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Lanchester Models With Discontinuities: An Application to Networked Forces

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ABSTRACT

We consider an extension of Lanchester's models of conflict by studying the effect of instantaneous, global reductions in effectiveness to include changes in the underlying process. We apply this extension to a hypothetical scenario based on the battle of Iwo Jima and discuss implications for networked forces.

INTRODUCTION

Deterministic equations of armed conflict, first introduced by Lanchester in 1916, are a simple yet useful analytical tool for analyzing conflict between two opposing forces (Lanchester 1916). Lanchester's model of armed conflict is studied by military services throughout the world. The aspects of the Lanchester's equations that are attractive for analysis include:

- *Simplicity*: Lanchester's model distills combat into a state variable and a single effectiveness parameter.
- *Transparency*: Lanchester's model is easily explained and readily understood, even by those with no background in mathematics.
- *Easy implementation*: Lanchester's model has compact solutions for the terminal conditions of a battle. If a trace of the battle is desired, various methods may be used to obtain a time-dependent solution. Nonlinear extensions, such as reinforcements, may be analyzed adequately in most applications with difference equations.

Lanchester Models

Generalized conflict model. In their most general form, differential equation models of armed conflict involving two forces, which we call Blue and Red are:

$$\frac{dR}{dt} = -\gamma_B$$
$$\frac{dB}{dt} = -\gamma_R, \qquad (1)$$

where

- B, R are state variables describing the number of combatants currently fighting for Blue and Red, respectively.
- γ_B, γ_R are functions, measuring the expected number of fighters on the opposite side attired by Blue and Red for an infinitesimal time period, respectively.

We review some choices for $\gamma_\diamond$, which lead to *Lanchester models*.

Aimed fire. If we let $\gamma_B = \beta_B B, \gamma_R = \beta_R R$, Equations (1) become:

$$\frac{dR}{dt} = -\beta_B B$$
$$\frac{dB}{dt} = -\beta_R R. \qquad (2)$$

These are the equations governing combat using *aimed fire*. A useful technique is to integrate Equations (2) to yield the *state equation*:

$$\frac{B_0^2 - B_t^2}{R_0^2 - R_t^2} = \frac{\beta_R}{\beta_B}, \qquad (3)$$

where B_0, R_0 represent the initial values of the Blue and Red forces, respectively. This equation says that the relationship between the square of the losses in any fixed time period is equal to the inverse ratio of the effectiveness parameters. It is for this reason the aimed fire law is commonly referred to as *Lanchester's Square Law*. As the equations model conflict between military forces, we restrict our analysis to the nonnegative real numbers.

Equations (3) lead to the *victory condition* for blue; let B_f, R_f denote the *stopping value* of B, R respectively. The stopping value includes the idea that cases in which forces fight "to the finish," such as the Greeks at Thermopile, US Army at Custer's Last Stand, and the Japanese army at Iwo Jima, are memorable because they are rare; most forces have "breakpoints" at which they will cease fighting and either withdraw or surrender. If:

$$\frac{B_0^2 - B_f^2}{R_0^2 - R_f^2} > \frac{\beta_R}{\beta_B}, \qquad (4)$$

Blue will have achieved victory in the sense that he will have inflicted sufficient casualties to reach Red's "breakpoint" while sustaining no more than his allowable casualties.

Finally, Equations (4) may be solved, see Washburn and Kress (2009) in closed form as functions of time, t:

$$B(t) = \frac{1}{2}\left(\left[B_0 - R_0\sqrt{\frac{\beta_B}{\beta_R}}\right]e^{t\sqrt{\beta_B\beta_R}}\right) + \frac{1}{2}\left(\left[B_0 + R_0\sqrt{\frac{\beta_B}{\beta_R}}\right]e^{-t\sqrt{\beta_B\beta_R}}\right)$$
$$R(t) = \frac{1}{2}\left(\left[R_0 - B_0\sqrt{\frac{\beta_R}{\beta_B}}\right]e^{t\sqrt{\beta_B\beta_R}}\right) + \frac{1}{2}\left(\left[R_0 + B_0\sqrt{\frac{\beta_R}{\beta_B}}\right]e^{-t\sqrt{\beta_B\beta_R}}\right) \quad (5)$$

Area fire. Alternately, we may let $\gamma_B = \beta_B BR$, $\gamma_R = \beta_R BR$,

$$\frac{dR}{dt} = -\beta_B BR$$
$$\frac{dB}{dt} = -\beta_R BR. \quad (6)$$

These equations govern combat using *area fire*. They assume that each soldier on the Red and Blue sides is shooting an ''area'' weapon, and that the effectiveness of the weapon is dependent both on the rate of fire (number of shooters) and target density, therefore the term BR appears in both equations

