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Realizable triples in dominator colorings

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THESIS

REALIZABLE TRIPLES IN DOMINATOR COLORINGS

by

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June 2007

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Given a graph *G* and its vertex set *V(G)*, the chromatic number, $\chi(G)$, represents the minimum number of colors required to color the vertices of *G* so that no two adjacent vertices have the same color. The domination number of *G*, $\gamma(G)$, is the minimum number of vertices in a set *S*, where every vertex in the set $V(G)-S$ is adjacent to a vertex in *S*. The dominator chromatic number of the graph, $\chi_d(G)$, represents the smallest number of colors required in a proper coloring of G with the additional property that every vertex dominates a color class. The ordered triple, (a,b,c), is realizable if a connected graph *G* exists with $\gamma(G)=a$, $\chi(G)=b$, and $\chi_d(G)=c$.

For every ordered triple, (a,b,c) of positive integers, if either (a) $a=1$ and $b=c\geq 2$ or (b) $2\leq a,b<c$ and $c\leq a+b$, previous work has shown that the triple is realizable. The bounds do not consider the case $a=b=c$. In an effort to realize all the ordered triples, we explore graphs and graph classes with $a=b=c=k\geq 2$.

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REALIZABLE TRIPLES IN DOMINATOR COLORINGS

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requirements for the degree of

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For every ordered triple, (a,b,c) of positive integers, if either (a) $a=1$ and $b=c\geq 2$ or (b) $2\leq a,b<c$ and $c\leq a+b$, previous work has shown that the triple is realizable. The bounds do not consider the case $a=b=c$. In an effort to realize all the ordered triples, we explore graphs and graph classes with $a=b=c=k\geq 2$.

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I. INTRODUCTION

A. BACKGROUND AND PURPOSE

Ever since Leonhard Euler took a mathematical approach to determine whether citizens in Konigsberg could walk across each of the city's seven bridges exactly once [1], mathematicians have applied graph theory to numerous problems. In today's modern world, some of the more commonly known uses of graph theory include networks. A network might be a collection of computers, telephones, or related technology interconnected by telecommunication equipment used to transmit or receive information. In addition, the definition of networks can expand to include groups of people. Graph coloring and domination are two areas of graph theory that have numerous applications to today's networks.

Network technology has rapidly evolved from those requiring a direct connection to those that are wireless, mobile and ad-hoc. As discussed in [2], the flexibility provided by networks such as satellite, radio, cellular, and sensor make them more efficient for today's modern world but are increasingly difficult to maintain at effective levels. Although graph coloring and domination are both still applicable to mobile ad-hoc networks, another topic might exist which better explains the behavior and properties of the modern network. This paper looks at one potential area, developed by combining graph coloring and domination. This area is called *dominator coloring*.

There has been research done on the dominator chromatic number for some of the more common graph classes [3, 4] and the relationship between a graph's dominator chromatic number and its chromatic and domination number [4]. As discussed in [4], an ordered triple, (a,b,c), can be used to represent the three parameters (domination number, chromatic number, and dominator chromatic number). Such a triple is *realizable* if there is a connected graph G where a represents the domination number, b the chromatic number, and c the dominator chromatic number of G. In [4] it was shown that a connected graph does exist that satisfies the requirements if either (a) $a=1$ and $b=c\geq 2$ or (b) $2\leq a,b<c$ and $c\leq a+b$. These bounds do not account for the case when $a=b=c$. In an attempt to realize all the triples (a,b,c), we want to show whether

or not graphs exist when all three parameters equal $k \geq 2$. This paper explores those graphs in which the domination number, chromatic number, and dominator chromatic number are equal.

B. TERMINOLOGY

A majority of the terms, definitions, and symbols used in the following paper are those found in [1]. Those terms and symbols that are not found in that text are referenced appropriately.

1. General Graph Overview

A *graph* G consists of a finite nonempty set *V* of elements called *vertices* and a set *E* of unordered pairs of distinct elements of *V* called edges. If $e = uv$ is an edge, vertices *u* and *v* are said to be *adjacent* and *e* is *incident* with both *u* and *v*. The *order* of a graph, $|G|$, is the number of vertices in *V(G)*. All references to graphs in this paper refer to *connected simple graphs*. A connected graph is a graph where every two vertices are adjacent and a simple graph is one where there are neither loops nor multiple edges between the same pair of vertices. The *degree* of a vertex is the number of edges incident with *v* and is denoted *deg v*. A *neighbor* of a vertex *v* is a vertex that is adjacent to *v*. The *open neighborhood*, *N(v)*, of the vertex *v* is the set of all neighbors of *v*. The *closed neighborhood*, *N[v]*, is defined by $N[v] = N(v) \cup \{v\}$. The *diameter* of a graph, $diam(G)$, is the greatest distance between any two vertices of *G*. Two graphs, *G* and *H*, are *equal* if their vertex sets and edge sets are equal and they are *isomorphic* ($G \cong H$) if the vertices of *G* and *H* can be labeled in a manner so that the two graphs are equal. Given *V(G)* and *S*, where *S* is a subset of *V(G)*, the notation $V(G) - S$ denotes the removal of the vertices of *S* from the set *V(G)*.

There are several standard classes of graphs we will refer to in this paper. The *complete graph* on *n* vertices, K_n, is a graph defined as one in which every two vertices of *G* are adjacent. If *G* is a graph of order $n \geq 3$ with vertices $v_1, v_2, ..., v_n$, and its edges are $v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1$, then *G* is called a *cycle* on *n* vertices and denoted C_n. A *bipartite* graph is one where $V(G)$ can be partitioned into two subsets *U* and *W* (referred to as *bipartite sets*) in such a way that every edge of G joins a vertex of U and a vertex of W. A *complete bipartite* graph, $K_{a,b}$ ($|U| = a$ and $|W| = b$), is one in which every vertex of U is adjacent to every vertex of W. A *star* is a complete bipartite graph where either $a = 1$ or $b = 1$

A *coloring* is an assignment of "colors" (usually integers) to the vertices of a graph. Graph coloring originated with a problem that Francis Guthrie explored in 1852 dealing with the minimum number of colors required to color a map so that no two adjacent regions have the same color. Guthrie proposed that one only needed four colors when coloring the countries on a map where no adjacent countries have the same color. This proposal became known as the Four Color Conjecture and its proof challenged graph theorists until 1976, when Wolfgang Haken and Kenneth Appel used a computer to assist in completing the proof. Given a graph G with $V(G)$ and $E(G)$, a coloring is a function $\theta : V(G) \to C$ from the set of vertices to a set C of colors. A *proper coloring* is one in which no two adjacent vertices are assigned the same color. A graph is *k-colorable* if it has a proper coloring with k colors and it is *k-chromatic* if it is k-colorable but not (k-1)-colorable. If G is k-chromatic, then we can partition $V(G)$ into k independent subsets, V_1, V_2,..., V_k, called *color classes*. The smallest number of colors in any proper coloring of a graph G is the *chromatic number* of G and is denoted by $\chi(G)$. For example, Figure 1 shows the graph C_4 with three different colorings.

