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FUNDAMENTAL CIRCULATION FUNCTIONS FOR DETERMINATION OF OPEN BOUNDARY CONDITIONS

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1 Introduction

One of the difficult problems in shallow water modeling is the uncertainty of the open boundary condition (OBC). At open boundaries where the numerical grid ends, the fluid motion should be unrestricted. Ideal open boundaries are transparent to motions. Two approaches, local-type and inverse-type are available for determining OBC. The local-type approach determines the OBC from the solution of the governing equations near the boundary. The problem now becomes selection from a set of *ad hoc* OBCs. Since any *ad hoc* OBC will introduce inaccuracies into a numerical solution (Chapman, 1985). Without any *ad hoc* OBC, the inverse-type approach, such as adjoint method (Seiler, 1993) and Jacobian matrix method (Chu et al., 1997), can determine the OBC from a "best" fit between model solutions and interior observations. The advantage of using the inverse-type approach is the well posedness and the use of observational data. The disadvantages that may restrict its use are: problems of stable integration if the data contain noises (Jacobian matrix method), requirement of large amounts of computer time and memory, ocean-model dependency (adjoint method), and difficulty in deriving the adjoint equation when the model contains rapidly changing processes, such as ocean mixed layer dynamics.

With the Boussinesq approximation, a three-dimensional velocity vector field can be represented by two scalar functions (Eremeev et al. 1992). The establishment of velocity data for an ocean domain from limited number of observational data with noise filtration becomes the determination of the spectral coefficients for the basis functions (Chu et al., 1999a,b). Along this path, I propose two-step determination of unknown open boundary conditions of the Boussinesq flow in coastal modeling: (a) pattern, and (b) magnitude. This is done through decomposing the horizontal components (u, v) into rotational and divergence parts, and representing the two parts by streamfunction ψ and velocity potential ϕ. The two functions (ψ, ϕ), satisfying the Poisson equation, can be expanded into two series of eigenfunctions $\{\Psi_n\}$ and $\{\Phi_n\}$. For an ocean basin (or global ocean), the two series of eigenfunctions $\{\Psi_n\}$ and $\{\Phi_n\}$ are pre-determined. Thus, we call them the fundamental circulation functions (FCFs), which determine the flow pattern at the open boundary. The magnitude of the velocity at the open boundary is determined at each time step.

2 Flow Decomposition

The continuity equation of the Boussinesq flow is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1}$$

We may decompose the horizontal velocity (u, v) into rotational and divergence parts,

$$u = -\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \tag{2}$$

where ψ, and ϕ are called the streamfunction, and velocity potential, respectively. The two scalar functions (ψ, ϕ) satisfy the following Poisson equations,

$$\nabla^2 \psi = \zeta, \quad \nabla^2 \phi = D. \tag{3}$$

Here $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, is the two-dimensional Laplacian operator; $\zeta \equiv \partial v/\partial x - \partial u/\partial y$, is the vertical relative vorticity; $D \equiv \partial u/\partial x + \partial v/\partial y$, is the horizontal divergence.

3 Fundamental Circulation Functions (FCFs)

The advantage of the decomposition is the existence of two series $\{\Psi_n\}$, $\{\Phi_n\}$ for a given ocean basin (or the global ocean) with rigid boundaries (Γ), which are eigenfunctions of ∇^2 with the Dirichlet boundary condition

$$\nabla^2 \Psi_k = \lambda_k \Psi_k, \quad \Psi_k|_\Gamma = 0 \tag{4}$$

and with the Neumann boundary condition

$$\nabla^2 \Phi_k = \mu_k \Phi_k, \quad \frac{\partial \Phi_k}{\partial n}|_\Gamma = 0 \tag{5}$$

Here, $\{\lambda_k\}$ and $\{\mu_k\}$ are corresponding eigenvalues; 'n' indicates the outward normal direction of the rigid boundary Γ. Notice that the eigenfunctions $\{\Psi_k\}$and $\{\Phi_k\}$ are function of x, y, z, and totally determined by the shape of the rigid boundaries of the ocean basin (or the global ocean). We may call $\{\Psi_k, \Phi_k\}$the fundamental circulation functions (FCFs).

