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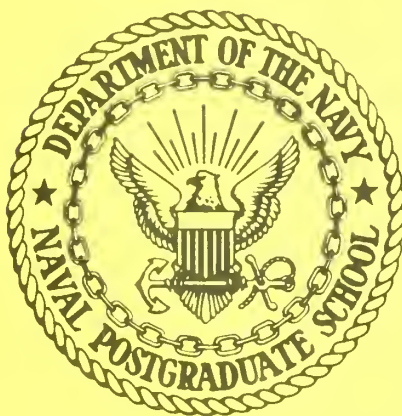
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ASYMPTOTIC REPRESENTATION OF  
STIRLING NUMBERS OF THE SECOND KIND

by

W. E. Bleick and Peter C. C. Wang

9 February 1977

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Prepared for:

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Statistics and Probability Branch

Arlington, VA 22217

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ABSTRACT:

The distribution of the Stirling numbers  $S(n,k)$  of the second kind with respect to  $k$  has been shown by Harper [Ann. Math. Statist., 38 (1967), 410-414] to be asymptotically normal near the mode. A new single-term asymptotic representation of  $S(n,k)$ , more effective for large  $k$ , is given here. It is based on Hermite's formula for a divided difference and the use of sectional areas normal to the body diagonal of a unit hypercube in  $k$ -space. A proof is given that the distribution of these areas is asymptotically normal. A numerical comparison is made with the Harper representation for  $n=200$ .

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1. Introduction.

Previous asymptotic representations of Stirling numbers  $S(n,k)$  of the second kind have been of two types. One type has been a complete infinite series expansion as given by Hsu [1], and by Bleick and Wang [2] and [3]. A second type has been the single-term representation of  $S(n,k)$  given by Harper [4] as the normal distribution approximation

$$(1) \quad S(n,k) \sim \frac{B_n}{\sigma\sqrt{2\pi}} \exp[-(k-\mu)^2/2\sigma^2]$$

where the mean  $\mu$  and the variance  $\sigma^2$  are expressed in terms of the Bell numbers  $B_n$  by

$$(2) \quad \mu = B_{n+1}/B_n - 1$$

and

$$(3) \quad \sigma^2 = B_{n+2}/B_n - (B_{n+1}/B_n)^2 - 1 .$$

The purpose of this note is to give a new single-term asymptotic representation based on Hermite's formula for a divided difference, and to compare it with that of Harper.

2. Use of Hermite's formula.

A Stirling number  $S(n,k)$  of the second kind is defined as the  $k$ th difference of  $z^n$  at  $z=0$  divided by  $k!$ . By [5,p.10] we find that this divided difference can be represented by a formula of Hermite as the repeated definite integral

$$(4) \quad S(n,k) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{k-1}} (d^k_{u_1} z^n / du_1^k) dt_k$$

where  $u_1 = t_1 + t_2 + \dots + t_k$ . We imagine that  $t_1, t_2, \dots, t_k$  constitute a set of

rectangular Cartesian coordinates and impose an orthogonal transformation of coordinates to  $u_1, u_2, \dots, u_k$ . The volume of the space over which the integration in (4) is performed is a portion of a unit hypercube in  $k$ -space. If we allow the coordinate  $u_1$  to vary along the body diagonal of the hypercube from 0 at one vertex to  $k$  at the opposite vertex, the sectional areas normal to the diagonal cut by the hyperplane  $u_1 = t_1 + t_2 + \dots + t_k$  from the domain of integration define a positive function  $g(u_1, k)$  even with respect to the argument  $u_1 - k/2$ . We take the integral of  $g(u_1, k)$  to be

$$(5) \quad \int_0^k g(u_1, k) du_1 = 1/k!$$

to agree with the volume of the space over which the integration in (1) is performed. We drop the  $u_1$  subscript henceforth. Noting that  $g(u, k) = 0$  for  $k < u < 0$ , we find that

$$(6) \quad g(u, 1) = 1 \quad \text{for } 0 \leq u \leq 1 \quad ,$$

$$(7) \quad 2!g(u, 2) = (1 - |u-1|) \quad \text{for } 0 \leq u \leq 2 \quad ,$$

and

$$(8) \quad 3!g(u, 3) = \begin{cases} (3/2 - |u-3/2|)^2/2 & \text{for } 1/2 \leq |u-3/2| \leq 3/2 \\ 3/4 - (u-3/2)^2 & \text{for } 1 \leq u \leq 2 \quad . \end{cases}$$

