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AN EXPONENTIAL AUTOREGRESSIVE-MOVING AVERAGE PROCESS EARMA(p,q): DEFINITION AND CORRELATIONAL PROPERTIES

by

A. J. Lawrance

and

P. A. W. Lewis

JANUARY 1978

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# AN EXPONENTIAL AUTOREGRESSIVE-MOVING AVERAGE PROCESS EARMA(p,q): DEFINITION AND CORRELATIONAL PROPERTIES

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## ABSTRACT

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## 1. INTRODUCTION

The first-order autoregressive sequence, EAR(1), was introduced by Gaver and Lewis (1975-1978) with the primary aim of generalizing the Poisson model for point processes to one in which the intervals between events were correlated but still had, marginally, exponentially distributed intervals. The EAR(1) sequence is a simple random linear combination of independent exponential random variables whose properties are relatively simple to derive. This is in contrast to previous attempts to generalize the Poisson process via Markov dependence which led to intractable models (see e.g. Wold, 1948 and Cox, 1955).

In Lawrance and Lewis (1977) another sequence of dependent exponential random variables was introduced. This sequence, called EMA(1), was again a random linear combination of independent exponential random variables, but had the dependency properties of a first order moving average process. The first-order moving average and autoregressive processes were combined by Jacobs and Lewis (1977) to form the EARMA(1,1) sequence. Jacobs and Lewis (1977) gave stationary initial conditions and mixing properties of the sequences, these results applying to the EAR(1) and EMA(1) processes as special cases.

In the present paper we extend these results and describe a mixed pth-order autoregressive, qth-order moving average process with exponential marginal distributions which we denote as EARMA(p,q). The process is again a random linear combination of independent exponential variables, and as such is simple to generate on a computer; it will thus be useful, for example, in simulation studies of queues with correlated interarrival times or service times (see Jacobs, 1978). The process is not unique, but its correlation sequence $\{\rho_k\}$ does satisfy equations like the Yule-Walker equations which arise in the study of linear processes (see e.g. Feller, 1966, or Box and Jenkins 1970).

It is perhaps well to reiterate the essential difference between the EARMA(p,q) process and the ARMA(p,q) process; this is that the EARMA(p,q) process is defined to have an exponential marginal distribution. It is not known how one would pick the error sequence in the ARMA(p,q) sequence to make it have, even approximately, marginal exponential distributions. In fact the marginal distributions would tend to be approximately normally distributed for most error sequences (Mallows, 1967); the catch in this result is that the distribution of the error sequence be independent of the parameters of the moving average and autoregression. This is not so for the EARMA(p,q) process.

This paper will be limited to definitions and to description of the correlational properties of the EARMA(p,q) process. In Section 2 the EAR(p) model is introduced and an explicit solution for the required error process for autoregression of order 2 is given. In Section 3 we describe the extensions of the EMA(1) model to the EMA(q) model; these are relatively straightforward and are indicated in Lawrance and Lewis (1977). The general

EARMA(p,q) model is introduced in Section 4, and specific results are obtained for the EARMA(1,1), EARMA(2,1) and EARMA(1,2) models in Section 5.

In deriving correlational properties it is assumed that the EARMA(p,q) process is stationary. The question of stationarity, stationary initial conditions and mixing properties will be considered elsewhere, as will be questions of distributions of sums of the dependent variables and spectra of point processes with EARMA(p,q) interval structure. There are also open questions of estimation of parameters and fitting to data.

We note too that there is a mild degeneracy to the EAR(p) process in that one obtains runs in which the variables are scaled versions of the previous variables. This disappears when the moving average component is introduced. Another drawback is that, unlike the ARMA(p,q) model, only positive valued serial correlations can be obtained from the EARMA(p,q) model and while much data appears to be of this type (see e.g. Lewis and Shedler, 1977), it is a drawback. This can be overcome by considering antithetic processes but this aspect of the model is beyond the scope of the present paper.

## 2. THE EXPONENTIAL AUTOREGRESSIVE EAR(p) MODEL

### 2a. Definition.

The standard linear, first-order autoregressive model for a stationary sequence of random variables $\{X_i\}$ is defined by the equation

$$X_i = \rho X_{i-1} + \varepsilon_i , \qquad i = 0, \pm 1, \pm 2, \ldots , \qquad (2.1)$$

where $\rho$ is a constant which is less than 1 in absolute value and $\{\varepsilon_i\}$ is a sequence of independent and identically distributed random variables. Gaver and Lewis (1975-1978) showed that if the $\{X_i\}$ sequence were to have an exponential marginal distribution with parameter $\lambda$, then the parameter $\rho$ should be greater than or equal to zero and less than one, and $\varepsilon_i$ should be zero with probability $\rho$ and an exponential($\lambda$) random variable, $E_i$, with probability $1-\rho$. Thus

$$X_i = \rho X_{i-1} + \varepsilon_i \qquad i = 0, \pm 1, \pm 2, \ldots ,$$

$$= \begin{cases} \rho X_{i-1} & \text{w.p. } \rho , \\ \rho X_{i-1} + E_i & \text{w.p. } 1-\rho , \end{cases} \qquad i = 0, \pm 1, \pm 2, \ldots , \qquad (2.2)$$

where $\{E_i\}$ is an i.i.d. sequence of exponential($\lambda$) random variables. Note that for this EAR(1) model the distribution of the $\varepsilon_i$ depends on $\rho$, the multiplicative weight of $X_{i-1}$.

This violates an assumption which is implicit in many applications of (2.1), the so-called AR(1) model. In particular standard results showing that the $\{X_i\}$ sequence becomes a normal process as $\rho \to 1$ for any $\{\varepsilon_i\}$ sequence are invalid; in the EAR(1) process the $X_i$'s always have, by construction, an exponential($\lambda$) marginal distribution.

Generalization of the usual higher-order autoregressive models AR(p) based on extensions of (2.1) to higher order autoregressive exponential processes is difficult. This is because it is not possible to solve the defining equation for the distribution of the $\varepsilon_i$'s, if it exists. We present here a different type of p-th order autoregressive models with exponential marginal distributions. They share with the AR(p) models the same correlation structure, are p-th order Markov processes, and are (autoregressively) functions of at least one of the previous $p$ variables.

The second-order model, EAR(2), takes the form

$$X_i = \begin{Bmatrix} \alpha_1 X_{i-1} & \text{w.p.} & 1-\alpha_2 \\ \\ \alpha_2 X_{i-2} & \text{w.p.} & \alpha_2 \end{Bmatrix} + \varepsilon_i \quad , \qquad (2.3)$$

where $\alpha_1$ and $\alpha_2$ are constants $(0 < \alpha_1, \alpha_2 < 1)$ and we show later that the distribution of the $\varepsilon_i$ is uniquely determined by the requirement that the $X_i$'s have exponential marginal distributions. The second-order autoregressive nature of the model is evident; $X_i$ is always a function of one of the previous two values $X_{i-1}$ and $X_{i-2}$. This is in contrast to the AR(2) model in which $X_i$ is a function of a linear combination of $X_{i-1}$ and $X_{i-2}$.