The two scalar functions (ζ, D) can be expanded into two series of FCFs,

$$\zeta = \sum_k \zeta_k \Psi_k, \quad D = \sum_k D_k \Phi_k \tag{6}$$

where the spectral coefficients for (ζ, D) are given by

$$\zeta_k = \int\int \zeta \Psi_k dx dy, \quad D_k = \int\int D\Phi_k dx dy. \tag{7}$$

The streamfunction (ψ) and the velocity potential (ϕ) are given by

$$\psi(x, y, z, t) = \sum_k \lambda_k^{-1}(z)\zeta_k(z, t)\Psi_k(x, y, z)$$

$$\phi(x, y, z, t) = \sum_k \mu_k^{-1}(z) D_k(z, t)\Phi_k(x, y, z) \tag{8}$$

4 Velocity Field

Substitution of (8) into (2) leads to the spectral form of the horizontal velocities,

$$u = -\sum_k \lambda_k^{-1}(z)\zeta_k(z, t)\frac{\partial \Psi_k(x, y, z)}{\partial y}$$

$$+\sum_k \mu_k^{-1}(z) D_k(z, t)\frac{\partial \Phi_k(x, y, z)}{\partial x} \tag{9}$$

$$v = \sum_k \lambda_k^{-1}(z)\zeta_k(z, t)\frac{\partial \Psi_k(x, y, z)}{\partial x}$$

$$+\sum_k \mu_k^{-1}(z) D_k(z, t)\frac{\partial \Phi_k(x, y, z)}{\partial y} \tag{10}$$

The velocity field is decomposed into spectral form. There are two parts for each component: dynamically independent and dependent parts. The dynamically independent part includes $\Psi_k(x, y, z)$, $\Phi_k(x, y, z)$, $\lambda_k(z)$, and $\mu_k(z)$; and the dynamically dependent part contains $\zeta_k(z, t)$, and $D_k(z, t)$.

5 Two-Step OBC Determination

For a coastal region with an open boundary (B), we may use the following two steps to determine the unknown OBC: (a) to compute the FCFs for the ocean basin with rigid boundaries (or the global ocean) $\{\Psi_k, \Phi_m\}$ and associated eigenvalues $\{\lambda_k, \mu_m\}$, and (b) to determine the circulation at the open boundary through the determination of the spectral coefficients ζ_k and D_m.

From the initial conditions $u(x, y, z, t_0)$, $v(x, y, z, t_0)$, we may invert the spectral coefficients $\zeta_k(z, t_0)$, and $D_m(z, t_0)$ using (7). With the known $\zeta_k(z, t_0)$, and $D_m(z, t_0)$, use of (9) and (10) leads to the determination of (u, v) at the open boundaries, which are treated as the OBC for next time step, $u_B(x, y, z, t_1)$, and $v_B(x, y, z, t_1)$. With this determined OBC, we may integrate the coastal model to obtain the solutions at t_1: $u(x, y, z, t_1)$, $v(x, y, z, t_1)$. The same procedure is used to obtain the OBC at the time step t_2. Thus, we can step-by-step determine the unknown OBCs for the coastal model.

