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PERFORMANCE ANALYSIS OF A BUFFER
UNDER LOCKING PROTOCOLS

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his paper formulates and analyzes stochastic models of process communication in computer stems. Messages are entered in a buffer (mailbox) by a source, and removed by a sink, at tes that are allowed to differ. The source, following message entry, and the sink, following ffer depletion, leave the buffer for independent exponentially distributed periods of absence, with rate parameters λ and μ, respectively. Locking protocols are in effect, i.e., ssage entry and removal can not occur simultaneously. The decision of the source or sink riving to find the buffer active can be to wait until it is free, or to leave on another riod of absence. We apply the analysis to both the "wait" and "no wait" options.

study of the fluid-approximation model shows that renewal theory forms the proper basis of e analysis; the important relevant results, including renewal-reward theorems, are briefly viewed. These results along with renewal-theoretic arguments, especially those exploiting generative properties, are then applied to derivations of basic performance measures, e.g., ansforms or expectations of busy periods and steady-state buffer levels. In particular, the (OVER)

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19. ABSTRACT cont.

formulas bring out the dependence of the expectations on the variance of message lengths. Desirable extensions of the models, to be reported in the unabridged version of the paper, are sketched.

Performance Analysis of a Buffer under Locking Protocols

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1. Introduction

We examine the behavior of a buffer system that acts as an intermediate storage facility in a computer system. We speak of inputs from a *source* and outputs to a *sink* as being *messages* consisting of data. The function of the buffer is to hold temporary accumulations of messages that arise because of irregular or randomly occurring input and output.

The source alternates between states when it is at the buffer entering a message at rate r^+, and states when it is away from the buffer. The loading of messages requires random time periods denoted generically by T. After a message is entered, the source leaves the buffer for a random *rest period*, R, before returning. Symmetrically, the sink alternates between states when it is away from the buffer and states when it is at the buffer removing data at rate r^-. The sink always completely empties the buffer before leaving; it returns following a rest period S. The sink immediately leaves on another rest period whenever it encounters an empty buffer.

We assume that there is no limit to the amount of storage that can be made available to the buffer. However, several styles of interaction between the source and sink will be considered. Styles in the class of interest in this paper are called *locking protocols*. According to such protocols, source and sink activities can not occur simultaneously; one activity locks out the other. Within this class of protocols, variants are determined by the reaction of the source or sink wishing to enter or remove data when it encounters the buffer in use. The taxonomy of models to be studied is

(i) Source no wait, sink no wait (NW, NW)

(ii) Source no wait, sink wait (NW, W)

(iii) Source wait, sink no wait (W, NW)

(iv) Source wait, sink wait (W, W)

Under the no-wait options an arriving source or sink encountering a buffer in use is assumed to leave the buffer for another source or sink rest period that is stochastically identical to and independent of the previous one. Under the wait options an arriving source or sink simply waits for the start of the next rest period, at which time it immediately begins entering or removing data, respectively. Depending on the application, it may be assumed that the message of a source turned away from a busy buffer is lost, and a new message is generated in the next rest period, or it may be assumed that the source attempts to enter the same message after the next rest period. The wait/no-wait options are also known as blocking/non-blocking options.

A primary (NW, NW) application motivating this paper requires the source and sink to be taken as autonomous processes. The buffer may be thought of as a mailbox where from time to time the source deposits mail and the sink collects mail. Message-based operating systems provide an important example. The computer system may be centralized or distributed, with the degree of coupling arbitrary; the coupling influences parameter values rather than model structures. The assumption $r^+ = r^-$ would frequently apply; the source and sink operate on identical machines. This assumption usually simplifies formulas without affecting the underlying analytical approach.

In a (NW, W) or (W, W) application, the source messages may be viewed as the intermediate products of a complex computation being passed on from one sub-process to another. In general, applications requiring one or both wait options imply a greater input or output dependence between source and sink; synchronization may be one of the functions being performed, along with message passing.

This paper contributes to a substantial literature on the analysis of buffer models. While there appears to be no recent survey, the extensive reference lists in a book by Aven, Coffman and Kogan (1987) and a recent paper by Mitra (1987) will be useful to the interested reader. We remark that this literature focuses chiefly on systems in which the sink is a slave process; the sink is always at the buffer either removing data or ready to do so. A notable exception is the recent work of Mitra (1987), where a multiple-source multiple-sink system without locking is analyzed; each source alternates between rest states and states when it is entering data into the buffer along with any other sources currently at the buffer. Each sink visits the buffer after successive rest periods until it finds a non-empty buffer, at which point it remains at the buffer removing data, along with any other sinks currently at the buffer, until the buffer is empty.

The following questions concerning buffer behavior are of interest, and are addressed in later sections:

- What is the characteristic occupancy level, e.g., the mean, or the probability distribution, of the buffer contents $X(t)$, or waiting time at a fixed time t, or in the long run, i.e., as $t \to \infty$?
- What is the mean, or probability distribution, of the time of first passage from an idle state ($X(t) = 0$) to a state $x > 0$, or from a busy state, $x > 0$ to idleness? In particular, what is a busy period duration?

