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Abstract— **We consider a fairly general class of state-constrained nonlinear hybrid optimal control problems that are based on coordinatizing Sussmann's model. An event set generalizes the notion of a guard set, reset map, endpoint set as well as the switching set. We present a pseudospectral (PS) knotting method that discretizes the continuous-time variables of the problem. The discrete event conditions are imposed over the PS knots leading to a large, sparse, mixed-variable programming (MVP) problem. The Karush-Kuhn-Tucker conditions for the MVP are transformed in a manner that makes them closely resemble the discretized necessary conditions obtained from the Hybrid Minimum Principle. A set of closure conditions are introduced to facilitate commuting the operations of dualization and discretization. An immediate consequence of this is a Hybrid Covector Mapping Theorem that provides an order-preserving transformation of the Lagrange multipliers associated with the discretized problem to the discretized covectors associated with the hybrid optimal control problem.**

I. Introduction

A fairly large class of complex control problems can be described under a unified framework of hybrid optimal control [4]. Solving a hybrid optimal control problem is an extremely challenging task since even a smooth, nonlinear, ordinary (i.e. non-hybrid) optimal control problem is still widely considered to be quite difficult to solve [26]. A significant source of difficulty arises from a need to obtain feedback solutions by solving the Hamilton-Jacobi-Bellman (HJB) equations. As is well-known [1], [5], [8], the HJB approach is beset with fundamental problems, such as the nonsmoothness of the value function [8] and the famous "curse of dimensionality". An alternative approach is the Hybrid Minimum (Maximum) Principle [23], [24], [25]. Although this approach is more tractable than the HJB approach, it generates only open-loop controls. The Minimum-Principle approach is also fraught with fundamental computational problems due to the fact that the costates are adjoint to the state perturbation equations [5]. In other words, the dynamics-adjoint equation pair typically generate a numerically sensitive multi-point boundary value problem that may produce such wild trajectories as to exceed the numerical range of the computer [5]. To overcome this difficulty, direct methods have been employed to solve smooth optimal control problems arising in engineering applications [2], [9], [10]. The main advantage of direct methods is that they facilitate solving the optimal control problem without requiring a development of the necessary conditions such as the adjoint equations or complex switching conditions in dual space. An extension of direct collocation methods for solving hybrid optimal control problems has been recently proposed by von Stryk and his colleagues [6], [7], [12], [27], [28] by incorporating integer programming techniques with sparse nonlinear programming. While significant research still needs to be done to develop efficient algorithms to solve the resulting large, sparse, mixed-integer nonlinear programming problem, it is possible, in principle, to achieve feedback solutions by predictive control techniques. In many applications, such as interplanetary spaceflight [22], the time-constants are large enough that feedback solutions, via real-time computation, are quite feasible with current hardware [19]. In this spirit, we present a Legendre pseudospectral (PS) knotting method as a means to verify the optimality conditions associated with a purportedly optimal hybrid trajectory. A key element of the Legendre PS method is the imposition of closure conditions [17], [18] which allows one to commute the operations of dualization and discretization so that the transformed Lagrange multipliers can be related to the discretized covectors associated with the Minimum Principle. This notion was exploited in [18] for the "smooth" optimal control problem formulation and the current paper extends this concept to hybrid optimal control problems.

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II. Hybrid System

Although there are many ways to model a hybrid system, we adopt Sussmann's model [23], [24], [25] as it is readily amenable to an application of the Hybrid Minimum Principle. Since our focus is largely practical applications, we coordinatize Sussmann's coordinate-free descriptions by way of functional inequalities. Except for the state- and control functions, $\boldsymbol{x}(\cdot)$ and $\boldsymbol{u}(\cdot)$, all functions are assumed to be piecewise differentiable; however, note that inequalities on the functions imply the inclusion of nonsmooth objects. We consider problems defined over a finite horizon and hence the time-dependent relations are assumed to hold for almost all t over this horizon. With these preliminaries in mind, let $\mathcal{Q}$ be a given finite set of cardinality, $N_Q \in \mathbb{N}$. The members of $\mathcal{Q}$ are called locations. For each $q \in \mathcal{Q}$, consider a continuous-time controlled dynamical system,

$$\dot{\boldsymbol{x}}(t) = \mathbf{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), q) \tag{1}$$

where $\mathbf{f}(\cdot, q) : \mathbb{R}^{N_x^q} \times \mathbb{R}^{N_u^q} \to \mathbb{R}^{N_x^q}$ is a controlled vector field indexed by $\mathcal{Q}$, $N(\cdot) \in \mathbb{N}$, while $\boldsymbol{x} \in \mathbb{R}^{N_x^q}$ and $\boldsymbol{u} \in \mathbb{R}^{N_u^q}$ are the continuous-time state and control variables respectively. Similarly, we define a (hybrid) path constraint as,

$$\mathbf{h}(\boldsymbol{x}(t), \boldsymbol{u}(t), q) \leq \mathbf{0} \tag{2}$$

where $\mathbf{h}(\cdot, q) : \mathbb{R}^{N_x^q} \times \mathbb{R}^{N_u^q} \to \mathbb{R}^{N_h^q}$, and $\leq \mathbf{0}$ implies a component-wise inequality. Although practical problems [20] have two-sided inequality constraints with lower and upper bounds, it suffices to consider a one-sided inequality like (2) for theoretical purposes since a two-sided inequality may easily be transformed to a one-sided form. Note also that (2) includes a coordinatization of the invariant set [14] (or the domain), $Inv : \mathcal{Q} \to 2^{\mathbf{X}}$, given by,

$$Inv(q) = \left\{ x \in \mathbb{R}^{N_x^q} : \mathbf{h}(x, u, q) \leq \mathbf{0} \right\} \tag{3}$$

with $\mathbb{X} = \mathbb{R}^{N_x^q}$. Now, let (x, u) and (x', u') denote the continuous-time state and control variables associated with two