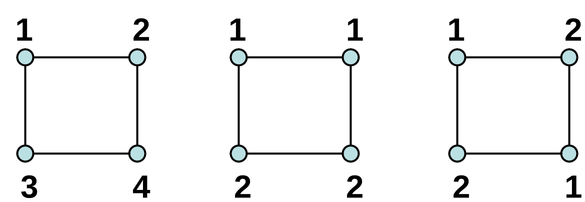

Figure 1. Examples of Graph Coloring

Using numbers to represent colors, the first graph in Figure 1 shows a 4-coloring of C_4. The second graph shows an improper coloring: two pairs of adjacent vertices have the same color. The third graph shows a 2-coloring of C_4, which is the smallest number of colors that result in a proper coloring. Therefore, $\chi(C_4)$=2. A modern application uses graph coloring to solve problems dealing with allocation of resources, such as channel assignments. Two radio or television transmitting stations can conflict if a message sent by the two stations can be received at the same place. Graph coloring can help identify and resolve these conflicts [2, 5]. For example, a simple network might have a structure similar to the graph in Figure 1, with each vertex representing some node in the network and the edges showing which nodes will conflict with the other if both are used simultaneously. By assigning a proper coloring to the network, nodes that conflict with one another are assigned different colors, with the total number of colors representing the number of required channels for this network to work properly.

Claude Berge began studying domination in graphs in 1958 [6], with Oystein Ore coining the term in 1962 [7]. A vertex v in a graph G is said to *dominate* itself and all of its neighbors, that is v dominates $N[v]$. A set S of vertices of G is a *dominating set* of G if every vertex of G is dominated by some vertex in S. More precisely, S is a dominating set of G if every vertex in $V(G)-S$ is adjacent to some vertex in S. A *minimum dominating set* is a dominating set of minimum cardinality. The *domination number* of G, $\gamma(G)$, is the number of vertices in a minimum dominating set. Consider the three copies of a graph in Figure 2.

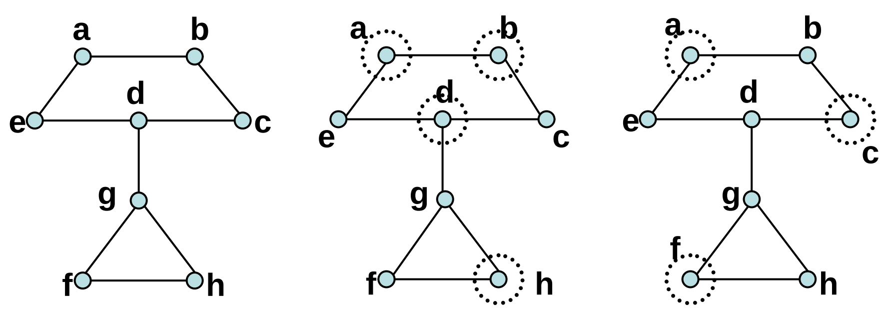

Figure 2. Examples of Dominating Sets of a Graph.

The sets $S_1 = \{a, b, d, h\}$ and $S_2 = \{a, c, f\}$ are dominating sets of G, while S_2 is a minimum dominating set. An example of domination can be seen in finding the minimum number of soldiers required to secure key terrain on an objective. Using the first graph in Figure 2, have each vertex represent key terrain features and an edge between two vertices signify the visibility of one terrain feature to another. Since the cardinality of the minimum dominating set is three, then three soldiers are required to secure the objective.

With respect to networks, we can apply domination in graphs toward the *clustering problem* [2]. Mobile ad-hoc networks face constant changes in their network topology. Clustering implements a hierarchy in these networks. A *connectivity graph* is a graph where the vertices represent the nodes in a network and the edges represent communication links between the nodes. The clustering problem divides the vertex set of a graph into subsets in such a way that the induced subgraph of each subset has a relatively small diameter. Within each subset, a vertex is chosen as the *cluster-head*. A new connectivity graph can be constructed using only the cluster-heads, where an edge exists between two cluster-heads if there is an edge between any of the vertices in the cluster-head's subset. In terms of domination in a graph, each vertex in the minimum dominating set becomes a cluster-head. Suppose the graph in Figure 2 depicts a connectivity graph for a network. The vertices in S_2 are the cluster-heads and the clusters are formed by the closed neighborhood of each vertex in S_2.

2. Dominator Coloring

In [3], a *dominator coloring* of G is defined to be a proper coloring in which every vertex dominates a color class. There are two cases by which a vertex dominates a color class. The vertex is either adjacent to all the vertices of one color class or is the only vertex in its color class, by which it will dominate its own color class. The *dominator chromatic number*, $\chi_d(G)$, is the minimum number of colors that allows a dominator coloring of G.

With respect to the chromatic number and domination number, previous research has shown that the dominator chromatic number is greater than or equal to either parameter and bounded above by their sum: $\chi(G), \gamma(G) \leq \chi_d(G) \leq \chi(G) + \gamma(G)$ [3, 4]. For completeness, we include a version of the proof shown by Gera in [4].

Proposition 1. Given a graph, *G*, then $\chi_d(G) \geq \chi(G)$ and $\chi_d(G) \geq \gamma(G)$.

Proof: Since a dominator coloring is also a proper coloring of *G*, it follows that $\chi_d(G) \geq \chi(G)$. For each color class *i*, $1 \leq i \leq k$, let v_i be a vertex of color class *i*. Define *S* to be a subset of *V(G)*, where *S* contains exactly one vertex of each color class. Let $x \in V(G)$. Since *x* dominates the color class *i*, for some *i* $1 \leq i \leq k$, it follows that x is dominated by v_i. Therefore, *S* is a dominating set, and $\chi_d(G) \geq \gamma(G)$. □

We construct a dominator coloring of the graph in Figure 3 in order to better illustrate the concept.

