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***Abstract*— As a result of significant progress in pseudospectral methods for real-time dynamic optimization, it has become apparent in recent years that it is possible to present a unified framework for both controller and observer design. In this paper, we present such an approach for nonlinear systems. The method can be applied to a wide variety of nonlinear systems. The convergence of the proposed computational method is guaranteed under verifiable conditions. Several numerical examples are also presented to demonstrate the efficiency of the proposed computational framework.**

I. INTRODUCTION

The past decade has witnessed many successful applications of computational methods in modern control system design. For instance, in recognizing that analytic solutions are extremely difficult to obtain in the area of nonlinear optimal control, numerical methods have been advanced as a practical technique for obtaining a solution. For this reason, many computational methods have been developed for solving nonlinear optimal control problems, e.g. [2], [1], [5], [4]. In recently years, computational methods have been applied to other control areas as well. For example, various numerical methods have been proposed for nonlinear observer design [21], [17], [20]. With the rapid progress in numerical methods and computer technology, it is not surprising to foresee more and more interaction between computational methods and control system design. In this paper, we present a unified framework for both nonlinear controller and observer design based on pseudospectral (PS) methods.

A unified design method is very desirable for sophisticated control systems because it can significantly reduce the design cost. Here, by "unify", we mean a method that is portable across heterogeneous systems and can be applied to achieve different objectives. Many existing control methods can only be applied to a specific type of systems. A simple change in the control plant, for instance, adding a constraint, can result in a redesign of the entire feedback control. On the other hand, a complicated control system may involve several heterogeneous subsystems. Each subsystem requires a different design method which is very inconvenient and expensive. Therefore, a unified design method that can be applied to a wide variety of systems will greatly reduce the design cost and save the design time. Even for a fixed nonlinear system, the change of the objectives, for example, stabilization, tracking, estimation, etc., can result in designing different controllers. A single method that achieves multiple purposes is much more desirable than conventional methods. At the implementation level, almost all modern control systems inevitably rely on computer software to realize various control algorithms. Common software for different systems and different objectives can significantly reduce the complexity of the software and save enormous cost associated with software verification and software management. In this paper, we present a *unified computational* framework. It has the potential to deal with a wide variety of nonlinear systems for both controller and observer design.

The unified framework we proposed is based on PS discretization and optimization. The idea is to formulate various problems (control/observation) as a dynamic optimization problem. Then, by applying PS discretization, it is approximated by a nonlinear programming, which can be solved by an efficient spectral algorithm and off-the-shelf optimization software. This is the approach used in DIDO [31], a MATLAB application package for solving dynamic optimization problems. In recent years, PS methods have been successfully applied to solve a wide variety of optimal control problems, [6], [26], [12], [32], [34], [15], [19], [23], [16], [27]. As a result of its success, PS methods are now part of OTIS [22], NASA's software package for solving trajectory optimization problems. One of the main reasons for the popularity of PS methods is that they offer an exponential convergence rate for the approximation of analytic functions while providing Eulerian-like simplicity. Thus, for a given error bound, PS methods generate a significantly smaller-scale optimization problem when compared to traditional discretization methods. This property is particularly attractive for control applications as it places real-time computation within easy reach of modern computational power [30]. As another advantage of PS methods demonstrated in [30], they can be applied to a variety of nonlinear optimal control problems. The main purpose of this paper is to show that PS computational methods can not only be used in optimal control but also in nonlinear observer design. Therefore, it serves as a good candidate for a unified computation framework that achieves aforementioned purposes.

II. PS DISCRETIZATION

Pseudospectral methods were largely developed in the 1970s for solving partial differential equations arising in fluid

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dynamics and meteorology [3], and quickly became "one of the big three technologies for the numerical solution of PDEs" [33]. During the 1990s, PS methods were introduced for solving optimal control problems [7], [8]; and since then, have gained considerable attention. In this section we briefly present Legendre pseudospectral discretization for deferential-algebraic equations.