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locations, $q, q' \in \mathcal{Q}$. When it is not an empty, the Event Set, $\mathbb{E}(q,q')$, is defined as,

$$\begin{aligned}\mathbb{E}(q,q') &= \{(\boldsymbol{x},\boldsymbol{u},\tau,\boldsymbol{x}',\boldsymbol{u}',\tau') : \\ &\quad \mathbf{e}(\boldsymbol{x},\boldsymbol{u},\tau,\boldsymbol{x}',\boldsymbol{u}',\tau',q,q') \leq \mathbf{0}\} \end{aligned} \tag{4}$$

where $\mathbf{e}(\cdot,q,q') : \mathbb{R}^{N_x^q} \times \mathbb{R}^{N_u^q} \times \mathbb{R} \times \mathbb{R}^{N_x^{q'}} \times \mathbb{R}^{N_u^{q'}} \times \mathbb{R} \to \mathbb{R}^{N_e^{qq'}}$ is called the event function [20] associated with q and q'. Similar to the reasons argued for the path constraints, we consider one-sided inequalities without loss in generality. In an event set, the clock is allowed to be reset (i.e. allowing $\tau \neq \tau'$) so that we can treat the endpoint constraint set in exactly the same manner as a switching set [24]. The clock resets also allow us to efficiently handle certain computational complexities as described in [20] and illustrated in [22]. The event set generalizes the notion of a Guard Set, $\mathcal{G} : \mathcal{Q} \times \mathcal{Q} \to 2^{\mathbb{X}}$, and Reset Map, $\mathcal{R} : \mathcal{Q} \times \mathcal{Q} \times \mathbb{X} \to 2^{\mathbb{X}}$, as these are given by,

$$\begin{aligned}\mathcal{G}(q,q') &= \left\{\boldsymbol{x} \in \mathbb{R}^{N_x} : (\boldsymbol{x},\boldsymbol{u},\tau,\boldsymbol{x}',\boldsymbol{u}',\tau') \in \mathcal{S}(q,q')\right\} \\ \mathcal{R}(q,q',\boldsymbol{x}) &= \left\{\boldsymbol{x}' \in \mathbb{R}^{N_x} : (\boldsymbol{x},\boldsymbol{u},\tau,\boldsymbol{x}',\boldsymbol{u}',\tau') \in \mathcal{S}(q,q')\right\}\end{aligned}$$

where $\mathcal{S}(q,q') \subset \mathbb{E}(q,q')$ is the Switching Set,

$$\begin{aligned}\mathcal{S}(q,q') &= \{(\boldsymbol{x},\boldsymbol{u},\tau,\boldsymbol{x}',\boldsymbol{u}',\tau') \in \mathbb{E}(q,q') : \\ &\quad \tau = \tau' \in \mathbb{R}\}\end{aligned}$$

If $\mathcal{S}(q,q') \neq \emptyset$, then $(q,q') \in \mathcal{Q} \times \mathcal{Q}$ is an edge of a digraph whose vertices are given by $\mathcal{Q}$. Finally, for each $q \in \mathcal{Q}$, we associate a running cost,

$$F(\cdot,q) : \mathbb{R}^{N_x^q} \times \mathbb{R}^{N_u^q} \to \mathbb{R}$$

while for any pair, (q,q'), we associate an event cost,

$$E(\cdot,q,q') : \mathbb{E}(q,q') \to \mathbb{R} \cup \{\infty\}$$

that takes the value ∞ whenever $\mathbb{E}(q,q') = \emptyset$. In a practical (computational) setting, we handle the evaluation, ∞, for the switching set $\mathcal{S}(q,q')$, by simply setting the corresponding element of the adjacency matrix to zero (see [21]) so that a transition from q to q' is disallowed.

III. Hybrid Bolza Problem (Problem $\mathcal{H}$)

Let $\mathbf{q} = [q^0, q^1, \ldots, q^{N_s}]$ be a discrete-variable matrix that represents a finite sequence of locations where $q^i \in \mathcal{Q}$ for $i \in \mathcal{N}_s = \{0,1,\ldots,N_s\}$ and $N_s \in \mathbb{Z}_+$ is the number of switches. Let $\mathbf{a} = [a_0, a_1, \ldots, a_{N_s}]$ and $\mathbf{b} = [b_0, b_1, \ldots, b_{N_s}]$ be real-valued matrices representing finite sequences of real numbers associated with $\mathbf{q}$ such that $[a_i, b_i]$, $a_i \neq b_i$, are compact subsets of $\mathbb{R}$. We define the initial time, $t_0 = a_0$, and the final time as $t_f = b_{N_s}$. Usually, we will have $a_{i+1} = b_i$ (as in the case of a switch), but it is not necessary to make this assumption. The freedom in not making this assumption is particularly helpful in practical problem solving via discretization as noted in [20] and exploited in [22]. Let $\boldsymbol{x}(\cdot) : t \mapsto (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots \boldsymbol{x}^{N_s})$ and $\boldsymbol{u}(\cdot) : t \mapsto (\boldsymbol{u}^0, \boldsymbol{u}^1, \ldots \boldsymbol{u}^{N_s})$ represent the continuous-time state and control functions associated with $\mathbf{q}$. The tuple, $(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), \mathbf{q}, \mathbf{a}, \mathbf{b}, N_s)$, is called a primal execution. Following Sussmann [25] we define $\tilde{+}$, as,

$$i\tilde{+}1 = \begin{cases} i+1 & i < N_s \\ & if \\ 0 & i = N_s \end{cases} \tag{5}$$

