), \quad (11)$$

$$\frac{dp_3}{dt} = -\frac{\partial H}{\partial y}(t,\underset{\sim}{x},\underset{\sim}{p},\phi_D^*) = \phi_D^* a_1 p_1 + (1-\phi_D^*)a_2 p_2. \quad (12)$$

Boundary conditions for the dual variables (also frequently called transversality conditions) are discussed below. When $t_f < t_{max}$, the following transversality condition also holds

$$H(t=t_f,\underset{\sim}{x},\underset{\sim}{p},\phi_D^*) = 0. \quad (13)$$

When $x_1, x_2 > 0$, the maximum principle yields the extremal control law [34], [41]

$$\phi_D^*(t) = \begin{cases} 1 & \text{for } v(t) > 0, \\ \\ 0 & \text{for } v(t) < 0, \end{cases} \quad (14)$$

[||] Taylor apparently is the only person to apply these important results to variational problems in operations research. See [41] for discussion of previous applications.

where $v(t) = (-p_1)a_1 - (-p_2)a_2$. In [34] we showed that there are no

singular subarcs (see Chapter 8 in [8]) in the solution.

Without loss of generality, let us consider a constrained subarc

on which $x_1(t) = 0$ for $t_1 \leq t \leq t_f$ (and $x_2, y > 0$ for $t < t_f$). Since

$\frac{dx_1}{dt} = 0$, the control is clearly $\phi_D^*(t) = 0$ for $t_1 \leq t \leq t_f$. The require-

ment that $\frac{\partial H}{\partial \phi} = 0$ yields the following relationship between dual variables

on the constrained subarc[||]

$$a_1 p_1(t) = a_2 p_2(t). \qquad (15)$$

The multiplier $\eta_1(t)$ is determined from the condition that $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$,

and this yields

$$\eta_1(t) = \frac{p_3(t)}{a_1}(a_1 b_1 - a_2 b_2). \qquad (16)$$

The interpretation of $\eta_1(t)$ (see [41] for a further discussion) is the

rate of marginal return to Y for keeping $x_1 = 0$. Thus, (intuitively)

Y tries to annihilate X_1 only when it profits him to do so. Further-

more, the requirement that $\eta_1(t) \geq 0$ when $x_1 = 0$ for a finite interval

of time yields that we must have

$$a_1 b_1 \geq a_2 b_2, \qquad (17)$$

since it may be shown that $p_3(t) > 0$ for $t < t_f$. The nonrestrictive

assumption that $a_1 b_1 > a_2 b_2$ (i.e. $R > 1$) implies that it is nonoptimal

to have $x_2 = 0$ for a finite interval of time.

Furthermore, when the necessary conditions of optimality are expressed

in Speyer and Bryson's format [32] (see also [19]), the corner conditions

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[||]The development of (15) requires a slightly different argument when $t = t_f$
and $y(t_f) = 0$. See [41] for a further discussion of this point.

(see pp. 125-126 of [8]) take a particularly simple form for a first order SVIC: the adjoint variables are continuous across all corners (both interior to and on the boundary of the state space). In other words

$$\underset{\sim}{p}(t_c^-) = \underset{\sim}{p}(t_c^+), \tag{18}$$

where t_c^- denotes the time just before the corner (i.e. a left-hand limit). We also have that

$$H(t_c, \underset{\sim}{x}(t_c), \underset{\sim}{p}(t_c^-), \phi_D^*(t_c^-)) = H(t_c, \underset{\sim}{x}(t_c), \underset{\sim}{p}(t_c^+), \phi_D^*(t_c^+)). \tag{19}$$

On entry to a constrained subarc with $x_1(t) = 0$ for $t_1 \leq t \leq t_f$, (19) yields

$$a_1 p_1(t_1^-) = a_1 p_1(t_1) = a_2 p_2(t_1^-) = a_2 p_2(t_1). \tag{20}$$

Let us finally consider the boundary conditions for the dual variables at $t = t_f$. The nonrestrictive assumption that $a_1 b_1 > a_2 b_2$ yields that no extremals lead to S_8. The three terminal states S_1, S_2, and S_3 may be discussed collectively. In all three cases the length of the battle is equal to t_{max}. Then, according to the results presented in [42], we have

for S_1, S_2, and S_3:

$$p_1(t_f) = -p + v_1, \quad p_2(t_f) = -q + v_2, \quad p_3(t_f) = r > 0, \tag{21}$$

where

$$\nu_i \begin{cases} = 0 \quad \text{for} \quad x_i(t_f) > 0, \\ \geq 0 \quad \text{for} \quad x_i(t_f) = 0 \quad \text{but} \quad x_i(t) > 0 \quad \text{for} \quad t < t_f, \\ \text{unrestricted for} \quad x_i(t_f) = 0 \quad \text{and} \quad x_i(t) = 0 \\ \text{for} \quad t_i \leq t \leq t_f \quad \text{with} \quad t_i < t_f. \end{cases} \qquad (22)$$

The latter condition that, for example, the multiplier ν_1 is unrestricted when the system is on a constrained subarc for a finite interval of time is because the boundary of the state space is "absorbing" (i.e. the state constraint $x_1 \geq 0$ essentially acts like a terminal equality constraint as far as the determination of boundary conditions for the adjoint variables [42]). If there were replacements in the model (7) so that the boundary of the state space would not be "absorbing," then we would have $\nu_i \geq 0$ for $x_i(t_f) = 0$.

For S_4, S_5, and S_6 the duration of the battle t_f is determined by the terminal equality constraint $y(t_f) = 0$ when $t_f < t_{max}$ so that the transversality condition (13) yields $p_3(t_f) = 0$. When $t_f = t_{max}$, additional analysis is required, and this is discussed in Section 4.4 below. Then, again according to the results presented in [42], we have for S_4, S_5, and S_6:

$$p_1(t_f) = -p + \nu_1, \quad p_2(t_f) = -q + \nu_2, \quad p_3(t_f) = 0, \qquad (23)$$

where the multipliers ν_i for $i = 1,2$ are again given by (22).

For S_7: $x_1(t_f) = x_2(t_f) = 0$, $y(t_f) > 0$, $t_f \leq t_{max}$, we have [8]

$$p_1(t_f) = -p + \nu_1, \quad p_2(t_f) = -q + \nu_2, \quad p_3(t_f) = r > 0, \qquad (24)$$

since t_f is determined by the (equality) terminal constraints $x_1(t_f) = 0$ and $x_2(t_f) = 0$. Since these are equality constraints, the multipliers

ν_1 and ν_2 are unrestricted in sign. Since t_f is unspecified, the transversality condition (13) with $\phi_D^*(t_f) = 0$ yields that $-p_2(t_f)a_2y = 0$ so that $p_2(t_f) = 0$ and $\nu_2 = q$. The condition (15) which, in particular, holds at $t = t_f$ yields that $p_1(t_f) = 0$. Thus, we have

for $S_7[x_1(t_f) = 0$ before $x_2(t_f) = 0$, $y(t_f) \quad 0]$:

$$p_1(t_f) = 0, \quad p_2(t_f) = 0, \quad p_3(t_f) = r. \tag{25}$$

4.4. Synthesis of Extremal Control.

For each terminal state, extremals may be synthesized by combining the conditions which must hold on a constrained subarc and the extremal control law (14) with a backwards integration of the adjoint equations (10), (11) and (12). The boundary conditions for the adjoint variables given in Section 4.3 and the corner conditions (18) and (19) are used in this backwards sweep process. It is convenient to use the switching function $v(t) = (-p_1)a_1 - (-p_2)a_2$ in synthesizing extremals. Using (10) and (11), we readily find that for $t < t_f$

$$\frac{dv}{dt} = p_3(t)(-a_1b_1+a_2b_2) < 0, \tag{26}$$

since $p_3(t) > 0$ for $t < t_f$.

Details in the synthesis of extremals are similar to those presented in [34]-[38], [41], and [43],[||] and hence they are omitted. The treatment in [37] is most similar to the problem at hand. Details for $\delta \geq 1$ and for $0 \leq \delta < 1$ are different.