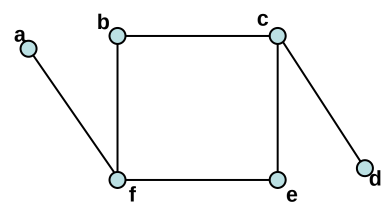

Figure 3. An Example of Dominator Coloring

Example 1. The graph *G* in Figure 3 has $\chi_d(G) = 4$.

To see this, we can partition the vertices of *G* into two partite sets, $S = \{a, b, d, e\}$ and $R = \{c, f\}$, therefore *G* is bipartite and has $\chi(G) = 2$. Assign the color 1 to *S* and the color 2 to *R*. Since G is bipartite, *V(G)* can only be partitioned into two subsets. These two subsets represent the color classes of *G*, which makes this proper coloring unique for *G* up to isomorphism. No matter how we assign colors to *R* and *S,* each will require a single and unique color. From Proposition 1, we know $\chi_d(G) \geq \chi(G)$. With the current coloring, no vertex dominates its own color class (it is not the only one of its color class) and it is not adjacent to all the vertices of another color class. Two colors will not work for a dominator coloring, therefore $\chi_d(G) \geq 3$.

Now, define a coloring on *G* where vertices *a*, *b*, *d,* and *e* have the color 1, the vertex *c* the color 2, and the vertex *f* the color 3. In this coloring, each of the vertices *c* and *f* dominate the color class 2 and 3, respectively, which are their own color class. Since the colors of *c* and *f* are not repeated, and since $\{c, f\}$ is a dominating set, each remaining vertex dominates the color class of *c* or *f*. It follows that $\chi_d(G) \leq 3$ and so $\chi_d(G) = 3$. ◊

For a finite graph, finding the dominator chromatic number is not relatively straightforward. An example is the Petersen graph. We proved $\chi_d(\text{Petersen}) = 5$, using a computer-assisted exhaustive proof. In fact, in [3] the authors showed that finding the dominator coloring of an arbitrary graph is NP-complete. In complexity theory, an *NP-complete problem* is a problem that can be solved nondeterministically in polynomial time and all other problems can be transformed to it in polynomial time [8]. These problems are recognized as being computationally difficult. For completeness, we include a version of the proof.

Theorem 1 [3]**.** Dominator chromatic number is NP-complete.

Proof: Dominator chromatic number is in NP. We can verify an assignment of colors to the vertices of a graph is a proper coloring and that every vertex dominates some color class. In order to show that the dominator chromatic number problem is NP-complete, we transform the chromatic number problem, which is NP-complete [9], to the dominator chromatic number problem. Consider an arbitrary graph, *G*, with $\chi(G) = k$, where $k \in \mathbb{N}$. Construct the graph *H* by adding a vertex *v* to *G* and an edge from *v* to every vertex in *V(G)*. Since every vertex of *G* is adjacent to *v*, assign *v* the color class $k+1$. The result is a proper coloring of *H*, where $\chi(H) = k+1$. Since *v* is the only vertex in its color class, *v* dominates its own color class. Furthermore, all the vertices in the set $V(H) - v$ dominate the color class $k+1$. And so it is also a proper dominator coloring with $\chi_d(H) = k+1$. Now, we have *H* with a dominator coloring using *k+1* colors. Since *v* is adjacent to every vertex in *G*, it must be the only vertex of the color *k+1* in this coloring. The removal of *v* results in a minimum proper coloring of *G* with *k* colors [3]. □

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II. ANALYSIS OF PAIRS

We have no generalized construction for graphs that have domination number, chromatic number, and dominator chromatic number equal. The first step is to find classes of graphs in which two of the three parameters are equal, i.e. $\gamma(G)=\chi(G)$, $\chi_d(G)=\chi(G)$, or $\chi_d(G)=\gamma(G)$. The purpose behind this step is for us to gain insight into the graph's structure that allow equality, and then apply that insight to develop a graph that has all three parameters equal.

A. THE CASE $\gamma(G)=\chi(G)$

First, we shall look at graphs satisfying $\gamma(G)=\chi(G)$. Observe that K_n with a pendant attached to each vertex satisfies this equality. This graph is known as the *corona* of K_n and K_1, denoted $Cor(K_n)$. The corona of the two graphs, K_n and K_1, is the graph formed from one copy of K_n and $|V(K_n)|$ copies of K_1, where each vertex in K_n is adjacent to a copy of K_1 [10]. Figure 4 shows an example of $Cor(K_3)$ and $Cor(K_4)$.

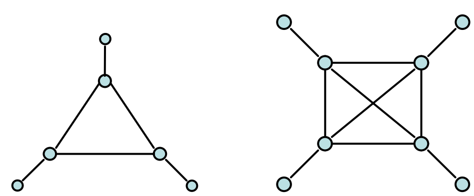

Figure 4. $Cor(K_3)$ and $Cor(K_4)$

Proposition 2. If G is $Cor(K_n)$, then $\gamma(G)=\chi(G)=n$.

Proof: Since G contains a copy of K_n, $\chi(G)\geq n$. Let H be the subgraph of G that represents the copy of K_n. Given $v_i\in V(H)$, for $1\leq i\leq n$, assign the color $i\in\mathbb{N}$, which results in n colors being used to color H. To each pendant attached to v_i, assign the color $i+1$ $(1\leq i\leq n-1)$ with the pendant adjacent to v_n having the color 1. This provides a proper coloring of G and $\chi(G)\leq n$. Thus $\chi(G)=n$. Next, note that $V(H)$ is a dominating set, so $\gamma(G)\leq n$. On the other hand, in order for each pendant to be dominated, either the pendant vertex or its neighbor must be in the dominating set. Suppose S_i is the set that includes v_i and its pendant vertex $(1 \le i \le n)$. For $1 \le i \le n$, all S_i are disjoint and at least n vertices are required for the dominating set, establishing $\gamma(G) \ge n$. Therefore, $\gamma(G) = n$. □

B. THE CASE $\chi(G) = \chi_d(G)$

Note that $\chi(G) = \chi_d(G)$ for K_n. Since every pair of vertices is adjacent, $\chi(K_n) = n$. In [2], the observation was made that for any complete graph, $\chi_d(K_n) = n$. We include a proof for completeness.

Proposition 3. The complete graph, K_n, has $\chi_d(K_n) = n$.