It appears that such questions are often answered conveniently by thinking of the buffer process as alternating between idle periods of generic duration I, and busy periods of duration B. In fact, it is often possible to proceed by restricting the analysis to the *busy-period process*, $\{X_b(t), t \geq 0\}$, where the latter refers to the content process $\{X(t), t \geq 0\}$ over periods during which the latter is positive. Under convenient and natural initial assumptions the sequences of idle periods and busy periods are composed of mutually independent random variables, and are themselves independently and identically distributed, and so renewal-theoretic arguments and theorems (particularly the *key renewal theorem*) can be applied; see Smith (1955, 1958), Feller (1971) or Karlin and Taylor (1975). First-passage time results can be found by exploiting the ideas of terminating renewal processes and large deviations: see Feller (1971), and also Gaver and Jacobs (1986) in a different context. Similar and complementary results can often be obtained, and more intuitively, by application of renewal-reward theorems derived from the strong law of large numbers; see Ross (1983).

2. The Model

We present below a standard probabilistic formulation, but specialize it in later sections in order to obtain explicit solutions.

Message loading periods form a sequence of independent, identically distributed (i.i.d.) random variables $\{T_i, i = 1, 2, \ldots\}$ with distribution function (d.f.) $F(t)$ and density $f(t)$. Between-message rest times, or message interarrival times $\{R_i; i = 1, 2, \ldots\}$ are also i.i.d. random variables with d.f. $G(t)$ and density $g(t)$. Idle times, $\{I_i\}$, form a sequence of i.i.d. random variables, although often it is natural to think of them as being the same as source rest times. Sink interarrival times are i.i.d. with d.f. $K(t)$ and density $k(t)$. All processes are assumed independent.

In later sections, the above model, with $G(t)$ and $K(t)$ exponential, gives rise to simple formulas that promise insights. The phase-type formulations, *cf* Neuts (1987), among others, should also provide explicit, although awkward formulas in terms of elementary functions; we do not consider such elaborations here.

In what follows we adopt the following notational conventions. $F^*(\theta)$ denotes the Laplace-Stieltjes transform of the d.f. $F(t)$, $F^{(n)}(t)$ denotes the d.f. of a sum of n i.i.d. random variables with d.f. $F(t)$, and $\bar{F}(t) = 1 - F(t)$ denotes tail probabilities.

Regenerative Structure and Renewal-Theoretic Results — Let $X(t) \geq 0$ denote the state of the buffer process (e.g., the number of bits stored) at time t. The state space is $\mathcal{R}^+$, so we deal in the usual fluid approximation. The process $\{X(t), t \geq 0\}$ has a regenerative property, describable in terms of a *cycle* sequence $\{C_i\}$, made up of i.i.d. random variables, $C_i = I_i + B_i$, I_i being the i^{th} idle period and B_i the ensuing busy period, during which the buffer contents are positive. The sequence $\{C_i\}$ comprises a renewal process. Let $\{N(t), t \geq 0\}$ be the number of cycle completions in $(0, t]$, and let the renewal function of $\{N(t)\}$ be denoted by

$$n(t) = \sum_{j=1}^{\infty} \Pr\left\{\sum_{i=1}^{j} C_i \leq t\right\} = \sum_{j=1}^{\infty} F_c^{(j)}(t),$$

where $F_c^{(j)}(t)$ denotes the d.f. of the sum of j cycles. Let $H(t)$ be the d.f. of idle periods.

For $x > 0$,

$$p_b((dx),t) = \Pr\{x \le X_b(t) \le x + dx \mid X_b(0) = 0\}$$

(2.1)

$$= \Pr\{x \le X(t) \le x + dx\,,\ X(\tau) > 0\ \forall \tau : 0 < \tau \le t \mid X(0) = 0\}$$

provides the probability element for the busy period process. It is convenient to say, informally, that $p_b((dx),t)$ gives the probability that $X(t)$ is at x. Then we can express the probability that $X(t)$ is at x at any time, given $X(0) = 0$ and an idle period just beginning, as

$$p((dx),t) = p_c((dx),t) + \int_0^t p((dx),t-\tau)F_c(d\tau)\,, \tag{2.2}$$

where

$$p_c((dx),t) = \begin{cases} 1 - H(t)\,, & x = 0\,, \\ \int_0^t p_b((dx),t-\tau)\,H(d\tau)\,, & x > 0\,, \end{cases} \tag{2.3}$$

which represents the probability that $X(t)$ is at x during the first cycle. Note that (2.2) is formally solved in terms of the renewal function as

$$p((dx),t) = p_c((dx),t) + \int_0^t p_c((dx),t-\tau)\,n(d\tau)\,, \tag{2.4}$$

the first term being the probability that $X(t)$ is at x and t belongs to the first cycle, and the second being the probability that $X(t)$ is at x and t belongs to some later cycle. For the point mass at $x = 0$, we write

$$p_0(t) = \Pr\{X(t) = 0 \mid X(0) = 0\} = \overline{H}(t) + \int_0^t \overline{H}(t-\tau)n(d\tau)\,. \tag{2.5}$$