Equations (6) may be similarly integrated to yield a state equation:

$$\frac{B_0 - B_t}{R_0 - R_t} = \frac{\beta_R}{\beta_B} \quad (7)$$

Contrasting with (3), these equations are commonly referred to as Lanchester's *linear law*. The accompanying victory condition for blue is:

$$\frac{B_0 - B_f}{R_0 - R_f} > \frac{\beta_R}{\beta_B} \quad (8)$$

Mixed effects. Deitchman (1962) proposed a *mixed* model of combat to describe an 'ambush' case where a surprised defender uses area fire against an attacker using aimed fire. Here $\gamma_B = \beta_B BR$, $\gamma_R = \beta_R R$, yielding

$$\frac{dR}{dt} = -\beta_B BR$$
$$\frac{dB}{dt} = -\beta_R R. \quad (9)$$

Integrating, we may find the state equation,

$$\frac{\beta_B}{2}\left[B_0^2 - B_t^2\right] = \beta_R[R_0 - R_t]. \quad (10)$$

Are Lanchester Models the Correct Approach for Networked Combat?

Before modifying Lanchester Models to consider cases motivated by network effects, we pause to ask if this is the right thing to be doing in the first place. Lanchester square models are supported by Engel (1954) followed by a Samz (1971) analysis of the Battle of Iwo Jima. Other models have been used to fit the same data, notably the *Log-Linear* model (see Hartley 2001). Hartley concludes that ''the fits (of the Square Law vs. Log-Linear model) are comparable, but neither fit is exceptionally good.''

By using simple Lanchester models to analyze a complex situation like combat, now made more complex with the introduction of confounding issues such as a network, we seek to create a model that is not necessarily predictive, but rather useful for evaluating alternatives. The contributions of our effort are:

- To provide an example of how network effects may be incorporated into deterministic models of combat.
- To provide a fast, transparent model to use for verifying complex simulation models.

In short, we seek to incorporate ideas about the quality and resilience of the network with ideas about the size and effectiveness of the fighting force in order to put network capabilities in the same space as tactical capabilities.

LANCHESTER MODIFIED FOR DISCONTINUOUS SHOCKS

We explore modifications to understand the effect that a single, discontinuous shock would have to one side's (Blue's) effectiveness. This shock is a global reduction in effectiveness, and although it may come from several different sources, the scenario we have in mind is damage to the network in a highly connected force. While Blue's network is in operation, he has effectiveness function γ_{BN}. Blue's single network is subject to a single, irrevocable ''shock'' (degradation to include failure) at time t^*.

Adjustments to Lanchester models have been proposed before, notably by Bracken (2001), who introduces d and $1/d$ parameters to indicate the advantage (disadvantage) to the attacker (defender) in the Ardennes Campaign, where the roles of attacker and defender changed during the engagement.

Approach

We take advantage of the property that our proposed choices for $\gamma_\diamond$ lead to time-independent solutions; therefore we may solve for B_* or R_*, the force level on either side associated with the time of network loss. Once any of the three * parameters is known, the others may be recovered directly. In all cases our model has the following form:

$$\frac{dR}{dt} = -\gamma_{BN} \quad t < t^*$$
$$\frac{dR}{dt} = -\gamma_B \quad t \geq t^*$$
$$\frac{dB}{dt} = -\gamma_R. \quad \forall_t$$

In this context, B_*(R_*) is the Blue (Red) force level when the instantaneous transition from γ_{BN}, representing a *networked* force, occurs to γ_B, representing a *nonnetworked* force, and Blue still meeting his victory condition. We assume in all cases that the following inequality holds:

$$\gamma_{BN}(t \mid B) \geq \gamma_B(t \mid B),$$

meaning that Blue is more effective before the shock than after.

By definition, B_* is restricted to be between B_0 and B_f, however we will see that it is possible to solve for values outside of this range. We understand that values of $B_* > B_0$ mean that Blue would have met its victory condition without the network, i.e., it could fail in the first instant of the battle. Similarly, values of $B_* < B_f$ or undefined indicate that Blue will fail to meet its victory condition with an ''invulnerable'' network. An algebraic proof for the Aimed Fire → Aimed Fire case is presented in the Appendix.

We wish to understand how long Blue's network must stay in operation to achieve a commander's desired outcome. To this end we treat t^* deterministically as the latest time that the network may be lost and still avoid Blue defeat, in the sense of (4); future work will treat it more properly as a random variable. Suppose that a commander knows the initial forces, B_0, R_0 and either knows or assumes breakpoints, B_f, R_f. He also knows $\gamma_{BN}, \gamma_B, \gamma_R$. He does not know how long his network needs to remain in operation to achieve these aims; solving for t^* informs this question.