Proof: Since $\chi_d(K_n) \ge \chi(K_n)$, we have that $\chi_d(K_n) \ge n$. At most n colors are required on n vertices, so we have $\chi_d(K_n) \le n$. It follows that $\chi_d(K_n) = n$. □

C. THE CASE $\gamma(G) = \chi_d(G)$

Note that $\gamma(G) = \chi_d(G) = 2$ if G is the complete bipartite graph. For all values greater than two, we can construct a graph G satisfying $\gamma(G) = \chi_d(G)$. Consider the following construction. Starting with two copies of $K_{3,3}$, place an edge from one vertex in the first copy to a vertex in the second copy. Figure 5 shows the resulting graph.

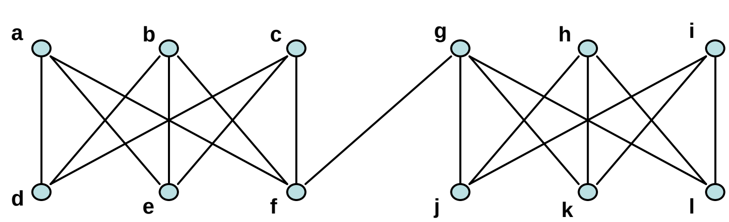

Figure 5. Graph Satisfying $\gamma(G) = \chi_d(G) = 4$

Example 2. The graph in Figure 5 has $\gamma(G) = \chi_d(G) = 4$.

We now present an argument to support the above claim. With respect to domination, we know that for each copy of $K_{3,3}$ there are two partite sets of vertices, where each vertex in one set is adjacent to all three vertices in the other partite set. It follows that for one copy of $K_{3,3}$ there are two vertices in any minimum dominating set *S*. For the graph in Figure 5, we consider the case where $g \in S$. Because *g* is adjacent to *f*, the set *S* would also dominate vertex *f*. The remaining undominated graph is a $K_{2,3}$, which requires two additional vertices for domination. Therefore, $\gamma(G) \geq 4$. Select the vertex set $S = \{a, d, g, j\}$. The vertex *a* dominates the vertices *a, d, e,* and *f,* the vertex *d* dominates *a, b, c,* and *d*, the vertex *g* dominates *f, g, j, k,* and *l*, and the vertex *j* dominates *g, h, i,* and *j*. The set *S* is a dominating set for *G,* so $\gamma(G) \leq 4$. Therefore, $\gamma(G) = 4$.

For the dominator chromatic number, by Proposition 1 we have $\chi_d(G) \geq 4$. Partition the vertices of *G* into four sets: $V_1 = \{a,b,c\}$, $V_2 = \{d,e,f\}$, $V_3 = \{g,h,i\}$, and $V_4 = \{j,k,l\}$. Assign each vertex set V_i the color *i* $(1 \leq i \leq 4)$. Each set V_i is one of the bipartite sets in a copy of $K_{3,3}$, so each vertex in V_i dominates the color class of the vertices in the other bipartite set to which it is adjacent. This is a proper dominator coloring, so $\chi_d(G) \leq 4$. Therefore, $\chi_d(G) = 4$ and $\chi_d(G) = \gamma(G) = 4$. ◊

The previous proof establishes $\chi_d(G) = \gamma(G) = 4$ for the graph that has two copies of $K_{3,3}$ joined by a single edge between one vertex from each copy. We generalize this construction as follows. We can attach an additional copy of $K_{3,3}$ to the graph in Figure 5 with an edge from any vertex, *v*, in *G* where *deg v* = 3. By adjoining additional copies of $K_{3,3}$ in this manner, we can increase the dominator chromatic number and domination by two for every copy. For all $i \in \mathbb{Z}^+$, let $G_i \cong K_{3,3}$ and let u_i and v_i denote an adjacent pair of vertices in G_i $(1 \leq i \leq n)$. Define H_n by

$$H_n = \begin{cases} G_1, & \text{if n = 1;} \\ H_{n-1} \cup G_n \cup \{u_{n-1}, v_n\}, & \text{if n > 1.} \end{cases}$$

Proposition 4. Given the graph H_n defined above, $\gamma(H_n) = \chi_d(H_n) = 2n$ for all $n \in \mathbb{Z}^+$.

Proof: We proceed by induction on $n \geq 1$. Since $H_1 \cong K_{3,3}$, we can select one vertex from each bipartite set for our dominating set establishing $\gamma(H_1) = 2$. Assign one bipartite set the color class 1 and the other bipartite set the color class 2. This is a proper dominator coloring since each vertex in one bipartite set is adjacent to every vertex in the other, therefore $\chi_d(H_1) = 2$.

Assume that $\gamma(H_k) = \chi_d(H_k) = 2k$ for all $k \geq 1$. Consider H_{k+1}, and we show $\gamma(H_{k+1}) = \chi_d(H_{k+1}) = 2k+2$. Since $H_{k+1} = H_k \cup G_{k+1} \cup \{u_k, v_{k+1}\}$, it follows that in addition to the *2k* vertices that form a dominating set for H_k, at least two more are needed since no vertex of $G_k - \{v_k\}$ is dominated by any vertex of the $2k$ vertices in the dominating set already formed. Thus $\gamma(H_{k+1}) \geq \gamma(H_k) + 2 \geq 2k+2$. For a dominating set, S, of H_{k+1}, select one vertex from each of the $k+1$ partite sets found in each G_i, where $i = 1, 2, .., k+1$. S is a dominating set for H_{k+1} that has two elements for every G_i. Therefore $\gamma(H_{k+1}) \leq 2k+2$, and it follows that $\gamma(H_{k+1}) = 2k+2$. For the dominator chromatic number, we know $\chi_d(H_{k+1}) \geq \gamma(H_{k+1})$, so it follows that $\chi_d(H_{k+1}) \geq 2k+2$. Assign each partite set of each G_i $(1 \leq i \leq k+1)$ a different color. The result is a proper dominator coloring. There are two colors for every copy of G_i $(1 \leq i \leq k+1)$, and so $\chi_d(H_{k+1}) \leq 2k+2$. Therefore, $\gamma(H_{k+1}) = \chi_d(H_{k+1}) = 2k+2 = 2(k+1)$. □

We have now established the case $\gamma(H_n) = \chi_d(H_n) = 2n,\ n \geq 1$. In order to cover the odd cases, we need to construct a graph G that satisfies $\gamma(G) = \chi_d(G) = 2n+1$. Initially, we desire a graph that satisfies $\gamma(G) = \chi_d(G) = 3$. Figure 6 shows such a graph, which we call J_1 for further reference.