There are two interesting aspects, moreover, that we encountered in synthesizing extremals. These are

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[||] In some of these references the non-negativity of the force levels (i.e. SVIC's) have been treated by means other than Speyer and Bryson's approach [8]. The basic principles of working backwards from the end, however, are the same in all applications.

(a)   when $0 \leq \delta < 1$ and a switch in the target type upon which all   Y-

      fire is concentrated occurs without the annihilation of a target type,

      the switching time depends upon the initial force levels and possibly

      the valuation of   Y   survivors, and

(b)   when $P^O = (x_1^o, x_2^o, y_0)$ is such that when $\delta < 1$ an extremal leads

      to S_4^S (i.e. we reach S_4 with a switch in tactics) with $t_f(S_4^S)$

      $< t_{max}$,   we can possibly also steer the system to an end point with

      $y(t_f = t_{max}) = 0$ without violating any necessary conditions of optimality.

Let us first discuss the dependence of the non-annihilation switching

time on force levels and valuation of   Y   survivors.   Such a switch in

fire distribution only happens for $\delta < 1$.   Let us compare the situations

for extremals leading to S_1 and S_4.   In both cases we have

$$\phi_D^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_f - \tau_1, \\ 0 & \text{for } t_f - \tau_1 < t \leq t_f, \end{cases} \tag{27}$$

where $x_1(t = t_f - \tau_1) > 0$.   It is convenient to introduce the "backwards time"

τ defined by $\tau = t_f - t$.   Then when $\delta < 1$,   we have $\phi_D^*(\tau) = 0$ for

$0 \leq \tau \leq \tau_1$ where τ_1 denotes the backwards time of the first switch in

fire distribution.   For $S_4 [x_1(t_f) > 0, \ x_2(t_f) > 0, \ y(t_f) = 0, \ t_f < t_{max}]$,

it may be shown using (10)-(12), (14), (23), and (26) that[||]

$$\tau_1(S_4) = \frac{1}{\sqrt{a_2 b_2}} \cosh^{-1} z, \tag{28}$$

where $z = (R-\delta)/(R-1)$.   For $S_1 [x_1(t_f) > 0, \ x_2(t_f) > 0, \ y(t_f) = 0,$

$t_f = t_{max}]$,   it may be shown that

[||] Further details of the results summarized in this section are to be found
in [43].   To keep the paper at hand from being too long, we have omitted
them.

$$\tau_1(S_1) = \frac{1}{\sqrt{a_2 b_2}} \ln\left(\frac{z+\sqrt{z^2+\alpha^2-1}}{1+\alpha}\right), \qquad (29)$$

where

$$\alpha = \frac{r}{q}\sqrt{\frac{b_2}{a_2}}. \qquad (30)$$

The following theorem is of interest (see [36] for a similar result).

THEOREM 1: Assume that $R > 1$ and $\delta < 1$.

Then,

$$\tau_1(S_1) < \tau_1(S_4).$$

A proof of Theorem 1 is given in [43]. Furthermore, it is readily shown that $\lim_{r \to +\infty} \tau_1(S_1) = 0$. Thus, when $\delta < 1$, the switching time $t = t_f - \tau_1(S_1)$ along extremals leading to S_1 explicitly depends on the value Y places upon the survival of his own forces. The higher he values Y-force survivors, the longer Y forces concentrate their fire on X_1 when $\delta < 1$. For extremals leading to S_4, the transversality condition (13) yields that Y-force survivors have zero value. Intuitively, we see that firing longer at X_1 prolongs the length of battle for those cases when $y(t_f) = 0$, since $a_1 b_1 > a_2 b_2$. However, for extremals leading to S_4 this is not an optimal tactic.

Let us therefore consider the case when $t_f = t_{max}$ for S_4. We just discussed above the possibility when $R > 1 > \delta$ of prolonging the length of battle along an extremal leading to S_4 by firing longer at X_1. Using (27), it may be shown that

$$y(t_f) = y(t_f - \tau_1)\cosh\sqrt{a_2 b_2}\ \tau_1 - \frac{(b_1 x_1(t_f - \tau_1) + b_2 x_2^o)}{\sqrt{a_2 b_2}} \sinh\sqrt{a_2 b_2}\ \tau_1, \qquad (31)$$

where

$$\tau_1 = \tau_1(a) = \frac{1}{\sqrt{a_2 b_2}} \ln\left(\frac{z + \sqrt{z^2 + a^2 - 1}}{1 + a}\right), \tag{32}$$

and

$$a = \frac{(r+\nu)}{q}\sqrt{\frac{b_2}{a_2}}, \tag{33}$$

where ν is the multiplier corresponding to the terminal constraint $y(t_f) = 0$. Then, the following lemma may be established [43].

> LEMMA 1: Consider an extremal leading to S_4 with $y(t_f)$ given by (31) and t_f defined by $y(t_f) = 0$. Then $\frac{\partial y(t_f)}{\partial t_f} < 0$ if and only if $a_1 b_1 y_0^2 < s^2$.

In [43] it is whown that by increasing the implicit valuation of Y survivors (i.e. ν in (33)) the length of battle may be extended until $t_f = t_{max}$. However, this is not an optimal policy. This situation in which a special case (here $t_f = t_{max}$ for S_4) requires an inordinate amount of analysis unfortunately has arisen in all problems that we have studied.

4.5. Obtaining an Optimal Policy.

After extremals have been synthesized, domains of controllability for extremals may be obtained as shown in [34]. It then remains to apply steps (d) and (e) of the solution procedure given in Section 4.1. A computer program written in FORTRAN has been developed to assist in the determination of an optimal policy. This computer program does the following: for a given point in the initial state space, we determine to which terminal states extremals lead. Then, the payoff corresponding to each extremal is computed. The optimal path (and hence the optimal policy) is readily obtained by determining which extremal yields the largest return to Y.

In the above fashion, the optimal fire distribution policy may be obtained as an open-loop control. After this has been obtained, it is a straightforward matter to express the optimal policy as a closed-loop control. In doing this, it is convenient to cite the principle of optimality [1] (a special case of Isaacs' tenet of transition [17] (see also [2])), i.e. every subarc of an optimal trajectory is itself an optimal trajectory.

5. Determination of an Optimal Policy for Stochastic Problem.

In this section we discuss how an optimal fire distribution policy (expressed as a closed-loop control) may be determined for (8). Using the formalism of dynamic programming, we develop the fundamental functional equation for the optimal expected value function. This is a sufficient condition of optimality: a control which leads to the satisfying of this equation is an optimal policy (see [29]). An analytic solution is developed to the fundamental functional equation for very small numbers of combatants. Finite difference methods are used, however, to generate a numerical approximate solution, since a general solution (for arbitrary numbers of combatants) has not been obtained to the fundamental functional equation.

5.1 Development of Fundamental Functional Equation.

Let $S(\tau, m_1, m_2, n)$ denote the optimal expected-value function (see [12]). Then

$$S(\tau, m_1, m_2, n) = \underset{\phi_S \in \Phi}{\text{maximum}} \; E_{m,\tau} [rN(\tau=0) - pM_1(\tau=0) - qM_2(\tau=0)], \quad (34)$$

where

the system state is m_1, m_2, n at time τ (i.e. $M_1(\tau) = m_1$, etc),

Φ is the class of admissible controls (i.e. ϕ_S must always be chosen from the set of rational numbers $\{0, \frac{1}{n(\tau)}, \frac{2}{n(\tau)}, \ldots, 1\}$),

$\tau = t_f - t$ is the "backwards time" from the end of battle (which

begins at $t = 0$),

$E_{\underset{\sim}{m},\tau}$ denotes mathematical expectation given that $\underset{\sim}{m}(\tau) = (m_1(\tau),$

$m_2(\tau), n(\tau))$,

casualties occur in a random fashion between t and t_f.

In other words, $S(\tau, m_1, m_2, n)$ is the maximum return that we get on the

average when we start with force levels m_1, m_2, and n at $t = t_f - \tau$,

follow an optimal policy $\phi_S^*(s, m_1, m_2, n)$ (chosen from the class of

admissible policies ϕ) for $t \leq s \leq t_f$, and casualties occur in a

random fashion.