In terms of Laplace transforms on t, with $n^*(s) = \int_0^\infty e^{-st}\,n(dt)$,

$$p^*((dx),s) = \int_0^\infty e^{-st}p((dx),t)dt = \frac{H^*(s)p_b^*((dx),s)}{1 - n^*(s)} \tag{2.6}$$

$$p_0^*(s) = \int_0^\infty e^{-st} p_0(t)dt = \frac{[1 - H^*(s)]/s}{1 - n^*(s)}, \tag{2.7}$$

which can occasionally be inverted explicitly, given a nice form of the busy-period transform $p_b^*((dx), s)$. One can also interpret $sp^*((dx), s)$ as the probability that $X(z) = x$, with z having the density se^{-st}, i.e., the probability of state x when viewing the system at a random (exponential) instant.

Long-run $(t \to \infty)$ information is often available from the key renewal theorem, or from Tauberian/Abelian theorems for Laplace transforms; see Feller (1971) or Widder (1946). For example, if $p_b((dx), t)$ is directly Riemann integrable, then

$$p(dx) = \lim_{t \to \infty} p((dx), t) = \frac{1}{E[C]} \int_0^\infty p_b((dx), t)dt. \tag{2.8}$$

For $p_b((dx), t)$ to be directly Riemann integrable, it is sufficient that it be positive, non-increasing and integrable; see Røss (1983). These conditions will be verifiable in particular cases. For another useful result let $\psi(x)$ be any function such that

$$\psi_b(t) = \int_0^\infty \psi(x) p_b((dx), t) \tag{2.9}$$

is directly Riemann integrable. Then

$$\lim_{t \to \infty} E[\psi(X(t))] = \frac{\psi(0) \cdot E[I] + \int_0^\infty \psi_b(t)dt}{E[C]}. \tag{2.10}$$

In particular, the long-run expected buffer content is

$$\lim_{t \to \infty} E[X(t)] = \frac{\int_0^\infty \left(\int_0^\infty x p_b((dx), t) \right) dt}{E[C]} = \frac{\int_0^\infty E[X_b(t)]dt}{E[C]}. \tag{2.11}$$

Because of positivity (Fubini's theorem),

$$\int_0^\infty E[X_b(t)] \mid X_b(0)dt = E\left\{ \int_0^B X_b(t)dt \right\},$$

B being the length of a busy period, so to calculate the long-run expected buffer contents it is enough to calculate the expected area under the random function $X_b(t)$, $0 \le t \le B$. As will be seen, such expectations are often straightforward for specific buffer models. In addition,

$$\lim_{t \to \infty} \Pr\{X(t) \le x\} = \int_0^\infty \frac{\Pr\{X_b(t) \le x\} dt}{E[C]}$$

(2.12)

$$= \frac{E\left\{\int_0^B \chi(X_b(t), x) dt\right\}}{E[C]},$$

where the indicator function $\chi(X_b, x) = 1$ if $X_b \le x$, and $\chi(X_b, x) = 0$ otherwise. Interpret the numerator as the expected time in a cycle during which buffer contents do not exceed x.

Cumulative Process or Renewal-Reward and Strong Law Results — Suppose a buffer process operates over a time t, and view each cycle as carrying with it a *reward*, V_i, so that $\{C_i, V_i\}$ is a sequence of i.i.d. pairs of random variables. In the present context, rewards are illustrated by

- the total bit-seconds of delay incurred by message components present in the buffer over the busy period,

$$V_i = \int_0^{B_i} X_b(t) dt ;$$

- the total message-seconds of delay at the buffer over the busy period,

(2.13)
$$V_i = \int_0^{B_i} M_b(t) dt ,$$

  where $M_b(t)$ denotes the number of messages (or parts thereof) in the buffer at time t after the start of a busy period;

- the total number of seconds during which there were fewer than x bits present in the buffer during the busy period

(2.14)
$$V_i = \int_0^{B_i} \chi(X_b(t), x) dt .$$

Now let $V(t) = \sum_{i=1}^{N(t)} V_i$ denote the total reward accumulated in $(0, t]$. Then the following result is available. If V and C are a typical reward and its cycle length, and if $E[V] < \infty$ and $E[C] < \infty$, then it follows from the strong law of large numbers that as $t \to \infty$

(a) $\dfrac{V(t)}{t} \to \dfrac{E[V]}{E[C]}$ with probability one, and

(b) $\dfrac{E[V(t)]}{t} \to \dfrac{E[V]}{E[C]}$.