The third-order model is given by

$$X_i = \begin{Bmatrix} \alpha_1 X_{i-1} & \text{w.p.} & 1-\alpha_2 \\ \alpha_2 X_{i-2} & \text{w.p.} & \alpha_2(1-\alpha_3) \\ \alpha_3 X_{i-3} & \text{w.p.} & \alpha_2\alpha_3 \end{Bmatrix} + \varepsilon_i \,. \tag{2.4}$$

The p-th order model is similarly constructed and may be written

$$X_i = \begin{Bmatrix} \alpha_1 X_{i-1} & \text{w.p.} & a_1 \\ \alpha_2 X_{i-2} & \text{w.p.} & a_2 \\ \vdots & \vdots & \vdots \\ \alpha_p X_{i-p} & \text{w.p.} & a_p \end{Bmatrix} + \varepsilon_i \,, \tag{2.5}$$

where

$$a_\ell = \prod_{j=2}^{\ell} \alpha_j (1-\alpha_{\ell+1}), \qquad \ell = 2,\ldots,p-1$$

$$\text{and} \qquad a_1 = (1-\alpha_2), \qquad a_p = \prod_{j=2}^{p} \alpha_j \,. \tag{2.6}$$

The mixing probabilities and the weights on the autoregressed variables are to some extent a matter of choice (other parametrizations are clearly possible) and we have been guided by two considerations; having a minimum number of parameters, preferably the same number of parameters as the order of the autoregression, and by the need for the autoregression in the EAR(p) model to reduce in order by one when the last coefficient, $\alpha_p$, is set to zero. In the present parametrization this implies the weak restriction that it is not possible to suppress intermediate autoregressions, i.e. dependence of $X_i$ on $X_{i-1}$ in a EARMA(2) model.

With regard to the question of parametrization, one could in (2.3) replace the probabilities $a_1 = (1-\alpha_2)$ and $a_2 = \alpha_2$ by $(1-p)$ and $p$ and there is then no need for the weights $\alpha_1$ and $\alpha_2$ to be less than or equal to one. But if they are not, the process will not reach equilibrium unless $p$ is suitably chosen, i.e. be stable. Again, even if the process is stable it is not clear yet that the additional parameter adds any generality to the process. We consequently consider only the parametrizations given in (2.6).

## 2b. The error sequence $\{\varepsilon_i\}$

We now obtain the distribution of the i.i.d. $\{\varepsilon_i\}$ sequence which will ensure that the $\{X_i\}$ sequence in the EAR(2) model has an exponential marginal distribution. Let $\phi_{X_i}(s)$ and $\phi_{\varepsilon_i}(s)$ be the Laplace-Stieltjes transforms of the marginal distributions of the $X_i$'s and the $\varepsilon_i$'s:

$$\phi_{X_i}(s) = E(e^{-X_i s}) \ ; \qquad \phi_{\varepsilon_i}(s) = E(e^{-\varepsilon_i s}) \ . \tag{2.7}$$

Then from Equation (2.3) we have

$$\phi_{X_i}(s) = [(1-\alpha_2)\ \phi_{X_{i-1}}(\alpha_1 s) + \alpha_2 \phi_{X_{i-2}}(\alpha_2 s)]\ \phi_\varepsilon(s)\ , \qquad (2.8)$$

where we have used the fact that expectation of the mixture of two dependent random variables is the mixture of the expectations of the marginal random variables, here $X_{i-1}$ and $X_{i-2}$. Thus we avoid the joint Laplace-Stietljes transform which comes in when one tries to solve the usual linear AR(2) equations to obtain an exponential process. Assuming marginal stationarity for the process, we have

$$\phi_X(s) = [(1-\alpha_2)\ \phi_X(\alpha_1 s) + \alpha_2 \phi_X(\alpha_2 s)]\ \phi_\varepsilon(s)\ . \qquad (2.9)$$

To show that such an error sequence $\{\varepsilon_i\}$ exists we solve (2.9) directly and invert the transform. We have for the EAR(2) model, using the key requirement that the marginal distribution of the $X_i$'s be exponential($\lambda$), and thus $\phi_X(s) = \lambda/(\lambda+s)$, that

$$\phi_\varepsilon(s) = \frac{\phi_X(s)}{(1-\alpha_2)\ \phi_X(\alpha_1 s) + \alpha_2 \phi_X(\alpha_2 s)} \qquad (2.10)$$

$$= \frac{(\lambda+\alpha_1 s)(\lambda+\alpha_2 s)}{(\lambda+s)[(1-\alpha_2)(\lambda+\alpha_2 s) + \alpha_2(\lambda+\alpha_1 s)]} \qquad (2.11)$$

Then by a partial fraction expansion

$$\phi_{\varepsilon}(s) = \pi_0 + \pi_1 \frac{\lambda}{\lambda+s} + \pi_2 \frac{\lambda}{\lambda+Ss} \quad ,$$

where $S = (1+\alpha_1-\alpha_2)\alpha_2$. Using the fact that $\alpha_1$ and $\alpha_2$ are probabilities, it is easily verified that $\pi_0$, $\pi_1$, and $\pi_2$ are positive, and since their sum is equal to 1, we have $0 \leq \pi_0, \pi_1, \pi_2 \leq 1$. Thus $\varepsilon$ is a convex mixture of a discrete component and two exponentials, and thus has a proper distribution. This distribution is also unique, by the unicity theorem for Laplace-Stieltjes transforms. The complete specification of $\varepsilon_i$ is, for $i = 0, \pm 1, \pm 2, \ldots$

$$\varepsilon_i = \begin{cases} 0 & \text{w.p.} \quad \alpha_1/\{1+\alpha_1-\alpha_2\}, \\ E_i & \text{w.p.} \quad (1-\alpha_1)(1-\alpha_2)/[1-\alpha_2(1+\alpha_1-\alpha_2)] \ , \\ E_i/S & \text{w.p.} \quad (1-\alpha_2)(\alpha_1-\alpha_2)^2/[(1+\alpha_1-\alpha_2)(1-S)] \ , \end{cases} \qquad (2.12)$$

where $\{E_i\}$ is again an i.i.d. sequence of exponential($\lambda$) random variables. It is obvious from (2.12) that the mean and variance of $\varepsilon_i$ depend on $\alpha_1$ and $\alpha_2$, the multiplicative weights of $X_{i-1}$ and $X_{i-2}$ respectively, as well as on $\lambda$.

As in the EAR(1) model there is a non-zero probability of $\varepsilon_i$ being zero; otherwise it is $E_i$ or a scaled version of $E_i$. The higher order models can in principle be similarly treated, although above the third order there will be difficulty with the partial fraction expansion.