We consider that casualties occur as a Markov process with discrete

state space (or discontinuous Markov process). Specifically, we assume

that

(1) the attrition process is a continuous parameter Markov chain with

stationary transition probabilities corresponding to a deterministic

Lanchester square-law attrition process; this is equivalent to

assuming

(a) the future occurrences of casualties depend only on the state

of the system at t and not on past history,

(b) the transition probabilities depend on only the state of the

system,

(c) $\text{Prob}\begin{bmatrix} \text{one } X_1 \text{ casualty} \\ \text{in interval } \Delta t \end{bmatrix} = \phi a_1 n \Delta t,$

$\text{Prob}\begin{bmatrix} \text{one } X_2 \text{ casualty} \\ \text{in interval } \Delta t \end{bmatrix} = (1-\phi) a_2 n \Delta t,$

$\text{Prob}\begin{bmatrix} \text{one } Y \text{ casualty} \\ \text{in interval } \Delta t \end{bmatrix} = (b_1 m_1 + b_2 m_2) \Delta t,$

where $\phi a_1 n$ is X_1 casualty rate, etc.,

(d) $\text{Prob}\begin{bmatrix} \text{more than one casualty} \\ \text{in interval } \Delta t \end{bmatrix} = 0((\Delta t)^2),$

where $0(x)$ denotes dependence on x such that $\lim\limits_{x \to 0} \dfrac{0(x)}{x} =$ const.,

(2) the Y-forces have perfect information as to the state of the system at t and the expected casualty rates,

(3) the Y-forces can instantaneously shift fire from any target at any time,

(4) the length of the battle is known.

Then, we have

<u>state variables</u>: $M_1(t), M_2(t), N(t),$

<u>decision (or control) variable</u>: $\phi_S,$

where

$$\phi_S \in \Phi = \left\{0, \frac{1}{n(t)}, \frac{2}{n(t)}, \ldots, \frac{n(t)-1}{n(t)}, 1\right\}.$$

To be more precise $\phi_S = \phi_S(t, m_1, m_2, n)$ is a closed-loop (or feedback) control.

To develop the <u>fundamental functional equation</u> for the optimal expected-value function, we begin by considering any interval of "backwards time" of length $\Delta\tau$ which occurs from $\tau - \tau\Delta$ to τ. There are five exhaustive and mutually exclusive possibilities for random events to occur in such an interval. These are

(1) one X_1 casualty occurs,

(2) one X_2 casualty occurs,

(3) one Y casualty occurs,

(4) no casualty occurs,

(5) more than one casualty occurs.

Let us now examine each of these cases and develop expected returns.

(1) <u>One X_1 casualty occurs in $\Delta\tau$</u>:

By our assumptions above, we have for the probability of occurrence of this event

$$\text{Prob[one } X_1 \text{ casualty occurs in } \Delta\tau] = \phi a_1 n\Delta\tau.$$

Given that one X_1 casualty is realized in the interval from τ to $\tau - \Delta\tau$, the optimal fire distribution policy for Y will consider the maximum expected value for the return functional as casualties continue to occur randomly from $\tau - \Delta\tau$ to $\tau = 0$. This maximum expected value is $S(\tau-\Delta\tau, m_1(\tau-\Delta\tau), m_2(\tau-\Delta\tau), n(\tau-\Delta\tau))$ where $m_1(\tau-\Delta\tau) = m_1(\tau)-1$, $m_2(\tau-\Delta\tau) = m_2(\tau)$, and $n(\tau-\Delta\tau) = n(\tau)$.

(2) <u>One X_2 casualty occurs in $\Delta\tau$</u>:

Similarly, we have that

$$\text{Prob[one } X_2 \text{ casualty occurs in } \Delta\tau] = (1-\phi)a_2 n\Delta\tau,$$

with the optimal expected-value function $S(\tau-\Delta\tau, m_1(\tau), m_2(\tau)-1, n(\tau))$. Events (3) through (5) are analyzed in a similar fashion.

Now, by the standard dynamic programming argument which combines the probabilities of events (1) through (5) above with the maximum expected return to be achievable given these events occur, we obtain the expression

$$S(\tau,m_1,m_2,n) = \underset{\substack{0 \le \phi_S \le 1 \\ \phi_S \in \Phi}}{\text{maximum}}\{[1-\Delta\tau\{\phi_S a_1 n+(1-\phi_S)a_2 n+b_1 m_1+b_2 m_2\}]S(\tau-\Delta\tau,m_1,m_2,n)$$

$$+\phi_S a_1 n\Delta\tau S(\tau-\Delta\tau,m_1-1,m_2,n) + (1-\phi_S)a_2 n\Delta\tau S(\tau-\Delta\tau,m_1,m_2-1,n)$$

$$+ (b_1 m_1+b_2 m_2)\Delta\tau S(\tau-\Delta\tau,m_1,m_2,n-1)\}. \qquad (35)$$

Rearranging terms in (35) and taking the limit as $\Delta\tau \to 0$, we obtain the <u>fundamental functional equation</u> for the optimal expected-value function

for $m_1, m_2, n > 0$:

$$\frac{dS}{d\tau}(\tau, m_1, m_2, n) = (b_1 m_1 + b_2 m_2)\{S(\tau, m_1, m_2, n-1) - S(\tau, m_1, m_2, n)\}$$

$$+ n \max_{\substack{0 \leq \phi_S \leq 1 \\ \phi_S \in \Phi}} [\phi_S a_1 \{S(\tau, m_1-1, m_2, n) - S(\tau, m_1, m_2, n)\}$$

$$+ (1-\phi_S)a_2 \; S(\tau, m_1, m_2-1, n) - S(\tau, m_1, m_2, n)\}], \qquad (36)$$

with the boundary condition at $t = t_f$

$$S(\tau=0, m_1, m_2, n) = rn - pm_1 - qm_2, \qquad (37)$$

where $m_1, m_2,$ and n are integers and

$$\Phi = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{(n-1)}{n}, 1\}. \qquad (38)$$

Special forms of (36) in which $m_1 = 0$, etc., will be given later.

More concisely, we could have said that (36) results from combination of the well-known formalism of dynamic programming with the retrospective (backward) probabilistic evolution of the system over time (c.f. [13], [22]). It should be noted that (36) is a special case of an equation given by Kushner in 1962 [21].

If we take (36) to be the basic equation for $S(\tau, m_., m_2, n)$, then (35) may be considered to be the simplest finite difference approximation to it, i.e. the result of applying the well-known Euler's method to (36) (see pp. 130-131 of [15]). (Of course, a method employing a higher order

approximation scheme (see pp. 132-140 of [15]) may be necessary under many circumstances.) We will find this point of view convenient when we consider developing a solution to (36).

Alternatively, we could have taken a discrete parameter Markov chain as our basic combat model. It is readily shown that an optimal policy exists for this latter formulation (see Theorem 1 on pp. 88-89 of [22]), and that a policy which yields the maximum in (35) is an optimal policy (see Theorem 2 on p. 89 of [22]).

