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Yan, Hui; Ross, I. Michael; Fahroo, Fariba

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Optimal Feedback Control Laws by Legendre Pseudospectral Approximations

Hui Yan[1]
Fariba Fahroo[2]
I. Michael Ross[3]

Abstract

We develop state feedback control laws for linear time-varying systems with quadratic cost criteria by an indirect Legendre pseudospectral method. This method approximates the linear two-point boundary value problem to a system of algebraic equations by way of a differentiation matrix. The algebraic system is solved to generate discrete linear transformations between the states and controls at the Legendre-Gauss-Lobatto points. Since these linear transformations involve simple matrix operations, they can be computed rapidly and efficiently. Two methods are proposed: one that circumvents solving the differential Riccati equation by a discrete solution of the boundary value problem, and another that generates a predictor feedback law without the use of transition matrices. Thus our methods obviate the need for solving the time-intensive backward integration of the matrix Riccati differential equation or inverting ill-conditioned transition matrices. A numerical example illustrates the techniques and demonstrates the accuracy and efficiency of these controllers.

1 Introduction

Orthogonal polynomials have been used extensively in solving optimal control problems. In particular, their use in solving linear time-varying (LTV) optimal control problems has been widespread. Hwang and Chang [1] used shifted Legendre polynomials whereas Chou and Horng [2] used Chebyshev polynomials for solving LTV problems. More recently, Razzaghi [3] employed a Fourier series method for solving this class of problems. The common approach in all these papers is to first expand the state and control variables as a generalized Fourier series with the appropriate orthogonal functions as the basis functions. Then, the orthogonality of these functions is used to arrive at simplified expressions for forward and backward integration matrices. These matrices, in turn, are used to express the state transition matrices in the optimal control law in terms of unknown coefficients of expansion.

Another approach has been to use orthogonal polynomials in the context of pseudospectral methods [4]-[6]. Through the use of a spectral differentiation matrix, the optimal control problem is transformed to a nonlinear programming problem. Thus, it is apparent that for linear systems with quadratic cost criteria, the optimal control problem can easily be transformed to a quadratic programming (QP) problem (a quadratic cost function subject to linear algebraic constraints)[7]. This method is in sharp contrast to prior work on using orthogonal polynomials which rely on approximating the two-point-boundary value problem (TPBVP) derived from the necessary conditions. Recently, Lu [8] approximated the related receding-horizon problem for LTV systems to a QP problem; Based on Simpson-trapezoid approximations for the integral and Euler-type approximations for the derivatives he approximated the LTV systems to a QP and then derived analytic control laws. Whereas Elnagar et al [7] chose to solve their QP problem numerically, Lu used the analytic solution. In both methods, by using a direct approach (avoiding the solution of the necessary conditions), one avoids the pitfalls of the indirect methods such as integrating the Riccati equation, but in Elnagar's approach, the solution maybe not be as accurate as the indirect methods, and in Lu's method, finding higher order control laws for step by step replacements for the states can be too tedious.

Recently, Fahroo and Ross [9] proposed the Indirect Psuedospectral Method for solving optimal control problems. In this method, the TPBVP arising from the necessary conditions is solved by spectral collocation. For general nonlinear problems the resulting set of algebraic equations that approximate the boundary-value problem are nonlinear and an iterative technique is necessary. However, for LTV systems with a quadratic cost function the algebraic system is linear. Thus, well-known methods from linear algebra can be used to solve

[1]NRC Research Fellow, Naval Postgraduate School, Monterey, CA 93943, E-mail: hyan@nps.navy.mil.
[2]Assistant Professor, Department of Mathematics, Code Ma/Ff, Naval Postgraduate School, Monterey, CA 93943, E-mail: ffahroo@nps.navy.mil.
[3]Associate Professor, Department of Aero/Astro, Code AA/Ro, Naval Postgraduate School, Monterey, CA 93943, E-mail: imross@nps.navy.mil.

the TPBVP. We propose two different ways of computing the feedback laws. In one technique, we solve for the values of states and costates at the collocation points, and in the process we calculate the optimal feedback law at these points. To find the values of the optimal states and controls at any time different from the collocation points, an interpolation scheme can be used. All the computations can be performed off-line once, and we will show that even with a small number of collocation points, the results are highly accurate.