## 2c. Correlation structure

The correlation structure of the stationary EAR(p) models can be obtained by the usual device of multiplying the defining equations for $X_i$ by $X_{i-r}$, for $r = 1,2,\ldots,$ and taking expectations. What results are difference equations which are entirely analogous to the Yule-Walker type equations obtained for the standard AR(p) model.

Thus taking the EAR(2) case as a typical example, we have from (2.3) that

$$E(X_i X_{i-r}) = (1-\alpha_2)[\alpha_1 E(X_{i-1}X_{i-r}) + E(X_{i-r})\ E(\varepsilon_i)]$$

$$+ \alpha_2[\alpha_2 E(X_{i-2}X_{i-r}) + E(X_{i-r})\ E(\varepsilon_i)]\ . \qquad (2.13)$$

Using the fact that $E(X_i) = \lambda^{-1}$, $\mathrm{var}(X_i) = \lambda^{-2}$, because the process has an exponential marginal distribution, and from (2.3) that

$$E(\varepsilon) = (1-\alpha_2)(1-\alpha_1+\alpha_2)\ E(X)\ , \qquad (2.14)$$

we obtain for the correlations $\rho_r = \mathrm{Corr}(X_i, X_{i-r})$ the equation,

$$\rho_r = \alpha_1(1-\alpha_2)\rho_{r-1} + \alpha_2^2\rho_{r-2} \qquad (r \geq 1) \qquad (2.15)$$

with $\rho_r = \rho_{-r}$ and $\rho_0 = 1$. For the general EAR(p) process there is the corresponding equation

$$\rho_r = \alpha_1 a_1 \rho_{r-1} + \alpha_2 a_2 \rho_{r-2} + \cdots + \alpha_p a_p \rho_{r-p} \qquad (r \geq 1). \qquad (2.16)$$

Equation (2.15) is a system of second order difference equations from which we can obtain the following results.

(i) The solution of the difference equation (2.15) is (see, for example, Box and Jenkins, 1970, pp. 58-59)

$$\rho_r = \gamma_1 z_1^r + \gamma_2 z_2^r \qquad (r \geq 1) \qquad (2.17)$$

$$= \frac{z_1(1-z_2^2)z_1^r - z_2(1-z_1^2)z_2^r}{(z_1-z_2)(1+z_1 z_2)}, \qquad (2.18)$$

where $z_1$ and $z_2$ are reciprocals of the roots of the characteristic equation

$$1 - \alpha_1(1-\alpha_2)B - \alpha_2^2 B^2 = 0 ,$$

and the roots are real since $\alpha_1^2(1-\alpha_2)^2 + 4\alpha_2^2 \geq 0$ . Also $0 \leq z_2 < z_1 < 1$. An implication of these results is that the serial correlations are positive and eventually decay geometrically, i.e. like $\gamma_1 z_1^r$. We have assumed that $\alpha_2 > 0$; otherwise we have the EAR(1) model.

(ii) The correlations $\rho_1$ and $\rho_2$ can be uniquely defined in terms of the parameters $\alpha_1$ and $\alpha_2$, and vice versa; this follows from (2.15) for $r = 1$ and $r = 2$. We have

$$\rho_1 = \frac{\alpha_1}{1+\alpha_2} \; ; \qquad \rho_2 = \alpha_1(1-\alpha_2)\rho_1 + \alpha_2^2 \; ; \tag{2.19}$$

and

$$\alpha_2 = \left(\frac{\rho_2-\rho_1^2}{1-\rho_1^2}\right)^{1/2} \; ; \quad \alpha_1 = \left(1 + \frac{\rho_2-\rho_1^2}{1-\rho_1^2}\right)^{1/2} \rho_1 \tag{2.20}$$

if $\alpha_2 \neq 0$. If $\alpha_2 = 0$ the model reduces to the EAR(1) model of Gaver and Lewis (1975-1978), and $\rho_1 = \alpha_1$. Equation (2.20) may be used to obtain Yule-Walker estimates of $\alpha_1$ and $\alpha_2$ from estimates of the first two serial correlations.

(iii) If $\alpha_2 \neq 0$ then, unlike the EAR(1) case in which $\rho_2 = \rho_1^2$, we have $\rho_2 > \rho_1^2$. This can be seen from (2.19), which can be written as $\rho_2 = \rho_1^2 + \alpha_2^2(1-\rho_1^2) \geq \rho_1^2$. Note too that there are values of $\alpha_1$ and $\alpha_2$ for which $\rho_2 > \rho_1$; thus the additional degree of autoregression produces, at least for the first two serial correlations, a broader correlation structure than is possible with the EAR(1) process $(\alpha_2 = 0)$.

(iv) One way to measure the amount of correlation which is introduced into the sequence $\{E_i\}$ by the autoregression is by an index of dispersion (Cox and Lewis, 1966, p. 71). This is just the limiting value of the variance of the sum of $k$ adjacent to $X_i$'s, standardized by its value for independent exponential variates:

$$J = \lim_{k \to \infty} \frac{\mathrm{var}(X)}{\{E(X)\}^2} \left\{ 1 + 2 \sum_{j=1}^{k-1} (1 - \frac{j}{k}) \rho_j \right\}$$

$$= 1 + 2 \sum_{j=1}^{\infty} \rho_j \, , \tag{2.21}$$

and is proportional to the initial point of the spectrum of the process $\{X_i\}$. For the EAR(2) process this is, using (2.17),

$$J = 1 + \frac{2\gamma_1 z_1}{(1-z_1)} + \frac{2\gamma_2 z_2}{(1-z_2)} \, . \tag{2.22}$$

This becomes very large as the roots $z_1$ and $z_2$ approach 1, indicating that the process has very long term dependence in it.

Some other properties of the EAR(2) process which are of interest are that the regression of $X_i$ on $X_{i-1}$ and $X_{i-2}$ is linear in the given values $x_{i-1}$ and $x_{i-2}$ of $X_{i-1}$ and $X_{i-2}$:

$$E(X_i | X_{i-1} = x_{i-1}; X_{i-2} = x_{i-2})$$

$$= (1-\alpha_2)\alpha_1 x_{i-1} + \alpha_2^2 x_{i-2} + \lambda(1-\alpha_2)(1-\alpha_1+\alpha_2) \, , \tag{2.23}$$

and that the conditional correlations of $X_i$ and $X_{i-\ell}$, given $X_{i-1}, \ldots, X_{i-\ell-1}$, are zero for $\ell = 3,4,\ldots$ .

We note too that the EAR(2) model, like the EAR(1) model, is slightly degenerate in that one obtains runs of $X_i$'s which are fixed multiples of the previous $X_{i-1}$ or $X_{i-2}$.

Joint distributions, higher-order joint moments and partial sums of the $\{X_i\}$ process will be considered elsewhere.