This operation simply allows us to wrap indices since q^i, for $i = 0, 1, \ldots, N_s$ is equal to $q^{i\tilde{+}1}$, for $i = N_s, 0, 1, \ldots, N_s - 1$. All the point-wise conditions for the hybrid problem, including the boundary conditions, can be succinctly evaluated as

$$\begin{aligned}&(\boldsymbol{x}^i(b_i), \boldsymbol{u}^i(b_i), b_i, \boldsymbol{x}^{i\tilde{+}1}(a_{i\tilde{+}1}), \boldsymbol{u}^{i\tilde{+}1}(a_{i\tilde{+}1}), a_{i\tilde{+}1}, \\ &\quad q^i, q^{i\tilde{+}1}) \in \mathbb{E}(q^i, q^{i\tilde{+}1}) \qquad \forall\, i \in \mathcal{N}_s\end{aligned} \tag{6}$$

The hybrid Bolza problem is to find a primal execution that minimizes the cost function,

$$\begin{aligned}&J[\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), \mathbf{q}, \mathbf{a}, \mathbf{b}, N_s] = \\ &\sum_{i=0}^{N_s} \Big(E(\boldsymbol{x}^i(b_i), \boldsymbol{u}^i(b_i), b_i, \boldsymbol{x}^{i\tilde{+}1}(a_{i\tilde{+}1}), \boldsymbol{u}^{i\tilde{+}1}(a_{i\tilde{+}1}), \\ &\quad a_{i\tilde{+}1}, q^i, q^{i\tilde{+}1}) + \int_{a_i}^{b_i} F(\boldsymbol{x}(t), \boldsymbol{u}(t), q^i)\, dt\Big)\end{aligned} \tag{7}$$

subject to the dynamics, (1), the path constraints, (2), and the event conditions, (6). Any primal execution that satisfies (1), (2) and (6) is called a primal feasible execution.

IV. First-Order Necessary Conditions (Problem $\mathcal{H}^{\boldsymbol{\lambda}}$)

A rigorous development of the necessary conditions for Problem $\mathcal{H}$ is given by Sussmann [23], [24], [25]. The first-order necessary conditions can be articulated as a generalized equation in primal-dual space. This equation can be obtained by applying the generalized Lagrange Multiplier Rule to Problem $\mathcal{H}$. For the purposes of brevity, we simply state these conditions and note some key points related to the Lagrangian for Problem $\mathcal{H}$, as it plays a central role in both theory and computation. There are a number of other "Lagrangians" associated with the problem as will be apparent shortly. In the following, we also limit our discussions to normal extremals, i.e. assume that the normality condition holds.

For each $q \in \mathcal{Q}$, we define the control Hamiltonian function, H, as the real-valued function,

$$H(\boldsymbol{\lambda}, \boldsymbol{x}, \boldsymbol{u}, q) = F(\boldsymbol{x}, \boldsymbol{u}, q) + \boldsymbol{\lambda}^T \mathbf{f}(\boldsymbol{x}, \boldsymbol{u}, q) \tag{8}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^{N_x^q}$ is a covector that satisfies the adjoint equation,

$$-\dot{\boldsymbol{\lambda}}(t) = \frac{\partial \overline{H}[t]}{\partial \boldsymbol{x}} \tag{9}$$

where the notation $H[t]$ is used as a shorthand for $H(\boldsymbol{\lambda}(t), \boldsymbol{x}(t), \boldsymbol{u}(t), q)$. The D-form of the Lagrangian of the Hamiltonian, $\overline{H}$, is defined as [13],

$$\overline{H}(\mu, \boldsymbol{\lambda}, \boldsymbol{x}, \boldsymbol{u}, q) = H(\boldsymbol{\lambda}, \boldsymbol{x}, \boldsymbol{u}, q) + \boldsymbol{\mu}^T \mathbf{h}(\boldsymbol{x}, \boldsymbol{u}, q) \tag{10}$$

where $\boldsymbol{\mu} \in \mathbb{R}^{N_h^q}$ satisfies the complementarity condition,

$$\mathbf{0} \leq \boldsymbol{\mu}(t) \perp -\mathbf{h}[t] \geq \mathbf{0} \tag{11}$$

where the notation, "$\perp$," means that $\boldsymbol{\mu}^T(t)\mathbf{h}[t] = \mathbf{0}$ in addition to the stated inequalities. Note that the covectors, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are implicit functions of q similar to the state or control variables. The gradient normality condition associated with the Hamiltonian minimization condition is simply given by,

$$\frac{\partial \overline{H}}{\partial \boldsymbol{u}} = \mathbf{0} \tag{12}$$

Equations (11) and (12) are essentially the Karush-Kuhn-Tucker (KKT) conditions associated with the minimization of the Hamiltonian. Now, for each $q, q' \in \mathcal{Q}$, we define an event Lagrangian, $\overline{E}$, associated with the pair, $(E, \mathbf{e})$, as,

$$\begin{aligned}\overline{E}(\boldsymbol{\nu}, \boldsymbol{x}, \boldsymbol{u}, \tau, \boldsymbol{x}', \boldsymbol{u}', \tau', q, q') &= E(\boldsymbol{x}, \boldsymbol{u}, \tau, \boldsymbol{x}', \boldsymbol{u}', \tau', q, q') \\ &\quad + \boldsymbol{\nu}^T \mathbf{e}(\boldsymbol{x}, \boldsymbol{u}, \tau, \boldsymbol{x}', \boldsymbol{u}', \tau', q, q')\end{aligned} \tag{13}$$

