1972-08-28

A geometric view of some simple pursuit type games

Franke, Richard H.

Monterey, California. Naval Postgraduate School

NPS-53FE72081A

NAVAL POSTGRADUATE SCHOOL
Monterey, California

A GEOMETRIC VIEW

OF SOME

SIMPLE PURSUIT TYPE GAMES

BY

RICHARD FRANKE

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral M. B. Freeman, Superintendent

M. U. Clauser, Provost

Abstract:

Two-person zero-sum games of pursuit/evasion and target attack/defense are considered. The geometric solution of these games, and some variations of them, is given. Several more complex games are discussed.

This task was supported by: Advanced Modular Concepts Division
Naval Electronics Laboratory Center
San Diego, California 92152

NPS-53FE72081A

28 August 1972

A Geometric View of Some Simple Pursuit Type Games

Richard Franke

1. Rules, definitions.

By a simple pursuit type game, we shall mean a game in which the players move with simple motion in the plane. That is, the coordinate velocities are given by

$$\dot{x} = w \cos \phi(t)$$

$$\dot{y} = w \sin \phi(t)$$

where w is a constant, and $\phi(t)$ is an arbitrary (integrable) function. We note that paths with sharp corners are permitted under the assumption of simple motion. The speed of the player is w at all times.

The object of one player, P, will be to "capture" the second, E, by causing the distance between them to be less than P's "capture radius," $\ell \geq 0$. The capture radius may be zero.

We want to consider this as a two player, zero sum game, so there must be defined some quantity which one player seeks to minimize, and the other to maximize. This quantity is called the "payoff." For example, the payoff in the simplest pursuit game is time-to-capture, which P wants to minimize and E wants to maximize.

We will assume that the pursuer, P, moves at least as fast as the evader, E, since otherwise E can outrun P, and reach <u>any</u> point he desires (if $\ell = 0$). Consider the set of points which E can reach before capture by P, regardless of the motion of P. This will be called the "safe region," and its boundary will be called the "BSR". "Safe region" may be a misnomer, since it is not a safe region in the sense that E may not be captured in it; if he plays poorly he could be captured in it.

For zero capture radius ($\ell = 0$), and P moving with speed w, E with speed v, $w > v$, the BSR is a circle. It is the set of points $X = (x,y)$ such that $v|X_p - X| = w|X_e - X|$, where $X_p = (x_p,y_p)$ and $X_e = (x_e,y_e)$ are the starting points of P and E, respectively. The equation of the circle (known as the Appolonius circle) is

$$\left(x - \frac{x_e-\alpha^2x_p}{1-\alpha^2}\right)^2 + \left(y - \frac{y_e-\alpha^2y_p}{1-\alpha^2}\right)^2$$

$$= \frac{\alpha^2(x_p^{\,2}+y_p^{\,2}) - (x_e^{\,2}+y_e^{\,2})}{1-\alpha^2} + \frac{(x_e-\alpha^2x_p^{\,2})^2 + (y_e-\alpha^2y_p)^2}{(1-\alpha^2)^2}$$

where $\alpha = v/w$.

If the capture radius is $\ell > 0$, then the BSR becomes the oval

$$\alpha[|X_p-X| - \ell] = |X_e-X|.$$

For $\alpha = 1$ and $\ell = 0$, the BSR is a straight line, the perpendicular bisector of the line segment from P to E. If $\alpha = 1$ and $\ell > 0$, the BSR is the branch of a hyperbola, with P and E at the foci. It is, of course, the branch nearest to E.

2. The Simple Pursuit Game.

Let the initial positions of P and E be given as X_p and X_e. The payoff is time to capture, with E seeking to maximize. It is clear that E should head for the point of the BSR which is furthest from P, since that will maximize P's travel time hence the time to capture. Thus, E heads directly away from P along the line through their initial positions, and P pursues along this line. If one player plays in less than optimal fashion, the other takes maximum advantage of such a goof by continually employing his optimal strategy, i.e., heading directly toward (or away from) the other player. We assume, of course, that no prior knowledge of the non-optimal action by either is known by the other player. A player knowing the other's optimal strategy cannot employ that to his advantage, a characteristic of two-player, zero-sum games.