## 3. THE EXPONENTIAL MOVING AVERAGE EMA(q) MODEL

The EAR(1) model of Gaver and Lewis (1975-78) led to the development by Lawrance and Lewis (1977) of a corresponding first-order exponential moving average model; this took the form, in the backward case, of

$$X_i = \begin{cases} \beta E_i & \text{w.p. } \beta \\ \beta E_i + E_{i-1} & \text{w.p. } (1-\beta), \end{cases} \qquad (0 \leq \beta \leq 1;\ i = 0, \pm 1, \pm 2, \ldots) \qquad (3.1)$$

where the $\{E_i\}$ is again a sequence of i.i.d. exponential($\lambda$) variables. The $X_i$'s have an exponential marginal distribution and are only serially dependent for lag one; this model is highly tractable and a full account of the statistically useful properties was obtained. The forward model is defined as a random mixture of $\beta E_i$ and $\beta E_i + E_{i+1}$, instead of $\beta E_i$ and $\beta E_i + E_{i-1}$.

Lawrance and Lewis (1977) pointed out briefly that extensions to second-order moving-average models are possible; thus we replace $E_{i-1}$ in (3.1) by another EMA(1) variable, a random linear combination of $\beta_1 E_{i-1}$ and $\beta_1 E_{i-1} + E_{i-2}$, which will still be exponentially distributed and independent of the $E_i$ variable.

Thus the second order backward model, EMA(2), becomes

$$X_i = \begin{cases} \beta_2 E_i & \text{w.p. } \beta_2 \\ \beta_2 E_i + \beta_1 E_{i-1} & \text{w.p. } (1-\beta_2)\beta_1 \\ \beta_2 E_i + \beta_1 E_{i-1} + E_{i-2} & \text{w.p. } (1-\beta_2)(1-\beta_1) \end{cases}, \quad (3.2)$$

where $0 \leq \beta_1, \beta_2 \leq 1$; $i = 0, \pm 1, \pm 2, \ldots$ . The serial dependency of this model clearly stops at the second lag and

$$\rho_1 = (1-\beta_2)\ \beta_1\{1 - (1-\beta_2)\beta_1\}; \quad \rho_2 = \beta_2(1 - \beta_1)(1 - \beta_2). \quad (3.3)$$

This model reduces to the independence case and the EMA(1) model for various values of $\beta_1$ and $\beta_2$. The general EMA(q) model takes the form

$$X_i = \begin{cases} \beta_q E_i & \text{w.p. } b_{q+1} \\ \beta_q E_i + \beta_{q-1} E_{i-1} & \text{w.p. } b_q \\ \quad \cdots\cdots & \quad \cdots\cdots \\ \beta_q E_i + \beta_{q-1} E_{i-1} + \cdots + \beta_1 E_{i-q+1} & \text{w.p. } b_2 \\ \beta_q E_i + \beta_{q-1} E_{i-1} + \cdots + \beta_1 E_{i-q+1} + E_{i-q} & \text{w.p. } b_1 \end{cases} \quad (3.4)$$

for $0 \leq \beta_1, \beta_2, \ldots, \beta_q \leq 1$; $i = 0, \pm 1, \pm 2, \ldots$ and

$$b_i = \begin{cases} \beta_q & i = q+1, \\ (1-\beta_q) \cdots (1-\beta_i)\beta_{i-1} & q \geq i \geq 2, \\ (1-\beta_q) \cdots (1-\beta_i) & i = 1. \end{cases} \quad (3.5)$$

Note that the $\beta_i$'s can be obtained uniquely from the $b_i$'s; there are q+1 $b_i$'s, but only q $\beta$'s, since the sum of the $b_i$'s is equal to one. It is simple to see that the $\{X_i\}$ have exponential marginal distributions. The serial correlations for this model clearly have the cut-off property associated with moving average schemes; they can be obtained without recourse to difference equations. Premultiplication of (3.4) by $X_{i-r}$ $(r \geq 1)$ and then taking an expectation gives

$$\begin{aligned} E(X_i X_{i-r}) = \beta_q (b_{q+1} + \cdots + b_1)\, E(E_i X_{i-r}) \\ + \beta_{q-1}(b_q + \cdots + b_1)\, E(E_{i-1} X_{i-r}) \\ + \cdots \\ + \beta_1 (b_2 + b_1)\, E(E_{i-q+1} X_{i-r}) + b_1 E(E_{i-q} X_{i-r}) . \end{aligned} \tag{3.6}$$

This simplifies since there are the relations

$$\beta_i (b_{i+1} + \cdots + b_1) = b_{i+1} \qquad (q \geq i \geq 1) . \tag{3.7}$$

Thus on converting (3.6) to covariances we have

$$\mathrm{Cov}(X_i, X_{i-r}) = \sum_{m=0}^{q} b_{q+1-m}\, \mathrm{Cov}(E_{i-m}, X_{i-r}) . \tag{3.8}$$

The covariances on the right-hand side of (3.8) follow from (3.4) as

$$\mathrm{Cov}(E_{i-m}, X_i) = \begin{cases} b_{q-m+1}\, \mathrm{Var}(E_{i-m}) & 0 \leq m \leq q \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Thus (3.8) becomes

$$\mathrm{Cov}(X_i, X_{i-r}) = \sum_{m=r}^{q} b_{q+1-m}\, b_{q-m+r+1}\, \mathrm{Var}(E_{i-m+r}) \quad (1 \le r \le q) \qquad (3.10)$$

with

$$\rho_r^{(q)} = \mathrm{Corr}(X_i, X_{i-r}) = \begin{cases} \sum_{v=1}^{q-r+1} b_v b_{v+r} & (1 \le r \le q) \\ 0 & (q+1 \le r < \infty). \end{cases} \qquad (3.11)$$

Thus the serial correlations are just lagged products of the $b_i$ sequence and the formula (3.11) is completely analogous to the formula for the serial correlations of the standard MA(q) process; see Box and Jenkins, 1970, p. 68. It can be seen from (3.11) that all the correlations are nonnegative and it may further be shown that they are bounded above by 1/4. Note too that since the $\rho_r^{(q)}$'s are lagged products of the $b_i$ sequence, it is not possible to determine the $b_i$'s uniquely from the $\rho_r^{(q)}$'s. Therefore it is not possible to uniquely determine the $\beta_i$'s from the $\rho_r^{(q)}$'s.

Consider now the index of dispersion J, defined at (2.21). It can easily be shown, from (3.11), that for the EMA(q) process, this is given by

$$J = 2 - \sum_{\rho=1}^{q+1} b_\ell \qquad (3.12)$$

This is maximized when all $b_\ell$s are equal and thus $\beta_\ell = 1/(1+\ell)$, $\ell = 1,2,\ldots,q$. These values of $\beta_\ell$ have the property that they give equal weights to the q+1 possible linear combinations which can make up an $X_i$, that is $b_i = 1/(1+q)$, i = 1,2,..., q+1.