In the second technique, we generate a discrete linear transformation from the initial state to the initial costate. We will show that this linear transformation is numerically very efficient and hence can be computed on-line. This generates a linear feedback law for the controls when the "initial" time, τ_0 is replaced by the current time, τ, and the final time, τ_f, by "time-to-go," $T = \tau_f - \tau$. By way of a numerical example, we show the accuracy, efficiency and effectiveness of both techniques which are based on pseudospectral approximations of the underlying equations.

2 Problem Formulation

Consider the LTV system

$$\dot{\mathbf{x}} = A(\tau)\mathbf{x} + B(\tau)\mathbf{u}, \quad \mathbf{x}(0) = \tilde{\mathbf{x}}_0. \tag{1}$$

Here, $\mathbf{x}(\tau) \in R^n$ and $\mathbf{u}(\tau) \in R^m$ are the state and control vectors, respectively. The time-varying dynamics and control matrices, $A(\tau)$ and $B(\tau)$ are of dimensions $n \times n$ and $n \times m$, respectively. The optimal feedback control problem is to determine $\mathbf{u}(\mathbf{x}, \tau)$ satisfying Eqs. (1), while minimizing the cost functional

$$J = \frac{1}{2}\mathbf{x}^T(\tau_f)P_f\mathbf{x}(\tau_f) + \frac{1}{2}\int_{\tau_0}^{\tau_f} \left[\mathbf{x}^T(\tau)Q(\tau)\mathbf{x}(\tau) + \mathbf{u}^T(\tau)R(\tau)\mathbf{u}(\tau)\right] d\tau, \tag{2}$$

where $P_f(\tau)$ and $Q(\tau)$ are symmetric positive semidefinite matrices, and $R(\tau)$ is a $m \times m$ symmetric positive definite matrix. The Hamiltonian for this system is

$$\mathcal{H} = \frac{1}{2}\left[\mathbf{x}^T(\tau)Q(\tau)\mathbf{x}(\tau) + \mathbf{u}^T(\tau)R(\tau)\mathbf{u}(\tau)\right] + \boldsymbol{\lambda}^T(\tau)\left[A(\tau)\mathbf{x}(\tau) + B(\tau)\mathbf{u}(\tau)\right], \tag{3}$$

where $\boldsymbol{\lambda}(\tau)$ is the costate vector that satisfies the dynamics

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}} = -\left[Q(\tau)\mathbf{x}(\tau) + A^T(\tau)\boldsymbol{\lambda}(\tau)\right] \tag{4}$$

with the transversality condition

$$\boldsymbol{\lambda}(\tau_f) = P_f\mathbf{x}(\tau_f). \tag{5}$$

From the Minimum Principle, the necessary optimality condition

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = \mathbf{0} \tag{6}$$

yields the optimal control

$$\mathbf{u}(\tau) = -R^{-1}(\tau)B^T(\tau)\boldsymbol{\lambda}(\tau) = F(\tau)\boldsymbol{\lambda}(\tau) \tag{7}$$

where $F(\tau) = -R^{-1}(\tau)B^T(\tau)$. Substituting Eq.(7) into Eq.(1) and including Eq. (4), we have the following linear two-point boundary-value problem

$$\begin{bmatrix}\dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}}\end{bmatrix} = \begin{bmatrix} A(\tau) & B(\tau)F(\tau) \\ -Q(\tau) & -A^T(\tau)\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ \boldsymbol{\lambda}\end{bmatrix} \tag{8}$$

with $\mathbf{x}(\tau_0) = \tilde{\mathbf{x}}_0$, and $\boldsymbol{\lambda}(\tau_f) = P_f\mathbf{x}(\tau_f)$. In principle, Eq.(7) is an open-loop controller. To generate closed-loop control by way of solving the linear two-point boundary-value problem, there are two well-known solution methods. In the backward sweep method inspired by Eq.(5), the problem is defined as finding $P(\tau)$ such that

$$\boldsymbol{\lambda}(\tau) = P(\tau)\mathbf{x}(\tau) \tag{9}$$

It is straightforward to show [10] that, $P(\tau)$, satisfies the differential Riccati matrix equation