We digress a moment to consider the implications of the assumptions of simple motion. First, the constant speed is no restriction, since if either player uses a lower speed, he clearly "loses" compared to his payoff by using maximum speed.

The fact that sudden changes in direction can be employed is somewhat disturbing, although we note that the optimal paths are straight lines. Thus, except for a possible change in direction initially, this poses no particular problem. If the initial range is large compared to the turn radius, simple motion is a good approximation. The exception would be if the evader is maneuverable enough to "sidestep" the pursuer. This then becomes a very complicated problem to solve, as can be noted by browsing through [1], and noting that a large portion is devoted to the solution of the "homicidal chauffeur" game.

3. Variations.

3.1 Guarding a target.

In this game the payoff is the closest approach, by E, to a target area (or point) before he is captured. We shall give the solution geometrically and then present all the equations necessary to direct the pursuer and evader in their optimal paths.

Again, consider the BSR. It is clear that the closest E can come to the target is the distance from the target to the BSR. E heads directly for the point of the BSR closest to the target. P also heads for this point to prevent E from making a closer approach to the target. For $\alpha = 1$, and $\ell = 0$, we have the following plan, Figure 1, for a typical game. C denotes the capture point if both players move in optimal fashion.

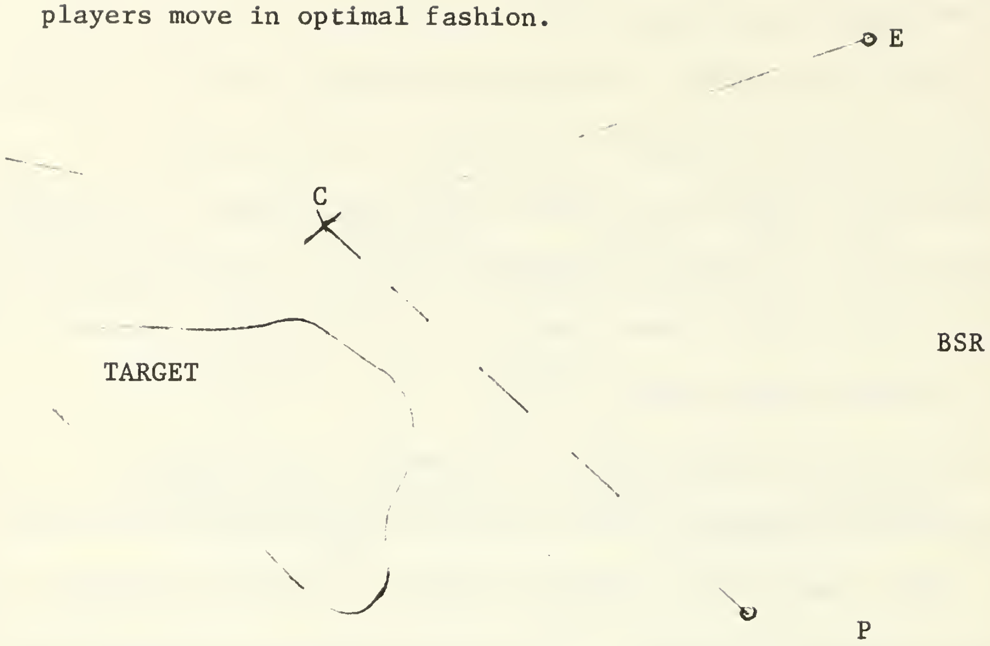

Figure 1: Guarding a target, $\alpha = 1$

We now consider the algebraic solution, as derived from the geometric one, where the target is the origin and $\ell = 0$. Let E and P start at (X_e,Y_e), (X_p,Y_p), and define the following quantities.

(ρ_e,ϕ_e) - polar coordinates of E

(ρ_p,ϕ_p) - polar coordinates of P,

$$\rho_e = \sqrt{x_e^2+y_e^2}$$

$$\phi_e = \tan^{-1}\frac{y_e}{x_e}$$

$$\rho_p = \sqrt{x_p^2+y_p^2}$$

$$\phi_p = \tan^{-1}\frac{y_p}{x_p}$$

$d = \sqrt{(x_e-x_p)^2 + (y_e-y_p)^2}$, the initial distance between P and E.