The maximum values of $J$ are then, as $q$ increases, 1.5, 1.666, 1.750, 1.8000 and generally, $1 + q/(1+q)$ with 2 as limiting value; thus beyond a certain point, increasing the order of the moving average (which can conceptually go to infinity), has little effect. This implies that the over all dependence in the process, as expressed by $J$, is bounded and that very high values of $q$ do not substantially increase dependence.

A convenient notation for the EMA(q) sequence $\{X_i\}$ is $M_i^{(q)}$, meaning that $X_i$ has a moving average structure of order q over $E_i, E_{i-1}, \ldots, E_{i-q}$ using the parameters $\beta_q, \beta_{q-1}, \ldots, \beta_1$. In this notation, $M_i^{(q)}$ can be expressed in terms of $M_i^{(q-1)}$ by the recursion

$$M_i^{(q)} = \beta_q E_i + I_i^{(q)} M_{i-1}^{(q-1)}, \qquad q = 1,2,\ldots \tag{3.13}$$

where $\{I_i^{(q)}\}$ is an independent sequence of binary variables taking value zero with probability $b_{q+1} = \beta_q$.

## 4. THE EXPONENTIAL AUTOREGRESSIVE-MOVING AVERAGE PROCESSES EARMA(p,q).

### 4a. Definitions

We have defined both autoregressive processes and moving-average processes in exponential variables of any specified orders, p and q. Here we bring them together into a single process, EARMA(p,q), although it will be seen that the method of combination is not unique. We will then have a process of great flexibility in modelling dependent exponential variables, bearing favorable comparison with the standard ARMA(p,q) process in modelling dependent Gaussian variables. Jacobs and Lewis (1977) linked the two first order exponential processes EMA(1) and EAR(1), giving an EARMA(1,1) mixed model and obtained the serial correlations, some higher order explicit results and discussed central limit and mixing properties. For the general mixed process we shall give two types of model but restrict ourselves to their correlational properties, and in particular derive the difference equations satisfied by the serial correlations; these are similar but not identical to those of the standard ARMA process. The special process EARMA(2,1) and EARMA(1,2) will be considered in more detail.

In seeking exponentially distributed mixed autoregressive-moving average processes we will work from the pure (backward) moving average process EMA(q) given in (3.4). One reason why the exponential moving average and exponential autoregressive models are appealing and tractable is that they are expressed in terms of independent exponential variables. If this property is to be carried over into the mixed models, then the autoregressive contribution should enter without violating this feature; thus, to construct the EARMA(p,1) process we replace the $E_{i-1}$ variable in the EMA(1) of (3.1) by $A_{i-1}^{(p)}$, an EAR(p) variable. This is independent of $E_i$ in (3.1) because it is a function only of $E_{i-1}, E_{i-2}, \ldots$ . The defining equation for the EARMA(p,1) process is thus

$$X_i = \begin{cases} \beta E_i & \text{w.p. } \beta \\ \beta E_i + A_{i-1}^{(p)} & \text{w.p. } 1-\beta \end{cases} \qquad (0 \leq \beta \leq 1;\ i=0,\pm 1,\pm 2,\ldots) \tag{4.1}$$

which is the model treated by Jacobs and Lewis (1977) when $p = 1$. Similarly, (3.4) leads, on replacing $E_{i-q}$ by $A_{i-q}^{(p)}$ to the EARMA(p,q) process, with equation

$$X_i = \begin{cases} \beta_q E_i & \text{w.p. } b_{q+1} \\ \beta_q E_i + \beta_{q-1} E_{i-1} & \text{w.p. } b_q \\ \vdots & \\ \beta_q E_i + \beta_{q-1} E_{i-1} + \cdots + \beta_1 E_{i-q+1} & \text{w.p. } b_2 \\ \beta_q E_i + \beta_{q-1} E_{i-1} + \cdots + \beta_1 E_{i-q+1} + A_{i-q}^{(p)} & \text{w.p. } b_1 \end{cases} \tag{4.2}$$

for $i = 0, \pm 1, \pm 2, \ldots$, where the $b_i$'s are defined at (3.5) and $0 \leq \beta_1, \ldots, \beta_q \leq 1$. Writing $X_i^{(p,q)}$ as a variable in the EARMA(p,q) process based on the moving average parameters $\beta_q, \beta_{q-1}, \ldots, \beta_1$, the mixed process can be defined recursively as

$$X_i^{(p,q)} = \begin{cases} \beta_q E_i & \text{w.p. } \beta_q \\ \beta_q E_i + X_{i-1}^{(p,q-1)} & \text{w.p. } 1-\beta_q \end{cases} \qquad (i=0,\pm 1,\pm 2,\ldots) \tag{4.3}$$

This class of models will sometimes be written as $\text{EARMA}^-(p,q)$ to signify that it is based on a backward moving average.

Consider further the structure of the mixed model; for instance, $X_i$ depends on $E_i, E_{i-1}, \ldots$ and not on $E_{i+1}, E_{i+2}, \ldots$, paralleling the standard model ARMA(p,q) model. In contrast to the standard model it is also possible that the autoregressive aspect could be absent in stretches of the process when one of the pure moving average selections is chosen each time. Dependency would still be retained in the model by the moving average part (apart from the $q = 1$ case of course); while this is not at all unnatural there can be other situations when it is desired to always have autoregressive dependency. Such considerations lead to alternative mixed models; initial concern at the non-uniqueness of these models is best allayed by realizing that there is nothing unique about the standard Gaussian mixed ARMA(p,q) models. In an alternative formulation of the general mixed model, to be denoted by $\text{EARMA}^+(p,q)$, the shifted form of the forward moving average, briefly mentioned in Section 3 is used. The retention of independence between successive terms in the model then leads to the $\text{EARMA}^+(p,1)$ process as

$$X_i = \begin{cases} \beta_1 A_{i-1}^{(p)} & \text{w.p. } \beta_1 , \\ \beta_1 A_{i-1}^{(p)} + E_i & \text{w.p. } 1-\beta_1 ; \end{cases} \qquad (i = 0, \pm 1, \pm 2, \ldots) \tag{4.4}$$

and to the EARMA$^+$(p,q) as

$$X_i = \begin{cases} \beta_q A_{i-q}^{(p)} & \text{w.p.} \quad b_{q+1}, \\ \beta_q A_{i-q}^{(p)} + \beta_{q-1} E_{i-q+1} & \text{w.p.} \quad b_q, \\ \vdots & \\ \beta_q A_{i-q}^{(p)} + \beta_{q-1} E_{i-q+1} + \cdots + \beta_1 E_{i-1} & \text{w.p.} \quad b_2, \\ \beta_q A_{i-q}^{(p)} + \beta_{q-1} E_{i-q+1} + \cdots + \beta_1 E_{i-1} + E_i & \text{w.p.} \quad b_1, \end{cases} \tag{4.5}$$