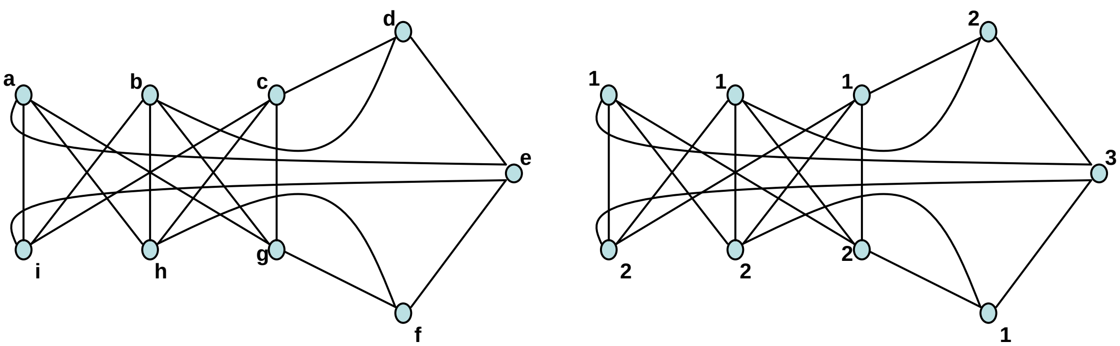

Figure 6. The Labeled Graph J_1 and its Dominator Coloring

Proposition 5. The graph J_1 in Figure 6 satisfies $\gamma(J_1) = \chi_d(J_1) = 3$.

Proof: First, we show $\gamma(J_1) = 3$. The vertices *c*, *e*, and *h* form a dominating set and it follows that $\gamma(J_1) \leq 3$. To verify that $\gamma(J_1) \geq 3$, it is necessary to show that there is no dominating set with exactly two vertices in it. Because of the symmetry of the graph, there are five cases. For each case, let *S* be a minimum dominating set.

CASE I: Assume $a \in S$. Then *a* dominates *a*, *e*, *g*, *h*, and *i*. In order for $|S| = 2$, then the vertices *b*, *c*, *d*, and *f* must share a common neighbor. They do not, so $|S| \geq 3$.

CASE II: Assume $b \in S$. Then *b* dominates *b*, *d*, *g*, *h*, and *i*. Since *a*, *c*, *e*, and *f* do not share a common neighbor, *S* requires at least 2 more vertices and $|S| \geq 3$.

CASE III: Assume $c \in S$. Then vertices *a*, *b*, *e*, and *f* require domination. These vertices do not have a common neighbor, therefore $|S| \geq 3$.

CASE IV: Assume $d \in S$. Vertices *a*, *f*, *g*, *h*, and *i* need to be dominated. There is no common neighbor and *S* requires at least two more vertices. Hence, $|S| \geq 3$.

CASE V: Assume $e \in S$. Then *b*, *c*, *g*, and *h* require domination. But *b* is not adjacent to *c* and *h* is not adjacent to *g*, so *S* requires at least two more vertices. Therefore $|S| \geq 3$.

Each case has shown that $\gamma(J_1) \geq 3$. Since it was previously shown that $\gamma(J_1) \leq 3$, we have $\gamma(J_1) = 3$.

With respect to the dominator chromatic number, we know from Proposition 1 that $\chi_d(J_1) \geq \chi(J_1)$. Since J_1 has an odd cycle, it follows that $\chi_d(J_1) \geq 3$. The second graph in Figure 6 shows a proper dominator coloring which establishes $\chi_d(J_1) \leq 3$. Therefore, $\chi_d(J_1) = 3$. □

Using J_1 we can construct a graph that provides a basis for a graph that satisfies $\gamma(G) = \chi_d(G) = 2n+1$ for an arbitrary *n*. Figure 7 shows the resulting graph.

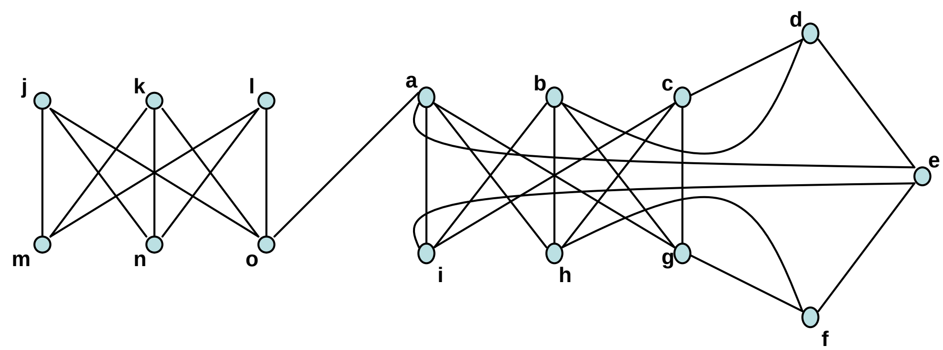

Figure 7. Graph Satisfying $\gamma(G) = \chi_d(G) = 5$

Lemma 1. The graph, *G*, in Figure 7 has $\gamma(G) = \chi_d(G) = 5$.

Proof: To show $\gamma(G) \geq 5$, consider first the subgraph J_1 of *G*. From Proposition 5, we know that at least three vertices of J_1 are required in any dominating set *S*. In the case where $a \in S$, the set S would also dominate vertex *o*. The remaining undominated graph is a $K_{2,3}$, which requires two additional vertices for domination. Therefore, $\gamma(G) \geq 5$. Let *S* = *{c, e, h, k, n}*. The set *S* is a dominating set for *G* and $\gamma(G) \leq 5$. Therefore $\gamma(G) = 5$. With respect to the dominator chromatic number, we know $\chi_d(G) \geq \gamma(G)$ which produces $\chi_d(G) \geq 5$. For the subgraph J_1 in *G*, assign colors as depicted in the second graph of Figure 6. Assign the color 4 to the vertices *j*, *k*, and *l* in *G* and the color 5 to the vertices *m*, *n*, and *o*. This assignment gives a proper dominator coloring and establishes that $\chi_d(G) \leq 5$. Therefore $\chi_d(G) = 5$. □

We now prove by induction that there is a graph, H_n^*, with $\gamma(H_n^*) = \chi_d(H_n^*) = 2i+1$. For all $n \geq 2$, let $G_n \cong K_{3,3}$ and let u_n and v_n denote an adjacent pair of vertices in G_n. Define $H_n^* = H_{n-1} \cup J_1 \cup \{u_{n-1}, v_n\}$.

Proposition 6. Given the graph H_n^* defined above, $\gamma(H_n^*) = \chi_d(H_n^*) = 2n+1$.