We note that the arctangent values are not principal values, but rather are determined by the signs of the coordinates of E and P.

For $\alpha < 1$, let (h,k) be the center of the BSR (a circle).

Then

$$h = \frac{x_e - \alpha^2 x_p}{1 - \alpha^2}$$

$$k = \frac{y_e - \alpha^2 y_p}{1 - \alpha^2}$$

and the radius r, is given by $r^2 = \dfrac{\alpha^2 \rho_p^2 - \rho_e^2}{1 - \alpha^2} + h^2 k^2 = \dfrac{\alpha^2 - d^2}{1 - \alpha^2}$

Let the polar coordinates of the center of the BSR be denoted by (ρ_a, ϕ_a), then

$$\rho_a = \sqrt{h^2 + k^2}$$

$$\phi_a = \tan^{-1} \frac{h}{k}$$

Let the "expected capture point," that is, the point of the BSR closest to the origin, be denoted by C, with coordinates (x_c, y_c). Then, the polar coordinates of C, (ρ_c, ϕ_c) are simply $(\rho_a - r, \phi_a)$, or in cartesian coordinates

$$x_c = (\rho_a - r) \cos \phi_a = \frac{(\rho_a - r)h}{\rho_a} = (1 - \frac{r}{\rho_a})h$$

$$y_c = (\rho_a - r) \sin \phi_a = \frac{(\rho_a - r)k}{\rho_a} = (1 - \frac{r}{\rho_a})k$$

For $\alpha = 1$ the BSR is the line

$$\frac{x_e - x_p}{d} x + \frac{y_e - y_p}{d} y = \frac{\rho_e^2 - \rho_r^2}{2d} .$$

The point of that line closest to the origin is at an angle $\phi_c = \tan^{-1} \dfrac{y_p - y_e}{x_p - x_e}$ at a distance $\rho_c = \dfrac{\rho_e^2 - \rho_p^2}{2d}$. Transforming to rectangular coordinates, we have

$$x_c = \frac{\rho_e^2-\rho_p^2}{2d}\cos\phi_c = \frac{\rho_e^2-\rho_p^2}{2d}\cdot\frac{x_e-x_p}{d} = \frac{(\rho_e^2-\rho_p^2)(x_e-x_p)}{2d^2}$$

$$y_c = \frac{\rho_e^2-\rho_p^2}{2d}\sin\phi_c = \frac{(\rho_e^2-\rho_p^2)(y_e-y_p)}{2d^2}$$

Now if we let (σ_e,θ_e) denote the range and direction from (x_e,y_e) to (x_c,y_c), we find

$$\sigma_e = \sqrt{(x_c-x_e)^2+(y_c-y_e)^2}\ ,$$

$$\theta_e = \tan^{-1}\frac{y_c-y_e}{x_c-x_e}$$

Similarly we find

$$\sigma_p = \frac{\sigma_e}{\alpha}$$

$$\theta_p = \tan^{-1}\frac{y_c-y_p}{x_c-x_p}$$

Figure 2 shows the pertinent data for a typical case.