Consider the following controlled differential equation with path constraints

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

$$h(x(t), u(t)) \leq 0. \tag{2}$$

We assume the time interval to be fixed at $[-1, 1]$ in order to facilitate a simpler bookkeeping in using the Legendre pseudospectral method. If the physical time domain of the problem is not $[-1, 1]$ but some fixed finite interval, it can always be projected to the computational domain $[-1, 1]$ by a simple linear transformation [6]. Note that, PS methods can also be applied to infinite time horizon; see [9], [28] for details. The end-point condition is

$$e(x(-1), x(1)) = 0. \tag{3}$$

We assume that the state $x(t)$ belongs to Sobolev space $W^{m,\infty}$ with $m \geq 2$. More specifically there is a constant $C > 0$ and an integer $m \geq 2$ such that

$$\sum_{i=0}^{m} \left|\left| \frac{d^{(i)}x(t)}{dt} \right|\right|_{L^\infty(-1,1)} \leq C \tag{4}$$

where $d^{(i)}/dt$ denotes the i-th order distribution derivatives [3]. Note that, if $x(t)$ is C^1 and if $\dot{x}(t)$ has bounded derivative everywhere except for a finite many points on the closed interval $t \in [-1, 1]$, then condition (4) is automatically satisfied. On the other hand, by Sobolev's Imbedding Theorems [3], any function $x(t)$ satisfying the aforementioned condition must have continuous $(m-1)$-th order classical derivatives on $[-1, 1]$. Therefore, this condition requires the state $x(t)$ be at least continuously differentiable. The condition can be further relaxed to cover the situation where $x(t)$ is only continuous but piecewise C^1. The interesting readers are referred to [18] for details.

In the Legendre PS approximation, the basic idea is to approximate $x(t)$ by N-th order Lagrange polynomials $x^N(t)$ based on the interpolation at the Legendre-Gauss-Lobatto (LGL) quadrature nodes, i.e.

$$x(t) \approx x^N(t) = \sum_{k=0}^{N} x^N(t_k)\phi_k(t),$$

where t_k are LGL nodes defined as,

$$t_0 = -1, \;\; t_N = 1$$
$$t_k, \text{ for } k = 1, 2, \ldots, N-1, \text{ are the roots of } \dot{L}_N(t)$$

where $\dot{L}_N(t)$ is the derivative of the N-th order Legendre polynomial, $L_N(t)$. The Lagrange interpolating polynomial $\phi_k(t)$ is defined by

$$\phi_k(t) = \frac{1}{N(N+1)L_N(t_k)} \frac{(t^2-1)\dot{L}_N(t)}{t - t_k}. \tag{5}$$

It is readily verifiable that $\phi_k(t_j) = 1$, if $k = j$ and $\phi_k(t_j) = 0$, if $k \neq j$. The derivative of the i-th state $x_i(t)$ at the LGL node t_k can be approximated by

$$\dot{x}_i(t_k) \approx \dot{x}_i^N(t_k) = \sum_{j=0}^{N} D_{kj} x_i^N(t_j), \; i = 1, 2, \ldots, N_x$$

where $(N+1) \times (N+1)$ differentiation matrix D is defined as

$$D_{ik} = \begin{cases} \frac{L_N(t_i)}{L_N(t_k)} \frac{1}{t_i - t_k}, & \text{if } i \neq k; \\ -\frac{N(N+1)}{4}, & \text{if } i = k = 0; \\ \frac{N(N+1)}{4}, & \text{if } i = k = N; \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Let $\bar{x}_k = x^N(t_k)$, $k = 0, 1, \ldots, N$. In a standard PS method, the continuous differential equation is approximated by the following nonlinear algebraic equations

$$\sum_{i=0}^{N} \bar{x}_i D_{ki} - f(\bar{x}_k, \bar{u}_k) = 0, \qquad k = 0, 1, \ldots, N \tag{7}$$

where $\bar{u}_k$ is taken to be analogous to $\bar{x}_k$. This discretization is used in [6], [27] for optimal control problems. However, it has been shown in [13] that a feasible solution to (7) may not exist; hence, to guarantee feasibility of the discretization, the following relaxation is used

$$\left|\left| \sum_{i=0}^{N} \bar{x}_i D_{ki} - f(\bar{x}_k, \bar{u}_k) \right|\right|_\infty \leq (N-1)^{\frac{3}{2}-m}, \tag{8}$$

Deferring a justification of this relaxation, note that when N tends to infinity, the difference between conditions (7) and (8) vanishes, since m, by assumption, is greater than or equal to 2. Throughout the paper, we use the "bar" notation to denote discretized variables. Note that the subscript in $\bar{x}_k$ denotes an evaluation of the approximate state, $x^N(t) \in \mathbb{R}^{N_x}$, at the node t_k whereas $x_k(t)$ denotes the k-th component of the exact state. The endpoint conditions and constraints are approximated in a similar fashion

$$||e(\bar{x}_0, \bar{x}_N)||_\infty \leq (N-1)^{\frac{3}{2}-m} \tag{9}$$

$$h(\bar{x}_k, \bar{u}_k) \leq (N-1)^{\frac{3}{2}-m} \cdot \mathbf{1}, \quad k = 0, \ldots, N \tag{10}$$

where $\mathbf{1}$ denotes $[1, \ldots, 1]^T$.