5.2. <u>On the Analytic Solution of the Fundamental Functional Equation.</u>

The first task in determining an optimal fire distribution policy (which requires obtaining the solution to (36) and (37) is to develop the entire system of equations (c.f. equations (2) through (4)). We must, therefore, develop the form that (36) takes at the boundary of the system, i.e. $m_1 = 0$ or $m_2 = 0$ or $n = 0$, where the fire distribution problem no longer exists. When $n = 0$, arguments similar to the above lead to for $n = 0$, $m_1 \geq 0$, $m_2 \geq 0$,

$$\frac{dS}{d\tau}(\tau,m_1,m_2,0) = 0 \quad \text{with} \quad S(\tau=0,m_1,m_2,0) = -m_1 p - m_2 q,$$

and hence

$$\text{for} \quad n=0, m_1 \geq 0, m_2 \geq 0: \quad S(\tau,m_1,m_2,0) = -m_1 p - m_2 q. \quad (39)$$

Similarly,

$$\text{for} \quad m_1=0, m_2=0, n \geq 0: \quad S(\tau,0,0,n) = nr, \quad (40)$$

$$\text{for} \quad m_1=0, m_2>0, n>0: \quad \frac{dS}{d\tau}(\tau,0,m_2,n) = b_2 m_2 \{S(\tau,0,m_2,n-1)$$

$$- S(\tau,0,m_2,n)\} + a_2 n \{S(\tau,0,m_2-1,n) - S(\tau,0,m_2,n)\}, \quad (41)$$

for $m_1 > 0, m_2 = 0, n > 0$: $\frac{dS}{d\tau}(\tau, m_1, 0, n) = b_1 m_1 \{S(\tau, m_1, 0, n-1)$

$$- S(\tau, m_1, 0, n)\} + a_1 n\{S(\tau, m_1-1, 0, n) - S(\tau, m_1, 0, n)\}. \qquad (42)$$

Equations (36) through (42) are the complete system of equations for the optimal expected-value function in the optimal control of the Lanchester stochastic process.

For $m_1 > 0$, $m_2 > 0$, $n > 0$ the optimal fire distribution policy is determined by the maximization operation in (34), and hence

$$\phi_S^*(\tau, m_1, m_2, n) = \begin{cases} 1 & \text{for } W(\tau, m_1, m_2, n) > 0, \\ 0 & \text{for } W(\tau, m_1, m_2, n) < 0, \end{cases} \qquad (43)$$

where we shall refer to $W(\tau, m_1, m_2, n)$ as the "switching function." It is defined by

for $m_1 > 0$, $m_2 > 0$, $n > 0$,

$$W(\tau, m_1, m_2, n) = a_1\{S(\tau, m_1-1, m_2, n) - S(\tau, m_1, m_2, n)\}$$

$$- a_2\{S(\tau, m_1, m_2-1, n) - S(\tau, m_1, m_2, n)\}. \qquad (44)$$

Let us observe that at the end of the battle at $t = t_f$, we may combine (37), (43), and (44) to obtain

$$\phi_S^*(\tau=0, m_1, m_2, n) = \begin{cases} 1 & \text{for } a_1 p > a_2 q, \\ 0 & \text{for } a_1 p < a_2 q, \end{cases} \qquad (45)$$

which is similar to results for the optimal control of the deterministic process (7) (see, for example, (14), (21), and (22)).

It should be noted that equations (36) through (42) have the same form as those for the Lanchester square-law attrition stochastic process

(i.e. equations (2) through (4) when the attrition rates are given by (6)).[||]

A general solution has not been obtained to these equations. Nevertheless,

it is of value to develop a partial solution. For example, since we use

finite difference methods to generate an approximate solution (see Section

5.3 below), it is desirable to check the adequacy of the approximation (in

particular, the "time step size" used in the numerical propagation of the

approximate solution by "marching ahead in time"). This is easily done by

comparing the approximate solution, denoted as $\hat{S}$, to the exact analytic

solution, denoted as S. Hence, a partial analytic solution is useful.

Careful consideration of (36) through (42) reveals that there are

restrictions on the order in which the optimal expected-value functions

$S(\tau, m_1, m_2, n)$ for $m_1 = 0, 1, 2, \ldots,$ etc., can be computed. In particular

an admissible sequence for building up the solution through $S(\tau, 1, 1, 1)$

is shown below in Table III.

m_1	m_2	n
0	0	0
1	0	0
0	1	0
0	0	1
1	1	0
0	1	1
1	0	1
1	1	1

Table III.

Admissible Order for Computing Optimal Expected-Value
Functions (admissible order is from top to bottom).

[||] We note that (36) becomes a first order system of ordinary differential
equations for $S(\tau, m_1, m_2, n)$ when ϕ_S^* as determined by (43) is used. Solving
for $S(\tau, m_1, m_2, n)$ for $m_1 = 0, 1, 2, \ldots,$ etc., we can then determine ϕ_S^* by
(43). The synthesis of an optimal control by combination of the control law
(43) with integration of a system of differential equations is similar to
that for deterministic optimal control problems.

We readily successively compute using (39) through (42)

$$S(\tau,0,0,0) = 0, \quad S(\tau,1,0,0) = -p, \quad S(\tau,0,1,0) = -q,$$

$$S(\tau,0,0,1) = r, \quad S(\tau,1,1,0) = -p - q,$$

$$S(\tau,0,1,1) = \left(\frac{b_2 r - a_2 q}{a_2 + b_2}\right) e^{-(a_2+b_2)\tau} + \left(\frac{a_2 r - b_2 q}{a_2 + b_2}\right),$$

$$S(\tau,1,0,1) = \left(\frac{b_1 r - a_1 p}{a_1 + b_1}\right) e^{-(a_1+b_1)\tau} + \left(\frac{a_1 r - b_1 p}{a_1 + b_1}\right). \tag{46}$$

Using (46), equations (36) and (37) become for $m_1 = 1$, $m_2 = 1$, $n = 1$,

$$\frac{dS}{d\tau}(\tau,1,1,1) = -(b_1+b_2)\{S(\tau,1,1,1)+(p+q)\} + \underset{\phi_S=0 \text{ or } 1}{\text{maximum}}[\phi_S a_1\{S(\tau,0,1,1)-S(\tau,1,1,1)\}$$

$$+ (1-\phi_S)a_2\{S(\tau,1,0,1)-S(\tau,1,1,1)\}], \tag{47}$$

with

$$S(\tau=0,1,1,1) = r - p - q,$$

where $S(\tau,0,1,1)$ and $S(\tau,1,0,1)$ are given by (46).

Using (43), (44), and (45), we may readily solve (47). As for the deterministic formulation, there are two cases that must be distinguished

Case (1) $\quad a_1 p \geq a_2 q,$

Case (2) $\quad a_1 p < a_2 q.$

For Case (1): $a_1 p \geq a_2 q$, we have that $\phi_S^*(\tau,1,1,1) = 1$ for $0 \leq \tau \leq \tau_1$, where τ_1 denotes the "backwards time" of the first switch in the optimal fire distribution policy. Thus τ_1 is the smallest τ which satisfies $W(\tau=\tau_1,1,1,1) = 0$ with $W(\tau,1,1,1)$ given by (44).

for $0 \le \tau \le \tau_1$ when $a_1 p \ge a_2 q$ $(\phi_S^*(\tau,1,1,1) = 1)$

$$S(\tau,1,1,1) = \frac{a_1(b_2 r - a_2 q)}{(a_1+b_1-a_2)(a_2+b_2)} e^{-(a_2+b_2)\tau} + \left\{ \frac{[(b_1-a_2)(b_1+b_2)+a_1 b_1]r}{(a_1+b_1-a_2)(a_1+b_1+b_2)} \right.$$

$$\left. - \frac{a_1 p}{(a_1+b_1+b_2)} + \frac{a_1 a_2 q}{(a_1+b_1-a_2)(a_1+b_1+b_2)} \right\} e^{-(a_1+b_1+b_2)\tau}$$

$$+ \left\{ \frac{a_1 a_2 r}{(a_2+b_2)(a_1+b_1+b_2)} - \frac{(b_1+b_2)p}{(a_1+b_1+b_2)} - \frac{[(b_1+b_2)(a_2+b_2)+a_1 b_2]q}{(a_2+b_2)(a_1+b_1+b_2)} \right\}. \qquad (48)$$