where $\nu \in \mathbb{R}^{N_e^{qq'}}$ satisfies the complementarity condition,

$$0 \le \nu \perp -\mathbf{e}(x,u,\tau,x',u',\tau',q,q') \ge 0 \qquad (14)$$

According to the Sussmann's Hybrid Minimum (Maximum) Principle [23], a putative optimal execution satisfies the switching conditions,

$$\{\lambda^i(b_i), -\lambda^{i\dot{+}1}(a_{i\dot{+}1})\} = \left\{\frac{\partial \overline{E}[i]}{\partial x^i(b_i)}, \frac{\partial \overline{E}[i]}{\partial x^{i\dot{+}1}(a_{i\dot{+}1})}\right\} \qquad (15)$$

$$\{-H[b_i], H[a_{i\dot{+}1}]\} = \left\{\frac{\partial \overline{E}[i]}{\partial b_i}, \frac{\partial \overline{E}[i]}{\partial a_{i\dot{+}1}}\right\} \qquad (16)$$

where we have used the shorthand notation,

$$\begin{aligned}\overline{E}[i] &\equiv \overline{E}(\nu^i, x^i(b_i), u^i(b_i), b_i, x^{i\dot{+}1}(a_{i\dot{+}1}),\\ &\quad u^{i\dot{+}1}(a_{i\dot{+}1}), a_{i\dot{+}1}, q^i, q^{i\dot{+}1})\\ H[a_i] &\equiv H(\lambda(a_i), x(a_i), u(a_i), q^i)\end{aligned}$$

The switching conditions are essentially a generalization of the transversality conditions and the Hamiltonian value conditions. All of these conditions can be derived from the generalized Lagrange Multiplier rule after constructing the Lagrangian for the full problem,

$$\overline{J}[\nu(\cdot),\mu(\cdot),\lambda(\cdot),x(\cdot),u(\cdot),\mathbf{q},\mathbf{a},\mathbf{b},N_s] = \sum_{i=0}^{N_s}\left(\overline{E}[i] + \int_{a_i}^{b_i}(\overline{H}[t,i] - \lambda^T[t,i]\dot{x}[t,i])\,dt\right) \qquad (17)$$

whose importance in terms of a sufficiency condition will be apparent shortly. In the above equation, the notation, $\overline{H}[t,i]$ stands for $\overline{H}(\mu(t),\lambda(t),x(t),u(t),q^i)$.

Problem $\mathcal{H}^\lambda$ is now defined as finding a primal-dual execution, $\{\nu(\cdot),\mu(\cdot),\lambda(\cdot),x(\cdot),u(\cdot),\mathbf{q},\mathbf{a},\mathbf{b},N_s\}$ that is primal feasible and satisfies the adjoint equation, (9), the first order Hamiltonian minimization condition, (12), the switching conditions, (15) and (16), and the complementarity conditions, (11) and (14).

V. The Pseudospectral Knotting Method

The details of the pseudospectral (PS) knotting method are described in [16], [20]. Here we briefly summarize the main points of the PS method as it pertains to the hybrid system model developed in the previous sections. The goal of the PS knotting method is to solve Problem $\mathcal{H}$ by approximating it to a mixed variable programming (MVP) problem in a manner that permits the discretization to commute with dualization. This means that a putative optimal execution must automatically satisfy the discretized necessary conditions. Solving hybrid problems by this approach is far simpler than developing and solving for the necessary conditions.

A general PS method [3] consists of two major steps. In the first step, for each $i \in \mathcal{N}_s$, we select (N^i+1) cardinal functions, $\phi_l^i,\ l = 0,1,\ldots,N^i$, over the time interval, $[a_i,b_i]$, such that they satisfy the Kronecker delta condition,

$$\phi_l^i(t_k^i) = \delta_{lk} \quad k = 0,1,\ldots,N^i$$

where the grid points, $\pi^i = \{t_0^i, t_1^i, \ldots, t_{N^i}^i\}$, are called nodes. The nodes are chosen in a manner consistent with approximation theory (e.g. shifted Gauss points). The continuous-time primal and dual variables are approximated as Lagrange interpolants,

$$y(t^i) \simeq y^{N^i}(t^i) = \sum_{l=0}^{N^i} y_l^i \phi_l^i(t^i) \qquad (18)$$

where y_l^i are the values of $y^{N^i}(t_l^i)$ and $y(\cdot)$ denotes a generic continuous-time variable. Once these basis functions are chosen, various operations on the continuous-time variable are commuted with the approximation implied in (18). Thus, for example, integration is approximated as,

$$\int_{a_i}^{b_i} y(t)\,dt \simeq \sum_{l=0}^{N^i} y_l^i \int_{a_i}^{b_i} \phi_l^i(t)\,dt = \sum_{l=0}^{N^i} w_l^i y_l^i \qquad (19)$$

where w_l^i,

$$w_l^i := \int_{a_i}^{b_i} \phi_l^i(t)\,dt$$

form weights for a discrete 1-form (inner product). Similarly, we approximate the derivative of $y(t)$ by the derivative of the approximation,

$$\dot{y}(t) \simeq \dot{y}^{N^i}(t) = \sum_{l=0}^{N^i} y_l^i \dot{\phi}_l^i(t) \qquad (20)$$

In the second major step of the method, the equations of approximation are obtained obtained by projecting the problem equations (Problem $\mathcal{H}$ and $\mathcal{H}^\lambda$) over the node points. Since a discrete event occurs at a_i and b_i, we choose the ends of the grid, π^i, such that $t_0^i = a_i$ and $t_{N^i}^i = b_i$. These are the shifted Lobatto points. Since a switch occurs when $a_{i+1} = b_i$, we have double Lobatto points, $t_0^{i+1} = t_{N^i}^i$ over a switch. These double Lobatto points are called PS knots [20] and facilitate distinct left- and right-hand limits, precisely the type of conditions required for a switch. The derivatives of $y^{N^i}(t)$ over π^i are evaluated from (20), where the differentiation matrix,

$$D_{kl}^{N^i} = \dot{\phi}_l^i(t_k^i) \quad l,k = 0,1,\ldots,N^i$$

provides a rapid procedure for evaluating the derivatives at the node points.