Many control/estimation problems can be formulated as an optimization to some integer type of cost functions which measure the control performances or estimation errors. PS methods provide an accurate way to discretize it. Consider the following genetic nonlinear cost function

$$J[x(\cdot), u(\cdot)] = \int_{-1}^{1} F(x(t), u(t)) \, dt + E(x(-1), x(1)) \tag{11}$$

$J[x(\cdot), u(\cdot)]$ can be approximated by the Gauss-Lobatto integration rule,

$$J[x(\cdot), u(\cdot)] \approx \bar{J}^N(\bar{X}, \bar{U}) = \sum_{k=0}^{N} F(\bar{x}_k, \bar{u}_k) w_k + E(\bar{x}_0, \bar{x}_N)$$

where w_k are the LGL weights given by

$$w_k = \frac{2}{N(N+1)} \frac{1}{[L_N(t_k)]^2}, \qquad k = 0, 1, \ldots, N$$

and $\bar{X} = [\bar{x}_0, \ldots, \bar{x}_N]$, $\bar{U} = [\bar{u}_0, \ldots, \bar{u}_N]$.

III. PS Methods for Nonlinear Optimal Control and Estimation

In this section we show that Pseudospectral methods provide a powerful tool to tackle both control and estimation problem and serve as the engine for both PS controller and PS observer. This enabling technology facilitates a cost efficient output feedback design for nonlinear system. The idea is illustrates in the following diagram.

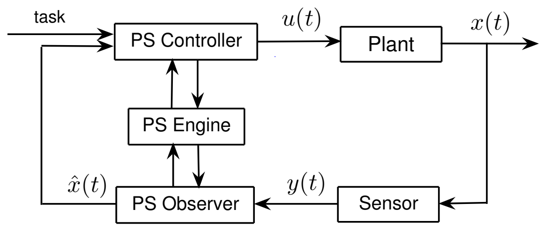

Fig. 1. Block diagram for the unified control/observer design framework by PS methods.

In the next two subsections we collect some key results in the literature of pseudospectral methods [12], [13], [14], [30] regarding nonlinear control and observer design. In Section IV we will provide an example of output feedback design by PS methods.

A. PS controller design

Problem B: Determine the state-control function pair, $t \mapsto (x,u) \in \mathbb{R}^{N_x} \times \mathbb{R}^{N_u}$, that minimize the cost function

$$J[x(\cdot),u(\cdot)] \;=\; \int_{-1}^{1} F(x(t),u(t))\,dt + E(x(-1),x(1))$$

subject to the dynamics,

$$\dot{x}(t) \;=\; f(x(t),u(t)) \tag{12}$$

endpoint conditions

$$e(x(-1),x(1)) \;=\; 0 \tag{13}$$

and path constraints

$$h(x(t),u(t)) \;\le\; 0 \tag{14}$$

It is assumed that $F : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \to \mathbb{R}$, $E : \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}$, $f : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \to \mathbb{R}^{N_x}$, $e : \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}^{N_e}$, and $h : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \to \mathbb{R}^{N_h}$, are continuously differentiable with respect to their arguments and their gradients are Lipschitz continuous over the domain. We assume that an optimal solution $(x^*(\cdot),u^*(\cdot))$ exists with the optimal state, $x^*(\cdot) \in W^{m,\infty}$, $m \ge 2$.

Applying Legendre Pseudospectral method to Problem B, we can discretized it to a finite dimensional nonlinear optimization problem summarized in the following

Problem B^N: Find $\bar{x}_k \in \mathbb{X}$ and $\bar{u}_k \in \mathbb{U}$, $k = 0,1,\ldots,N$, that minimize

$$\bar{J}^N(\bar{X},\bar{U}) \;=\; \sum_{k=0}^{N} F(\bar{x}_k,\bar{u}_k)w_k + E(\bar{x}_0,\bar{x}_N) \tag{15}$$

subject to

$$\left\|\sum_{i=0}^{N} \bar{x}_i D_{ki} - f(\bar{x}_k,\bar{u}_k)\right\|_\infty \;\le\; (N-1)^{\frac{3}{2}-m} \tag{16}$$

$$||e(\bar{x}_0,\bar{x}_N)||_\infty \;\le\; (N-1)^{\frac{3}{2}-m} \tag{17}$$

$$h(\bar{x}_k,\bar{u}_k) \;\le\; (N-1)^{\frac{3}{2}-m}\cdot\mathbf{1} \tag{18}$$

In Problem B^N, $\mathbb{X}$ and $\mathbb{U}$ are two compact sets representing the search region and containing the continuous optimal solution $(x^*(t),u^*(t))$. Problem B^N can be solved by an appropriate globally-convergent algorithm [24], [2], such as for example, a sequential-quadratic programming method. This approach has been successfully used in solving an impressive array of problems (see for example, [30], [6], [32]).