$$\dot{P} = -PA - A^TP + PBR^{-1}B^TP - Q, \qquad P(\tau_f) = P_f \tag{10}$$

Thus, Eq.(7) together with Eq.(10) forms a continuous feedback control law. It is well-known that this method is potentially unstable and numerically intensive. This is a critical issue in perturbation guidance where the Accessory Minimum Problem associated with nonlinear optimal control problems needs to be solved online and repeatedly [10]. In the sampled-data feedback approach, the goal is to find the transformation that maps $\mathbf{x}(\tau_0)$ to $\boldsymbol{\lambda}(\tau_0)$, so that $\mathbf{u}(\tau_0)$ can be obtained from Eq.(7). Replacing τ_0 and the parameters at τ_0 by those at τ, the most recent sample time, a continuous feedback law is easily generated. The important issue is, therefore, a computationally stable and rapid method for arriving at this transformation. In the method based on transition matrices [10], this transformation is obtained as follows: Denote by $X(t)$ and $\Lambda(t)$ the transition matrices that satisfy the state and costate equations with the boundary conditions

$$X(\tau_f) = I, \text{ the identity matrix, } \Lambda(\tau_f) = P_f$$

then we have

$$\mathbf{x}(\tau) = X(\tau)[X(\tau_0)]^{-1}\mathbf{x}(\tau_0), \tag{11}$$

$$\boldsymbol{\lambda}(\tau) = \Lambda(\tau)[X(\tau_0)]^{-1}\mathbf{x}(\tau_0), \tag{12}$$

from which we get

$$P(\tau_0, \tau) = \Lambda(\tau)[X(\tau_0)]^{-1} \tag{13}$$

Replacing τ_0 by the current time, τ, we get,

$$P(\tau) = \Lambda(\tau)[X(\tau)]^{-1} \tag{14}$$

This method is potentially ill-conditioned since the required integrations are unstable in either the forward or backward directions.

In order to avoid the problems associated with either of these methods, in this paper we suggest an efficient discretization technique from which $P(\tau)$ can be rapidly generated in the sense that no forward or backward integrations are explicitly required. Our method is *fundamentally* approximate in the sense that we rely on accurately discretizing the equations by a differentiation matrix. The discretization technique is based on a pseudospectral scheme which will be described in the next section. We present two methods that are similar in spirit to the two outlined above but without some of their associated problems. In the first method based on a discretized version of the backward sweep method, we seek to find the mapping,

$$\boldsymbol{\lambda}(\tau_i) = P(\tau_i)\mathbf{x}(\tau_i) \tag{15}$$

that approximates Eq.(9) at the shifted Legendre-Gauss-Lobatto (LGL) points (defined in the next section) in the interval, $\tau \in [\tau_0, \tau_f]$. The optimal control is given by

$$\mathbf{u}(\tau_i) = -R^{-1}(\tau_i)B^T(\tau_i)P(\tau_i)\mathbf{x}(\tau_i) \tag{16}$$

As in the usual practical implementation of the backward sweep method, the sequence of gain matrices,

$$K_i = -R^{-1}(\tau_i)B^T(\tau_i)P(\tau_i) \tag{17}$$

may be interpolated for values of τ in between the LGL points.

In the second method based on the sample data feedback law, we seek to find a mapping such that

$$\boldsymbol{\lambda}(\tau_i) = \mathcal{L}_i\mathbf{x}(\tau_0) \tag{18}$$

where, as before, the subscript i denotes the LGL collocation points. The sampled data optimal control is expressed as

$$\mathbf{u}(\tau_i, \mathbf{x}(\tau_0)) = -R^{-1}(\tau_i)B^T(\tau_i)\mathcal{L}_i\mathbf{x}(\tau_0) \tag{19}$$

Of course, we are only interested in $\mathcal{L}_0$ in the sense that the feedback controller is obtained by replacing $\mathbf{x}(\tau_0)$ by the most recent state, $\mathbf{x}$. In the following sections we describe and derive a numerically efficient method for computing $P(\tau_i)$ and $\mathcal{L}_0$. We also derive a functional relationship for $\mathcal{L}_0$ and expound on Eq.(19).