Proof: With respect to domination, we proceed by induction on n. The base case, $n = 2$, is shown in Lemma 1. Assume for $k \geq 2$, $\gamma(H_k^*) = 2k+1$ and prove $\gamma(H_{k+1}^*) = 2(k+1)+1$. Let G_1 be the first copy of $K_{3,3}$ in H_{k+1}^*. Let $H' = H_{k+1}^* - G_1$. There exists a vertex, say v, in G_1 that is adjacent to a vertex, say u, in H' such that $uv \in E(H_{k+1}^*)$. By the induction hypothesis, $\gamma(H') = 2k+1$. Since v is the only vertex of G_1 that can be dominated by some vertex of H', consider $G_1 - \{v\} \cong K_{2,3}$. Since $\gamma(K_{2,3}) = 2$, it follows that $\gamma(H_{k+1}^*) = \gamma(H') + \gamma(K_{2,3}) = 2k+1+2 = 2(k+1)+1$.

For $\chi_d(H_n^*)$, since $\chi_d(H_n^*) \geq \gamma(H_n^*)$ it follows that $\chi_d(H_n^*) \geq 2n+1$. Assign to each bipartite set of each of the G_n's in H_{n-1} a different color. The result is a proper dominator coloring for H_{n-1}, with $\chi_d(H_{n-1}) = 2(n-1)$. In the copy of J_1, assign the vertices a, b, c, and f the color class $2(n-1)+1$, the vertices d, g, h, and i the color class $2(n-1)+2$, and the vertex e the color class $2(n-1)+3$. The vertices a, b, and c dominate the color class $2(n-1)+2$. The vertices g, h, and i dominate the color class $2(n-1)+1$, and the vertices d, e, and f dominate the color class $2(n-1)+3$. Three additional colors are required and it follows that $\chi_d(H_n^*) \geq 2(n-1)+3 \geq 2n+1$. Therefore, $\chi_d(H_n^*) = 2n+1$. □

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III. REALIZABLE TRIPLES

A. PREVIOUS RESULTS

In [4], Gera presented and proved Theorem 2.

Theorem 2. For each ordered triple of integers (a,b,c), where either (a) $a=1$ and $b=c\geq 2$ or (b) $2\leq a,b<c$ and $c\leq a+b$, there is a connected graph G with $\gamma(G)=a,\ \chi(G)=b,$ and $\chi_d(G)=c$.

The graph G is constructed from K_b, where $V(K_b)=\{u_1,u_2,...,u_b\}$, by first adding a pendant to each vertex $u_1,u_2,...,u_{a-k}$ $(0\leq k<a)$. Next we add k copies of P_2, whose vertices are v_i and w_i, together with k additional edges u_bv_i $(0\leq i\leq k)$. An example of the graph is in Figure 8.

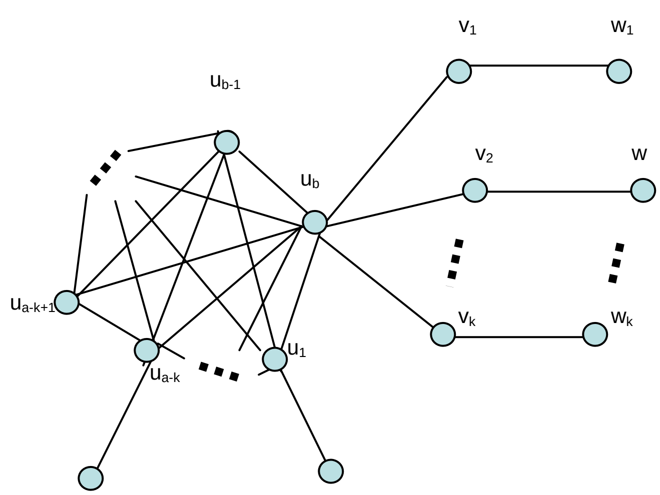

Figure 8. The Graph G in Theorem 2 [4]

B. SMALL CASES

We will look at some small cases where the three parameters equal. One can easily show that C_4 has a dominator chromatic number, chromatic number, and domination number of two. For the chromatic number, the third graph in Figure 1 shows a 2-coloring of C_4 and so $\chi(C_4)\leq 2$. Since it is not possible to color the graph with 1 color, we have $\chi(C_4)=2$. With respect to the domination number, we can select any vertex and it dominates itself and its neighbors but not the fourth vertex. Therefore, $\gamma(C_4) = 2$. Finally, using the same third graph in Figure 1, each vertex dominates a color class and the dominator chromatic number is 2. The graph C_4 is also the case $n = 2$ of the complete bipartite graph $K_{n,n}$. The following theorem generalizes this result.

Theorem 3. Let G be a graph and $a,b \geq 2$. Then $\chi(G) = \gamma(G) = \chi_d(G) = 2$ if and only if $G \cong K_{a,b}$.

Proof: ($\Rightarrow$) Since $\chi(G) = 2$, we know that G is bipartite and there exist two partite sets of vertices, I and J. We know that G is not a star because $\gamma(G) = 2$ and $|I| \geq 2$ and $|J| \geq 2$. Since $\chi_d(G) = 2$, every vertex in I must dominate J and every vertex in J must dominate I otherwise we need one more color. Therefore, $G \cong K_{a,b}$.

($\Leftarrow$) Suppose $G = K_{I,J}$ with order n > 3. Since G is bipartite, $\chi(G) = 2$. Select vertices $x \in I, y \in J$. Since G is complete bipartite, x dominates all the vertices in J and y dominates all the vertices in I, so $\gamma(G) \leq 2$. By choosing only one vertex, one vertex set will not be dominated and it follows that $\gamma(G) = 2$. Assign to the vertices in I the color blue and to the ones in J the color red. Because G is a complete bipartite graph, every vertex in either partite set is adjacent to all the vertices in the other partite set, so each vertex dominates a color class and $\chi_d(G) \leq 2$. Since at least two colors are needed, $\chi_d(G) = 2$. □

Theorem 3 tells us that the complete bipartite graph is only case where $\chi(G) = \gamma(G) = \chi_d(G) = 2$. The next step is to explore whether or not a graph exists with all three parameters equal to k for all $k > 2$. We begin by looking at the case $k = 3$. We will now show that the graph J_1, from Chapter II, meets this requirement.

Proposition 7. The graph J_1 in Figure 6 satisfies $\gamma(J_1) = \chi(J_1) = \chi_d(J_1) = 3$.