In the following we list some fundamental results regarding the existence and convergence of the Legendre PS method. The proofs of these results can be found in [12], [13].

Theorem 1: Given any feasible solution, $t \mapsto (x,u)$, for Problem B, suppose $x(\cdot) \in W^{m,\infty}$ with $m \ge 2$. Then, there exists a positive integer N_1 such that, for any $N > N_1$, Problem B^N has a feasible solution, $(\bar{x}_k,\bar{u}_k)$. Furthermore, the feasible solution satisfies $\bar{u}_k = u(t_k)$ and

$$\|x(t_k) - \bar{x}_k\|_\infty \;\le\; L(N-1)^{1-m}, \tag{19}$$

for all $k = 0,\ldots,N$, where t_k are LGL nodes and L is a positive constant independent of N.

Theorem 1 shows that Problem B^N is well-posed with a nonempty feasible set as long as a sufficient number of nodes are chosen. Therefore, an optimal solution always exists. More importantly, (19) shows the existence of a feasible discrete solution around any neighborhood of the continuous trajectory. The convergence of PS methods for nonlinear optimal control problems has been proved in [12], [13] in a way similar in spirit to Polak's theory of consistent approximations [24].

Let $(\bar{x}_k^*,\bar{u}_k^*)$, $k = 0,1,\ldots,N$, be an optimal solution to Problem B^N. Let $x^N(t) \in \mathbb{R}^{N_x}$ be the N-th order interpolating polynomial of $(\bar{x}_0^*,\ldots,\bar{x}_N^*)$ and $u^N(t) \in \mathbb{R}^{N_u}$ be any interpolant of $(\bar{u}_0^*,\ldots,\bar{u}_N^*)$, i.e.

$$x^N(t) = \sum_{k=0}^{N} \bar{x}_k^*\phi_k(t), \qquad u^N(t) = \sum_{k=0}^{N} \bar{u}_k^*\psi_k(t)$$

where $\phi_k(t)$ is the Lagrange interpolating polynomial defined by (5) and $\psi_k(t)$ is any continuous function such that $\psi_k(t_j) = 1$, if $k = j$ and $\psi_k(t_j) = 0$, if $k \ne j$. Note that $u^N(t)$ is not necessarily a polynomial. Typically, we use linear or spline functions for interpolating $(\bar{u}_0^*,\ldots,\bar{u}_N^*)$. Now consider a sequence of Problems B^N with N increasing from N_1 to infinity. Correspondingly, we get a sequence of discrete optimal solutions $\{(\bar{x}_k^*,\bar{u}_k^*), k = 0,\ldots,N\}_{N=N_1}^\infty$ and their interpolating function sequence $\{x^N(t),u^N(t)\}_{N=N_1}^\infty$.

Definition 1: A continuous function $\rho(t)$ is called the uniform accumulation point of a function sequence $\{\rho^N(t)\}_{N=0}^\infty$, $t \in [-1,1]$, if there is a subsequence of $\{\rho^N(t)\}_{N=0}^\infty$ that uniformly converges to $\rho(t)$.

Assumption 1: x_0^∞ is an accumulation point of the first element (i.e. $k = 0$) of the sequence, $\{\bar{x}_k^*, k = 0, \ldots, N\}_{N=N_1}^\infty$.

Theorem 2: Let $\{(\bar{x}_k^*, \bar{u}_k^*), 0 \le k \le N\}_{N=N_1}^\infty$ be a sequence of optimal solutions of Problem B^N satisfying Assumption 1, and $(x^N(t), u^N(t))_{N=N_1}^\infty$ be their interpolating function sequence. Let the pair of continuous functions, $(q(t), u^\infty(t))$, be any uniform accumulation point of the sequence $(\dot{x}^N(t), u^N(t))_{N=N_1}^\infty$. Then, $u^\infty(t)$ is an optimal control to the original continuous Problem B, and $x^\infty(t) = \int_{-1}^t q(\tau)d\tau + x_0^\infty$ is the corresponding optimal trajectory.