Further, from a central limit theorem on $\{C_i\}$, we have as $t \to \infty$

(c) $\dfrac{V(t) - tE[V]/E[C]}{\sqrt{t\rho}} \sim \mathcal{N}(0, 1)$ (the standardized normal distribution),

i.e., for large t $V(t)$ is approximately normally distributed with

$$\rho = \frac{Var[V]}{E[C]} + \frac{Var[C]E[V]}{(E[C])^3} - \frac{2(Var[C]\,Var[V])^{1/2}\,Corr(C, V)E[V]}{(E[C])^2},$$

provided the above quantities are finite and other natural conditions hold; *cf* Smith (1955), and Ross (1983) for (a) and (b).

3. The Buffer with No Source or Sink Waiting

We analyze the (NW, NW) system under exponential assumptions for the source and sink rest periods:

(3.1) $$g(t) = \lambda e^{-\lambda t}, \quad t \ge 0$$

(3.2) $$k(t) = \mu e^{-\mu t}, \quad t \ge 0.$$

The process of buffer contents, $\{X(t)\}$, evolves as follows. Suppose $X(0) = 0$; then after an idle period of duration I, a busy period begins; buffer contents at time t following the beginning of the busy period, and before it ends, are $X_b(t)$. At some time after the beginning of the busy period, the sink arrives. If it arrives while the source is at rest, it is said to be *effective* and immediately begins depleting the buffer; otherwise, the sink commences a new (i.i.d.) rest period. We let L denote the time from the start of the busy period to the moment buffer emptying begins, and we let

$S(L)$ denote the time required to empty the buffer, once emptying begins. A message arrival during $S(L)$ is turned away; the source commences a new (i.i.d.) rest period, at the end of which it returns with the message. (Observe that L is not simply a sink interarrival time, i.e., it does not have the density (3.2), but must be derived from scratch.) At busy period termination a new idle period of duration I begins, and so the process continues. For this model, I has the distribution of a rest period, i.e., it has the density (3.1).

We now discuss various random quantities characterizing the above processes; see Fig. 1.

(a) For the time from the beginning of a busy period to an effective sink arrival (SA), we write

$$L = T_1 + R_1' + T_2 + R_2' + \cdots + T_{M-1} + R_{M-1}' + T_M + R_{M-1}'' , \tag{3.3}$$

where M denotes the number of messages in the buffer at the time of an effective sink arrival, and R' (respectively, R'') is a rest period that is not (respectively, is) interrupted by an SA. By the exponential assumptions R' and R'' are equal in distribution. We look upon these modified source-rest periods as i.i.d. exponentials with parameter $\lambda+\mu$; at the end of such a period, depletion begins with probability $\mu/(\lambda+\mu)$, and a new message begins entering with probability $\lambda/(\lambda+\mu)$. Since M is geometrically distributed,

$$\Pr\{M = m\} = \left(\frac{\lambda}{\lambda+\mu}\right)^{m-1} \left(\frac{\mu}{\lambda+\mu}\right), \quad m \geq 1 ,$$

we can represent L as

$$L = \begin{cases} T_1 + R_1'' , & \text{when } M = 1 \text{ (first SA effective),} \\ & \text{i.e. with probability } \mu/(\lambda+\mu), \\ T_1 + R_1' + L^* & \text{when } M > 1 \text{ (first SA ineffective),} \\ & \text{i.e. with probability } \lambda/(\lambda+\mu), \end{cases} \tag{3.4}$$

with L^* independent of L and having the same d.f. Now introduce transforms to find

$$\phi_L(\theta) = E\{e^{-\theta L}\} = F^*(\theta) \int_0^\infty (\lambda+\mu) e^{-(\lambda+\mu+\theta)t} dt \left[\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \phi_L(\theta)\right] ,$$

so that

(3.5) $$\phi_L(\theta) = \frac{\mu F^*(\theta)}{\mu+\theta+\lambda[1-F^*(\theta)]}.$$

From (3.5), we obtain

(3.6) $$E[L] = \mu^{-1}[1+(\lambda+\mu)E(T)],$$

and

$$\mu E[L] \sim \frac{E[T]+E[R]}{E[R]} \quad as \quad \mu \to 0,$$

as is intuitive. It is possible to show that $\hat{L} = L/E[L]$ will tend weakly to the unit exponential d.f. as $\mu \to 0$.