In the Legendre PS method, which is the focus of the current paper, the grid points are the shifted Legendre-Gauss-Lobatto (LGL) points where the "shift" is achieved by mapping the physical domain, $[a_i,b_i] \ni t^i$, to a computational domain, $[-1,1] \ni \tau$, by the affine transformation,

$$\tau(t^i) = \frac{2t^i - (b_i + a_i)}{(b_i - a_i)}$$

where we have abused notation in using τ to imply both the transformation as well as the transformed variable. The LGL weights and the differentiation matrix,

$$w_k^i := \frac{b_i - a_i}{N^i(N^i+1)}\frac{1}{[L_{N^i}(\tau_k^i)]^2} \quad k = 0,1;\ldots,N^i$$

$$D_{kl}^{N^i} := \frac{2}{b_i - a_i}\begin{cases}\frac{L_{N^i}(\tau_k^i)}{L_{N^i}(\tau_l^i)}\cdot\frac{1}{\tau_k^i - \tau_l^i} & k \ne l\\ -\frac{N^i(N^i+1)}{4} & k = l = 0\\ \frac{N^i(N^i+1)}{4} & k = l = N^i\\ 0 & \text{otherwise}\end{cases}$$

with $\tau_k^i, k = 0,1,\ldots,N^i$ denoting the LGL nodes [3] satisfy a discrete form of integration by parts that is explicitly used in the derivation of the main theorem of this paper (details omitted, but please see [18]) . The integration-by-parts lemma can be summarized as,

Lemma 1: For each $i \in \mathcal{N}_s$, the elements of the Differentiation Matrix, $D_{kl}^{N^i}$, and the LGL weights, w_k^i, satisfy a discrete integration by parts condition,

$$w_k^i D_{kl}^{N^i} + w_l^i D_{lk}^{N^i} = 0 \qquad k,l = 1,\ldots,N^i - 1 \tag{21}$$

In addition, the Lobatto terms satisfy a normality condition, $2w_0 D_{00} = -1$, and $2w_{N^i}^i D_{N^i N^i}^{N^i} = 1$. Finally, $\sum_{k=0}^{N^i} w_k^i = (b_i - a_i)$.

For a proof of this, see [11].

In the following sections we will denote by $[y_k^i]$ the collection of the discretized continuous-time variable for $k = 0,\ldots,N^i$, $i = 0,\ldots N_s$.

VI. Discretized Primal Problem (Problem $\mathcal{H}^N$)

For $i = 0,\ldots,N_s$, let

$$x^{N^i}(t^i) = \sum_{l=0}^{N^i} x_l^i \phi_l^i(t^i) \qquad u^{N^i}(t^i) = \sum_{l=0}^{N^i} u_l^i \phi_l^i(t^i)$$

Following the procedure outlined in Section V, Problem $\mathcal{H}$ can be approximated as the sparse MVP (Problem $\mathcal{H}^N$) of finding the discretized primal execution, $\left\{[x_k^i],[u_k^i],\mathbf{q},\mathbf{a},\mathbf{b},N_s\right\}$ that minimizes,

$$\begin{aligned} J^N([x_k^i],[u_k^i],\mathbf{q},\mathbf{a},\mathbf{b},N_s) = \sum_{i=0}^{N_s} \Big[& E(x_{N^i}^i, u_{N^i}^i, b_i, \\ & x_0^{i\bar{+}1}, u_0^{i\bar{+}1}, a_{i\bar{+}1}, q^i, q^{i\bar{+}1}) \\ & + \sum_{l=0}^{N^i} F(x_l^i, u_l^i, q^i) w_l^i \Big] \end{aligned} \tag{22}$$

subject to,

$$\mathbf{f}(x_k^i, u_k^i, q^i) - \sum_{l=0}^{N^i} D_{kl}^{N^i} x_l^i = \mathbf{0} \tag{23}$$

$$\mathbf{h}(x_k^i, u_k^i, q^i) \leq \mathbf{0} \tag{24}$$

$$\mathbf{e}(x_{N^i}^i, u_{N^i}^i, b_i, x_0^{i\bar{+}1}, u_0^{i\bar{+}1}, a_{i\bar{+}1}, q^i, q^{i\bar{+}1}) \leq \mathbf{0} \tag{25}$$

for $k = 0,\ldots,N^i$ and $i = 0,\ldots,N_s$. Any discretized primal execution that satisfies (23)-(25) is called discrete primal feasible.

In a practical implementation of this MVP, it may be necessary to define an explicit algebra associated with the set $\mathcal{Q}$. For example, if $\mathcal{Q}$ is generated by some finite automaton, then the discrete dynamics that generates $\mathbf{q}$ can be easily added to the definition of Problem $\mathcal{H}^N$. Inspired by the work of von Stryk and his colleagues [27], [28], we illustrate this point by way of using a binary control variable as follows.

