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Development of coupling technique for LBM and FEM for FSI application

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Abstract

Purpose – To develop a technique to couple the lattice Boltzmann method (LBM) and the finite element method (FEM) to solve fluid-structure interaction (FSI) problems.

Design/methodology/approach – The FEM was applied to structural analysis while the LBM was applied to fluid flow analysis. The two techniques were coupled in a staggered manner through interface boundary conditions.

Findings – In order to demonstrate the developed coupling technique, various FSI examples were analyzed and presented. The coupling technique was useful to solve FSI problems.

Originality/value – To the best knowledge of the author, there have been few efforts to couple the two techniques to solve the fluid and flexible structure interaction problems.

Keywords Finite element analysis, Fluid dynamics, Flow, Structural analysis

Paper type Research paper

Introduction

Coupled multiphysics problems are important in real engineering applications. Because of the complexity of the nature, numerical techniques such as the finite element method (FEM) have been applied to coupled problems. There is an extensive amount of literature in the subject field. As a result, it is not the intention of the author to include those here. A few of early works are found in the literature (Zienkiewicz and Newton, 1969; Zienkiewicz *et al.*, 1981; Lewis *et al.*, 1984). Among the coupled problems, fluid-structure interaction (FSI) is one of common applications. FSI examples include, but not limited to, flow over aircraft wing, bridge, buildings and civil structures; underwater explosion propagating toward submerged structures (Kwon and Fox, 1993; Kwon *et al.*, 1994; Kwon and Cunningham, 1998); flow inside pipes; blood flow in artery; flow over a bundle of pipes; vibration of turbine and compressor blades, etc. As a result, numerical techniques have been also developed for FSI problems. Some of them used FEM for both fluid and structure analyses (Newton, 1980; Zienkiewicz *et al.*, 1983; Bathe *et al.*, 1995; Kwon and McDermott, 2001), and some others used coupled FEM and the boundary element method (Everstine and Henderson, 1990; Giordano and Koopmann, 1995). Most of those studies considered potential flow for FSI. Viscous flow was considered in blood flow using FEM (Dubini *et al.*, 1995; Tienfuan *et al.*, 1997).

The lattice Boltzmann method (LBM) has been developed and applied to fluid flow applications since late 1980s. (McNamara and Zenetti, 1988; Qian, 1993; Chen, 1993; Cali *et al.*, 1992; Chen and Doolen, 1998) The technique was proved to be very efficient and powerful for such applications. For example, problems such as multiphase flows (Flekoy, 1993; Swift *et al.*, 1996), turbulent flow (Soe *et al.*, 1998), and thermal flow

(Peng *et al.*, 2003) could be handled effectively using the LBM. On the other hand, the FEM has been utilized predominantly for structural applications. However, to the best knowledge of the author, there were few efforts to couple the two techniques to solve the fluid and flexible structure interaction problems. An application of the LBM to FSI was the case with flow around rigid structures as appeared in artificial heart-valve geometries (Krafczyk *et al.*, 2001). As a result, this paper discusses coupling of the LBM for the fluid domain and the FEM for the flexible structural domain. The next section describes the development of the LBM. Then, the coupling technique between LBM and FEM is presented. Finally, some numerical examples are presented to demonstrate the coupled technique, and the summary is followed.

Lattice Boltzmann method

Because LBM is a relatively new technique, this section describes the technique. The LBM was originated from lattice gas (LG) automata (Frisch *et al.*, 1986), which are discrete particle kinetics based on discrete time and lattice spaces. The evolution equation for the LG automata is expressed as:

$$f_i(\mathbf{x} + \mathbf{e}_i \Delta t, t + \Delta t) - f_i(\mathbf{x}, t) = \Omega_i(f(\mathbf{x}, t)) \quad (i = 0, 1, \dots, n) \quad (1)$$

where $f_i(\mathbf{x}, t)$ denotes the probability of finding a particle at lattice site $\mathbf{x}$ and time t , which moves along the i th lattice direction with the local particle velocity $\mathbf{e}_i$. Furthermore, Δt is the time increments, and Ω_i is the collision operator for the rate of change of f_i resulting from collision, and it depends only on the local value of $f_i(\mathbf{x}, t)$. In the above and following equations, the bold face indicates a vector quantity.

Another way to derive LBM is from the Boltzmann equation:

$$\frac{\partial f_i}{\partial t} + \mathbf{e}_i \cdot \nabla f_i = \Omega_i \quad (2)$$

The left side of equation (1) represents a forward discretization of the Boltzmann operator given at the left side of equation (2). If the time and space increments are chosen such that $\Delta x / \Delta t = |\mathbf{e}_i|$, equation (1) is expressed as:

$$f_i(\mathbf{x} + \mathbf{e}_i, t + 1) - f_i(\mathbf{x}, t) = \Omega_i(f(\mathbf{x}, t)) \quad (i = 0, 1, \dots, n) \quad (3)$$

In those equations, the local particle velocity $\mathbf{e}_i$ is discrete in the given lattice. For example, for a 2D lattice as shown in Figure 1, the velocities for the nine possible directions are:

Figure 1.
2D lattice with nine points
showing discrete velocity
vectors

$$\mathbf{e}_i = \begin{cases} (0, 0) & i = 0 \\ (\cos\{(i-1)\pi/2\}, \sin\{(i-1)\pi/2\}) & i = 1, 2, 3, 4 \\ (\sqrt{2}\cos\{(i-1)\pi/2 + \pi/4\}, \sqrt{2}\sin\{(i-1)\pi/2 + \pi/4\}) & i = 5, 6, 7, 8 \end{cases} \quad (4)$$

For the BGK model (Bhatnagar *et al.*, 1954), the collision operator is expressed as:

$$\Omega_i = -\frac{1}{\tau} (f_i - f_i^{\text{eq}}) \quad (5)$$

where τ is the relaxation time and f_i^{eq} denotes the local equilibrium distribution. This local equilibrium is derived from the Maxwell-Boltzmann equilibrium distribution. Using its quadratic expansion, the local equilibrium distribution for fluid flow is given as:

$$f_i^{\text{eq}} = \rho w_i \left[1 + \frac{\mathbf{v} \cdot \mathbf{e}_i}{c_s^w} + \frac{(\mathbf{v} \cdot \mathbf{e}_i)^2 - c_s^2 \mathbf{v} \cdot \mathbf{v}}{2c_s^4} \right] \quad (6)$$

in which ρ is the fluid density, $\mathbf{v}$ is the fluid velocity, c_s is the lattice speed of sound. In addition, w_i is the weighting parameter for each velocity direction, and it is given for the 2D lattice shown in Figure 1:

$$w_i = \begin{cases} 4/9 & i = 0 \\ 1/9 & i = 1, 2, 3, 4 \\ 1/36 & i = 5, 6, 7, 8 \end{cases} \quad (7)$$

The fluid density ρ and momentum density $\rho \mathbf{v}$ are expressed as:

$$\rho = \sum_i f_i \quad (8)$$

and

$$\rho \mathbf{v} = \sum_i f_i \mathbf{e}_i \quad (9)$$

Furthermore, the fluid pressure p and the kinematic viscosity ν are expressed as:

$$p = \rho c_s^2 \quad (10)$$

and

$$\nu = c_s^2 \left(\tau - \frac{\Delta t}{2} \right) \quad (11)$$

In equilibrium, the conservation of mass and momentum is satisfied at each lattice:

$$\sum_i \Omega_i = 0 \quad (12)$$

and

$$\sum_i \Omega_i \mathbf{e}_i = 0 \quad (13)$$

Coupling of LBM and FEM

One of the boundary conditions at the fluid-structure interface is given as

$$v = \frac{\partial \mathbf{u}}{\partial t} \quad (14)$$

where $\mathbf{u}$ is the structural displacement vector at the fluid-structure boundary and this equation states the continuity of velocity at the boundary assuming no slip condition. Furthermore, the continuity of traction at the fluid-structure boundary is expressed as:

$$\sigma_{kl}^f n_l = \sigma_{kl}^s n_l \quad (15)$$

in which σ_{kl}^f is the stress tensor in fluid which is computed as:

$$\sigma_{kl}^f = -p\delta_{kl} + \mu(v_{k,l} + v_{l,k}) \quad (16)$$

σ_{kl}^s is the stress at the structure wall, and n_l is the normal unit vector at the interface. In equation (16), p is the pressure, and μ is the viscosity.

Coupling of LBM to FEM for FSI application was undertaken in the staggered manner. In other words, the LBM was applied to the fluid domain using the velocity boundary conditions obtained from the FEM at the fluid-structure interface. Then, the fluid traction was computed from the LBM at the fluid-structure boundary. The traction was applied to the structural finite element analysis. This solution cycle continued until the solutions for the fluid and structure became compatible at the interface boundaries.

A procedure to apply the fluid-structure interface velocity boundary condition to the LBM is described below. First, of all, the so-called bounce-back scheme was applied to the fluid-structure boundary lattice points of the LBM. This means that when a particle distribution hits a boundary lattice point, the particle distribution scatters back to the node it came from. Then, the local particle distribution f_i was further modified as follows to maintain the velocity continuity at the fluid-structure boundary. Let $\dot{u}_x$ and $\dot{u}_y$ be the structural velocity components along the x - and y -axis at the fluid-structure interface. The particle distribution is revised as follows:

$$\begin{aligned} f_2 &\leftarrow f_2 + \frac{\dot{u}_x}{3} \\ f_3 &\leftarrow f_3 + \frac{\dot{u}_y}{3} \\ f_4 &\leftarrow f_4 - \frac{\dot{u}_x}{3} \\ f_5 &\leftarrow f_5 - \frac{\dot{u}_y}{3} \\ f_6 &\leftarrow f_6 + \frac{\dot{u}_x}{12} + \frac{\dot{u}_y}{12} \\ f_7 &\leftarrow f_7 - \frac{\dot{u}_x}{12} + \frac{\dot{u}_y}{12} \\ f_8 &\leftarrow f_8 - \frac{\dot{u}_x}{12} - \frac{\dot{u}_y}{12} \\ f_9 &\leftarrow f_9 + \frac{\dot{u}_x}{12} - \frac{\dot{u}_y}{12} \end{aligned} \quad (17)$$

By doing so, the local fluid mass was conserved at the lattice points lying at the fluid-structure interface and the velocity continuity condition was enforced between the fluid and structure.