We note that τ_1 might be equal to $+\infty$, i.e. we never switch. Assuming that a switch in targets does occur, however, let us denote $S(\tau=\tau_1,1,1,1)$ by S_0 where, as we recall, τ_1 is the smallest τ which satisfies $W(\tau=\tau_1,1,1,1) = 0$. Then, we have that $\phi_S^*(\tau,1,1,1) = 0$ for $\tau_1 < \tau \le \tau_2$, where τ_2 denotes the "backwards time" of the second switch in the optimal fire distribution policy. Then, we have

for $\tau_1 < \tau \le \tau_2$ when $a_1 p \ge a_2 q$ $(\phi_S^*(\tau,1,1,1) = 0)$

$$S(\tau,1,1,1) = \frac{a_2(b_1 r - a_1 p)}{(a_2+b_2-a_1)(a_1+b_1)} \left\{ e^{-(a_1+b_1)\tau} - e^{(a_2+b_2)(\tau_1-\tau)-a_1\tau_1-b_1\tau} \right\}$$

$$+ \left\{ S_0 - \frac{a_1 a_2 r}{(a_1+b_1)(a_2+b_2+b_1)} + \frac{[(b_1+b_2)(a_1+b_1)+a_2 b_1]p}{(a_1+b_1)(a_2+b_2+b_1)} + \frac{(b_1+b_2)q}{(a_2+b_2+b_1)} \right\} e^{(a_2+b_2+b_1)(\tau_1-\tau)}$$

$$+ \left\{ \frac{a_1 a_2 r}{(a_1+b_1)(a_2+b_2+b_1)} - \frac{[(b_1+b_2)(a_1+b_1)+a_2 b_1]p}{(a_1+b_1)(a_2+b_2+b_1)} - \frac{(b_1+b_2)q}{(a_2+b_2+b_1)} \right\}. \qquad (49)$$

Again, we note that τ_2 might be equal to $+\infty$, i.e. we might never redistribute fire a second time. Assuming that a second switch in fire distribution does occur, we have $\phi_S^*(\tau,1,1,1) = 1$ for $\tau_2 < \tau \le \tau_3$. We have not carried out the computation of $\dot{S}(\tau,1,1,1)$ past τ_2.

For Case (2): $a_1 p < a_2 q$, the results are symmetric to the above (interchange the roles of X_1 and X_2) and hence are omitted.

Although the above constitutes a complete development for $S(\tau,1,1,1)$ (and hence $\phi_S^*(\tau,1,1,1)$ via $W(\tau,1,1,1)$), these results are complex enough that it is not immediately clear how $\phi_S^*(\tau,1,1,1)$ changes over time and/or depends on model parameters.[||]

5.3. Development of Numerical Solution.

With the advent of modern high-speed digital computers, finite difference methods of obtaining an approximate solution are commonly used when an analytic solution cannot be obtained to equations like (36) through (42). Euler's method (see pp. 130–131 of [15]) yields the simplest finite difference approximation for (36). Let us denote the approximation to the optimal expected value function as $\hat{S}$. We shall compute values for this approximation at discrete points in time separated by a constant amount $\Delta\tau$. We let $\tau = \ell\Delta\tau$ so that $t_f = L\Delta\tau$. Then (36) may be approximated by

for $m_1 > 0$, $m_2 > 0$, $n > 0$:

$$\hat{S}((\ell+1)\Delta\tau,m_1,m_2,n) = \{1-(\Delta\tau)(b_1 m_1 + b_2 m_2)\}\hat{S}(\ell\Delta\tau,m_1,m_2,n) +$$

$$(\Delta\tau)(b_1 m_1 + b_2 m_2)\hat{S}(\ell\Delta\tau,m_1,m_2,n-1) + n(\Delta\tau) \underset{\substack{0\leq\hat{\phi}_S\leq 1 \\ \hat{\phi}_S \in \Phi}}{\text{maximum}}[\hat{\phi}_S a_1\{\hat{S}(\ell\Delta\tau,m_1-1,m_2,n)$$

$$- \hat{S}(\ell\Delta\tau,m_1,m_2,n)\} + (1-\hat{\phi}_S)a_2\{\hat{S}(\ell\Delta\tau,m_1,m_2-1,n) - \hat{S}(\ell\Delta\tau,m_1,m_2,n)\}], \qquad (50)$$

for $\ell = 0,1,\ldots,L-1$ with boundary condition (37) and also (38). Similar approximations may be developed for (41) and (42).

[||] We recall that for the deterministic formulation when $x_1(t_f) > 0$ and $x_2(t_f) > 0$, the conditions $a_1 p \geq a_2 q$ and $a_1 b_1 > a_2 b_2$ implied that $\phi_D^*(\tau,x_1,x_2 y) = 1$ for the entire battle.

As noted above, consideration of (36) through (42) yields that there are restrictions on the order in which the optimal expected-value function S (or its approximation $\hat{S}$) is computed. The computation of $\hat{S}((\ell+1)\Delta\tau, m_1, m_2, n)$ depends upon the quantities shown in Figure 1 below.
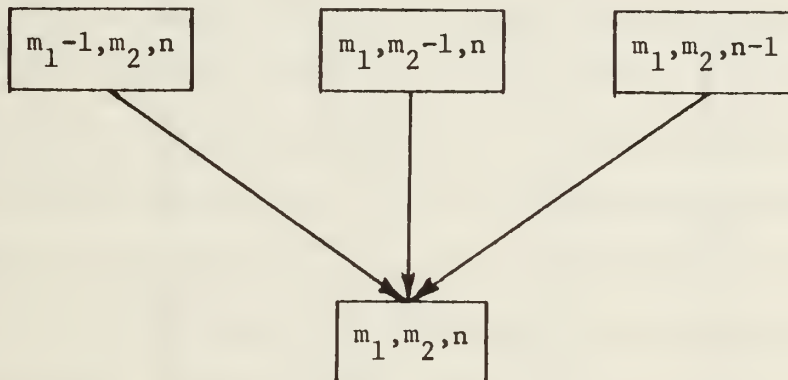
Figure 1. Dependence of Optimal Expected-Value Function on Discrete State Variables.

Based on the dependence depicted in Figure 1, the solution can be "built-up" as shown in Table IV.

It remains to discuss the adequacy of the finite difference approximation (50). It is well-known (see pp. 130-145 in [15]) that Euler's method yields a finite difference approximation for such a system of differential equations that is both consistent and stable so that the approximate solution $\hat{S}$ can be guaranteed to converge to the exact analytic solution S as $\Delta\tau \to 0$ (and $L \to \infty$) [28]. However, $\Delta\tau$ must not be too large in order to keep the truncation error satisfactorily small. Moreover, the time step size $\Delta\tau$ is also limited by the fact quantities like $(\Delta\tau)(b_1 m_1 + b_2 m_2)$ or $a_1 n\Delta\tau$ or $a_2 n\Delta\tau$ in (50) represent probabilities and hence must be less than one. In our computational work[||] we have used a

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[||] A computer program has been written in FORTRAN for this purpose.

Table IV. Admissable Order for Computing Optimal
Expected-Value Functions.

m_1	m_2	n	m_1	m_2	n
0	0	0	3	1	0
1	0	0	3	2	0
0	1	0	1	3	0
0	0	1	2	3	0
1	1	0	3	3	0
0	1	1	0	3	1
1	0	1	0	3	2
1	1	1	0	1	3
2	0	0	0	2	3
0	2	0	0	3	3
0	0	2	3	0	1
2	1	0	3	0	2
1	2	0	1	0	3
2	2	0	2	0	3
0	2	1	3	0	3
0	1	2	3	1	1
0	2	2	3	2	1
2	0	1	3	1	2
1	0	2	3	2	2
2	0	2	1	3	1
2	1	1	2	3	1
1	2	1	3	3	1
1	1	2	1	3	2
2	2	1	2	3	2
1	2	2	1	1	3
2	1	2	2	1	3
2	2	2	3	1	3
3	0	0	1	2	3
0	3	0	1	3	3
0	0	3	2	2	3
			3	3	2
			2	3	3
			3	2	3
			3	3	3
			4	0	0
			0	4	0

etc.

Note: Admissible order is top to bottom, starting with
column (composed of m_1, m_2, n) on left.

time step size which yields agreement in the fourth decimal place to the right of the decimal point when $\hat{S}$ is compared to the exact analytic solution S in the special cases (such as (48) and (49)) when the latter has been obtained.

6. Comparison of Results from Deterministic and Stochastic Formulations.

In this section we compare the structures of the optimal fire distribution policy between the deterministic control problem (7) and the stochastic control problem (8). Before presenting this comparison, it seems appropriate to discuss some general methodological considerations.