3 The Legendre Pseudospectral Method

The basic idea of this method is to seek polynomial approximations for the state, costate and control functions in terms of their values at the Legendre-Gauss-Lobatto (LGL) points. Then the LTV systems with quadratic criteria are reduced to solving a system of algebraic equations. Based on the algebraic equations, the analytical control laws can be derived. In the numerical approximation of the optimal control problem, since the collocation (LGL) points lie in the computational interval $[-1, 1]$, the problem is transformed to this interval by the linear transformation for $t \in [t_0, t_N] = [-1, 1] : \tau = \frac{(\tau_f - \tau_0)t + (\tau_f + \tau_0)}{2}$ It follows that Eqs. in (8), and the boundary conditions can be replaced by

$$\dot{\mathbf{x}} = \frac{\tau_f - \tau_0}{2}\left[A(\tau)\mathbf{x} + B(\tau)F(\tau)\boldsymbol{\lambda}\right], \tag{20}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\tau_f - \tau_0}{2}\left[Q(\tau)\mathbf{x}(\tau) + A^T(\tau)\boldsymbol{\lambda}(\tau)\right] \tag{21}$$

$$\mathbf{x}(-1) = \tilde{\mathbf{x}}_0, \tag{22}$$

$$\boldsymbol{\lambda}(1) = P_f\mathbf{x}(1) \tag{23}$$

In the Legendre pseudospectral method, the LGL collocation points are closely related to the Legendre polynomials which are orthogonal over the interval $[-1, 1]$ with the weight function $\alpha(t) = 1$. Let $L_N(t)$ be the Legendre polynomial of degree N on the interval $[-1, 1]$. In the Legendre collocation approximation [4]-[7] of Eqs.(20)-(23), we use the LGL points, which have a fixed value at the first and last nodes, and therefore, are most suited for solving boundary value problems. These points $t_l, l = 0, \ldots, N$ are given by

$$t_0 = -1, \quad t_N = 1,$$

and for $1 \le l \le N-1$, t_l are the zeros of $\dot{L}_N$, the derivative of the Legendre polynomial, L_N. As described in the previous section, we start by approximating the continuous state and control variables by Nth degree polynomials of the form

$$\mathbf{x}(t) \approx \mathbf{x}^N(t) = \sum_{l=0}^{N} \mathbf{x}_l\phi_l(t), \tag{24}$$

$$\boldsymbol{\lambda}(t) \approx \boldsymbol{\lambda}^N(t) = \sum_{l=0}^{N} \boldsymbol{\lambda}_l\phi_l(t), \tag{25}$$

where, for $l = 0, 1, \ldots, N$

$$\phi_l(t) = \frac{1}{N(N+1)L_N(t_l)}\frac{(t^2-1)\dot{L}_N(t)}{t - t_l}, \tag{26}$$

are the Lagrange polynomials of order N which interpolate the functions at the LGL points. It follows that

$$\mathbf{x}^N(t_l) = \mathbf{x}_l, \; \boldsymbol{\lambda}^N(t_l) = \boldsymbol{\lambda}_l. \tag{27}$$

To carryout the approximation of the state equations, we impose the condition that the approximations above satisfy the differential equations at the LGL collocation points. To express the derivative $\dot{\mathbf{x}}^N(t)$ in terms of $\mathbf{x}^N(t)$ at the collocation points t_k, we differentiate

(24) and evaluate the result at t_k to obtain a matrix multiplication of the following form:

$$\dot{\mathbf{x}}^N(t_k) = \sum_{l=0}^{N} \mathbf{x}_l \dot{\phi}_l(t_k) = \sum_{l=0}^{N} D_{kl}\mathbf{x}_l, \qquad (28)$$

where $D_{kl} = \dot{\phi}_l(t_k)$ are entries of the $(N+1)\times(N+1)$ differentiation matrix $\mathbf{D}$

$$\mathbf{D} := [D_{kl}] := \begin{cases} \frac{L_N(t_k)}{L_N(t_l)}\cdot\frac{1}{t_k - t_l} & k \neq l \\ -\frac{N(N+1)}{4} & k = l = 0 \\ \frac{N(N+1)}{4} & k = l = N \\ 0 & \text{otherwise} \end{cases} \qquad (29)$$