We define the operation $*$ over the Cartesian product, $\mathcal{Q}\times\{0,1\}$, as,

$$q * 0 = \emptyset \quad q * 1 = q \qquad \forall\, q \in \mathcal{Q}$$

Let Q be a row matrix whose columns are the N_Q elements of $\mathcal{Q}$. Let $\Delta \in \{0,1\}^{N_Q \times (N_s+1)}$ with the property that

$$\sum_{n=1}^{N_Q} \Delta_{n,(i+1)} = 1 \text{ or } 0 \qquad \text{for all } i = 0,\ldots,N_s \tag{26}$$

We define $q * \Delta$ as a termwise operation so that each column of $q * \Delta$ contains no more than one element $q \in \mathcal{Q}$. We define $q + \emptyset = \emptyset + q = q$ so that the $*$ operation is extended to the product $Q * \Delta$ in the usual sense of a matrix operation. Thus, Δ, is a discrete controller that determines the sequence of locations $\mathbf{q}$ by way of the equation,

$$\mathbf{q} = Q * \Delta, \quad \Delta \in \mathbb{U}_D \subset \{0,1\}^{N_Q \times (N_s+1)} \tag{27}$$

where $\mathbb{U}_D$ is the set of allowable discrete controls that represent a transcription of the switching sets, $\mathcal{S}(q,q')\ \forall\ (q,q') \in \mathcal{Q}\times\mathcal{Q}$. In incorporating (27) under Problem $\mathcal{H}^N$, we now treat Δ as a discrete controller while $\mathbf{q}$ takes the role of the discrete state.

VII. Discretized Necessary Conditions (Problem $\mathcal{H}^{\lambda N}$)

For $i = 0,\ldots,N_s$, let

$$\lambda^{N^i}(t^i) = \sum_{l=0}^{N^i} \lambda_l^i \phi_l^i(t^i) \qquad \mu^{N^i}(t^i) = \sum_{l=0}^{N^i} \mu_l^i \phi_l^i(t^i)$$

Following a procedure outlined in Section V, it can be shown that, for $i = 0,\ldots,N_s$ and $k = 0,\ldots,N^i$, the discretized necessary conditions are given by,

$$\frac{\partial \overline{H}}{\partial u_k^i}\left(\mu_k^i, \lambda_k^i, x_k^i, u_k^i, q^i\right) = \mathbf{0} \tag{28}$$

$$\mathbf{0} \leq \mu_k^i \perp -\mathbf{h}[k,i] \geq \mathbf{0} \tag{29}$$

$$\frac{\partial \overline{H}}{\partial x_k^i}\left(\mu_k^i, \lambda_k^i, x_k^i, u_k^i, q^i\right) + \sum_{l=0}^{N^i} D_{kl}^{N^i} \lambda_l^i = \mathbf{0} \tag{30}$$

$$\mathbf{0} \leq \nu^i \perp -\mathbf{e}[i] \geq \mathbf{0} \tag{31}$$

$$\lambda_{N^i}^i - \frac{\partial \overline{E}[i]}{\partial x_{N^i}^i} = \mathbf{0} \tag{32}$$

$$\lambda_0^i + \frac{\partial \overline{E}[i]}{\partial x_0^i} = \mathbf{0} \tag{33}$$

$$H\left(\lambda_{N^i}^i, x_{N^i}^i, u_{N^i}^i, q^i\right) = -\frac{\partial \overline{E}[i]}{\partial b_i} \tag{34}$$

$$H\left(\lambda_0^i, x_0^i, u_0^i, q^i\right) = \frac{\partial \overline{E}[i]}{\partial a_i} \tag{35}$$

where the notation, $\mathbf{h}[k,i]$, $\overline{E}[i]$ and $\overline{\mathbf{e}}[i]$ are used as a shorthand for,

$$\begin{aligned} \mathbf{h}[k,i] &:= \mathbf{h}(x(t^i(\tau_k^i)), u(t^i(\tau_k^i)), q^i) \\ \overline{E}[i] &:= \overline{E}(\nu^i, x_{N^i}^i, u_{N^i}^i, b_i, x_0^{i\bar{+}1}, u_0^{i\bar{+}1}, a_{i\bar{+}1}, q^i, q^{i\bar{+}1}) \\ \mathbf{e}[i] &:= \mathbf{e}(x_{N^i}^i, u_{N^i}^i, b_i, x_0^{i\bar{+}1}, u_0^{i\bar{+}1}, a_{i\bar{+}1}, q^i, q^{i\bar{+}1}) \end{aligned}$$

Thus, Problem $\mathcal{H}^{\lambda N}$ can now be defined as a mixed variable, mixed complementarity problem of finding $\{\nu^i, [\mu_k^i], [\lambda_k^i], [x_k^i], [u_k^i], \mathbf{q}, \mathbf{a}, \mathbf{b}, N_s\}$ that satisfy (23) - (25) and (28) - (35). It is already apparent that Problem $\mathcal{H}^N$ is significantly easier to solve than Problem $\mathcal{H}^{\lambda N}$.

VIII. KKT Conditions (Problem $\mathcal{H}^{N\lambda}$)

The KKT conditions for Problem $\mathcal{H}^N$ can be generated quite easily after constructing the Lagrangian. As noted in Section V and elsewhere [15], [17], we can use the discrete weights to construct the 1-form so that we can define the Lagrangian as,

$$\begin{aligned} & \overline{J^N}[\widetilde{\nu}^i, [\widetilde{\mu}_k^i], [\widetilde{\lambda}_k^i], [x_k^i], [u_k^i], \mathbf{q}^i, \mathbf{a}, \mathbf{b}, N_s] = \\ & \sum_{i=0}^{N_s} \Bigg[\widetilde{E}[i] + \sum_{k=0}^{N^i} F(x_k^i, u_k^i, q^i) w_k^i + (\widetilde{\lambda}_k^i)^T w_k^i \mathbf{f}(x_k^i, u_k^i, q^i) \\ & + \sum_{k=0}^{N^i} (\widetilde{\mu}_k^i)^T w_k^i \mathbf{h}(x_k^i, u_k^i, q^i) - \sum_{k=0}^{N^i} (\widetilde{\lambda}_k^i)^T w_k^i \sum_{l=0}^{N^i} D_{kl}^{N^i} x_l^i \Bigg] \end{aligned} \tag{36}$$

where $\widetilde{\lambda}_k^i, \widetilde{\mu}_k^i$ and $\widetilde{\nu}^i$ are the Lagrange multipliers associated with (23) - (25) respectively, and,

$$\widetilde{E}[i] \;:=\; \overline{E}(\widetilde{\nu}^i, x_{N^i}^i, u_{N^i}^i, b_i, x_0^{i+1}, u_0^{i+1}, a_{i+1}, q^i, q^{i+1})$$