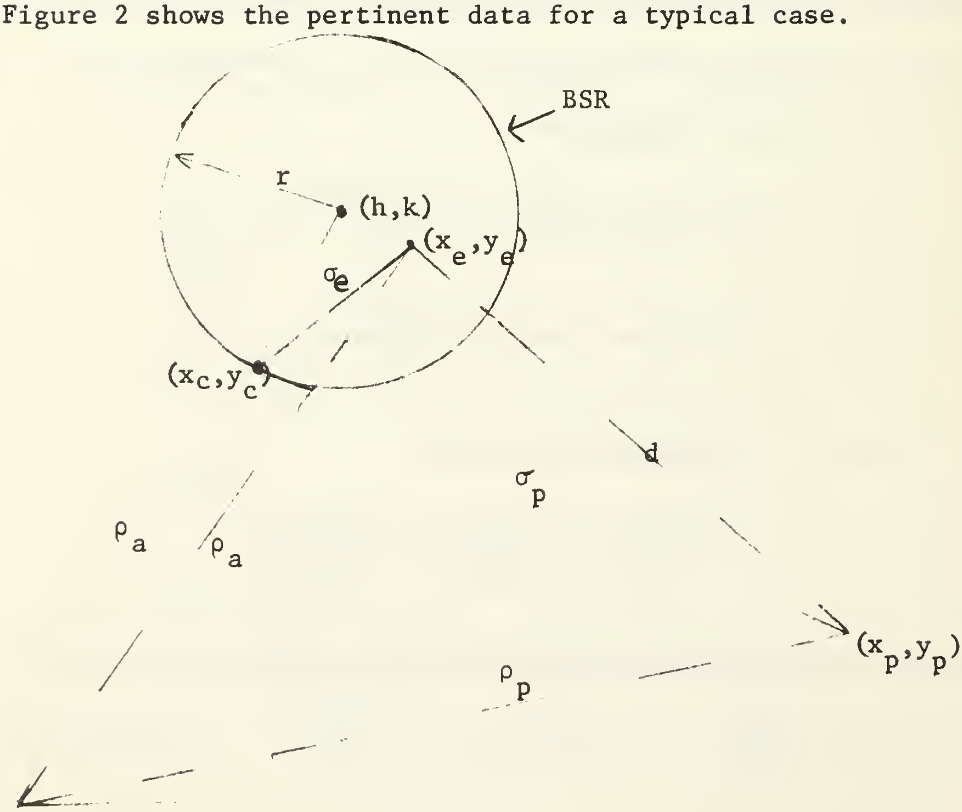

Figure 2: Defending a target, α=1/3

Figure 3 shows the paths when the defender acts optimally at all times, while the attacker simply heads directly toward the target. The increasing penalty he pays for non-optimal action is shown by the increasing distance of his closest possible approach.

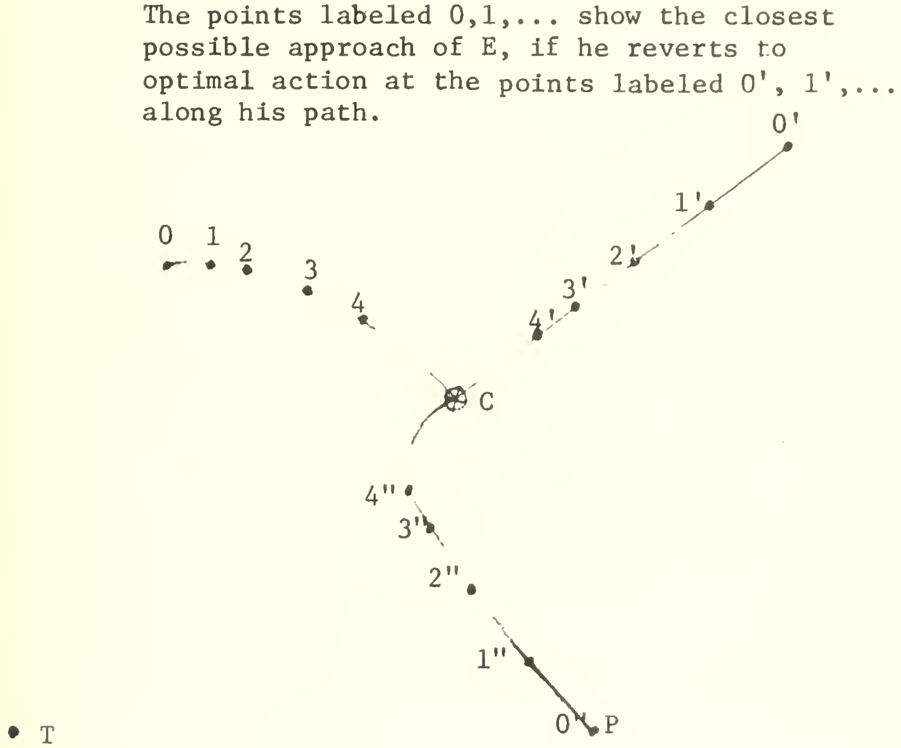

Figure 3: Guarding a target, α=1

3.2 Two Pursuers.

We consider the safe region for E from each of the two pursuers. Their intersection is his safe region in this game. The optimal action for all three participants is to head for the boundary point of the safe region which is furthest from the pursuers. See Figure 4 for a geometric example.