This result demonstrates a key property of PS discretization methods: *if the optimal solution of the discrete Problem B^N converges, it must converge to an optimal solution of the continuous Problem B*. Note that Assumption 1 is posed on the discrete solution only. It can be checked easily by standard numerical methods, for example, by matrix multiplication as demonstrated in [12]. Thus, under relatively mild conditions, Theorem 1-2 guarantee the existence and convergence of the discrete-time optimal solution to the continuous-time solution of the original problem. Therefore, the continuous nonlinear optimal control Problem B, boils down to a problem of sparse nonlinear programming that can be solved using an appropriate method. We use a spectral algorithm in conjunction with an SQP method [11] to solve these problems. This algorithm has been implemented in DIDO [31] which can simply be described as a minimalists approach to solving dynamic optimization problems. All simulations presented in this paper are programmed in MATLAB on a Pentium 4, 2.4GHz PC with 256MB of RAM. In the following we use an example to demonstrate the efficiency of the PS computational methods.

Example 1: Consider the finite time stabilization problem for the following system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^3 \\ \dot{x}_3 &= u \end{aligned} \tag{20}$$

with initial condition $x(0) = (1, 0.5, 1)$ and finial condition $x(3) = (0, 0, 0)$. The objective is to minimize

$$\int_0^3 u^2(t)dt$$

subject to the constraint $|u| \le 3$.

The linearization of (20) has an uncontrollable mode which is unstable. Therefore, by Brockett's necessary condition, there is no continuously differentiable state feedback controller that asymptotically stabilizes the system. One approach is to utilize non-smooth feedback design methods such as [25].

By proposed PS method, the optimal controller for system (20) can be easily calculated. A solution for N=64 nodes is shown in Figure 2. It clearly demonstrates the stability of the system under the action of the optimal control.

B. Nonlinear observer design by PS methods

In this subsection we briefly review a Pseudospectral nonlinear observer proposed in [14]. Consider the following nonlinear system with sampled output

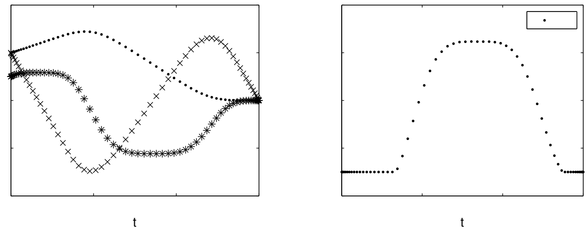

Fig. 2. Discrete optimal solution with 64 nodes

$$\dot{x} = f(x, t) \tag{21}$$

$$y_i = h(x(t_i)) \tag{22}$$

where state $x \in \mathbb{R}^{N_x}$ and output $y \in \mathbb{R}^{N_y}$. It is assumed that $f : \mathbb{R}^{N_x} \times \mathbb{R} \to \mathbb{R}^{N_x}$, $h : \mathbb{R}^{N_x} \to \mathbb{R}^{N_y}$, are continuously differentiable with respect to their arguments. $\{t_i\}_{i=0}^\infty$ is the sequence of sampling time with $\lim_{i\to\infty} t_i = \infty$. Correspondingly, y_i is the measurement of the output $y(t)$ at the sampled points t_i. The observer problem is to estimate the state $x(t)$ at the current sampling time t_p based on measurement $\{y_i\}_{i=0}^p$ only. The state trajectory $x(t)$ is assumed to lie in Sobolev space $W^{m,\infty}$ with $m \ge 2$. Note that we do not include the control input $u(t)$ in (21). In observer design $u(t)$ is usually assumed to be a known function, therefore a nonlinear system with control input can always be casted into the time-varying form (21).

Although the state $x(t)$ is not measured directly, we frequently have some information about it. For instance, $x(t)$ may only lie in a certain interval. Apparently, utilizing this information should help the design of the observer. For this reason we introduce the constraint

$$r(x(t)) \le 0 \tag{23}$$

where $r : \mathbb{R}^{N_x} \to \mathbb{R}^{N_r}$ is continuously differentiable with respect to x. One essential purpose of the constraint set, $\{x \mid r(x) \le 0\}$ is to capture any *a priori* known information. An important consequence of the constraint set is that, we can include nonlinear systems governed by differential-algebraic equations (DAE). Observer design for DEAs is such a challenge problem that cannot be dealt with by many existing results especially gain-based design. However, for the online optimization based methods, the appearance of the algebraic equations can simply be treated as constraint sets. Throughout the paper the following observability condition is assumed.