Any comparison between the two models should be guided be the purpose of the comparison. In the paper at hand our purpose is to consider whether the structure of the optimal fire distribution policy is the same for the two formulations. In other words, we would like to determine upon what groups of model parameters the optimal allocation rule depends and whether this depends upon the particular form of model adopted (here deterministic or stochastic). The things that can be compared between the two models are (1) the optimal fire distribution policy and (2) the optimal (expected) return. It is the opinion of the authors that the second criterion (i.e. optimal return) is only significant when there are differences between the optimal policies from the two models. Furthermore, there are two types of comparisons that we can make between the models: one is quantitative and the other is qualitative.

A direct quantitative comparison of the optimal policies[||] obtained from the two formulations is impossible: on the one hand for the deterministic

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[||]The only papers known to the authors in which a quantitative comparison between results for deterministic and stochastic optimal control problems is made are [48] and [49]. In both papers the state space is continuous in the stochastic problem.

model we have a piecewise differentiable battle trajectory, while on the
other hand for the stochastic model we have a discontinuous Markov process
describing the force levels. Thus, we have $\phi_D^*(t,x_1,x_2,y)$ for the deter-
ministic formulation with x_1, x_2, and y varying continuously over time,
and we have $\phi_S^*(t,m_1,m_2,n)$ for the stochastic formulation with m_1, m_2,
and n restricted to be non-negative integers and casualties occurring
randomly as a Markov jump process. The impossibility of directly comparing
$\phi_D^*(t,x_1,x_2,y)$ and $\phi_S^*(t,m_1,m_2,n)$ continuously over time should be apparent.

Nevertheless, we can still <u>qualitatively</u> compare the structures of
the two policies. There is, however, a difficulty in that $\phi_S^*(\tau,m_1,m_2,n)$
represents a conditional policy, i.e. the optimal policy given that the
system is in state (m_1,m_2,n) with "backwards time" τ remaining in the
battle. When a state transition occurs (randomly) to (m_1',m_2',n'), then
the optimal policy accordingly becomes $\phi_S^*(\tau,m_1',m_2',n')$. In comparing the
optimal policies this should be taken into account, since it does not seem
appropriate to compare $\phi_S^*(\tau,m_1^o,m_2^o,n_0)$ with m_1^o, m_2^o, and n_0 held con-
stant to $\phi_D^*(\tau,x_1,x_2,y)$ with x_1, x_2, and y changing (continuously)
over time. Since for the stochastic formulation it does not make sense
to consider an "average" optimal policy or the optimal policy for "average"
force levels, for comparison with the optimal policy for the deterministic
formulation we have considered a realization of the stochastic attrition
process in which the force levels are always "near to" those of the corre-
sponding deterministic process. In other words, we will compare $\phi_D^*(\tau,x_1,x_2,y)$
to $\phi_S^*(\tau,m_1,m_2,n)$ at selected values of x_1, x_2, and y. The force levels
in the deterministic model are rounded to integers to yield the values of
m_1, m_2, and n as follows: $m_1 = [x_1] + 1$ (and $m_1 = 0$ when $x_1 = 0$)

where [x] denotes "the greatest integer in x," i.e. [3.96] = 3.[||]

Moreover, in our comparison we will try to use the results obtained from the deterministic formulation to gain insight into the behavior of the optimal policy for the stochastic control problem. In other words, we will try to explain results from the stochastic formulation by considering the corresponding behavior for the deterministic formulation.

Numerical results have been generated using two FORTRAN programs run on an IBM 360-67 computer. The program which generates $\phi_D^*(t,x_1,x_2,y)$ (and also the force level trajectories) has been discussed in Section 4.5. The program which generates $\phi_S^*(t,m_1,m_2,n)$ performs the computations described in Section 5.3. The program for the stochastic formulation is limited by computer memory requirements. Results for all force levels are retained for two time steps. A battle with $m_1^o = 5$, $m_2^o = 5$, and $n_0 = 5$ requires 200,000 bytes of computer memory, and this increases exponentially with the force levels as Table IV indicates. Thus, most runs of the computer program for the stochastic formulation have been with the above as the upper limit for initial force levels, although we have run one case with $m_1^o = 9$, $m_2^o = 9$, and $n_0 = 9$ which required nearly 2,000,000 bytes of memory.

The above computer programs have been run for over fifteen different "parameter sets," typical examples of which are shown in Table V. In all cases we have chosen parameter values so that $a_1 b_1 > a_2 b_2$. The optimal policies for the deterministic and the stochastic formulations have been compared as discussed above. The results of these comparisons will now be summarized.

[||] This is done so that an interval process (time between casualties) of the casualty process will be "similar" in the deterministic and stochastic formulations.

Table V. Parameter Sets Used to Generate Numerical
Results Given in Tables VI through VIII.

Parameter Set	a_1	a_2	b_1	b_2	p	q	r
1	0.025	0.015	0.035	0.005	0.75	2.25	2.0
2	0.005	0.003	0.007	0.001	0.15	0.45	0.4
3	0.085	0.080	0.03	0.03	1.0	2.0	2.0

Note: For all the above parameter sets we have $a_1 b_1 > a_2 b_2$ and $a_1 p < a_2 q$.

The first thing to be pointed out is that the optimal fire distribu-
tion policy for both formulations has the property that ϕ^* is either
0 or 1 (almost everywhere in time).[||] For the deterministic formulation,
we have shown [34] that a <u>singular solution</u> is impossible and that ϕ_D^*
must be 0 or 1 except for at most one point in time. Although we have not
proved such a result for the stochastic formulation, we have never encountered
any exception to it in all our numerical computations. As we have discussed
above, two cases must be distinguished:

Case (1) $a_1 p \geq a_2 q$,

Case (2) $a_1 p < a_2 q$.

For Case (1): $a_1 p \geq a_2 q$, the optimal policy is apparently identical
for both formulations: $\phi_D^*(t,x_1,x_2,y) = \phi_D^*(t,m_1,m_2,n) = 1$ for $x_1 > 0$
(or $m_1 > 0$). We recall that this result has been proved for the determi-
nistic formulation. Although a proof has not been found, it apparently
is also true for the stochastic formulation. No exception has been encoun-
tered in all the cases for which numerical determinations have been made.

[||] See [36] for a discussion of why this is so and for an example of a similar
problem with a different attrition process for which ϕ^* may take on an
intermediate value, i.e. $0 < \phi^* < 1$ (see also [38]).

For Case (2): $a_1 p < a_2 q$, the optimal policies are similar but not identical. The basic structures are apparently essentially the same. As discussed above, the two policies have been compared at selected points along a deterministic trajectory by considering a corresponding realization of the stochastic process obtained by rounding the deterministic force levels. The time of such a comparison is rounded up to the next whole minute in the case of the occurrence of a casualty and to the next 0.01 minute in the case of a switch in fire distribution. Cases corresponding to over ten parameter sets have been considered; illustrative examples of such parameter sets are shown in Table V.

In Table VI we show some typical comparisons. Although not shown in Table VI, it should be noted that in all the cases numerically computed $\phi_S^*(\tau,m_1,m_2,n)$ had the property that for <u>constant</u> m_1, m_2, and n

$\phi_S^*(\tau,m_1,m_2,n) = 0$ for $0 \le \tau < \tau_1$ and $\phi_S^*(\tau,m_1,m_2,n) = 1$ for $\tau_1 > \tau$

where τ denotes the "backwards time." In Table VI we show the optimal policies for the two formulations for two parameter sets. The optimal policies are given at discrete points in time following the above discussion. These times correspond to a switching time in one of the formulations or the occurrence of a casualty in the "typical" realization of the stochastic process. The deterministic force levels x_1, x_2, and y from which m_1, m_2, and n have been determined are not shown in Table VI. The optimal returns for the two formulations are also shown.

The results shown in Table VI are typical and indicate (at least for all the cases so far computed) that there is no fundamental difference between the structures of the two optimal policies, at least where the deterministic battle does not terminate prematurely, i.e. $t_f = t_{max}$.||

|| Thus, these remarks apply to cases in which optimal deterministic trajectories lead to terminal states S_1, S_2, and S_3.