6 Rectangular Ocean Basin

6.1 Fundamental Circulation Functions

Taking a rectangular ocean basin as an example, the FCFs are given by

$$\Psi^{(i,j)} = \sin\frac{i\pi x}{L_x}\sin\frac{j\pi y}{L_y},$$

$$\Phi^{(i,j)} = \cos\frac{i\pi x}{L_x}\cos\frac{j\pi y}{L_y} \tag{11}$$

with the corresponding eigenvalues

$$\lambda^{(i,j)} = \mu^{(i,j)} = -\pi^2(\frac{i^2}{L_x^2} + \frac{j^2}{L_y^2}) \tag{12}$$

,

Here $i, j = 1, 2, ...;$ are integers, L_x and L_y are horizontal sizes of the rectangular. We may rearrange the two-dimensional array $\{\Psi^{(i,j)}, \Phi^{(i,j)}, \lambda^{(i,j)}, \mu^{(i,j)}\}$ into one-dimensional array $\{\Psi_k, \Phi_k, \lambda_k, \mu_k\}$. Figs. 1 and 2 show first few FCFs $\{\Psi_k\}$, and $\{\Phi_k\}$ for the rectangular ocean.

6.2 Example - Princeton Ocean Model (POM)

The method was tested for a flat bay centered at 35°N and bounded by one open and three rigid boundaries (Chu et al, 1999b). This bay expands 1000 km in both the north-south and east-west directions. The northern, southern, and western boundaries are rigid, and the eastern boundary is open. The Cartesian coordinate system is chosen with the origin at the southwest corner. The x-axis points towards the east, and the y-axis towards the north (Fig.3a). The circulation in the bay is modeled with the Princeton Ocean Model (POM) developed by Blumberg and Mellor (1987). We use the 2-D version of POM to illustrate the usefulness of this two-step method for determining the open boundary conditions.

The area depicted in Fig.3a is called **Domain A**, where the boundary conditions are known at the three rigid boundaries (northern, southern, and western), and unknown at the eastern boundary. The eastern boundary of Domain A is connected to a mirror image of Domain A (about $x = 1000$ km) forming a closed rectangular domain (Fig.3b), called **Domain B**. The POM model was integrated from the following initial conditions

$$(u, v, w) = 0,$$

$$T = 283\left[1 + \exp(\frac{z}{1000})\right] \quad [^{\circ}K], \qquad S = 35 \quad [psu] \tag{13}$$

under no surface heat or salinity fluxes and zonal surface pseudo wind stress varying with latitude

$$\frac{\tau_x}{\rho_0} = -10^{-4}\cos\frac{\pi y}{L_y}, \qquad [\mathrm{m}^2/\mathrm{s}^2] \tag{14}$$

The time step is chosen as 5 minutes. The horizontal resolution is 50-km. Bottom stress is parameterized by the quadratic drag relation. Horizontal kinematic viscosity is set to be zero. We integrate POM over Domain B with four rigid boundaries (known boundary conditions) for 150 days from the initial conditions (10) and surface forcing (11) and take the solution in the Domain B as the reference dataset for the error estimation.

Using the velocity data inside Domain A, we use (7) to obtain the spectral coefficients, and then use (9) and (10) to obtain the velocity filed at the open boundary. The inverted velocity at the open boundary is evaluated by the velocity field for the Domain B. Chu et al. (1999b) show a good agreement between the inverted and "observed" velocities at the open boundary.

7 Global Ocean FCFs

We plan to compute the FCFs for the global ocean. The lateral boundary geometry for any depth is obtained from the Naval Oceanographic Office's Digital Bathymetry Data Base with $5' \times 5'$ resolution (DBDB5). The calculated global FCFs $\Psi_k(x, y, z)$, $\Phi_k(x, y, z)$ with associated eigenvalues $\lambda_k(z)$, and $\mu_k(z)$ are useful not only in determining the open boundary conditions for coastal modeling, but also in analyzing the spectral components for global modeling.

8 Conclusions

(1) The proposed two-step method provides a useful scheme to obtain unknown open boundary values. The fundamental circulation functions (FCFs) are predetermined, and the spectral coefficients are determined step-by-step by inversion.

(2) The global FCFs are useful not only in determining the open boundary conditions for coastal modeling, but also in analyzing the spectral components for global modeling.