Assumption 2: There is a constant $\delta > 0$ such that for any $T \in [\delta, \infty]$ and any two trajectories $x^1(t)$, $x^2(t)$ of (21) satisfying constraint (23),

$$\int_{T-\delta}^T \|h(x^1(t)) - h(x^2(t))\|^2 dt = 0$$

implies $x^1(t) = x^2(t)$, for all $t \in [T - \delta, T]$.

Remark 3.1: For linear time-varying systems, Assumption 2 always holds if the system is uniformly observable. In the nonlinear case, if the system is uniformly observable in the sense of [10], Assumption 2 is automatically satisfied.

This assumption also covers systems which are not uniformly observable such as the Duffing system discussed later in the paper.

Let $T = t_p$ be the current sampling time. Consider the following optimization problem:

Problem E: Determine the function $z(t)$, that minimizes the cost function

$$J[z(\cdot)] = \int_{T-\delta}^{T} \|h(z(t)) - y(t)\|^2 dt \quad (24)$$

subject to the state equation

$$\dot{z}(t) = f(z(t), t) \quad (25)$$

and constraint

$$r(z(t)) \leq 0 \quad (26)$$

Based on Assumption 2, Problem 1 has a *unique* optimal solution $z^*(t) = x(t)$. Therefore, the current state $x(T)$ can be obtained by evaluating the solution of Problem 1 at the current sampling time T, i.e., $x(T) = z^*(T)$. Based on this fact, a moving horizon type of observer can be constructed [20], [21], [17]. That is, at every sampling point, Problem E is solved online; then moving the time window $[T-\delta, T]$ to the next sampling point, the problem is solved again. The design philosophy is quite simple; however, a successful implementation of this concept depends on a key assumption: Problem E can be solved online.

To apply Legendre pseudospectral method, first, we need to project the physical time domain $[T-\delta, T]$ in Problem E to the computational domain $[-1, 1]$. To this end, the following transformation is introduced

$$\tau = \frac{2t - 2T + \delta}{\delta} \quad (27)$$

Under (27), by applying the same PS discretization method presented before, Problem E can be approximated by the following nonlinear programming problem.

Problem $\mathrm{E^N}$ Find $\bar{z}_k \in \mathbb{R}^{N_x}$, $k = 0, 1, \ldots, N$, that minimize

$$\bar{J}^N = \frac{\delta}{2} \sum_{k=0}^{N} \|\bar{z}_k - \bar{y}_k\|^2 w_k \quad (28)$$

subject to

$$\left\| \sum_{i=0}^{N} \bar{z}_i D_{ki} - \frac{\delta}{2} f(\bar{z}_k, \frac{\tau_k \delta - \delta + 2T}{2}) \right\|_\infty \leq (N-1)^{\frac{3}{2}-m}, \quad (29)$$

$$r(\bar{z}_k) \leq (N-r)^{\frac{3}{2}-m} \cdot \mathbf{1}, \quad (30)$$

where τ_k are LGL nodes, $\bar{y}_k = y(\frac{\tau_k\delta - \delta + 2T}{2})$ and $0 \leq k \leq N$.

The existence and convergence results of Problem $\mathrm{E^N}$ can be proved under similar condition as in the previous section. Let $\{t_i\}_{i=0}^{\infty}$ be the sequence of sampling time with $\lim_{i\to\infty} t_i = \infty$. Denote $y_i = y(t_i)$, i.e., y_i is the measurement of the output $y(t)$ at the sampled points t_i. By the moving horizon strategy, during each sampling period the continuous-time optimization Problem E is solved online by the PS method. Then the estimation of the current state is given by the optimal solution of the discrete Problem $\mathrm{E^N}$. A pseudospectral observer is formulated as the following algorithm.

Initialization:

1. Select tuning parameters N, L and initial guess of $x(t_0)$. Here $N > 1$ and $L > 1$ are two positive integers. N presents the number of nodes used in the pseudospectral discretization and L is the number of data to be processed at each iteration. If the sampling period is ΔT, then the backward integration length $\delta = \Delta T \cdot L$.
2. Calculate the LGL nodes τ_k, LGL weights w_k, $k = 0, 1, \ldots, N$, and the differential matrix D.
3. Propagate the initial guess of $x(t_0)$ by the differential equation (21) to generate the guess of the state at the shifted LGL notes $\frac{(\tau_k+1)(t_L-t_0)+2t_0}{2}$. Denote the guess as $\hat{z}_k^-$, $0 \leq k \leq N$, where the superscript "−" means prediction or a priori estimation. Set $Z_{initial} = \{\hat{z}_k^-\}_{k=0}^N$. Here $Z_{initial}$ is the starting point/initial guess for the optimization software. It is different to the initial guess of $x(t_0)$.
4. Collect initial data $\{y_0, y_1, \ldots, y_{L-1}\}$ and set $p = L$.