Shock Action

There are two ways that a shock may change Blue's force, assuming that he is using aimed fire initially. A shock may simply change (reduce) the effectiveness of his aimed fire while he continues to employ aimed fire. Alternately, the shock may cause him to switch from aimed fire to area fire, fundamentally changing the dynamics of combat.

We assume in all cases that Red employs Aimed fire, and is invulnerable to a similar shock.

Aimed Fire → Aimed Fire Shocks

In this section, we consider the case where a shock to Blue's network occurs and he continues to use aimed fire. Lanchester's model incorporating network effects in this case is:

$$\frac{dR}{dt} = -\beta_{BN}B \quad t < t^*$$
$$\frac{dR}{dt} = -\beta_B B \quad t \geq t^*$$
$$\frac{dB}{dt} = -\beta_R R. \tag{11}$$

These equations may be considered to be two serial instances of Lanchester's original model, where the state of the combatants, B_{t^*}, R_{t^*} at t^*, the time the network is shocked, become the initial conditions to the nonnetworked battle.

Let B_*, R_* denote the force level on each side at the moment the network fails. We may solve for B_* by applying equation (3),

$$\frac{B_0^2 - B_*^2}{R_0^2 - R_*^2} = \frac{\beta_R}{\beta_{BN}}$$

$$\frac{B_*^2 - B_f^2}{R_*^2 - R_f^2} = \frac{\beta_R}{\beta_B}.$$

Expanding, we may write,

$$\beta_{BN} B_0^2 - \beta_{BN} B_*^2 = \beta_R R_0^2 - \beta_R R_*^2$$

$$\beta_B B_*^2 - \beta_B B_f^2 = \beta_R R_*^2 - \beta_R R_f^2.$$

Summing these equations gives

$$\beta_{BN} B_0^2 - \beta_{BN} B_*^2 + \beta_B B_*^2 - \beta_B B_f^2 = \beta_R R_0^2 - \beta_R R_f^2.$$

The $\beta_R R_*^2$ terms have canceled out, and we may solve for

$$B_* = \sqrt{\frac{\beta_{BN} B_0^2 - \beta_B B_f^2 - \beta_R (R_0^2 - R_f^2)}{\beta_{BN} - \beta_B}}. \qquad (12)$$

Once B_* is determined, equation (5) may be used to solve implicitly for t^*.

Aimed Fire → Area Fire Shocks

In this section, we consider the case where the loss of Blue's network causes him to shift from aimed fire to area fire. This changes not only the parameters, but also the functional form of γ_B. This model is specified as follows:

$$\frac{dR}{dt} = -\beta_{BN} B \quad t < t^*$$
$$\frac{dR}{dt} = -\beta_B BR \quad t \geq t^*$$
$$\frac{dB}{dt} = -\beta_R R. \qquad (13)$$

We take advantage of the serial nature of this conflict, and utilize the state equations (3), (10) and write:

$$\beta_{BN}\left[B_0^2 - B_*^2\right] = \beta_R\left[R_0^2 - R_*^2\right]$$

$$\frac{\beta_B}{2}\left[B_*^2 - B_f^2\right] = \beta_R\left[R_* - R_f\right].$$

Eliminating B_* by subtraction and significant algebraic manipulation yields:

$$R_* = \frac{\beta_{BN}}{\beta_B} + \sqrt{\frac{\beta_{BN}^2}{\beta_B^2} + R_0^2 - \frac{\beta_{BN}}{\beta_R}\left[B_0^2 - B_f^2\right] - 2\frac{\beta_{BN}}{\beta_B} R_f}. \qquad (14)$$

We may now use our solution for R^* to solve for B^* and t^*.

The Network Loss Function

We do not specify the mechanism by which the network is disabled. We sidestep this fundamental issue by treating the time of network loss, t^*, deterministically, as an unknown quantity that may be solved for from other model parameters, or as a known (or presumed) quantity.

Although we do not specify a mathematical form for network loss, we do have some intuition about it. First, if the threat to the network is an asymmetric one—to include cyberattacks but also physical ones like RF Jamming or Electromagnetic Pulse (EMP), the loss of the network may not be tied to a certain phase in hostilities. Secondly, if we know the distribution time of the shock, we may compute the probability of victory for Blue.

Great care should be taken in modeling the time of network loss, t^*.