Table VI.   Comparisons of Results from Deterministic

and Stochastic Optimal Control Problems

(Deterministic Trajectory Leads to Terminal State S1)

| Parameter Set 1 |

Elapsed Time, t (minutes)	Force Levels m_1	m_2	n	$\phi_D^*(t,x_1,x_2,y)$	Deterministic Optimal Return	$\phi_S^*(t,m_1,m_2,n)$	$S(t,m_1,m_2,n)$
0	2	5	3	1	-10.95	1	-8.93
13	2	5	2	1	-10.95	1	-11.16
18	1	5	2	1	-10.95	1	-9.12
31	1	5	1	1	-10.95	1	-10.96
35.39	1	5	1	1	-10.95	0	-10.79
41.28	1	5	1	0	-10.95	0	-10.54
$50=t_f=t_{max}$	1	5	1	0	-10.95	0	-10.00

| Parameter Set 2 |

Elapsed Time, t (minutes)	Force Levels m_1	m_2	n	$\phi_D^*(t,x_1,x_2,y)$	Deterministic Optimal Return	$\phi_S^*(t,m_1,m_2,n)$	$S(t,m_1,m_2,n)$
0	5	5	5	1	-2.06	1	-0.62
27	5	5	4	1	-2.06	1	-2.17
50	4	5	4	1	-2.06	1	-1.67
55	4	5	4	1	-2.06	0	-1.64
56	4	5	3	1	-2.06	0	-2.06
56.38	4	5	3	0	-2.06	0	-2.05
87	4	5	2	0	-2.06	0	-2.18
$100=t_f=t_{max}$	4	5	2	0	-2.06	0	-2.05

| Parameter Set 2 |

Elapsed Time, t (minutes)	Force Levels m_1	m_2	n	$\phi_D^*(t,x_1,x_2,y)$	$\phi_S^*(t,m_1,m_2,n)$
0	5	5	5	1	1
5.61	5	5	5	1	0
6.38	5	5	5	0	0
26	5	5	4	0	0
$50=t_f=t_{max}$	5	5	4	0	0

The reader should note that ϕ_S^* changes somewhat earlier in forward time from 1 to 0 than does ϕ_D^* (at least for the realization of the stochastic process considered here).

In cases in which the deterministic battle ends prematurely (i.e. the optimal trajectory leads to S_4, S_5, S_6, or S_7) more pronounced quantitative differences may occur. This is illustrated by the cases shown in Table VII. As noted above, the deterministic trajectory determines at which values of m_1, m_2, and t we look at ϕ_S^*. This should explain to the reader why the stochastic results shown in Table VII are not realizable.[||] Thus, for the first battle shown in Table VII, a realization of the stochastic battle would evolve differently (in structure) than the deterministic battle due to this difference in the optimal controls. The authors feel that this is due to the fact that Y marginally wins the deterministic battle, and thus in the stochastic model there is a fairly good probability[¶] at t much less than t_{max} that Y will lose the battle. In other words, there are some possible probabilistic trajectories which yield a reduced payoff to Y. These are weighted in the stochastic decision process, and Y consequently follows a more conservative policy for the stochastic formulation. For the case of the first battle shown in Table VII, Y essentially gives up his chances of winning to guarantee a given level of return. This phenomenon is similar to the "flypaper effect" noted by Whittle [48] in certain stochastic optimal control problems. In the second battle shown, Y achieves a clear-cut victory in the deterministic battle, and this phenomenon does not occur.

[||]A transition from $(m_1, m_2, n) = (3,5,5)$ to $(2,5,5)$ is impossible when $\phi_S^* = 0$.
[¶]This probability has not been explicitly determined.

Table VII.   Comparisons of Results from Deterministic

and Stochastic Optimal Control Problems

(Deterministic Trajectory Leads to Terminal State S7)

Parameter Set 3 t_{max} = 50 minutes

Elapsed Time, t (minutes)	m_1	m_2	n	$\phi_D^*(t,x_1,x_2,y)$	$\phi_S^*(t,m_1,m_2,n)$
0	3	5	5	1	0
3	2	5	5	1	0
5	2	5	4	1	0
6	1	5	4	1	0
8.59	0	5	4	0	0
11	0	5	3	0	0
13	0	4	3	0	0
18	0	3	3	0	0
21	0	3	2	0	0
24	0	2	2	0	0
31	0	1	2	0	0
40.1=t_f	0	0	2	0	0

Parameter Set 3

Elapsed Time, t (minutes)	m_1	m_2	n	ϕ_D^*	ϕ_S^{50*}	ϕ_S^{40*}	ϕ_S^{30*}	ϕ_S^{20*}
0	2	3	5	1	1	1	1	0
3	1	3	5	1	1	1	0	0
5.04	0	3	5	0	0	0	0	0
8	0	2	5	0	0	0	0	0
11	0	1	5	0	0	0	0	0
14	0	1	4	0	0	0	0	0
14.11=t_f	0	0	4	0	0	0	0	0

Note: ϕ_S^{40*} denotes $\phi_S^*(t,m_1,m_2,n)$ computed with t_{max} = 40 minutes.

In addition, in cases in which there is a premature termination in the deterministic formulation, the optimal policy for Y in the corresponding stochastic problem is affected by the length of the "perceived planning horizon." This effect is shown in the data for the second battle of Table VII in which optimal policies are given for stochastic battles of varying lengths. We see that when the deterministic battle ends near to the scheduled end of the stochastic battle, Y follows a more conservative policy in the stochastic battle. Since there is some chance that Y cannot annihilate the X forces in the "perceived length of battle," he follows a conservative policy of firing at X_2. This might, in fact, explain the results for the first battle. Other similar phenomena have been encountered in cases not shown here.

Finally, in Table VIII we show that the optimal policy followed by Y in a realization of the stochastic combat process may differ appreciably from that for the deterministic formulation if the realization does not "follow" the deterministic trajectory. It is seen that ϕ_S^* may repeatedly switch back and forth from 0 to 1 for certain realizations of the stochastic process. This is quite different than the corresponding behavior for the deterministic version.

7. Discussion.

In this section we discuss what we have learned from the above comparison. First and foremost, the authors feel that the deterministic formulation provides more insight into the structure of the optimal fire distribution policy. The explicit dependence of the optimal control upon various parameter groups (these are (1) $R = a_1 b_1 / (a_2 b_2)$, (2) $\delta = a_1 p / (a_2 q)$,

Table VIII. One Possible Dependence of Optimal Stochastic
Control on Realization of Casualties in
Stochastic Lanchester Attrition Process
(Deterministic Trajectory Leads to Terminal State S7; See Table VII.)

$\boxed{\text{Parameter Set 3}}$ t_f = 50 minutes

Elapsed Time, t (minutes)	Force Levels			$\phi_S^*(t, m_1, m_2, n)$
	m_1	m_2	n	
0	3	5	5	0
0.5	3	4	5	0
0.7	3	3	5	1
10.0	2	3	5	1
15.0	2	3	4	0
20.0	2	2	4	1
23.55	2	2	4	0
24.0	2	2	3	0
25.0	2	1	3	1
26.0	1	1	3	1
30.0	1	1	2	0
35.0	1	0	2	1

and (3) $\alpha = \frac{r}{q}\sqrt{\frac{b_2}{a_2}}$) is readily obtained for the deterministic optimal

control problem. This has not been true for the stochastic problem for

which only the dependence upon δ has been analytically obtained.

Let us now summarize the observed differences and similarities

between the structures of the optimal policies for the deterministic and

stochastic formulations. The similarities are: (1) optimal policy always

0 or 1, (2) same parameter groups $(R, \delta,$ and $\alpha)$ upon which optimal

policy depends, (3) optimal policy dependent upon force levels and

whether Y wins or loses, (4) in both models $\phi^* = 1$ for $x_1 > 0$ when

$\delta \geq 1$ and $R > 1$, and (5) $\phi^* = 0$ for $t \in (T-\tau_1,T]$ when $0 \leq \delta < 1 < R$;

furthermore $\tau_1 = \tau_1(\alpha).$[¶] The differences are: (1) in the stochastic

formulation the optimal policy actually implemented (i.e. followed) in a

battle depends upon the battle's probabilistic (forward) evolution (i.e.

the realization of the stochastic process) and the time remaining in the

prescribed duration battle, and (2) τ_1 is "greater in the stochastic

model" except for cases corresponding to premature termination in the

deterministic battle. Overall, we feel that an understanding of the

structure of an optimal policy is best developed by considering the

deterministic version of such a combat problem. For problems too complex

for analytic treatment, rules of thumb for approximating an optimal policy

are probably best obtained from deterministic formulations.