Proof: Proposition 5 establishes $\gamma(J_1) = \chi_d(J_1) = 3$, so we must show J_1 is 3-colorable. Since the graph contains an odd cycle (*c, d, e, f, g*), $\chi(J_1) \geq 3$. On the other hand, Figure 4 shows a 3-coloring of J_1. This 3-coloring of J_1 implies $\chi(J_1) \leq 3$. Therefore, $\chi(J_1) = 3$. □

The graph J_1 establishes the existence of a graph where all three parameters are equal to three, but we cannot easily generalize the method of construction. The next step is to determine whether or not there exists an algorithm to construct a graph G that satisfies $\chi(G) = \gamma(G) = \chi_d(G) \geq 4$. A technique developed by Jan Mycielski, which increases the chromatic number of a graph without introducing triangles, proved useful [11]. It involves the use of *shadow graphs* [1].

A shadow graph is constructed by adding a vertex, v', known as a *shadow vertex*, for each existing vertex, v, in the current graph. The shadow vertex v' is then adjacent to the neighbors of v. A vertex in G and its shadow vertex are not adjacent in the shadow graph and no two shadow vertices are adjacent. For example, in Figure 5 we start with K_3. The vertex set $\{a', b', c'\}$ is the set of shadow vertices of the graph. Vertex a' is adjacent to b and c, b' is adjacent to a and c, and c' is adjacent to a and b. The second graph in Figure 9 shows the construction of the shadow graph of K_3.

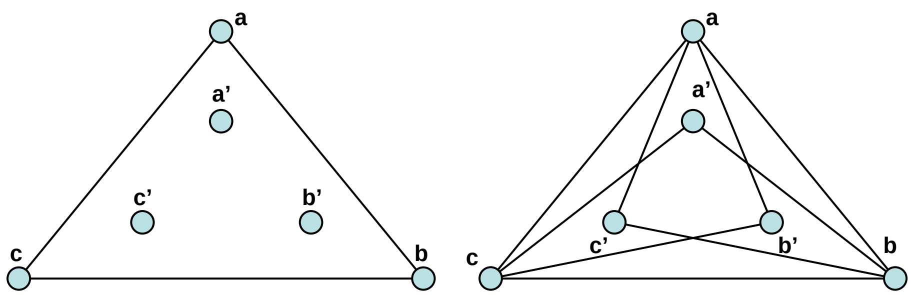

Figure 9. The Construction of the Shadow Graph of K_3

Mycielski's construction uses the shadow graph, but introduces an additional vertex [11]. This vertex is adjacent to all the shadow vertices in the graph, increasing the chromatic number of the graph without introducing triangles. With respect to the dominator chromatic number, the problem with introducing triangles into a graph lies in the structure of the triangle: $\chi_d(K_3) = \chi(K_3) = 3$ but $\gamma(K_3) = 1$. By introducing a triangle into a graph, it could potentially increase a graph's chromatic and dominator chromatic number more than its domination number.

As previously established in Theorem 3, C_4 represents the simplest case of a graph that has all three parameters equal to two. Mycielski's construction is intended to increase the chromatic number of a graph, but it will also have a desired impact on domination and the dominator chromatic number. Figure 10 shows the graph, J_2, obtained by applying Mycielski's construction to C_4.

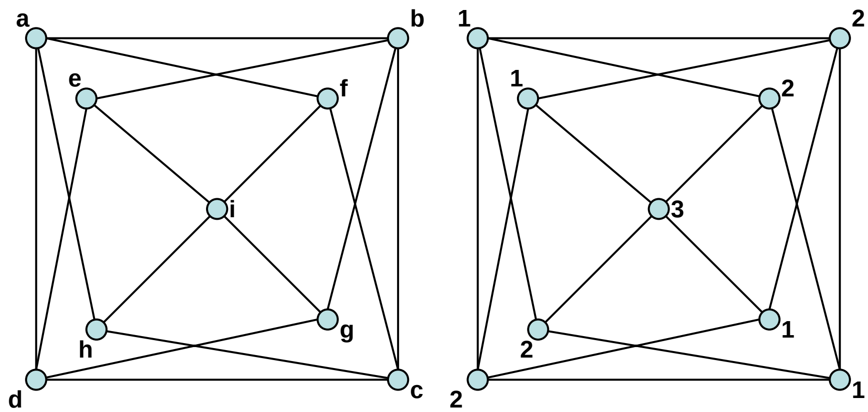

Figure 10. The Labeled Graph J_2 and its Proper Dominator Coloring

Proposition 8. The graph J_2 in Figure 10 satisfies $\gamma(J_2) = \chi(J_2) = \chi_d(J_2) = 3$.

Proof: First, graph J_2 is 3-colorable. It contains an odd cycle *(a, f, i, g, d)* making its chromatic number greater than or equal to three. The second graph in Figure 10 shows a 3-coloring of J_2, establishing $\chi(J_2) \leq 3$. Therefore, $\chi(J_2) = 3$.

To show $\gamma(J_2) = 3$, first observe that the vertices *a*, *c*, and *i* constitute a dominating set establishing $\gamma(J_2) \leq 3$. Because of the symmetry of the graph, without loss of generality select one of the vertices on the outer 4-cycle, say *a*. Vertex *a* dominates all the vertices except *c*, *e*, *g*, and *i*. Since the vertices *c*, *e*, *g*, and *i* do not share a common neighbor, the dominating set of J_2 cannot have just two vertices, thus $\gamma(J_2) \geq 3$. Therefore, $\gamma(J_2) = 3$.

In Figure 10, the vertices *e, f, g, h,* and *i* all dominate the color class 3. The vertices *a* and *c* dominate the color class 2 and the vertices *b* and *d* dominate the color class 1. This dominator coloring establishes $\chi_d(J_2) \leq 3$. From Proposition 1, we know that $\chi_d(J_2) \geq \gamma(J_2)$ which implies $\chi_d(J_2) \geq 3$. Therefore, $\chi_d(J_2) = 3$. □

And so Mycielski's construction, when applied to C_4, produces another graph, J_2, where all three parameters are equal to three. Moreover, we obtained J_2 using a graph from the only class of graphs where $\gamma(G)=\chi(G)=\chi_d(G)=2$. The next step is to determine whether the shadow graph of J_2, SJ_2, results in a graph with all three parameters equal to four. After applying Mycielski's construction to J_2, we find that $\gamma(SJ_2)=\chi(SJ_2)=4$, but $\chi_d(SJ_2)=5$. We applied the same construction to graphs other than C_4 and did not achieve the desired result. Although this technique does not provide a general algorithm for constructing the graphs where $\gamma(G)=\chi(G)=\chi_d(G)=k$ for all $k>3$, it does provide us a second graph that satisfies the criteria where all three parameters are equal.