Note that the weights are not used in defining $\widetilde{E}[i]$. Examining (36) and (17), it is very tempting to set $\lambda_k^i = \widetilde{\lambda}_k^i$, $\mu_k^i = \widetilde{\mu}_k^i$, and $\nu^i = \widetilde{\nu}^i$ since the approximation of (17) by way of (19) is exactly equal to (36). In general, this is not true; that is, the discretized covectors are not equal to the Lagrange multipliers associated with the discretized problem. This is because dualization and discretization are not commutative operations [17], [18].

When the necessary conditions arising from the stationarity of the Lagrangian are derived, they do not resemble the discretized necessary conditions. On the other hand, when Lemma 1 is used, it then becomes straightforward to show the following:

For $i = 0, \ldots, N_s$ and $k = 0, \ldots, N^i$, we get,

$$\frac{\partial \overline{H}}{\partial u_k^i}\left(\widetilde{\mu}_k^i, \widetilde{\lambda}_k^i, x_k^i, u_k^i, q^i\right) = \mathbf{0} \tag{37}$$

$$\mathbf{0} \le \widetilde{\mu}_k^i \perp_w -\mathbf{h}[k,i] \ge \mathbf{0} \tag{38}$$

where $\perp_w$ implies the discrete weighted complementarity condition for each $i \in \mathcal{N}_s$. For $i = 0, \ldots, N_s$ and $k = 1, \ldots, N^i - 1$, we have,

$$\frac{\partial \overline{H}}{\partial x_k^i}\left(\widetilde{\mu}_k^i, \widetilde{\lambda}_k^i, x_k^i, u_k^i, q^i\right) + \sum_{l=0}^{N^i} D_{kl}^{N^i} \widetilde{\lambda}_l^i = \mathbf{0} \tag{39}$$

Finally for $i = 0, \ldots, N_s$, we have

$$\mathbf{0} \le \widetilde{\nu}^i \perp -\mathbf{e}[i] \ge \mathbf{0} \tag{40}$$

$$\frac{\partial \overline{H}}{\partial x_{N^i}^i}\left(\widetilde{\mu}_{N^i}^i, \widetilde{\lambda}_{N^i}^i, x_{N^i}^i, u_{N^i}^i, q^i\right) + \sum_{l=0}^{N^i} D_{N^i l}^{N^i} \widetilde{\lambda}_l^i = \mathbf{c}_{N^i}^i \tag{41}$$

$$\widetilde{\lambda}_{N^i}^i - \frac{\partial \widetilde{E}[i]}{\partial x_{N^i}^i} = w_{N^i}^i \mathbf{c}_{N^i}^i \tag{42}$$

$$\frac{\partial \overline{H}}{\partial x_0^i}\left(\widetilde{\mu}_0^i, \widetilde{\lambda}_0^i, x_0^i, u_0^i, q^i\right) + \sum_{l=0}^{N^i} D_{0l}^{N^i} \widetilde{\lambda}_l^i = -\mathbf{c}_0^i \tag{43}$$

$$\widetilde{\lambda}_0^i + \frac{\partial \widetilde{E}[i]}{\partial x_0^i} = w_0^i \mathbf{c}_0 \tag{44}$$

$$\frac{1}{(b_i - a_i)} \sum_{l=0}^{N^i} w_l^i H\left(\widetilde{\lambda}_l^i, x_l^i, u_l^i, q^i\right) = -\frac{\partial \widetilde{E}[i]}{\partial b_i} \tag{45}$$

$$\frac{1}{(b_i - a_i)} \sum_{l=0}^{N^i} w_l^i H\left(\widetilde{\lambda}_l^i, x_l^i, u_l^i, q^i\right) = \frac{\partial \widetilde{E}[i]}{\partial a_i} \tag{46}$$

where $\mathbf{c}_0^i$ and $\mathbf{c}_{N^i}^i$ are arbitrary vectors in $\mathbb{R}^{N_x^i}$. Thus, Problem $\mathcal{H}^{N\lambda}$ can now be defined as a mixed variable, mixed complementarity problem of finding $\{\widetilde{\nu}^i, [\widetilde{\mu}_k^i], [\widetilde{\lambda}_k^i], [x_k^i], [u_k^i], \mathbf{q}, \mathbf{a}, \mathbf{b}, N_s\}$ that satisfy (23) - (25) and (37) - (46).