(b) Next, consider the maximum level reached during a busy period

(3.7) $$X(L) = r^+(T_1+T_2+\cdots+T_M).$$

In terms of transforms again, $\phi_{X(L)}(\theta) = E[e^{-\theta X(L)}]$ is given by

(3.8) $$\phi_{X(L)}(\theta) = E[\{F^*(r^-\theta)\}^M] = \frac{\mu F^*(r^+\theta)}{\mu+\lambda[1-F^*(r^-\theta)]},$$

from which

(3.9) $$E[X(L)] = \mu^{-1}(\lambda+\mu)r^-E[T],$$

and

(3.10) $$\mu E[X(L)] \sim r^+\lambda E[T] \quad as \quad \mu \to 0.$$

For μ small, we correctly anticipate

(3.11) $$E[X(L)] \approx r^- \frac{E[T]}{E[T]+E[R]} E[L].$$

These results are applied trivially to the buffer emptying time, by means of

(3.12) $$S(L) = X(L)/r^-.$$

(c) For an analysis of busy periods,

(3.13) $$B = S(L) + L\,,$$

we first refer to (3.3)-(3.7) and (3.12) to see that

(3.14) $$E[e^{-\theta_S S(L) - \theta_L L} \mid M, R'_1, \ldots, R'_{M-1}, R''_M] = \left(E[e^{-(\theta_S(r^+/r^-)+\theta_L)T}]\right)^M e^{-\theta_L(R'_1 + \cdots + R'_{M-1} + R''_M)}$$

Removing the rest-period conditions (recall R' and R'' are i.i.d.), we obtain

(3.15) $$E[e^{-\theta_S S(L) - \theta_L L} \mid M] = [F^*(\theta_S(r^+/r^-)+\theta_L)]^M \left(\frac{\lambda+\mu}{\lambda+\mu+\theta_L}\right)^M .$$

Then, we remove the condition on M,

$$\phi_{S(L),L}(\theta_S, \theta_L) = \sum_{m \geq 1} E[e^{-\theta_S S(L) - \theta_L L} \mid M = m] \left(\frac{\lambda}{\lambda+\mu}\right)^{m-1} \frac{\mu}{\lambda+\mu}\,,$$

to find a result similar in form to (3.5) and (3.8)

$$\phi_{S(L),L}(\theta_S, \theta_L) = \frac{\mu F^*(\theta_S(r^+/r^-)+\theta_L)}{\mu + \theta_L + \lambda[1-F^*(\theta_S(r^+/r^-)+\theta_L)]}\,.$$

By (3.13), we have $\phi_B(\theta) = E[e^{-\theta B}] = \phi_{S(L),L}(\theta, \theta)$, and hence

(3.16) $$\phi_B(\theta) = \frac{\mu F^*((r^+/r^- + 1)\theta)}{\mu+\theta+\lambda[1-F^*((r^+/r^- + 1)\theta)]}\,,$$

with

(3.17) $$E[B] = \frac{1}{\mu}\left[1 + \left(\frac{r^+}{r^-} + 1\right)(\lambda+\mu)E[T]\right]$$

(d) In order to evaluate the long-run expectation of buffer contents we proceed to find the expected area under $X(t)$ during a busy period. Begin with the filling portion; a glance at Fig. 1 shows that the area of random triangles and rectangles is

(3.18) $$A(L) = \frac{r^+}{2}(T_1^2 + T_2^2 + \cdots + T_M^2) + r^+[T_1 R'_1 + \cdots + (T_1+T_2+\cdots+T_{M-1})R'_{M-1} + (T_1 + \cdots + T_M)R''_M]\,,$$

but a closer look provides the more convenient form

(3.19)
$$A(L) = \begin{cases} r^+(T_1^2/2 + T_1R_1'') & \text{when } M=1 \text{ (first SA effective)} \\ r^+(T_1^2/2 + T_1R_1') + A(L)^* + (r^+T_1)L^* & \text{when } M>1 \text{ (first SA ineffective)}, \end{cases}$$

where the regenerative properties of $\{X(t)\}$ have been fully exploited; i.e., both $A(L)^*$ and L^* have the same distributions as $A(L)$ and L, respectively.

Now take expectations in (3.19),

(3.20)
$$E[A(L)] = r^+\left\{\frac{1}{2}E[T^2] + E[T]E[R']\right\}$$
$$+ \Pr\{M>1\}\{E[A(L)] + r^+E[T]E[L]\},$$

then substitute (3.6), $E[R'] = \frac{1}{\lambda+\mu}$, and $\Pr\{M>1\} = \frac{\lambda}{\lambda+\mu}$ to obtain

(3.21)
$$E[A(L)] = r^+\left(\frac{\lambda+\mu}{\mu}\right)\left\{\frac{1}{2}E[T^2] + \frac{1}{\mu}E[T](1+\lambda E[T])\right\}.$$

Turning now to the area under the emptying portion of a busy period, we have

(3.22)
$$A(S(L)) = \frac{1}{2}X(L)S(L) = \frac{1}{2r^-}X^2(L) = \frac{(r^+)^2}{2r^-}(T_1 + \cdots + T_M)^2$$