Main algorithm:

1. Collect the new measurement y_p.
2. Construct the spline function $y^s(t)$ of the data $\{y_{p-L}, y_{p-L+1}, \ldots, y_p\}$ such that $y^s(t_i) = y_i$ for all $p - L \leq i \leq p$. Set $\bar{y}_k = y^s(\frac{\tau_k\delta-\delta+2T}{2})$, where $\delta = t_p - t_{p-L}$ and $k = 0, 1, \ldots, N$. Here $\bar{y}_k$ is the reference signal in the cost function of Problem $\mathrm{E^N}$.
3. Apply nonlinear programming solver to Problem $\mathrm{E^N}$ with initial guess as $Z_{initial}$ to get the optimal solution $\{\bar{z}_k^*\}_{k=0}^N$. The estimation of the current state $x(t_p)$ is given by $\bar{z}_N^*$.
4. Propagate $\bar{z}_N^*$ by the differential equation (21) to get the prediction of the state at the next sampling time t_{p+1}. Denote the prediction as $\hat{x}_{p+1}^-$.
5. Construct the spline function $\hat{z}^s(t)$ of the data $\{\bar{z}_0^*, \ldots, \bar{z}_N^*, \hat{x}_{p+1}^-\}$.
6. Set $p = p + 1$ and $Z_{initial} = \{\hat{z}^s(\frac{\tau_k\delta-\delta+2T}{2})\}_{i=0}^N$.
7. Go to step 1.

Remark 3.2: The pseudospectral discretization of Problem E requires the measurement $y(t)$ at shifted LGL nodes, i.e., $y(\frac{\tau_k\delta-\delta+2T}{2})$. But in practice, the sampling time are normally pre-fixed. To overcome this difficulty, spline function $y^s(t)$ is introduced in Step 2 of the main algorithm. If the sampling rate is sufficiently fast, the difference between $y^s(t)$ and the true output $y(t)$ will be very small. Therefore, by the convergence results presented in the previous section, the optimal solution $\bar{z}_N^*$ is also close to $x(t_p)$ as long as the number of LGL nodes N is sufficiently large.

Remark 3.3: The proposed observer algorithm is a prediction-correction scheme. After Step 3 of the main algorithm, the current estimation $\bar{z}_N^*$ is used to generate a good prediction of the state at the next sampling time t_{p+1} by some numerical integration method like a Runge-Kutta method. Then, this prediction is used to form an initial guess for the optimization solver in the next iteration. The optimization performed in Step 3 of the main algorithm acts as a correction to the prediction provided by numerical integration. This prediction-correction scheme greatly reduces the running time, because the optimization at step $p+1$ only needs to be done locally in a small neighborhood around the initial guess.

To demonstrate the efficiency of the proposed observer algorithm, we consider designing an observer for a forced Duffing system. We choose this system because it is a nonlinear, time-varying, chaotic system with an unknown parameter. Even worse, the system is not uniformly observable, which renders many gain-based methods inapplicable.

Example 2:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.25x_2 + x_1(1-x_1^2) + \theta\cos(t) \\ y &= x_1 + 0.5x_2 \end{aligned} \tag{31}$$

where θ is an unknown parameter. In the simulation θ is set to 0.4. This choice of parameter makes the performance of the system chaotic. The sampling time is $t_p = 0.1p$, $p = 0, 1, 2, \cdots$, and the measured output is $y_p = y(t_p)$. By treating θ as an extra state with the dynamic $\dot{\theta} = 0$, we apply the proposed pseudospectral algorithm to construct an observer. In the simulation, we choose the initial condition of (31) to be $(x_1(0), x_2(0)) = (2, 1)$ which is unknown to the observer. The tuning parameters are set to $N = 15$, $L = 8$ and the guess of the initial condition is $(0, 0)$.

The results are demonstrated in Figure 3. Once the initial data $(y_0, \ldots, y_L)$ are collected, the PS observer provides accurate estimation of both the state and the unknown parameter. Actually, the estimation errors of x_1 and x_2 are within 10^{-4} while the error in θ is less than 10^{-3}. The average running time for each iteration is about 0.05 second.