9 ACKNOWLEDGMENTS

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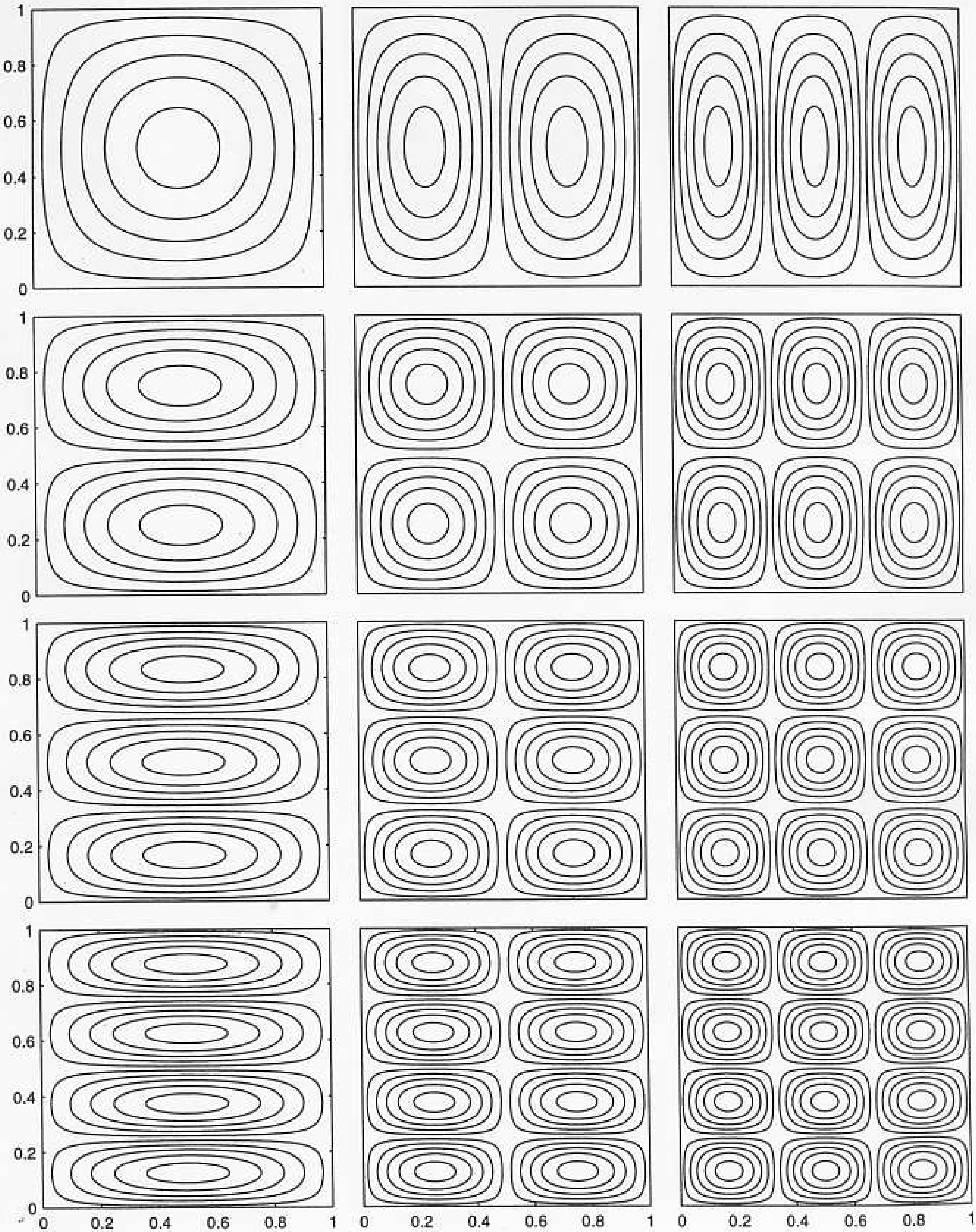