DERIVATION OF EFFECTIVENESS PARAMETERS, $\beta_\diamond$

Some discussion is required on the derivation of parameters. In military applications, these parameters typically come from data analysis or expert opinion. Expert opinion is adequate if the terminal conditions of the engagement are the focus of analysis because, as seen in equations (4), (7), (10) the effectiveness ratio is the key metric and statements such as ''we expect that Red is 75% as capable as Blue'' have meaning. We present below an alternate, engineering approach to the Lanchester parameters.

Aimed Fire

Consider a force of B Blue troops, each of whom are using aimed fire to engage some number of Red troops. Each individual blue soldier has some firing rate, η, and his weapon has some lethality, $p_{kill|hit}$, and some probability of hit, p_{hit}. We ''roll these up'' into a single parameter β by

$$\beta = \eta \cdot p_{kill|hit} \cdot p_{hit}$$

Now consider a large force and a single ''snapshot'' in time. The expected number of casualties inflicted on the opposing side during some small interval h is:

$$E[k(h)] = \eta\, p_{kill|hit} p_{hit} Bh = \beta Bh \tag{15}$$

Taking limits yields (2).

Network Loss on Aimed Fire

Using Equation (15) as a guide, we may understand how the loss of a network may affect Blue's effectiveness parameter. Loss of the network may lower p_{hit} because of lower quality or quantity of targeting data. It may also reduce η due to supply chain or targeting difficulties.

Degradation Effects on Aimed Fire

As the quality of targeting data decreases, the error associated with targeting increases. We present some analysis to show how the *transition* between aimed and area fire occurs.

Although error distributions are specific to the case being studied, in general, suppose that the impact distribution about an aimpoint X_0, Y_0 is multivariate normal, see Grimmett and Stirzaker (2001). Where the Y variable refers to ''Range Error,'' and the X term refers to ''azimuth error.'' The distribution of points is then:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2(1-\rho)}\left[\left(\frac{x-mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\frac{y-\mu_y}{\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

In the special case where the errors are equal and uncorrelated; $\sigma_X = \sigma_Y = \sigma, \rho = 0$, we say that the errors are *circularly distributed* and

$$\Pr\{R < r\} = 1 - e^{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2}. \tag{16}$$

A common metric for targeting accuracy is the Circular Error Probable (CEP), defined as the radius that encloses 50% of rounds fired. We may convert from CEP to σ by

$$\sigma = \frac{CEP}{\sqrt{\ln 4}} \approx \frac{CEP}{1.177}.$$

Equation (16) gives us some intuition as to how the loss of targeting data actually affects the Lanchester parameters. Figure 1 demonstrates the reduction in effectiveness as CEP increases.

Area Fire

Similarly, we may consider the same force of B troops using area fire. The situation here is slightly more complicated because we have to assume a finite area that the blue force is firing into. Again, each individual soldier has some firing rate η, his weapon has some lethality, $p_{kill|hit}$. A simple but useful model compares a lethal area to the total area of regard. We denote the lethal area per munition as A_L and the total area as A_T. Following the previous development, our equations become:

$$E[k(h)] = \eta\, p_{kill|hit} \frac{A_L}{A_T} BRh$$

Choosing between Aimed *and* Area *fires*. It is possible, likely even, that a Commander, losing a network, needs to decide between employing his now degraded weapons as area fire or aimed fire. It follows immediately from the preceding subsections that if

$$\frac{A_L}{A_R} R > p_{hit},$$

he should prefer area fire.

ANALYZING THE BATTLE OF IWO JIMA WITH A NETWORKED FORCE

We now take the model developed earlier and apply it to an historical scenario with one minor change. We consider the Battle of Iwo Jima (1945), with the Blue (US) side having access to a networked force. The network enables both communication between engaged units, as