[||] In [34] and [36] one can find further discussion of the structure of the optimal policy, including interpretation of such parameter groups. The reader may find the following interpretations useful for understanding the solution to the problem studied in the paper at hand. The quantity $a_1 b_1$ may be thought of as the rate of destroying X_1's kill capability against Y. It is a measure of strategic (long run) return. The quantity $a_1 p$ represents the rate of destruction of X_1 value by Y at the end of battle. Thus, it represents short run return. The quantity $r\sqrt{b_2}$ reflects the loss of Y value at the end of battle so that α measures the loss of Y value relative to that of X_2 at the end of battle.

[¶] Moreover, τ_1 depends upon $m_1, m_2,$ and n in the stochastic optimal control problem.

Finally, we would like to point out that there is a circumstance under which the stochastic formulation is to be preferred over the deterministic one. This is, namely, when there is a small number (approximately three or under) of each combatant type. As noted above, obtaining a numerical approximate solution to the optimal stochastic control problem is limited to small numbers of combatants due to computer memory requirements.[||] In such cases, however, of small numbers of combatants (and a stochastic attrition process), the stochastic formulation as a Markov chain is to be preferred when the required computer resources are available for the obvious reason that the deterministic differential equation model cannot adequately describe the situation. This point made comparison of results from the two formulations difficult.

8. Implications for Defense Planners.

The authors feel that the study of even the very simplest abstractions (idealizations) of tactical allocation structures as considered in this paper has yielded significant implications for defense planners and military operations analysts. First and foremost is the fact that study of such deterministic optimal control problems provides much more insight into the structure of optimal allocation policies than corresponding stochastic formulations. We feel that such deterministic formulations provide a better understanding of the effects of modelling assumptions on optimal military strategies derived from the mathematical models. This is, of course, essential for determining optimal (or near-optimal) solutions to real world problems that are far too complex to be solved by exact analytic methods.

[||] These grow exponentially as force levels increase because of the way in which a solution must be "built up." See Figure 1 and Table IV for illustrations of this point.

Moreover, one might apply general principles or rules of thumb developed from the study of such idealizations to higher resolution studies which, for example, might use computer simulation methods.

The study of the deterministic optimal control problem (7) in this paper yields several significant results which should be kept in mind by practitioners who perform more detailed computer simulation studies. These are

(1) Force levels do affect optimal strategies. Whether one "wins" or "loses" affects optimal strategies.

(2) Even the nature of the scenario (terminal control or prescribed duration conflict) may affect optimal strategies. This, if one develops "good" tactics for a 90 day compaign, such tactics need not be "good" if the conflict does not terminate at the prescribed time.

(3) The nature of the attrition process has a significant effect upon optimal strategies.[||]

Finally, the authors feel that the above results indicate that more basic research should be done on the termination of battles and wars[¶] as well as combat attrition theories. The demonstrated sensitivity of results obtained from optimization problems like the one considered here shows this.

[||] This result has been pointed out elsewhere [36], [38] and is partially based on the study of a similar problem [38].

[¶] Some work has been done in this direction [14], [33], [46], [47], although it does not appear to be widely known among practicing analysts.

APPENDIX. Explanation of Notation.

The symbols which are used in this paper are defined as follows:

a_1, a_2, b_1, b_2 = constant attrition-rate coefficients,

$A(m,n), B(m,n)$ = attrition rates of X and Y forces, respectively, in stochastic battle; it should be noted that

$$\text{Prob}\left[\begin{array}{l}\text{one X casualty in time}\\ \text{interval from t to } t + \Delta t\end{array}\right] = A(m,n)\Delta t,$$

$E_{\underset{\sim}{m},\tau}[\cdot]$ = conditional expectation (mathematical expectation of quantity in brackets at $\tau = 0$ given that at τ we have $\underset{\sim}{m}(\tau) = (m_1(\tau), m_2(\tau), n(\tau))$),

H = Hamiltonian function,

$M_1(t), M_2(t), N(t)$ = the numbers (a random variable) of X_1, X_2, and Y combatants, respectively, at time t,

m_1, m_2, n = realizations of the random variables $M_1(t)$, $M_2(t)$, and $N(t)$; initial values denoted as m_1^o, m_2^o, n_0,

p, q, r = utilities assigned to surviving X_1, X_2 and Y forces respectively,

$p_i(t)$ for $i = 1, 2, 3$, = dual variable corresponding to $x_i(t)$ $(x_3(t) = y(t))$,

$\underset{\sim}{p} = (p_1, p_2, p_3)$ (a vector),

$P(t,m,n) = \text{Prob}[M(t)=m, N(t)=n]$ = state probability,

$P^o = (x_1^o, x_2^o, y_0)$ = point in the initial state space,

$R = a_1 b_1 / (a_2 b_2)$,

$S(\tau, m_1, m_2, n)$ = optimal expected value function,

$\hat{S}$ = numerical approximation to $S(\tau, m_1, m_2, n)$,

S_i for $i = 1, \ldots, 8$ = the ith part of the terminal surface as defined in Table I.

$$s = s(x_1^o, x_2^o) = b_1 x_1^o + b_2 x_2^o,$$

t = time after beginning of battle,

t_1 = time at which X_1 is annihilated, i.e. $x_1(t_1) = 0$,

t_2 = first time at which $2b_1 x_1(t_2) x_2^o + b_2(x_2^o)^2 = a_2 y^2(t_2)$ for an extremal leading to S_6,

t_3 = last time at which fire is directed at X_1 for an extremal leading to S_3,

t_4 = time at which X_2 is annihilated (before X_1), i.e. $x_2(t_4) = 0$, for an extremal leading to S_8,

t_f = time at which battle ends,

t_{max} = maximum possible duration for battle, i.e. $t_f \leq t_{max}$,

$v = v(\tau) = a_2 p_2(\tau) - a_1 p_1(\tau)$,

$W(\tau, m_1, m_2, n)$ = "switching function" defined by equation (44),

x_1, x_2, y = average force strengths; with initial values x_1^o, x_2^o, y_0,

$$z = \cosh\sqrt{a_2 b_2}\ \tau_1\ (S_4) = \frac{R-\delta}{R-1}\ ,$$

$$\alpha = \frac{r}{q}\sqrt{\frac{b_2}{a_2}}\ ,$$

$$\delta = a_1 p / (a_2 q),$$

$\eta_i(t)$ for $i = 1,2$, = multiplier corresponding to state variable inequality constraint $x_1 \geq 0$,

ν_i for $i = 1,2$, = multiplier corresponding to state variable terminal inequality constraint $x_i(T) \geq 0$,

$\phi_D(\phi_S)$ = fraction of Y-fire directed at X_1 in deterministic (stochastic) formulation; extremal and optimal controls denoted as $\phi_D^*(\phi_S^*)$,

$$\Phi = \{0, \frac{1}{n(t)}, \frac{2}{n(t)}, \ldots, \frac{n(t)-1}{n(t)}, 1\} = \text{set of admissible controls in stochastic problem,}$$

τ = "backwards time" from the end of battle defined by $\tau = t_f - t$, i.e. the time remaining before the end of battle,

$\tau_1(S_i)$ = "backwards time" of the first switch in tactics for extremals leading to S_i.

Additionally, remarks similar to those for $\tau_1(S_i)$ above apply to $t_1(S_i)$, $t_f(S_i)$, etc.

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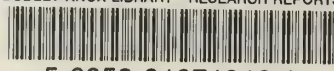|	No. of Copies
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