The state equations and the initial and terminal state conditions are discretized by first substituting Eqs.(24)-(25) and derivatives of the form (28) in Eqs.(20)-(23) and collocating at the LGL nodes, t_k. The state and costate equations are transformed into the following algebraic equations for $k = 0, \ldots, N$,

$$\sum_{l=0}^{N} D_{kl}\mathbf{x}_l - \frac{\tau_f - \tau_0}{2}(A_k\mathbf{x}_k + B_kF_k\boldsymbol{\lambda}_k) = \mathbf{0} \qquad (30)$$

$$\sum_{l=0}^{N} D_{kl}\boldsymbol{\lambda}_l + \frac{\tau_f - \tau_0}{2}(Q_k\mathbf{x}_k + A_k^T\boldsymbol{\lambda}_k) = \mathbf{0} \qquad (31)$$

where for a generic matrix $A(t)$, the notation A_k denotes $A(t_k)$. Also, the boldface $\mathbf{0}$ represents the zero vector of appropriate dimension. Writing these equations in block matrix notation, for

$$\mathbf{X} = [\mathbf{x}_0^T, \mathbf{x}_1^T, \ldots, \mathbf{x}_N^T]^T,\ \boldsymbol{\Lambda} = [\boldsymbol{\lambda}_0^T, \boldsymbol{\lambda}_1^T, \ldots, \boldsymbol{\lambda}_N^T]^T, \qquad (32)$$

we have

$$\tilde{A}_-\mathbf{X} - \frac{\tau_f - \tau_0}{2}\tilde{G}\boldsymbol{\Lambda} = \mathbf{0} \qquad (33)$$

$$\frac{\tau_f - \tau_0}{2}\tilde{Q}\mathbf{X} + \tilde{A}_+\boldsymbol{\Lambda} = \mathbf{0} \qquad (34)$$

Where $\tilde{A}_-, \tilde{A}_+, \tilde{G}, \tilde{Q}$ are $[n(N+1)\times n(N+1)]$ matrices whose (ij)th blocks are $n \times n$ matrices of the following form

$$[\tilde{A}_-]_{ij} = \begin{cases} D_{ij}I_n, & i \neq j \\ D_{ii}I_n - \frac{\tau_f - \tau_0}{2}A_i & i = j \end{cases} \qquad (35)$$

$$[\tilde{A}_+]_{ij} = \begin{cases} D_{ij}I_n, & i \neq j \\ D_{ii}I_n + \frac{\tau_f - \tau_0}{2}A_i^T & i = j \end{cases}$$

$$[\tilde{G}]_{ij} = \begin{cases} 0_n, & i \neq j \\ B_iF_i = -B_iR_i^{-1}B_i, & i = j \end{cases}$$

$$[\tilde{Q}]_{ij} = \begin{cases} 0_n, & i \neq j \\ Q_i & i = j \end{cases}$$

In the above, I_n and 0_n are the $n \times n$ identity and zero matrices, respectively. The initial and final conditions are

$$\mathbf{x}_0 = \tilde{\mathbf{x}}_0 \qquad (36)$$

$$\boldsymbol{\lambda}_N = P_f\mathbf{x}_N \qquad (37)$$

The goal is to solve Eqs.(33) and (34) subject to the transversality conditions Eqs. (36) and (37). Therefore, first we write the equations for the state and costate vectors $\mathbf{x}$ and $\boldsymbol{\lambda}$ in block form to have the block matrix form

$$\begin{bmatrix} \tilde{A}_- & -\frac{\tau_f - \tau_0}{2}\tilde{G} \\ \frac{\tau_f - \tau_0}{2}\tilde{Q} & \tilde{A}_+ \\ \tilde{P} & -\tilde{I} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \boldsymbol{\Lambda} \end{bmatrix} \equiv V\mathbf{Z} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \qquad (38)$$

In these equations $\mathbf{Z}^T = [\mathbf{X}^T, \boldsymbol{\Lambda}^T]$ and $\tilde{P}$ and $\tilde{I}$ are the following $n \times n(N+1)$ matrices

$$\tilde{P} = [0_n, \ldots, 0_n, P_f] \qquad (39)$$

$$\tilde{I} = [0_n, \ldots, 0_n, I_n] \qquad (40)$$

The matrix V in Eq.(38) is of dimension $n(2N+3) \times 2n(N+1)$ that may be rearranged in two useful ways. Consider first its partitioning as $V = [V_x \quad V_\lambda]$ where V_x and V_λ are each of dimensions $n(2N+3)\times n(N+1)$ so that we have