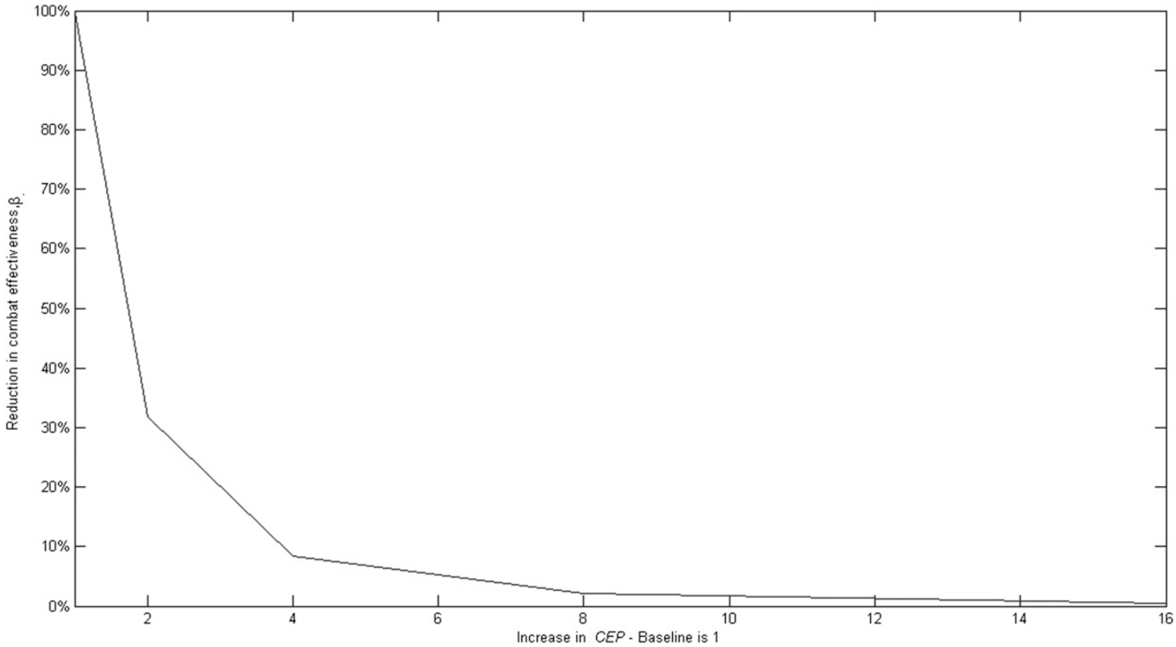

Figure 1. Reduction in β as a function of Circular Error Probable (CEP). As CEP increases, β diminishes super-exponentially.

well as communication with unmanned systems, which we credit the Blue side with having. The Blue network has an (unspecified) vulnerability which will cause it to be taken out of action at time t^*, with no possibility of repair.

Before considering the new model, we review the original battle.

Analysis of the Battle of Iwo Jima (1945)

Engel (1954), and Samz (1971) modify equations (2) to analyze the battle of Iwo Jima. Their model is:

$$\frac{dB}{dt} = P(t) - \beta_R R$$
$$\frac{dR}{dt} = -\beta_B B, \qquad (17)$$

where $P(t)$ is the pulsed arrival of Blue troops ashore. Their parameters are:

$$\beta_B = .0106$$
$$\beta_R = .0544 \quad R_0 = 21{,}500$$
$$P(0) = 54{,}000 \quad P(2) = 6{,}000 \quad P(5) = 13{,}000.$$

Casualties[a] for the Blue (US) side were 20,860, for a casualty rate of 28%. Iwo Jima was a true ''fight to the finish'' for the Japanese, whose casualty rate was 100%. A time trace of this battle is shown in Figure 2.

Iwo Jima, Networked (Present Day)

Suppose that a commander was to refight the battle of Iwo Jima with a modern, network enabled force. This commander believes his network invulnerable, and that his network negates the adversary's advantage for being ''dug in.'' Should he lose his networked advantage, his effectiveness reverts to the historical value. Furthermore, he determines that his tolerance for casualties is 10%. From Equation (17), he estimates his total landing force should be approximately 56,000 troops, which we distribute similarly to the 1945 commander. The parameters are:

$$\beta_R = .0544 \quad R_0 = 21{,}500$$
$$\beta_{BN} = .0544 \quad \beta_B = .0106$$
$$P(0) = 40{,}000 \quad P(2) = 5{,}000 \quad P(5) = 11{,}000$$