Consideration of the Laplace transforms of (6), (7) and (8) suggests that we conjecture the Laplace transform of  $k!g(u, k)$  to be

$$(9) \quad (1 - e^{-s})^k / s^k = e^{-ks/2} \left( \frac{\sinh s/2}{s/2} \right)^k$$

for all  $k$ . We demonstrate the truth of this conjecture later. On performing the integration in (4) over the variables  $u_2, u_3, \dots, u_k$  we find

$$(10) \quad S(n, k) = k! \binom{n}{k} \int_0^\infty u^{n-k} g(u, k) du \quad .$$

Using operation 82 of [6,p.10] on the Laplace transform of

$$(11) \quad k! \int_0^u u^m g(u,k) du$$

we find the mth moment of the  $k!g(u,k)$  distribution about  $u=0$  to be

$$(12) \quad \lim_{s \rightarrow 0} (-1)^m (d/ds)^m (1-e^{-s})^k / s^k .$$

It is now easy to demonstrate the truth of the conjecture (9) by showing, with the aid of the multinomial theorem, that (12) is the same as the repeated integral

$$(13) \quad \int_0^1 dt_1 \int_0^1 dt_2 \dots \int_0^1 (t_1+t_2+\dots+t_k)^m dt_k$$

over the volume of the hypercube.

Use of (12) and (5) shows the variance of the  $k!g(u,k)$  distribution to be

$$(14) \quad \sigma^2 = k/12 .$$

Using (14) the series

$$(15) \quad \exp(\sigma^2 s^2 / 2) = 1 + \frac{ks^2/24}{1!} + \frac{(ks^2/24)^2}{2!} + \dots$$

is the bilateral, but not  $s$  multiplied, Laplace transform of the normal distribution

$$(16) \quad (1/\sigma\sqrt{2\pi}) \exp(-t^2/2\sigma^2)$$

according to [7,p.2]]. The corresponding series for (9) multiplied by  $e^{ks/2}$ , or the bilateral Laplace transform of  $k!g(u,k)$  shifted left by  $k/2$ , is

$$(17) \quad (2/s)^k \sinh^k s/2 = [1 + \frac{s^2/4}{3!} + \frac{(s^2/4)^2}{5!} + \dots]^k .$$

The dominant  $k$  power term in the coefficient of  $(s^2/4)^n$  in (15) is

$k^n/6^n n!$ , and may be shown to be the same in the expansion of (17)

by the use of the recurrence formula 6.361 of [8,p.119]. This proves

that the  $k!g(u,k)$  distribution is asymptotically normal as  $k \rightarrow \infty$ . It is

remarkable that the normal distribution should arise in the purely



geometrical context of sectional areas normal to the body diagonal of a hypercube of high dimension.

On replacing  $k!g(u,k)$  in (10) by its Gaussian normal approximation of mean  $\mu=k/2$  and variance  $\sigma^2=k/12$  we find

$$(18) \quad S(n,k) \sim \frac{1}{\sigma\sqrt{2\pi}} \binom{n}{k} \int_0^{\infty} u^{n-k} \exp[-(u-k/2)^2/2\sigma^2] du$$

$$\sim \frac{1}{\sqrt{2\pi}} \binom{n}{k} \int_{-\infty}^{\sqrt{3k}} (k/2-\sigma t)^{n-k} e^{-t^2/2} dt .$$

### 3. Numerical example.

Table 1 compares the exact values of  $S(200,k)$  with the asymptotic approximations computed from the single-term representations (1) and (18). Harper's representation (1), which uses  $B_{200} = .62475 \cdot 10^{276}$ ,  $\mu=49.975$  and  $\sigma=3.0551$ , gives an excellent fit near the mode ( $k=50$ ), but (18) gives a much better fit for large values of  $k$ .

Table 1. Values of  $S(200,k)$

<u>k</u>	<u>Asymptotic from (1)</u>	<u>Exact</u>	<u>Asymptotic from (18)</u>
2	.23135 $10^{222}$	.80347 $10^6$	.69244 $10^{126}$
40	.39504 $10^{273}$	.24458 $10^{273}$	.42658 $10^{273}$
50	.81579 $10^{275}$	.81493 $10^{275}$	.15285 $10^{277}$
60	.37452 $10^{273}$	.53533 $10^{273}$	.29658 $10^{274}$
100	.49065 $10^{217}$	.22839 $10^{235}$	.27994 $10^{235}$
150	.13938 $10^{43}$	.30251 $10^{143}$	.30441 $10^{143}$
199	.16955 $10^{-241}$	.19900 $10^5$	.19900 $10^5$

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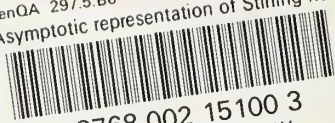
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