Focusing on $Y = T_1 + \cdots + T_M$, we write conditionally

$$Y = \begin{cases} T_1, & \text{if } M=1 \\ T_1 + Y^*, & \text{if } M>1, \end{cases}$$

so

$$Y^2 = \begin{cases} T_1^2, & \text{if } M=1 \\ T_1^2 + 2T_1Y^* + (Y^*)^2, & \text{if } M>1, \end{cases}$$

where Y and Y^* are independent and equal in distribution. Remove conditions to find

$$E[Y] = E[T] + \frac{\lambda}{\lambda+\mu}E[Y] = \frac{\lambda+\mu}{\mu}E[T],$$

$$E[Y^2] = E[T^2] + \frac{\lambda}{\lambda+\mu}(2E[T]E[Y] + E[Y^2])$$

$$= \frac{\lambda+\mu}{\mu}\left\{E[T^2] + \frac{2\lambda}{\mu}(E[T])^2\right\},$$

and hence

$$\text{(3.23)} \qquad E[A(S(L))] = \frac{(r^+)^2}{r^-}\,\frac{\lambda+\mu}{\mu}\left\{\frac{E[T^2]}{2} + \frac{\lambda}{\mu}(E[T])^2\right\}$$

We assemble previous results and deduce expected long-run buffer constants from the key renewal theorem or the strong law of large numbers (*cf* (2.10)):

$$\text{(3.24)} \qquad E[X] = \lim_{t\to\infty} E[X(t)] = \frac{E[A(L)] + E[A(S(L))]}{E[C]},$$

where $E[C] = \frac{1}{\lambda} - E[B]$. Substitution of (3.17), (3.21) and (3.23) yields

(3.25)

$$E[X] = \left(\frac{\lambda+\mu}{\mu}r^-\right)\frac{\frac{E[T^2]}{2}\left(\frac{r^+}{r^-}+1\right) + \frac{1}{\mu}E[T]\left[1+\lambda\left(\frac{r^+}{r^-}+1\right)E[T]\right]}{E[C]}$$

with

$$\text{(3.26)} \qquad E[C] = \frac{1}{\lambda} - \frac{1}{\mu}\left\{1 - \left(\frac{r^+}{r^-}+1\right)(\lambda-\mu)E[T]\right\}.$$

4. Results on Wait Protocols

In general, wait protocols complicate matters, however, the principles of the analysis remain the same. This can be seen in the following treatment of the source no-wait, sink wait protocol Because of space limitations, the analysis is necessarily somewhat condensed.

Figure 2 illustrates the two types of busy-period sample functions that can arise in the (NW, W) model: the peaked ones occur when an SA encounters an active source, and waits until it finishes, while the remainder occur when an SA encounters a source rest period The analysis of the random variables in the preceding section proceeds as follows.

(a) We can express the time until emptying begins, measured from the start of a busy period, as

$$L = \begin{cases} T'; & M=1, \text{ SA during first loading period} \\ T'' + R & M=1, \text{ SA during first source-rest period} \\ T'' + R' + L^{*}; & M>1, \end{cases} \tag{4.1}$$

where L^{*} is independently distributed as L,

$$\begin{aligned} \Pr\{t \le T' \le t+dt\} &= (1-e^{-\mu t})F(dt)/[1-F^{*}(\mu)] \\ \Pr\{t \le T'' \le t+dt\} &= e^{-\mu t}F(dt)/F^{*}(\mu), \end{aligned} \tag{4.2}$$

R' has the same distribution as in section 3, and M has the geometric distribution

$$\Pr\{M=m\} = \left(\frac{\lambda F^{*}(\mu)}{\lambda+\mu}\right)^{m-1}\left(1-\frac{\lambda F^{*}(\mu)}{\lambda+\mu}\right). \tag{4.3}$$

Note that $F^{*}(\mu) = E[e^{-\mu T}]$ is the probability of no SA during a message entry. Then the transform becomes

$$\begin{aligned} \phi_L(\theta) &= \int_0^\infty e^{-\theta t}(1-e^{-\mu t})F(dt) \\ &\quad + \int_0^\infty e^{-(\mu+\theta)t}F(dt)\left\{\int_0^\infty \mu e^{-(\mu+\theta)t}[1-e^{-\lambda t}]dt + \phi_L(\theta)\int_0^\infty \lambda e^{-(\lambda+\mu+\theta)t}dt\right\}, \end{aligned} \tag{4.4}$$

so after simplification,

$$\phi_L(\theta) = \frac{(\lambda+\mu+\theta)F^{*}(\theta) - (\lambda+\theta)F^{*}(\mu+\theta)}{\mu+(\lambda+\theta)[1-F^{*}(\mu+\theta)]}. \tag{4.5}$$

For the expected value we find

$$E[L] = \frac{(\lambda+\mu)E[T] + F^{*}(\mu)}{\mu+\lambda[1-F^{*}(\mu)]}, \tag{4.6}$$

and $\mu E[L] \sim 1$ as $\mu \to 0$, for there are no re-tries, and the time to begin emptying after an SA is asymptotically the exponential SA time. In fact, expansion of (4.6) shows that, as $\mu \to 0$,

$$E[L] \sim \frac{1}{\mu} + \frac{E[T^2]}{2E[T]}\left(\frac{E[T]}{E[T]+E[R]}\right),$$

which is intuitively appealing (note that $E[T^2]/(2E[T])$ is an expected residual message loading time and $E[T]/(E[T]+E[R])$ is asymptotically the fraction of L spent loading messages).