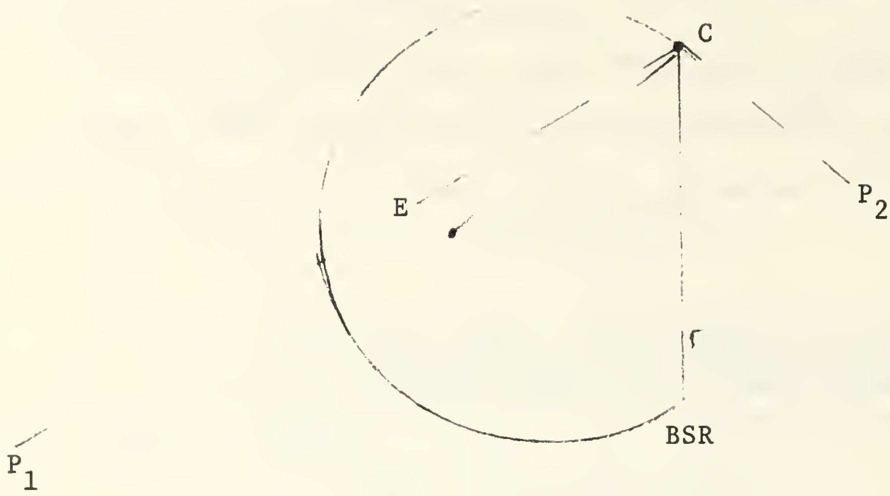

Figure 4: Two pursuers with $\alpha_1<1, \alpha_2=1$

3.3 Maneuverable Target

We consider a variation of the problem considered in 3.1, now allowing the target to be maneuverable. It is possible for T to be initially in E's safe region, but move out of it before E arrives.

It initially seems that the appropriate action for the target is to move away from E along the line through his own position and the point C, the capture point for the game outlined in 3.1. However, this is not necessarily the case, since by reducing the time to capture, E can reduce the amount by which T can increase his distance.

It is again clear the E should head directly for some point of the BSR, P should head for the same point, and T should move in a straight line through his initial position and the new capture point. This reduces the problem to a simple minimization problem; to which point of the BSR should E head in order to minimize the distance from himself to T, at the time he reaches the BSR? The necessary calculations to solve this problem have not been carried out. They should be straightforward, if tedious, although an analytical solution may not be possible in the general case.

3.4 Undetected Pursuer.

Suppose that E is attacking a target, and has been detected by P, but does not know that. P is to attempt to capture E. We assume that the probability of E having discovered he is being pursued is given as a function of distance between E and P. We assume that E and P act in optimal fashion in the target defense game, as soon as E discovers P.

Now, if E never discovers P, P's optimal course is to intercept E directly, assuming E is intent on attacking the target. If E knows P has discovered him at the outset, the game reduces to that of target defense.

Our assumptions have reduced this to a maximization problem; in this case, not so simple, however. Let Q(d) be the probability that E has discovered P by the time they are a distance d apart. Assume P moves with velocity $(w \cos \phi(t), w \sin \phi(t))$ along a apth P(t), and that E moves along E(t) until he discovers P.

Let $\mu(\phi;t)$ be the distance between P and E at time t, and let $\rho_c(\phi;t)$ denote the distance from the target to the capture point if E discovers P at time t. Then the expected distance to the target achieved by E is given by

$$\int_{\tau=0}^{\infty} \rho_c(\phi,\tau)dQ(\mu(\phi,\tau)).$$

Assuming that Q(t) and $\mu(\phi;t)$ are differentiable functions of t, we could write the above as

$$\int_{0}^{\infty} \rho_c(\phi;\tau)Q'(\mu(\phi;\tau))\ \mu'(\phi;\tau)d\tau\ .$$

The optimal path for P is determined by a function $\phi(t)$ which maximizes the above integral. Clearly this is a very difficult problem to solve.

One could also include a probability that E destroys the target by coming within a given distance, then seek to minimize the probability of E destroying the target.

The complexity of the problem dictates that its solution should yield a significantly better expected value than P can achieve by simply defending the target on the assumption that E may revert to optimal attack at any time. One quickly convinces himself that this is likely not the case, since if P starts near the target, compared to E, the headings which P could take do not vary significantly. That is, the angle between the straight intercept path, assuming E heads directly to T, and the optimal target defense path is quite small.