Figure 1 FCFs of streamfunction

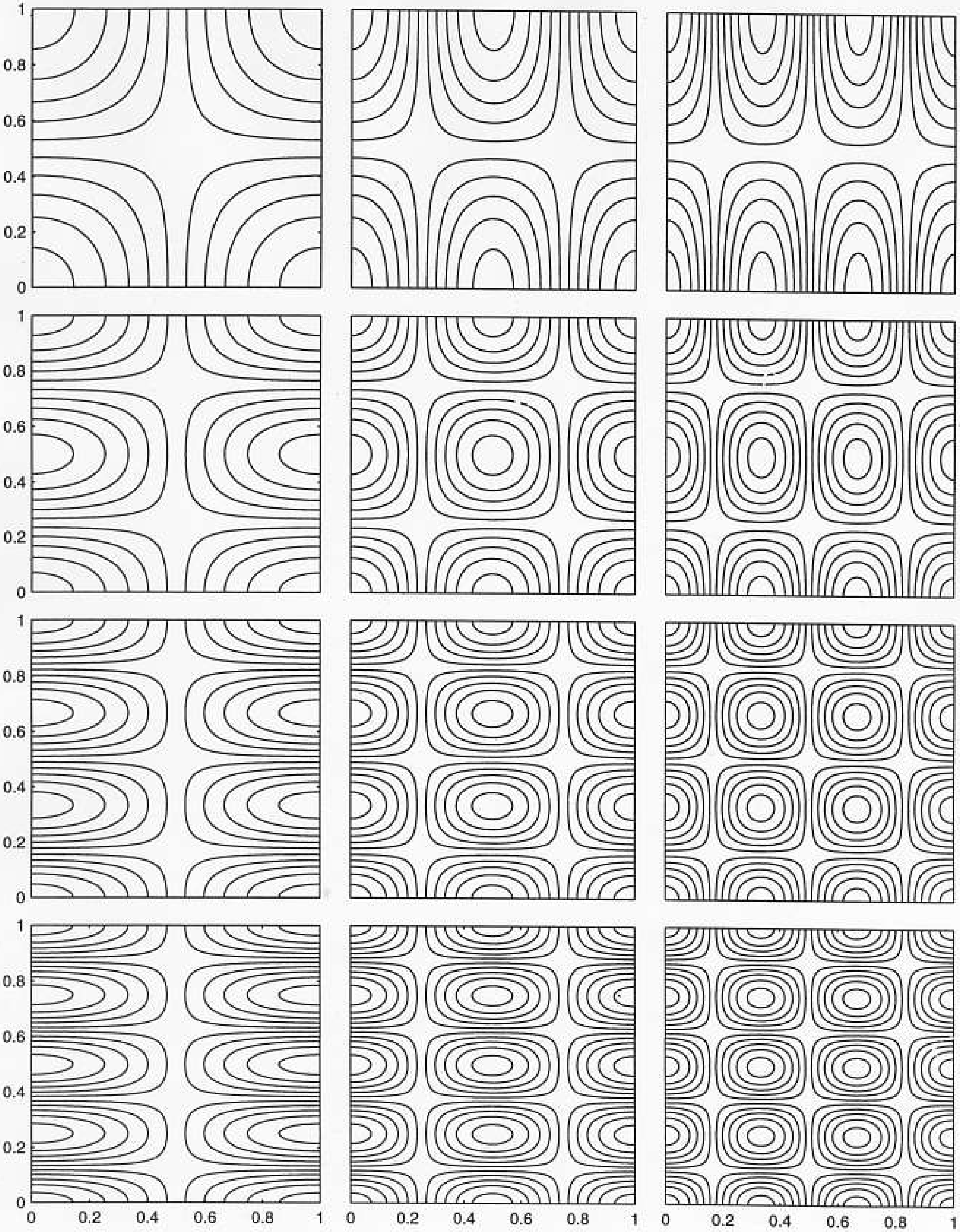

Figure 2 FCFs of velocity potential.