(b) For the maximum level reached in a busy period, we have in agreement with (3.7)

$$X(L) = r^+(T_1 + \cdots + T_M),$$

and consequently,

$$E[S(L)] = E[X(L)]/r^- = \frac{r^+}{r^-} E[T]E[M]$$

or

$$E[S(L)] = \frac{r^+}{r^-} \frac{(\lambda+\mu)E[T]}{\mu+\lambda[1-F^*(\mu)]}. \tag{4.7}$$

This together with (4.6) yields for $E[B] = E[L] + E[S(L)]$,

$$E[B] = \frac{(\lambda+\mu)\left(\frac{r^+}{r^-}+1\right)E[T] - F^*(\mu)}{\mu+\lambda[1-F^*(\mu)]} \tag{4.8}$$

and, as $\mu \to 0$,

$$E[B] \sim E[L]\left\{1 + \left(\frac{r^+}{r^-}\right)\frac{E[T]}{E[T]+E[R]}\right\}, \tag{4.9}$$

as would be anticipated for rare SA's.

(c) To obtain $E[X]$, we start, in close analogy with (3.19), by expressing $A(L)$ as

$$A(L) = \begin{cases} r^+(T_1')^2/2; & M=1, \text{ SA during first loading period} \\ r^+[(T_1')^2/2 + T_1'R_1']; & M=1, \text{ SA during first source-rest period} \\ r^+[(T_1')^2/2 + T_1'R_1'] + A(L^*) + (r^+T_1')L^*; & M>1. \end{cases}$$

Taking expectations, a calculation shows that

$$E[A(L)] = r^+(\lambda+\mu)\left\{\frac{E[T^2]/2}{\mu+\lambda[1-F^*(\mu)]} + \frac{E[Te^{-\mu T}][1+\lambda(E[T]+F^*(\mu)/(\lambda+\mu))]}{\{\mu+\lambda[1-F^*(\mu)]\}^2}\right\}. \tag{4.10}$$

and that as $\mu \to 0$,

$$\mu E[A(L)] \sim r^+ (E[L])^2 \frac{E[T]}{E[T] + E[R]}$$

Next, write

$$A(S(L)) = \frac{1}{2} \frac{(r^+)^2}{r^-} S_M^2 ,$$

where

$$S_M = \begin{cases} T_1'' ; & M=1, \text{ first SA during first loading period} \\ T_1' ; & M=1, \text{ first SA during first source-rest period} \\ T_1' + S_M^{\#} ; & M>1 . \end{cases}$$

A straightforward calculation leads to

$$E[S_M] = \frac{(\lambda+\mu)E[T]}{\mu+\lambda[1-F^*(\mu)]} ,$$

$$E[S_M^2] = (\lambda+\mu)\left\{\frac{E[T^2]}{\mu+\lambda[1-F^*(\mu)]} + \frac{2\lambda E[Te^{-\mu T}]E[T]F^*(\mu)}{\{\mu-\lambda[1-F^*(\mu)]\}^2}\right\},$$

so that

(4.11) $$E[A(S(L))] = \frac{(r^+)^2}{r^-}(\lambda+\mu)\left\{\frac{E[T^2]/2}{\mu-\lambda[1-F^*(\mu)]} - \frac{\lambda E[Te^{-\mu T}]E[T]F^*(\mu)}{\{\mu-\lambda[1-F^*(\mu)]\}^2}\right\},$$

whereupon $E[X] = E[A(L)] - E[A(S(L))]$ follows by substitution of (4.10) and (4.11).

We conclude with brief observations on the source-wait models. The burden added to the analysis of these models results from a new idle period d.f., which now has an atom at 0 (see Fig. 3). Clearly, $\Pr\{I=0\}$ is the probability that the source arrives with a message during $S(L)$. To preserve regenerative structure, it is convenient to deal with *composite* busy periods, and idle periods that are residual source rest periods, as in Section 3. A composite busy period, $\bar{B}$, consists (with probability 1) of a sequence of busy periods $B_i, B_{i+1}, \ldots, B_{i+j}$, where B_{l+1} begins when B_l ends, $i \le l \le j-1$, and B_i and B_{i+j} are preceded and followed, respectively, by idle periods of positive duration (in particular, with density $\lambda e^{-\lambda t}$); see Fig. 3. The statistics of composite busy periods are easily derived. For example, write

(4.12)
$$\bar{B} = \begin{cases} B, & \text{with probability } e^{-\lambda S(L)} \\ B + \bar{B}^{\#}, & \text{with probability } 1-e^{-\lambda S(L)}, \end{cases}$$

where $\bar{B}$ and $\bar{B}^{\#}$ are independent and equal in distribution. Then,

$$E[\bar{B}] = E[B] + E[\bar{B}]E[1-e^{-\lambda S(L)}]$$

and

(4.13)
$$E[\bar{B}] = \frac{E[B]}{E[e^{-\lambda S(L)}]} .$$

By lines already well established, an analysis of the (W,NW) system leads to an expression for $E[e^{-\lambda S(L)}]$ having a familiar form:

(4.14)
$$E[e^{-\lambda S(L)}] = \frac{\mu F^*(\lambda)}{\mu + \lambda[1-F^*(\lambda)]} .$$

Then $E[\bar{B}]$ is determined from (3.17) (with $r^+ = r^- = 1$), (4.13), and (4.14). The analysis of the renewal process $C_i = \bar{B}_i + I_i$, $i = 1, 2, \ldots$, continues in analogy with Section 3.