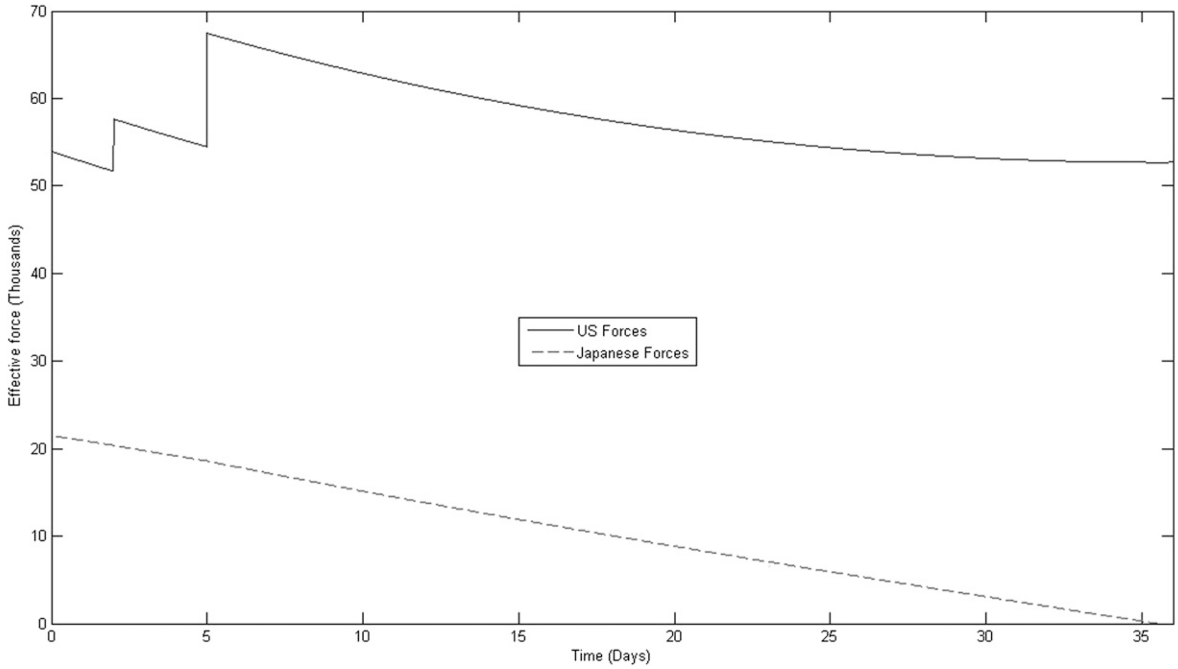

Figure 2. Time trace of the Battle of Iwo Jima, based on Engel's analysis of the battle. The three spikes in the Blue force level represent the three waves of landings on the Island.

It is important to note that the reinforcement schedule for Iwo Jima and the analyses of the battle are time based, not condition based. Condition-based reinforcement rules may lead to nonmonotone results (see Dewar et al. 1996).

We wish to know what the effect of loss of the network is on the Blue casualty rate. We may do so by analyzing the Blue casualties at the end of the battle as a function of t^* (Figure 3).

For instance, compare the time trace of the battle in Figure 2 with the time trace in Figure 4, which is the battle with $t^* = 3$. In the modified battle, Blue losses are approximately $23,000$ (41%) as opposed to his anticipated losses of 5,600, or 10%. Whereas Blue's network-based planning led him to arrive with a more lethal—and therefore, smaller—force, the unanticipated failure of his network led him to higher casualties than the original battle. At this point, we would have to ask if Blue would continue to fight with such a high proportion of his force lost, or if he would withdraw.

The takeaway for both analysts and decision makers from Figure 1 is that planning for an operation without adequately appreciating the fragility of the network may lead to unexpected—and negative—results. As a final exercise, we wish to explore the relationship between relative numerical strength and time of network loss, t^*. This both shows the flexibility of this approach as well as provides another illustration (see Figure 5).

CONCLUSION

This article is not an answer so much as a place to begin putting analysis to this important issue. We have:

- Demonstrated an analytical approach to incorporating the effects of network degradation in a kinetic battle.
- Shown, using a well-established historical context, that planning for combat using a networked force without planning for failure of the network may lead to disaster.
- Demonstrated how a Service may compare the value of network reliability with increased combat capability.

In this model, we have treated the failure of the network as a fixed event within a deterministic Lanchester model. We have also treated