$$V_x\mathbf{X} + V_\lambda\boldsymbol{\Lambda} = \mathbf{0} \qquad (41)$$

By rewriting this equation, we get the following expression for $\boldsymbol{\Lambda}$ in terms of $\mathbf{X}$:

$$\boldsymbol{\Lambda} = -V_\lambda \backslash V_x\mathbf{X} = \tilde{P}\mathbf{X} \qquad (42)$$

where the \ operator (inspired by MATLAB) denotes the least-squares solution. In MATLAB, matrix $\tilde{P}$ can be computed very efficiently by way of a QR decomposition. Comparing Eq. (42) to Eq. (15), it is apparent that we have found the transformation for our first method (Cf. Eq. (15)) as,

$$\begin{bmatrix} P(\tau_0) \\ \vdots \\ P(\tau_f) \end{bmatrix} = -V_\lambda \backslash V_x = \tilde{P} \qquad (43)$$

This matrix can be computed once off-line, and using the solutions to the system (38), we can write the optimal feedback law at the collocation points from Eq. (16):

$$\mathbf{u}(\tau_i) = -R^{-1}(\tau_i)B^T(\tau_i)P(\tau_i)\mathbf{x}(\tau_i)$$

For the second method, we partition V as $V = [V_0 \quad V_e]$ such that

$$V_0\mathbf{x}_0 + V_e\mathbf{X}_e = \mathbf{0} \tag{44}$$

where vector $\mathbf{X}_e$ is of dimension $n(2N+1) \times 1$ and is defined as

$$\mathbf{X}_e = [\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_N^T, \boldsymbol{\lambda}_0^T, \ldots, \boldsymbol{\lambda}_N^T]^T \tag{45}$$

Thus, V_0 and V_e are $[n(2N+3) \times n]$, $[n(2N+3) \times n(2N+1)]$ block matrices of V, respectively. We can solve Eq. (44) for $\mathbf{X}_e$ as

$$\mathbf{X}_e = -V_e \backslash V_0 \mathbf{x}_0 = W\mathbf{x}_0 \tag{46}$$

where, as before, the \ operator denotes the least-squares solution in MATLAB. As indicated in Eq. (46), $W \equiv -V_e \backslash V_0$ is a matrix of dimension $(2nN+n) \times n$. Since $\mathbf{Z} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{X}_e \end{bmatrix}$ we get

$$\mathbf{Z} = \begin{bmatrix} \mathbf{X} \\ \boldsymbol{\Lambda} \end{bmatrix} = \begin{bmatrix} I_n \\ W \end{bmatrix} \mathbf{x}_0 \equiv \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \mathbf{x}_0$$

where W_1 and W_2 are partitions of the $[I_n \quad W]$ matrix, each of dimension $n(N+1) \times n$ so that we have,

$$\mathbf{X} = W_1\mathbf{x}_0 \tag{47}$$

$$\boldsymbol{\Lambda} = W_2\mathbf{x}_0 \tag{48}$$

Comparing Eq. (48) to Eq. (18), it is apparent that

$$\begin{bmatrix} \mathcal{L}_0 \\ \vdots \\ \mathcal{L}_N \end{bmatrix} \equiv W_2 \tag{49}$$

Thus, $\mathcal{L}_0$ is simply the first $n \times n$ block of W_2. Note that for a given system, $\mathcal{L}_0$ is a function of N and $\tau_f - \tau_0$. Replacing τ_0 by the current time, τ, we have $\mathcal{L}_0 \equiv \mathcal{L}_0(N, T)$ where T is "time-to-go" defined as, $T = \tau_f - \tau$. Thus, a *continuous-data feedback* controller (Cf. Eq. 19) can now be written as

$$\mathbf{u}(\tau, \mathbf{x}) = -R^{-1}(\tau)B^T(\tau)\mathcal{L}_0(N, T)\mathbf{x} \tag{50}$$

It is important to note that in our methods, both $P(\tau_i)$ and $\mathcal{L}_0$ and hence the controllers are obtained without any explicit integration or construction of transition matrices.