5. Conclusions and Extensions

The models presented and analyzed represent in a simple way many situations encountered in computer science. To a degree, they resemble continuous polling models (*cf.* Coffman and Gilbert 1986), but they have their own unique features. Of course, there are many extensions to be considered, an important one being to recognize the finiteness of the buffer, i.e there is a capacity b so that $X(t) \le b < \infty$. This means that protocols must be established to deal with message inputs colliding with the capacity b. Possibilities include: (i) split such messages, sending the overflow and all subsequent messages before the next SA to an effectively unlimited secondary buffer, or (ii) reject such messages and close the buffer until the sink reappears; see Gaver and Jacobs (1980) for some partial but relevant results. Design questions concern the determination of b, as a function of source and sink rates and message statistics, so as to achieve a suitably small probability of overflow.

More complete information than the simple expectations exhibited here is desirable. Transforms of the time-dependent distributions of buffer contents are available, from which the tail behavior and its dependence on $F(t)$ can frequently be derived; the tool is large deviation theory. The present analysis can also be extended to multiple sources. Results have been obtained and will be reported in the full-length paper.

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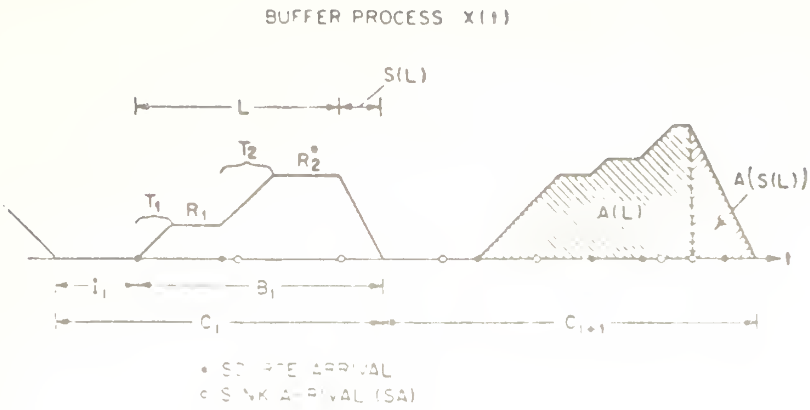

FIGURE 1 THE (NW, NW) BUFFER PROCESS ($r^* = \frac{1}{2} r^-$)

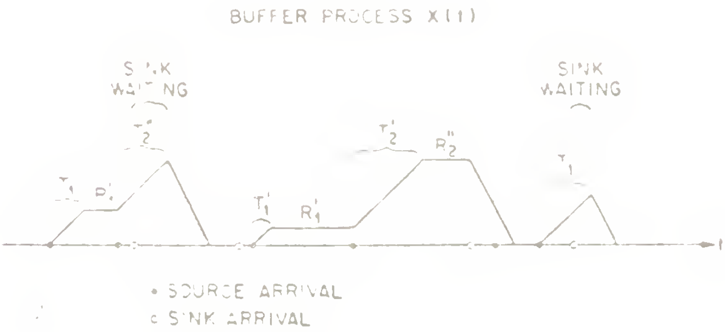

FIGURE 2 THE [illegible] ($r^* = \frac{1}{2} r^-$)

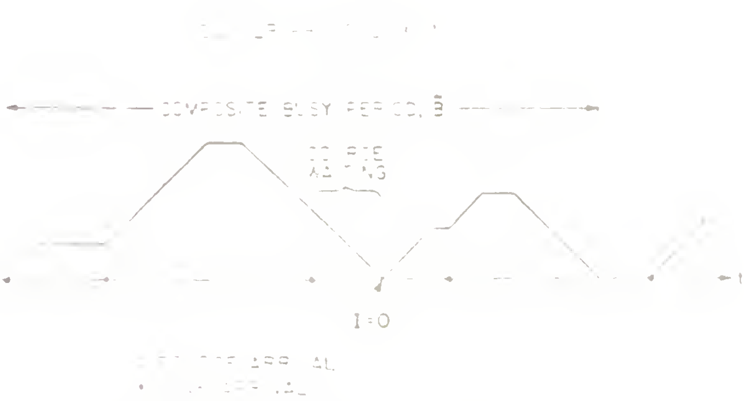

FIGURE 3 THE [illegible] BUFFER PROCESS [illegible]

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