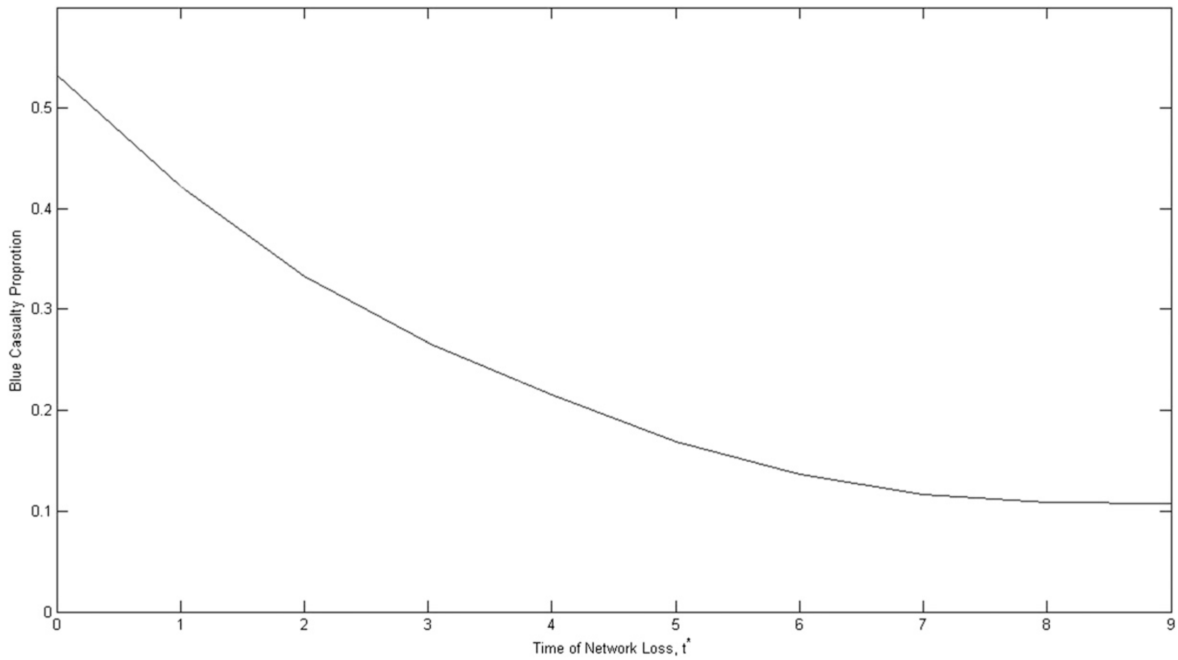

Figure 3. Blue's casualty rate (Red fights to the finish) as a function of the time of network loss, t^* in the modern Iwo Jima scenario, planned for 10% casualties with an invulnerable network. Loss of the network early in the battle has a dramatic effect on Blue's casualty rate.

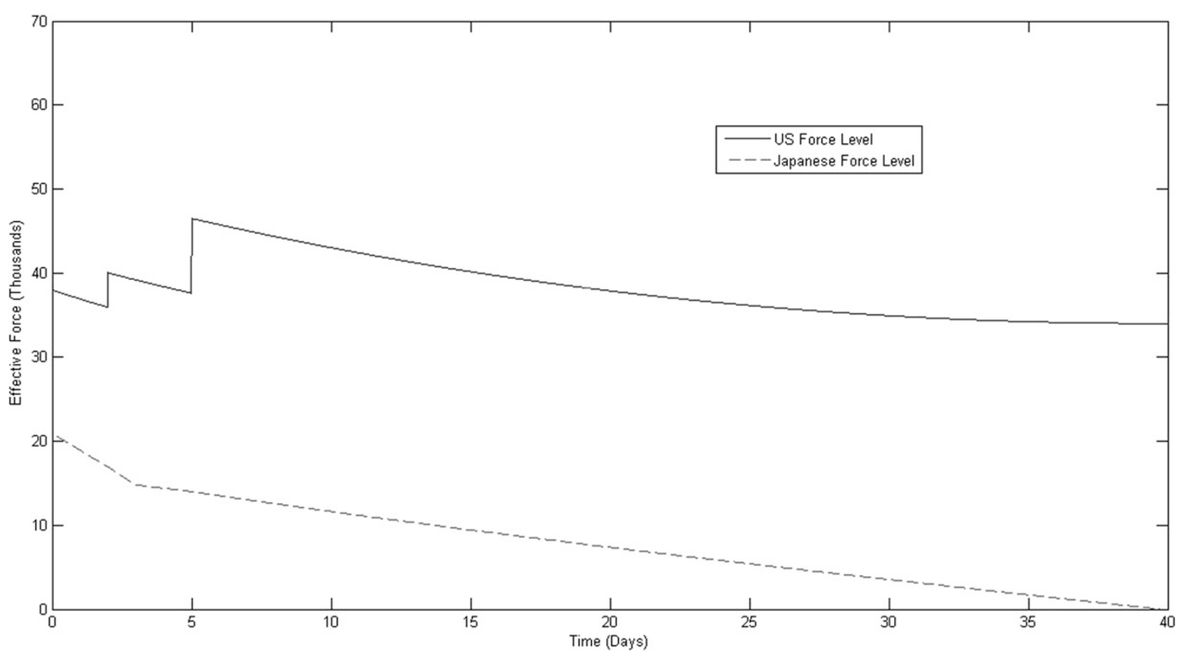

Figure 4. Time trace of modified battle of Iwo Jima with $t^* = 3$\$. The shock effect, changing Blue's effectiveness, is seen as an inflection point on Red's force level.