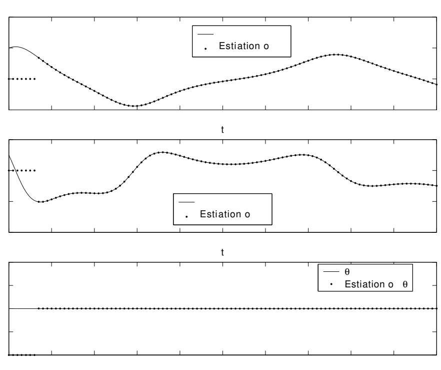

Fig. 3. PS observer for a Duffing system with uncertainty.

In much of observer theory, for example, Extended Kalman Filter, Unscented Kalman filter, Moving Horizon Observer, High-gain observer, the convergence of the estimation error is asymptotical. This means it will take some time for the estimated state to be close to the true state trajectory. In the proposed pseudospectral observer algorithm, the convergence of the estimated state is in finite time. As clearly demonstrated in Example 2, in the first step of the iteration, the error is already near zero. It takes virtually no time for convergence. The reason for this impressive property is very simple. In Problem E, the unique optimal solution is the unmeasured state, $x(t)$. Hence, from the convergence property presented in the previous section, at each iteration, if an optimal solution of Problem E^{N} is found, it must lie in an ϵ-neighborhood of $x(t)$.

The finite-time convergence property of the proposed algorithm is very attractive in practice, especially in the design of output feedback controllers, since the separation principle does not hold for nonlinear systems with an asymptotically convergent (even exponentially convergent) observer. In our pseudospectral observer, there are three tuning parameters: 1) backward integration length L; 2) number of the discretization nodes N; and 3) initial guess of the state $x(0)$. The guidelines to chose these parameters are explained in detail in [14]. Also the observer algorithm can be modified to incorporate measurement noise [14].

IV. An Output Feedback Example

Consider the problem in Example 1 with output feedback, i.e., stabilize system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^3 \\ \dot{x}_3 &= u \end{aligned} \tag{32}$$

with the information of output $y = x_1$ only. First construct the estimation Problem E with $\delta = 1$. The number of Legendre PS discretization nodes is chosen as $n = 16$ and the sampling period is set to 0.1. The solution to Problem E provide the estimation of the state denoted as $\hat{x}$.

Next, form the following finite horizonal optimal control Problem B

$$\text{Min.} \quad J[x(\cdot), u(\cdot)] = \int_{t_0}^{t_0+10} u^2(t)dt$$

subject to state dynamic (32), initial condition

$$x(t_0) = \hat{x}(t_0)$$

and end-point condition

$$x(t_0+10) = (0, 0, 0)$$

Note that the estimation of the state is used as the initial condition in the Problem B. In applying PS methods, 36 nodes is used to solve the nonlinear optimal control problem.

Problem E and Problem B are solved online in a receding horizon sense with all computational delay been considered. The feedback is implemented in a similarly way as [28], [29]. The only difference is that initially, control is set to 0 for $t \in [0, 1]$ so that the observer can collect enough data. For the sake of simplicity, we omit the details of the feedback implementation. Interesting readers are referred to [28], [29]. The result with initial condition $x(0) = (1, 0.5, 1)$ is shown in Figure 4 and the result with initial condition $x(0) = (-1, 0.1, 0.8)$ is shown in Figure 5.

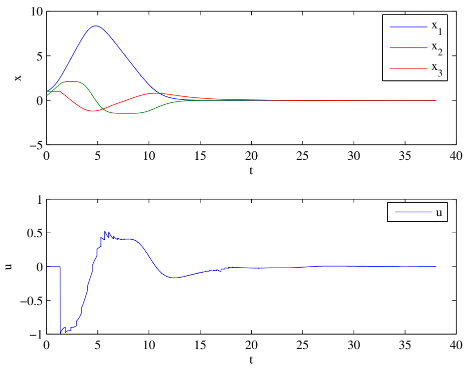

Fig. 4. Trajectory and the control.

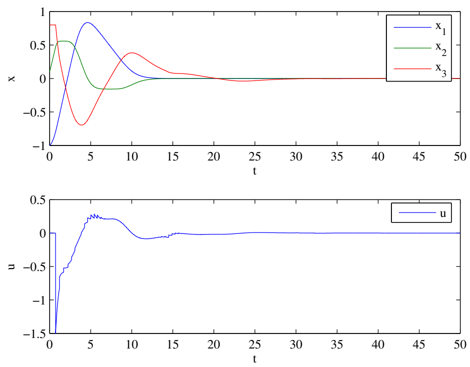

Fig. 5. Trajectory and the control.

Apparently, in both cases the states converges. The average running time for solving the optimal control problem is 0.388 second and the average running time for solving the estimation problem is 0.096 second.

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