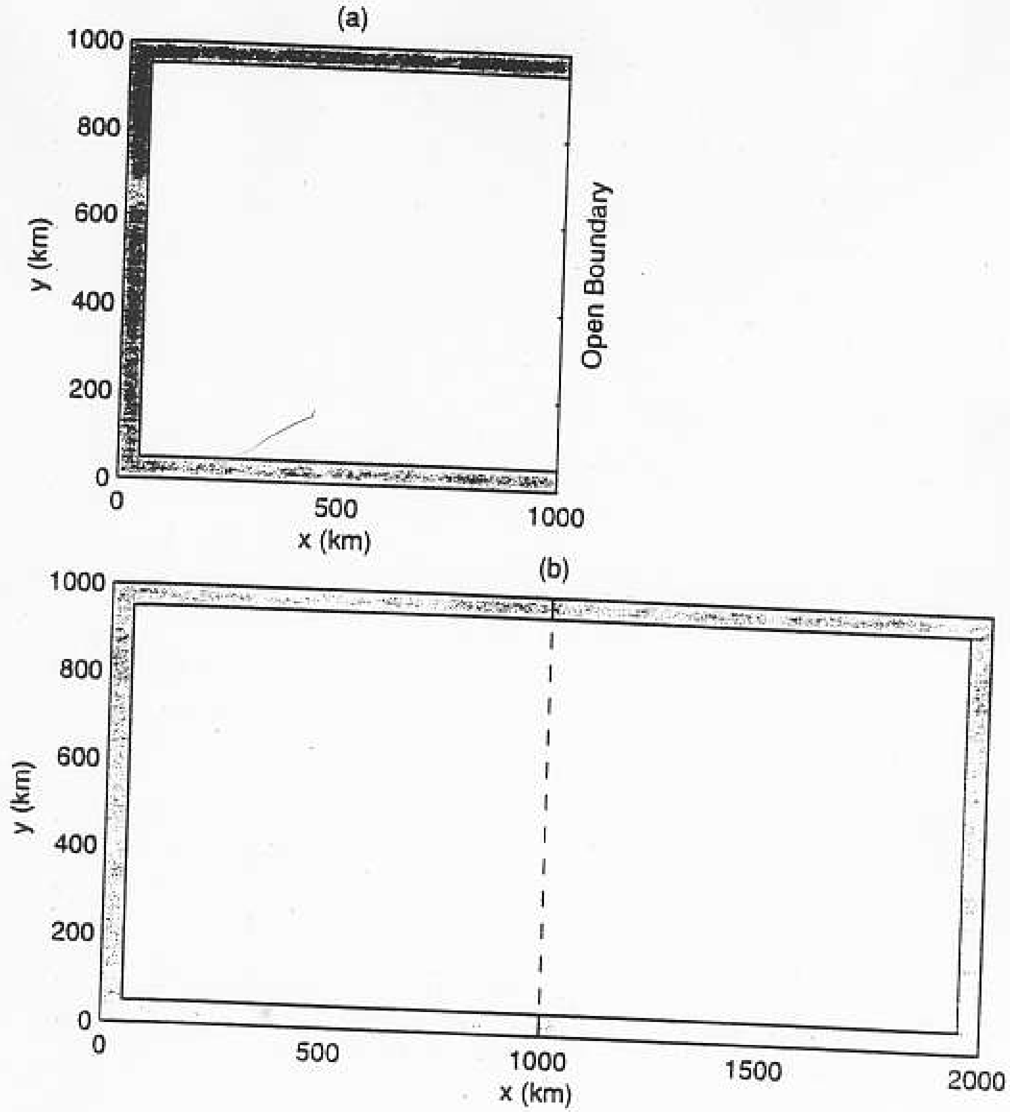
(a)
1000
800
600
400
200
0
y (km)
0
500
1000
x (km)
Open Boundary
(b)
1000
800
600
400
200
0
y (km)
0
500
1000
1500
2000
x (km)

Figure 3