4. A modified pursuit game.

Let us modify the pursuit game in the following way. We assume that P must pay a penalty to determine the position of E. That is, he must stop, wait Δt_1 seconds before determining E's position, and then wait Δt_2 seconds before starting out again. At other times we assume P moves with simple motion, as does E.

It is assumed that E knows all the rules of the game pertaining to P. We start the game at an instant where P has determined E's position. Payoff is time to capture.

We assume P and E move at speeds w and v, and that P has a capture radius $\ell > 0$. We later give a lower bound for ℓ which ensures the capture of E by P.

We note that if P has determined E's position and traveled in a straight line for t seconds, P moves a distance wt while E can move a distance $v(t+\Delta t_2)$. If P locates E, travels for t seconds and locates E again, P has traveled a distance wt, while E has traveled a distance $v(t + \Delta t_1 + \Delta t_2)$.

We will give a strategy for P. It is apparently not optimal, although it does enjoy a property of two-person zero-sum games: It can be announced in advance by P with penalizing himself. E's strategy is a choice of directions to move between times that P locates E's position.

Let L denote the last known position of E. Then our strategy is for P to proceed to L, take a new reading on E's position, and repeat. Now, it is possible for P to stop at a point near L, and be sure he has not increased the capture time. In fact he may decrease it if E moves other than directly away from P. E's optimal strategy cannot be to move directly away from P, since knowledge of that would allow P to capture with no stops, if E employed the strategy, thus decreasing the time to capture. It is expected that the optimal strategy for P involves, for each move, a probability distribution of stopping points. Likewise, the optimal strategy for E likely involves a probability distribution of directions to proceed between P's data points. When the number of times P must stop is large (4 or 5 or more, perhaps), it is anticipated that the expected capture time for the optimal strategy would not be significantly better than the time for the strategy we outline below.

We give the details of our strategy for P, and show it cannot be improved on by any strategy which requires the same number of stops. For a smaller number of stops it is possible for E to escape.

Suppose the initial distance between P and E is ρ_0. P travels to L, taking a time to travel of $\frac{\rho_0}{w}$, and determines E's position. Then E's distance is $\rho_1 = v(\frac{\rho_0}{w} + \Delta t_1 + \Delta t_2)$. Letting $\alpha = \frac{v}{w}$ and $T = \Delta t_1 + \Delta t_2$, we obtain $\rho_1 = \alpha\rho_0 + vT$, $\rho_2 = \alpha\rho_1 + vT = \alpha(\alpha\rho_0 + vT) + vT = \alpha^2\rho_0 + (\alpha+1)vT$. In general, $\rho_n = \alpha^n\rho_0 + (\alpha^{n-1}+\alpha^{n-2}+\ldots+1)vT = \alpha^n\rho_0 + \frac{1-\alpha^n}{1-\alpha}vT$. We see that $\lim_{n\to\infty} \rho_n = \frac{vT}{1-\alpha}$. Since E travels a distance $v\Delta t_1$ after P stops, and before E's position is determined, it is necessary that the capture radius, ℓ, satisfy the inequality

$$\begin{aligned}\ell > \frac{vT}{1-\alpha} - v\Delta t_1 &= \frac{v}{1-\alpha}[\Delta t_1 + \Delta t_2 - (1-\alpha)\Delta t_1] \\ &= \frac{v}{1-\alpha}[\Delta t_2 + \alpha\Delta t_1] \\ &= \frac{\alpha}{1-\alpha}[w\Delta t_2 + v\Delta t_1].\end{aligned}$$

Then, if $\ell > \frac{\alpha}{1-\alpha}(w\Delta t_2 + v\Delta t_1)$, we can determine the number of stops required before capture. We achieve capture after n moves where n is the smallest integer which satisfies the inequality

$$\rho_n - v\Delta t_1 < \ell, \text{ or}$$

$$\alpha^n\rho_0 + \frac{1-\alpha^n}{1-\alpha} vT - v\Delta T_1 < \ell.$$

This reduces to

$$\alpha^n(\rho_0 - \frac{vT}{1-\alpha}) < \ell - \frac{\alpha(v\Delta t_1 + w\Delta t_2)}{1-\alpha}.$$