for $i = 0, \pm 1, \pm 2, \ldots$ . It can be seen from (4.5) that this has the structure

$$X_i^{(p,q)} = \begin{cases} \beta_q A_{i-q}^{(p)} & \text{w.p.} \quad \beta_q \\ \beta_q A_{i-q}^{(p)} + X_i^{(q-1)} & \text{w.p.} \quad 1-\beta_q \end{cases} \qquad (i=0,\pm 1,\pm 2,\ldots) \tag{4.6}$$

where $X_i^{(q-1)}$ is a variable in the shifted forward moving average model of order q-1 using the parameters $\beta_{q-1}, \ldots, \beta_1$. Thus the autoregressive dependence is always present, and is lagged q values in arrears; the moving average variable gives greater flexibility to the initial form of the dependence, and differs for the two models. In EARMA$^-$(p,q) the most recent $E_i$ is always included; then with probability $(1-\beta_q)\beta_{q-1}$ a linear combination of $E_i$ and $E_{i-1}$ is included, and so on moving back; thus because of the certain addition of a new $E_i$ each time there cannot be runs of scaled values; further, this is a back progression, natural in many cases. The price of these features is that the model can exhibit patches of independence when only the $E_i$ is chosen, i.e. the autoregressive tail is not chosen. This cannot happen in the $EARMA^+(p,q)$ where the autoregressive dependency is always present; however, complicated but weak runs of scaled values are just possible in the $EARMA^+(p,q)$ model, arising from a low order autoregressive contribution sucessively being chosen on its own. Such a situation would be extremely rare. Our general feeling is that in practice there would not be much to choose from between the two types; a third type, more similar to $EARMA^+(p,q)$ than the other can be formed by interchanging the processes in (4.6) with a suitable shifting of scale. This is not considered here.

4b. Correlations for the backward mixed model EARMA(p,q)

We next derive equations satisfied by the serial correlations of the $EARMA^-(p,q)$ process, denoted here simply as EARMA(p,q). Multiplying each side of the defining equations (4.2) by $X_{i-r}$ $(r \neq 0)$ and taking expectations, gives

$$\begin{aligned} E(X_i X_{i-r}) = {} & \beta_q (b_{q+1} + \cdots + b_1)\, E(E_i X_{i-r}) \qquad r = \pm 1, \pm 2, \ldots \\ & + \beta_{q-1} (b_q + \cdots + b_1)\, E(E_{i-1} X_{i-r}) \\ & + \cdots + \beta_1 (b_2 + b_1)\, E(E_{i-q+1} X_{i-r}) \\ & + b_1 E(A_{i-q}^{(p)} X_{i-r}) \end{aligned} \tag{4.7}$$

This equation is not valid for $r = 0$ since the expectation of the mixture is not the mixture of the expectations when the variables are identical. Following equations (3.6), (3.7) and (3.8), the covariance form of (4.7) becomes

$$\operatorname{Cov}(X_i, X_{i-r})$$
$$= \sum_{m=0}^{q-1} b_{q+1-m} \operatorname{Cov}(E_{i-m}, X_{i-r}) + b_1 \operatorname{Cov}(A_{i-q}^{(p)}, X_{i-r}) \ . \tag{4.8}$$

It now becomes easier to work mainly in terms of correlations and to define

$$\rho_r = \operatorname{Cov}(X_i, X_{i-r}) \quad \text{and} \quad K_r = \operatorname{Corr}(E_i, X_{i-r}) \tag{4.9}$$

for $r = 0, \pm 1, \pm 2, \ldots$ . Since the $E_i$ and $X_i$ variables have the same marginal exponential distribution, (4.8) becomes

$$\rho_r = \sum_{m=0}^{q-1} b_{q+1-m} K_{r-m} + b_1 \operatorname{Corr}(A_{i-q}^{(p)}, X_{i-r}) \tag{4.10}$$

for $r = \pm 1, \pm 2, \ldots$ . To calculate the cross-covariances between the autoregressive and mixed process, we first note that

$$A_{i-q}^{(p)} = \left\{ \begin{array}{lll} \alpha_1 A_{i-1-q}^{(p)} & \text{w.p.} & a_1 \\ \alpha_2 A_{i-2-q}^{(p)} & \text{w.p.} & a_2 \\ \vdots & & \vdots \\ \alpha_p A_{i-p-q}^{(p)} & \text{w.p.} & a_p \end{array} \right\} + \varepsilon_{i-q}, \quad i = 0, \pm 1, \pm 2, \ldots \tag{4.11}$$

following the notation at (2.5) and (2.6). The $\varepsilon_i$-term in (4.11) has a distribution which depends only on $E_i$, such as was determined for $p = 2$ in Section 2. Multiplication of (4.11) by $X_{i-r}$ in order to calculate correlations leads to

$$\mathrm{Corr}(A_{i-q}^{(p)}, X_{i-r})$$

$$= \sum_{\ell=1}^{p} \alpha_\ell a_\ell \mathrm{Corr}(A_{i-\ell-q} X_{i-r}) + \mathrm{Cov}(\varepsilon_{i-q}, X_{i-r})/\mathrm{Var}(E) ,$$

$$r = 0, \pm 1, \pm 2, \ldots . \tag{4.12}$$

We now wish to substitute from (4.10) for the correlations in (4.12) and so obtain a difference equation for $\rho_r$. However we do not have (4.10) in the case $r = 0$. Thus in (4.12) when $i-r-(i-\ell-q) = q$ this substitution is not possible, that is when $\ell = r$ if $r \leq p$. In this case

$$\mathrm{Corr}(A_{i-q}^{(p)}, X_i) = b_1 , \tag{4.13}$$

as may be seen from (4.2). Thus (4.10) and (4.12) lead to

$$\rho_r - \sum_{m=0}^{q-1} b_{q+1-m} K_{r-m}$$

$$= \sum_{\ell=1}^{p}{}^{*} \alpha_\ell a_\ell \{\rho_{r-\ell} - \sum_{m=0}^{q-1} b_{q+1-m} K_{r-\ell-m} + \alpha_r a_r b_1^2$$

$$+ b_1 \mathrm{Cov}(\varepsilon_{i-q}, X_{i-r})/\mathrm{Var}(E) \tag{4.14}$$

for $r = 1,2,\ldots,p$; if $r > p$ then the term $\alpha_r a_r b_1^2$ is omitted. The asterisk denotes that the $\ell = r$ term is omitted from this summation when $r \le p$. The equation simplifies on defining

$$\rho_0 = b_1^2 \quad \text{and} \quad C_{r-q} = \mathrm{Cov}(\varepsilon_{i-q}, X_{i-r})/\mathrm{Var}(E) \; ; \tag{4.15}$$

equation (4.14) may then be written

$$\rho_r = \sum_{\ell=1}^{p} \alpha_\ell a_\ell \rho_{r-\ell} + \sum_{m=0}^{q-1} b_{q+1-m}\{K_{r-m} - \sum_{\ell=1}^{p*} K_{r-\ell-m}\} + b_1 C_{r-q}. \tag{4.16}$$

This is the desired general result.