A Numerical Example

Consider the linear time-varying system

$$\dot{x}(\tau) = \tau x(\tau) + u(\tau), \qquad 0 \le \tau \le 1 \tag{51}$$

$$x(0) = 1 \tag{52}$$

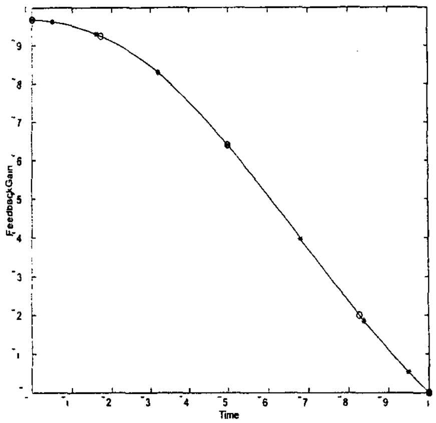

Figure 1: Comparison of the results for the feedback gain from the numerical integration (True Solution) and LGL method

with the cost functional

$$J = \frac{1}{2}\int_0^1 [x^2(\tau) + u^2(\tau)]d\tau. \tag{53}$$

The problem is to find the optimal control $u(\tau)$ which minimizes (53) subject to the constraints of the Eqs. (51) and (52). The optimal control is given by

$$u(\tau) = -w(\tau)x(\tau), \tag{54}$$

where $w(\tau)$ is the solution of the Riccati equation

$$\dot{w}(\tau) = -2\tau w(\tau) + w^2(\tau) - 1, \qquad w(1) = 0 \tag{55}$$

In Refs. [1]-[3], the optimal feedback law is approximated using the respective method and is compared against the numerical solution of Eq. (55). Using our method, we can solve for $w(t_k)$ from Eqs. (47)-(50). In Table 1 we show the solutions of the feedback gain for 4 and 8 LGL points and compare them against the numerical solution obtained by integrating the Riccati equation (55) and interpolating the results to get the values at t_k. The results show that we can get very accurate results for a low number of N in a very fast and efficient manner.

Figure 1 demonstrates the results. The solid line represents the "true" solution from numerical integration; 'ooo' represents the solution for 4 LGL points and the solution for 8 LG points is denoted by '***'. The second graph shows the accuracy of the first technique where the states, costates and the controls are computed for 8 LGL points, and then interpolation is used to find the values for the times in between. The curve-fitting or interpolation is performed for 20 points. The graph shows clearly the accuracy of results.

$w(t_k)$ by Control Law for $N = 4$	Exact $w(t_k)$	$w(t_k)$ by Control Law for $N = 8$	Exact $w(t_k)$
0.96883	0.96854	0.96854	0.96854
0.92411	0.92429	0.96277	0.96277
0.63851	0.63856	0.92936	0.92936
0.20009	0.20541	0.82909	0.82909
0.00001	0	0.63856	0.63856
		0.39677	0.39677
		0.18524	0.18524
		0.05258	0.05258
		0.00000	0

Table 1: Comparisons of the Solutions for the Riccati Equation

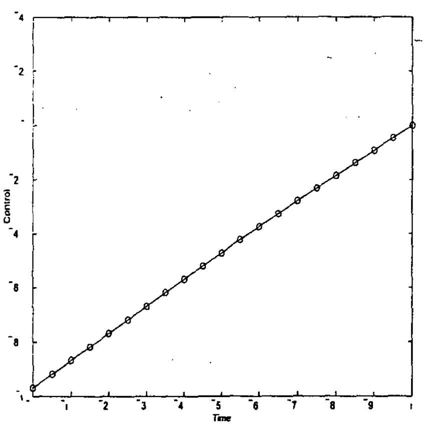

Figure 2: Comparison of the results for the optimal feedback control from the numerical integration (True Solution) and the interpolation technique

Conclusions

In this paper, we have shown that by using a Legendre pseudospectral discretization method the necessary conditions for an LTV system can be transformed to a coupled system of linear algebraic equations. We proposed two techniques for solving the resulting system to obtain the optimal control laws. Both techniques rely on simple linear algebra methods which avoid integration of differential Riccati equation (DRE), or using transition matrices. The second technique based on the sampled-feedback data has the potential of extension to problems such as receding-horizon control problems where continuous on-line computation of the DRE is required. For these problems, our technique offers an efficient and speedy calculation of the optimal feedback laws on-line without the need for integration of the DRE.

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