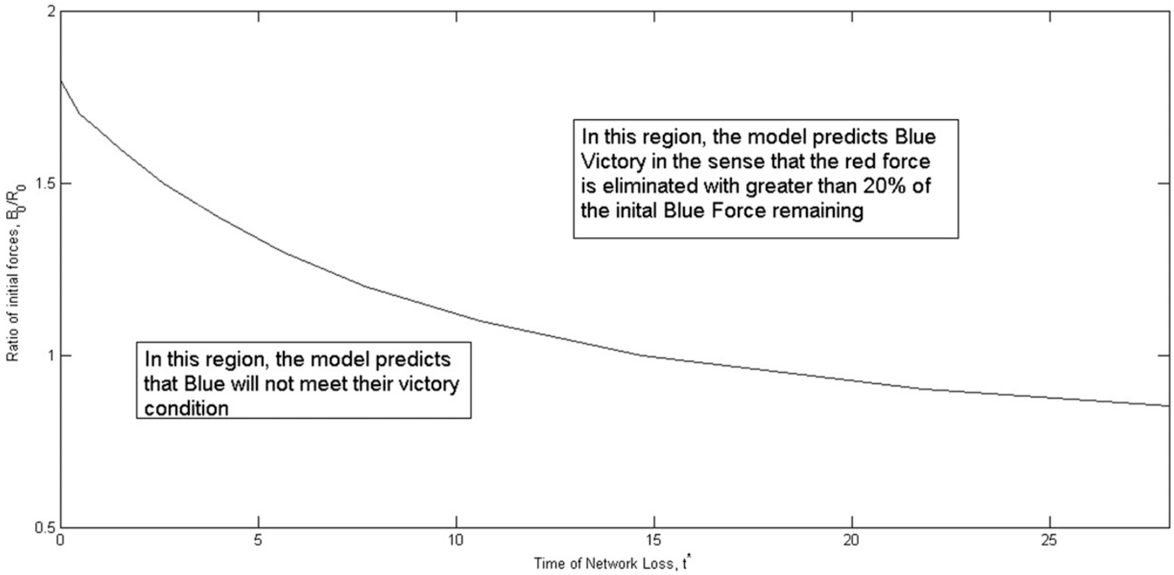

Figure 5. Comparison of initial force ratio and network loss for the case where $R_0 = 10{,}000$, $\beta_R = .03$, $R_f = 1$, $\beta_{BN} = .05$, $\beta_B = .01$, $\beta_f = B_0/5$. The region above the line represents those combinations of B_0 and t^* where Blue is expected to be victorious, in the sense that his losses are less than 20%.

network failure as an independent event from the kinetic battle.

Future work will be to relax both of these constraints, specifically by treating the effectiveness parameters, β_B, β_R as functions instead of constants. A similar idea has been used by Artelli and Deckro (2008) to introduce fatigue into a Lanchester-like battle.

NOTES

[a] Engel counts a casualty for the Blue side as being ''out of action,'' which includes wounded as well as killed.

AUTHOR STATEMENT

The views expressed in this paper are those of the author and do not reflect the official policy or position of the US Navy, Department of Defense or the US Government.

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APPENDIX: PROOF OF B_* PROPERTIES FOR AIMED FIRE CASE

Theorem. If Equation (12) results in $B_* \geq B_0$, Blue may achieve their desired victory condition or better without the network.

Proof. Recalling the victory condition (4), it is sufficient to demonstrate that:

$$\frac{B_0^2 - B_f^2}{R_0^2 - R_f^2} \geq \frac{\beta_R}{\beta_B} \Rightarrow B_* \geq B_0$$

The victory condition may be rewritten as:

$$\beta_B(B_0^2 - B_f^2) - \beta_R(R_0^2 - R_f^2) \geq 0.$$

Substituting and expanding yields:

$$\beta_B B_0^2 - \beta_B B_f^2 - \beta_{BN} B_0^2 + \beta_{BN} B_*^2 - \beta_b B_*^2 + \beta_B B_f^2 \geq 0.$$

Simplifying and regrouping results in:

$$\beta_B(B_0^2 - B_*^2) \geq \beta_{BN}(B_0^2 - B_*^2)$$

However, because $B_* \geq B_0$ and we are restricted to being nonnegative, $B_0^2 - B_*^2 \leq 0$. Multiplying by -1, we see are left with $\beta_B \leq \beta_{BN}$, which is true by definition and the proof is complete.

Theorem. If the formula for B_* returns an imaginary number, the victory conditions B_f, R_f are not achievable with an invulnerable network

Proof. Examining Equation (12), we see that B_* is imaginary iif

$$\beta_R(R_0^2 - R_f^2) > \beta_{BN} B_0^2 - \beta_B B_f^2,$$

which we recognize as (4) for Red's victory condition. Therefore,

$$\beta_R(R_0^2 - R_f^2) > \beta_{BN} B_0^2 - \beta_B B_f^2$$
$$\geq \beta_{BN} B_0^2 - (\beta_{BN} - \beta_N) B_*^2 - \beta_N B_f^2$$

For any allowable (nonnegative) choice of B_*.