Noting that $K_j = 0$ for $j \ge 1$, we see that for $r \ge p+q$, (4.16) reduces to an rth order difference equation

$$\rho_r = \alpha_1 a_1 \rho_{r-1} + \alpha_2 a_2 \rho_{r-2} + \cdots + \alpha_p a_p \rho_{r-p} \qquad (r \ge p+q) \; , \tag{4.17}$$

which is the same as (2.16) for the EAR(p) process. To calculate the initial $p+q-1$ serial correlations, $\rho_1, \ldots, \rho_{p+q-1}$, we need $K_0, K_{-1}, \ldots, K_{-p-q+2}$ and $C_0, C_{-1}, \ldots, C_{-q+1}$; explicit expressions for these quantities are given in the Appendix. The correlation structure of EARMA(p,q) processes is thus similar to that of the standard ARMA(p,q) processes; the only difference is that initial calculation of the first $p+q$ serial correlations are needed to start the difference equation (4.17), rather than the first $p$ as in the ARMA(p,q) case. Note finally that (4.16) is only strictly true when $p \ge 1$, $q \ge 1$, although similarities with the equations for the $(p = 0, q)$ case and $(p, q = 0)$ case are apparent.

These results all apply to the backward model, EARMA(p,q); for the forward $\text{EARMA}^+(p,q)$ model (see (4.5)) slightly different correlation equations apply. The main difference is that (4.8) is now

$$\text{Corr}(X_i, X_{i-r}) = b_{q+1}\, \text{Corr}(A_{i-q}^{(p)}, X_{i-r}) + \sum_{m=1}^{q} b_{q+1-m} K_{r-q+m} \,. \tag{4.18}$$

Using (4.12) we then find, corresponding to (4.16), that

$$\rho_r = \sum_{\ell=1}^{p} \alpha_\ell a_\ell \rho_{r-\ell} + \sum_{m=1}^{q} b_{q+1-m}\{K_{r-q+m} - \sum_{\ell=1}^{p_*} b_{r-\ell-q+m}\} + b_{q+1} C_{r-q} \tag{4.19}$$

for $r = 1,2,\ldots$ with $\rho_0 \equiv b_{q+1}^2$; as before explicit calculation of $K_0, K_{-1}, \ldots, K_{-p-q+2}$ and $C_0, C_{-1}, \ldots, C_{-q+1}$ are required to obtain the first p+q serial conditions. Further comparisons between the two types of model will be dealt with elsewhere.

## 5. SPECIAL CASES OF THE EARMA(p,q) PROCESS

We shall give the explicit versions of the correlation equations (4.16) in the cases $(p = 1, q = 1)$, $(p = 2, q = 1)$ and $(p = 1, q = 2)$; these are considered to be potentially the most useful.

(i) EARMA(1,1)

This model will be written in the notation

$$X_i = \begin{cases} \beta E_i & \text{w.p. } \beta \\ \beta E_i + A_{i-1} & \text{w.p. } 1-\beta \end{cases},$$

where

$$A_i = \rho A_{i-1} + \varepsilon_i, \qquad \varepsilon_i = \begin{cases} 0 & \text{w.p. } \rho \\ E_i & \text{w.p. } 1-\rho \end{cases}.$$

The correlation equations, from (4.16), are then

$$\rho_1 = \alpha_1 a_1 b_1^2 + b_2 K_1 + b_1 C_0 , \qquad \rho_r = \alpha_1 a_1 \rho_{r-1} , \quad (r \geq 2)$$

giving

$$\rho_1 = \rho(1-\beta)^2 + (1-\rho)\ \beta(1-\beta), \qquad \rho_r = \rho\rho_{r-1}, \qquad (r \geq 2).$$

This agrees with the result (2.4) of Jacobs and Lewis (1977).

(ii) EARMA(2,1)

Using (4.16) to obtain the results for $\rho_1$ and $\rho_2$ we have

$$\rho_1 = \alpha_1 a_1 b_1^2 + \alpha_2 a_2 \rho_{-1} + b_2\{K_1 - K_{-1}\} + b_1 C_0 ;$$

Thus

$(1-\alpha_2 a_2)\rho_1$

$$= \alpha_1 a_1 (1-\beta)^2 - \alpha_1 a_1 \beta(1-\beta)(\pi_1 + \pi_2 S^{-1}) + (\pi_1 + S^{-1}\pi_2)\ \beta(1-\beta),$$

and

$$\rho_2 = \alpha_1 a_1 \rho_1 + \alpha_2 a_2 (1-\beta)^2$$

with

$$\rho_r = \alpha_1 a_1 \rho_{r-1} + \alpha_2 a_2 \rho_{r-2} \qquad (r \geq 3).$$

(iii) EARMA(1,2)

In this case (4.16) gives

$$\rho_1 = \alpha_1 a_1 b_1^2 + b_3 K_1 + b_2 K_0 + b_1 C_{-1}\ ;$$

$$\rho_2 = \alpha_1 a_1 \rho_1 + b_3(K_2 - K_1) + b_2(K_1 - K_0) + b_1 C_0,$$

and

$$\rho_r = \alpha_1 a_1 \rho_{r-1} \qquad (r \geq 3)\ .$$

These simplify to

$$\rho_1 = \rho(1-\beta_1)^2(1-\beta_2)^2 + \beta_1\beta_2(1-\beta_2) + (\pi_1 + S^{-1}\pi_2)\ \beta_1(1-\beta_1)(1-\beta_2)^2\ ,$$

$$\rho_2 = \rho\rho_1 - \beta_1\beta_2(1-\beta_2) + (\pi_1 + S^{-1}\pi_2)(1-\beta_1)\ \beta_2(1-\beta_2)\ ,$$

and

$$\rho_r = \alpha_1 a_1 \rho_{r-1} \qquad (r \geq 3)\ .$$

## 6. FURTHER DEVELOPMENTS

There are many facets and properties of the EARMA(p,q) process which will be investigated in later papers. Some of these properties have been investigated for the EAR(1) process, the EMA(1) process and the EARMA(1,1) process by Gaver and Lewis (1975-78), Lawrance and Lewis (1977) and Jacobs and Lewis (1977) respectively. They include mixing conditions, infinite divisibility, stationary initial conditions, joint distributions, distributions of sums, spectra and higher order correlations. It should also be emphasized that all the serial correlations in the EARMA(p,q) process are positive; this aspect of the process can be broadened by considering antithetic processes and will be discussed elsewhere.