Thus

$$n > \frac{\ln\left\{\dfrac{\ell - \dfrac{\alpha(v\Delta t_1+w\Delta t_2)}{1-\alpha}}{\rho_0 - \dfrac{vT}{1-\alpha}}\right\}}{\ln \alpha}.$$

Thus the number of stops required is

$$n_0 = \text{integer part of } \frac{\ln\left[\dfrac{(1-\alpha)\ell - \alpha(v\Delta t_1 + w\Delta t_2)}{(1-\alpha)\rho_0 - vT}\right]}{\ln \alpha}$$

Now, the time to capture, T_c, satisfies the equation $[T_c - (n_o T + \Delta t_2)]w = vT_c + \rho_0 - \ell$ provided n_0 stops are required, and that E acts in an optimal way. Solving for T_c, we find

$$T_c = \frac{\rho_0 - \ell + (n_0 T + \Delta t_2)v}{w - v}$$

It is clear that if n_0 stops are required to achieve capture, there is no strategy for P which will assure him of capture in less time. This is seen by induction on the number of stops required, and noting that the time to capture cannot be decreased in any one step by more than the amount spent in taking that step, assuming E makes his best possible move. We also note that it is possible for E to escape if P stops fewer than n_0 times.

A more appropriate way for the pursuer to behave would require somewhat more information as to the where-abouts of E. Any change in the payoff which would require this additional stopping by P seems artificial. Thus it seems one could just as well impose an artificial constraint on P's motion. For example, we might want to make sure that P didn't get closer to E than a certain distance without knowing it. P's optimal strategy would probably again be dependent on a certain probability distribution of stopping points, but a pure strategy such as that given above could be given, and the time to capture bounded. This would again require the capture radius to satisfy a certain inequality, a somewhat larger capture radius being required than previously.

REFERENCES

1. Rufus Isaacs "Differential Games," John Wiley and Sons, Inc. New York, 1965.

2. W. Frye, "Game Theoretic formulation of some ASW problems." NEL report TM985, 1966

Distribution List

Defense Documentation Center Cameron Station Alexandria, Virginia	12
Library Naval Postgraduate School Monterey, California 93940	2
Naval Electronics Laboratory Center Research Library, Code 6700 San Diego, California 92152	2
NELC Mr. W. J. Dejka, Code 4300 San Diego, California 92152	8
NELC Mr. S. Snyder, Code 4000 San Diego, California 92152	1
NELC Mr. H. T. Mortimer, Code 0220 San Diego, California 92152	1
NELC Mr. M. Lamendola, Code 5200 San Diego, California 92152	1
NELC Mr. W. Loper, Code 5200 San Diego, California 92152	1
NELC Dr. R. Kolb, Code 3300 San Diego, California 92152	1
NELC Mr. E. E. McCown, Code 3100 San Diego, California 92152	1
NELC Mr. H. F. Wong, Code 3200 San Diego, California 92152	1
NELC Dr. P. C. Fletcher, Code 2000 San Diego, California 92152	1

Dean J. M. Wozencraft, Code 023 2
Dean of Research Administration
Naval Postgraduate School
Monterey, California 93940

Professor W. M. Woods, Code 53Wo 1
Chairman, Department of Mathematics
Naval Postgraduate School
Monterey, California 93940

Professor Richard Franke, Code 53Fe 5
Department of Mathematics
Naval Postgraduate School
Monterey, California 93940

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Postgraduate School Monterey, California	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	2b. GROUP

3. REPORT TITLE

A Geometric View of Some Simple Pursuit Type Games

4. DESCRIPTIVE NOTES *(Type of report and, inclusive dates)*

Technical Report, 1972

5. AUTHOR(S) *(First name, middle initial, last name)*

Richard H. Franke

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
28 August 1972	20	2

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	NPS-53FE72081A
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Naval Electronics Laboratory Center San Diego, California

13. ABSTRACT

Two-person zero-sum games of pursuit/evasion and target attack/defense are considered. The geometric solution of these games, and some variations of them, is given. Several more complex games are discussed.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Game theory						
Pursuit Games						
Zero-Sum Games						
Two person Games						