IX. Closure Conditions

Let $\chi := \{[x_k^i], [u_k^i], \mathbf{q}, \mathbf{a}, \mathbf{b}, N_s\}$ and $\Lambda := \{\nu^i, [\mu_k^i], [\lambda_k^i]\}$. We denote by $\mathbb{M}^{\lambda N}(\chi)$ the multiplier set corresponding to χ,

$$\mathbb{M}^{\lambda N}(\chi) := \{\Lambda : \Lambda \text{ satisfies (28) – (35)}\} \tag{47}$$

Similarly, we define, $\widetilde{\Lambda} := \left\{\widetilde{\nu}^i, [\widetilde{\mu}_k^i], [\widetilde{\lambda}_k^i]\right\}$ and $M^{N\lambda}(\chi)$ the multiplier set,

$$\mathbb{M}^{N\lambda}(\chi) := \left\{\widetilde{\Lambda} : \widetilde{\Lambda} \text{ satisfies (37) – (46)}\right\} \tag{48}$$

Clearly, $\mathbb{M}^{\lambda N}(\chi) \subset \mathbb{M}^{N\lambda}(\chi)$. That is, every solution to Problem $\mathcal{H}^{\lambda N}$ is also a solution to Problem $\mathcal{H}^{N\lambda}$ but not vice versa. Introducing the closure conditions,

$$\mathbf{c}_0^i = \mathbf{0} \tag{49}$$

$$\mathbf{c}_{N^i}^i = \mathbf{0} \tag{50}$$

$$H\left(\widetilde{\lambda}_0^i, x_0^i, u_0^i, q^i\right) = H\left(\widetilde{\lambda}_{N^i}^i, x_{N^i}^i, u_{N^i}^i, q^i\right) \tag{51}$$

$$= \frac{\sum_{l=0}^{N^i} w_l^i H\left(\widetilde{\lambda}_l^i, x_l^i, u_l^i, q^i\right)}{(b_i - a_i)} \tag{52}$$

we generate a new multiplier set,

$$\widehat{\mathbb{M}}^{N\lambda}(\chi) := \left\{\widetilde{\Lambda} \in \mathbb{M}^{N\lambda}(\chi) : \widetilde{\Lambda} \text{ satisfies (49) – (52)}\right\} \tag{53}$$

Obviously, $\widehat{\mathbb{M}}^{N\lambda}(\chi) \sim \mathbb{M}^{\lambda N}(\chi)$. Thus, the imposition of closure conditions on Problem $\mathcal{H}^{N\lambda}$ implies that every solution of the modified Problem $\mathcal{H}^{N\lambda}$ is also a solution to Problem $\mathcal{H}^{\lambda N}$.

X. The Hybrid Covector Mapping Theorem

Let $\mathbb{M}^{\lambda N}(\chi) \neq \emptyset$ and $\left\{\widehat{\nu}^i, [\widehat{\mu}_k^i], [\widehat{\lambda}_k^i]\right\} \in \widehat{\mathbb{M}}^{N\lambda}(\chi)$; then the bijection, $\widehat{\mathbb{M}}^{N\lambda}(\chi) \sim \mathbb{M}^{\lambda N}(\chi)$, is given by,

$$\lambda^N(t_k^i) = \widehat{\lambda}_k^i \quad \mu^N(t_k^i) = \widehat{\mu}_k^i, \quad \nu = \widehat{\nu} \tag{54}$$

The proof of this follows quite simply from the closure conditions. A schematic of the main results are depicted in Fig. 1.

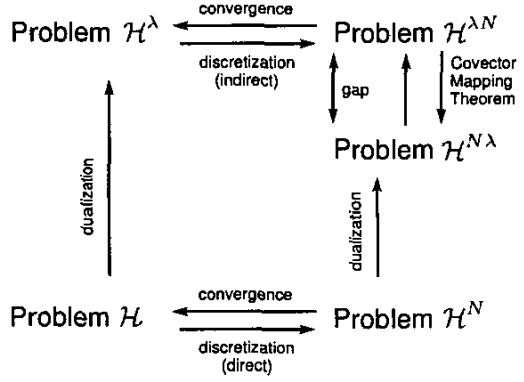

Fig. 1. Schematic of the Main Results: The gap denotes the set $\mathbb{M}^{N\lambda}(\chi) \backslash \mathbb{M}^{\lambda N}(\chi)$.

Remark 1: Although (54) offers Eulerian-like elegance, note that this equation was obtained only after imposing the closure conditions and defining the discrete Lagrangian as a weighted 1-form. No such additional conditions are necessary for Eulerian discretizations.

Remark 2: The grid π^i (see Sec.V) contains N^i points. Unlike a PS method, a forward Euler method does not collocate a derivative at the point $t_{N^i}^i$. Hence, derivative information across two adjacent grids, π^i and π^{i+1}, cannot be transferred at a discrete event even if double node points were defined as "Eulerian knots." A similar notion holds for a backward Euler method or Runge-Kutta Methods.

Remark 3: Since dualization and discretization are noncommutative operations (see Fig. 1), a solution to the MVP Problem $\mathcal{H}^{N\lambda}$ may be primal feasible but not satisfy the discretized necessary conditions (i.e. Problem $\mathcal{H}^{\lambda N}$). Such a spurious solution can be easily detected from the Hybrid Covector Mapping Theorem by

solving the system of generalized (set-valued) linear equations that define $\widehat{\mathbb{M}}^{N\lambda}(\chi)$ for a given primal execution, χ. If $\widehat{\mathbb{M}}^{N\lambda}(\chi) = \emptyset$, we have a spurious solution.

XI. Conclusions

It is far simpler to discretize and solve a hybrid optimal control problem than to solve for the necessary conditions resulting from the Hybrid Minimum Principle. Rather than use Eulerian discretizations that generate a linear convergence rate, the pseudospectral (PS) knotting method is proposed as an efficient higher-order method to solve hybrid problems. PS knots provide a simple method to handle switches, resets and other event conditions. Solving the PS-discretized hybrid problem may result in spurious solutions. The hybrid covector mapping theorem can be used to detect these spurious solutions by checking the optimality conditions over the node points.

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