Another important question which arises is whether there are non-normal distributions other than the exponential for which mixed autoregressive, moving average structures can be defined analogous to EARMA(p,q). The general question is difficult and has been considered by Gaver and Lewis (1975-78) for the first order-autoregressive process. However it is clear that by adding two independent EARMA(p,q) processes, say $\{X_i^{(1)}\}$ and $\{X_i^{(2)}\}$, we obtain a process which has Gamma(2) marginal distributions,

$$P\{X_i \leq X_i^{(1)} + X_i^{(2)} \leq x\} = \int_0^{2\lambda x} v\, e^{-v}\, dv ,$$

and ARMA(p,q) correlation structure. Some analysis shows that this process is generated directly as a mixture over two independent i.i.d. exponential($\lambda$) sequences, $\{E_i^{(1)}\}$ and $\{E_i^{(2)}\}$, possibly scaled. It would be interesting to extend this process to the fractional Gamma case, i.e. $\kappa$ not an integer but it is not clear whether this process exists or how to construct it.

For the first-order autoregressive case, it is simple to show that the random variable $\varepsilon_i$ defined at (2.12) is infinitely divisible and therefore that the solution of (2.4) for $\varepsilon_i$ exists when $X_i$ is Gamma($\kappa,\lambda$) and has a Laplace-Stieljes transform which is just (2.11) raised to the power $\kappa$,

$$E(e^{-s\varepsilon_i}) = \left(\rho + (1-\rho)\,\frac{\lambda}{\lambda+s}\right)^{\kappa}, \qquad \kappa > 0, \quad \lambda > 0 \qquad (6.1)$$

However it is difficult to invert this transform, or to generate the random variable on a computer unless $\kappa$ is an integer.

This should give an indication of some of the interesting theoretical questions which are raised by the EARMA process.

APPENDIX: Calculations of $\{K_{-r},\ r = 0,1,2,\ \ldots\}$ and $\{C_{-r},\ r = \ ,1,\ldots,q-1\}$ for EARMA(p,q) models

By definition

$$K_{-r} = \mathrm{Corr}(E_i, X_{i+r}) = \mathrm{Corr}(E_{i-r}, X_i) \quad . \tag{A.1}$$

By the usual process of multiplication and expectation it is found that

$$\mathrm{Cov}(E_{i-r}, X_i) = \begin{cases} b_{q-r+1}\ \mathrm{Var}(E_{i-r}) & 0 \le r \le q-1\ , \\ b_1\ \mathrm{Cov}(E_{i-r}, A^{(p)}_{i-q}) & q \le r < \infty\ , \end{cases} \tag{A.2}$$

and in terms of correlations

$$K_{-r} = \begin{cases} b_{q-r+1} & 0 \le r \le q-1\ , \\ b_1\ \mathrm{Corr}(E_{i-r}, A^{(p)}_{i-q}) & q \le r < \infty \quad . \end{cases} \tag{A.3}$$

The calculation of the cross-correlations between the independent exponential sequence and the derived autoregressive sequence proceeds in the usual way, and gives

$$\mathrm{Corr}(E_i, A^{(p)}_i) = \mathrm{Cov}(E_i, \varepsilon_i)/\mathrm{Var}(E) \equiv d_0 \ ; \tag{A.4}$$

$$\mathrm{Corr}(E_{i-j}, A^{(p)}_i) = \sum_{\ell=1}^{\min(j,p)} \alpha_\ell a_\ell\ \mathrm{Corr}(E_{i-j}, A^{(p)}_{i-\ell}) \ . \tag{A.5}$$

We only need (A.5) when $p \geq 2$ and for $j = 1,2,\ldots,p-2$. Recursively then (A.4) and (A.5) give

$$\mathrm{Corr}(E_{i-j}, A_i^{(p)})$$

$$= (\alpha_1 a_1) d_0 \qquad (j=1)$$

$$= \{(\alpha_1 a_1)^2 + (\alpha_2 a_2)\} d_0 \qquad (j=2)$$

$$= \{(\alpha_1 a_1)^3 + 2(\alpha_1 a_1)(\alpha_2 a_2) + (\alpha_3 a_3)\} d_0 \qquad (j=3) \qquad (A.6)$$

$$= \{(\alpha_1 a_1)^4 + 3(\alpha_1 a_1)^2 (\alpha_2 a_2) + 2(\alpha_1 a_1)(\alpha_3 a_3) + (\alpha_2 a_2)^2 + (\alpha_4 a_4)\} d_0 \ . \qquad (j=4)$$

Further expressions will be evident based on the fact that the sum of the suffices equals $j$, that all possible such groups of terms are present, and that the coefficient of a particular term is the number of distinct orders of a term of that type. By definition we have

$$C_{-r} = \mathrm{Cov}(\varepsilon_i, X_{i+r})/\mathrm{Var}(X) = \mathrm{Cov}(\varepsilon_{i-r}, X_i)/\mathrm{Var}(X) \ . \qquad (A.7)$$

As at (A.2) above we have

$$\mathrm{Cov}(\varepsilon_{i-r}, X_i) = b_{q+1-r}\, \mathrm{Cov}(\varepsilon_{i-r}, E_{i-r}) \ , \quad 0 \leq r \leq q-1. \qquad (A.8)$$

Hence

$$C_{-r} = b_{q+1-r}\, d_0 \quad . \tag{A.9}$$

The calculation of (A.8) depends on the form of the error random variable $\varepsilon_i$ as a function of the random variable $E_i$. For example, in the EAR(2) case,

$$d_0 = \pi_1 + \pi_2/S \ . \tag{A.10}$$

## REFERENCES


Box, G.E.P. and Jenkins, G. (1970). Time Series Analysis, Forecasting and Control, San Francisco: Holden Day.

Cox, D.R. (1955). Some statistical methods connected with series of events. J.R. Statist. Soc. B 17, 129-164.

Cox, D.R. and Lewis, P.A.W. (1966). The Statistical Analysis of Series of Events, Methuen. London; Wiley, New York; Dunod. Paris.

Feller, W. (1966). An Introduction to Probability Theory and its Applications, Volume 2. London, Wiley.

Gaver, D.G. and Lewis, P.A.W. (1975-78). First order autoregressive Gamma sequences and point processes. To appear.

Jacobs, P.A. and Lewis, P.A.W. (1977). A mixed autoregressive-moving average exponential sequence and point process EARMA(1,1). Adv. Appl. Prob. 9, 87-104.

Lawrance, A.J. and Lewis, P.A.W. (1977). An exponential moving-average sequence and point process, EMA(1). J. Appl. Prob. 14, 98-113.

Lewis, P.A.W. and Shedler, G.S. (1977). Analysis and modelling of point-processes in computer systems. To appear.

Mallows, C.L. (1967). Linear processes are nearly Gaussian. J. Appl. Prob. 4, 313-329.

Wold, H. (1948). Sur les processus stationnaires ponctuels. Colloques Int. Cent. Natn. Rech Scient. 13, 76-86.

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