C. LARGE CASES

Our next step is to find a class of graphs where $\gamma(G)=\chi(G)=\chi_d(G)=k$ for $k>3$. From our previous analysis, we know $\chi(K_n)=\chi_d(K_n)$ and, by Theorem 3, a complete bipartite graph, $K_{a,b}$, satisfies $\gamma(K_{a,b})=\chi(K_{a,b})=\chi_d(K_{a,b})=2$. Let R_α, where $\alpha\geq 2$, be the graph obtained in the following manner. Let α represent the number of disjoint copies of $K_{3,3}$ in R_α. Let U_i be one bipartite set of $K_{3,3}$ composed of three vertices in the ith copy and let W_i be the other bipartite set in the ith copy, where $1\leq i\leq\alpha$. Let $v_{2i-1}\in U_i$ and $v_{2i}\in W_i$. Define the set $V=\{v_1,v_2,...,v_{2\alpha}\}$ and add edges v_iv_j $(1\leq i,j\leq\alpha)$ to construct the complete graph $K_{2\alpha}$ on V. Figure 11 shows the graphs R_2 and R_3. As we show next, the class of graphs $\{R_\alpha | \alpha\geq 2\}$ satisfies $\gamma(R_\alpha)=\chi(R_\alpha)=\chi_d(R_\alpha)=2\alpha$.

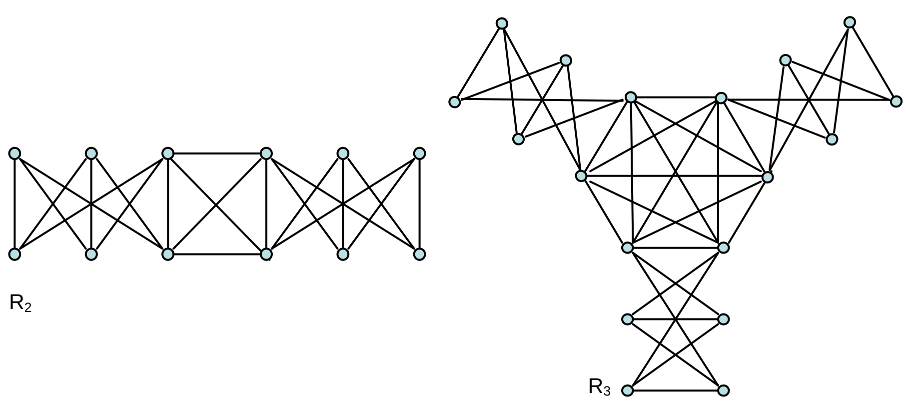

Figure 11. The Graphs R_2 and R_3

Theorem 4. For $\alpha \geq 2$, there is a class of graphs, $\{R_\alpha\}$, such that $\gamma(R_\alpha) = \chi(R_\alpha) = \chi_d(R_\alpha) = 2\alpha$.

Proof: Since $K_{2\alpha}$ is a subgraph of R_α, $\chi(R_\alpha) \geq 2\alpha$. Assign the color *2i-1* to vertices in U_i, and the color *2i* to the vertices in W_i. The result is a proper coloring of *G*, establishing $\chi(G) \leq 2\alpha$. Therefore, $\chi(R_\alpha) = 2\alpha$.

To determine the domination number, first choose $x \in U_i - \{v_{2i-1}\}$. Note that only elements of $W_i \cup \{x\}$ will dominate it. There remains a vertex $y \in U_i - \{v_{2i-1}, x\}$ that needs to be dominated. Since *y* is not adjacent to any element in U_i, it follows that a vertex from W_i is required in the dominating set. It follows that for each copy of $K_{3,3}$, two vertices are needed in the dominating set and $\gamma(R_\alpha) \geq 2\alpha$. Let *S* = *V*. For $1 \leq i \leq \alpha$, v_{2i-1} dominates the vertices in W_i and v_{2i} dominates the vertices in U_i. It follows that $\gamma(R_\alpha) \leq 2\alpha$, therefore $\gamma(R_\alpha) = 2\alpha$.

For $\chi_d(R_\alpha)$, Proposition 1 establishes $\chi_d(R_\alpha) \geq 2\alpha$. Refer to the proper coloring of R_α described above. Since U_i and W_i are bipartite sets in $K_{3,3}$, each vertex in U_i dominates the color class *2i* and each vertex in W_i dominate the color class *2i-1*, where $1 \leq i \leq \alpha$. As a result, $\chi_d(R_\alpha) \leq 2\alpha$ and $\chi_d(R_\alpha) = 2\alpha$. □

Theorem 4 establishes the existence of a class of graphs $\{R_\alpha\}$ where all three parameters are equal, but only for even values greater than or equal to four. The graphs J_1 and J_2 have all parameters equal to three, so we are concerned with finding a class of graphs that satisfies $\gamma(G) = \chi(G) = \chi_d(G) = 2k+1$ for $k \geq 2$. After attempting several constructions, we propose the following conjecture.

Conjecture 1. There is no graph, *G*, that satisfies $\gamma(G) = \chi(G) = \chi_d(G) = 2k+1$ for $k \geq 2$.

IV. TOPICS FOR FURTHER RESEARCH

A. APPLICATIONS

As a relatively new development, there are no specific applications to date for dominator coloring. Because it combines two topics, coloring and domination, that currently have applications to networks, the initial focus is on finding an application in that area. It is possible that applications exist in other areas.

B. OPEN QUESTIONS

There are several interesting questions that are still open for dominator colorings. With respect to this paper, the most immediate is proving or disproving Conjecture 1. If it is proved true, then we will know that there are some limitations on constructing graphs with these three parameters. If disproved, then we know all the triples are realizable.

Below are some possible open questions.

1. We found two graphs, J_1 and J_2, that satisfy $\gamma(G) = \chi(G) = \chi_d(G) = 3$. Are there other graphs that satisfy this condition? If so, can such a graph be used to construct a class of graphs that will disprove Conjecture 1?

2. Other than the class $\{R_\alpha\}$, are there graphs or graph classes that satisfy $\gamma(G) = \chi(G) = \chi_d(G) = 2\alpha$?

3. From a computer-assisted exhaustive proof, we know that the dominator chromatic number of the Petersen Graph is five. Is it possible to show $\chi_d(\text{Petersen}) = 5$ without the aid of a computer?

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