Another way to apply the fluid-structure interface velocity to the LBM is described below. In this approach, the bounce-back scheme is not applied to the interface lattice points. Instead, let $\Delta \dot{u}_x$ and $\Delta \dot{u}_y$ be the x - and y -components of the velocity difference between structure and fluid at the interface. Then, the particle distribution is revised as follows:

$$\begin{aligned}
 f_2 &\leftarrow f_2 + \frac{\Delta \dot{u}_x}{3} \\
 f_3 &\leftarrow f_3 + \frac{\Delta \dot{u}_y}{3} \\
 f_4 &\leftarrow f_4 - \frac{\Delta \dot{u}_x}{3} \\
 f_5 &\leftarrow f_5 - \frac{\Delta \dot{u}_y}{3} \\
 f_6 &\leftarrow f_6 + \frac{\Delta \dot{u}_x}{12} + \frac{\Delta \dot{u}_y}{12} \\
 f_7 &\leftarrow f_7 - \frac{\Delta \dot{u}_x}{12} + \frac{\Delta \dot{u}_y}{12} \\
 f_8 &\leftarrow f_8 - \frac{\Delta \dot{u}_x}{12} - \frac{\Delta \dot{u}_y}{12} \\
 f_9 &\leftarrow f_9 + \frac{\Delta \dot{u}_x}{12} - \frac{\Delta \dot{u}_y}{12}
 \end{aligned} \tag{18}$$

Comparing the two different approaches using numerical examples, both techniques resulted in quite comparable solutions.

When the traction was computed from the LBM using equation (16), then the finite element analysis is conducted using the following equation:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F\} + \{P\} \tag{19}$$

where $[M]$, $[C]$, and $[K]$ are the finite element mass, damping, and stiffness matrices, respectively. $\{u\}$ is the nodal displacement vector, and the superimposed dot denotes the temporal derivative. Furthermore, $\{F\}$ is the external force vector, and $\{P\}$ is the force vector resulting from the fluid-structure interaction as expressed below:

$$\{P\} = \sum \int_{\Gamma_{\text{int}}} [N]^T \{f\} d\Gamma \tag{20}$$

Here, $[N]$ is the matrix composed of finite element shape functions over the interface element boundary Γ_{int} , and $\{f\}$ is the traction vector. The summation is over the total number of finite element boundaries at the fluid-structure interface.

Numerical examples

Two-dimensional numerical examples were presented here. The first example was a flow between two parallel boundaries, one of which was assumed to be rigid while the

other was considered to be flexible as shown in Figure 2. The flexible boundary was modeled as a beam with clamped at both end points. The left and right side boundaries of the flow field were assumed to be periodic. In this example, the density ratio of the solid to fluid was assumed to be 100. The kinematic viscosity of the fluid was 1.35×10^{-2} . The beam rigidity was 83.3. All these values had consistent units. The domain size is shown in Figure 3. The fluid domain has 50×25 lattice points and the beam has 50 elements. As fluid flows, the flexible beam vibrates up and down. Figure 3 shows snapshots of the flow velocities as well as the displacements of the beam, indicated by bold faced arrows, at two different times for the flow with Reynolds number (Re) 16.5. This figure suggests that the beam vibration has multi-modal components.

Time history of the vibrational motion of the flexible beam is shown in Figures 4-6. Figure 4 shows the transverse displacement and velocity plots of the beam at the center, while Figures 5 and 6 show them at the left and right quarter points of the beam, respectively. In all plots, displacements and velocities were normalized with respect to the lattice grid size while time was normalized in terms of the time increment. When comparing Figures 5 and 6, the responses at the left and right sides of the beam are not similar due to inclusion of even order components of mode shapes.

As the flow speed increases, the vibrational amplitude of the beam increases. Comparing Figures 4 and 7 shows the increases of displacement and velocity at the center of the beam as the flow speed becomes double.

Figure 8 shows an example similar to Figure 2 except that there is a narrow flow section in the middle of the duct. The applied duct Re was 33 as before. The velocity profile and the beam displacement are shown in Figure 9 for an instant. The figure shows a vortex flow at the upper right side of the flow domain. Plotting and comparing the beam deflections at the left and right quarter locations, respectively, shows that the left quarter point has almost consistently downward deflections while the right quarter point has upward deflection (Figure 10).

The next example was flow between the rigid walls. A flexible beam structure was located between the two rigid walls as shown in Figure 11. The internal flexible structure was supported by vertical and horizontal rigid structures which are fixed in the positions. The material properties used for this example were the same as those in the previous example. The domain size was shown in Figure 12. The fluid domain has 50×50 lattice points while the structure has ten beam elements.

Figure 11 shows a snapshot of the flow over the flexible beam structure at an instant with duct Re 16.5. In addition, Figure 13 shows the transverse displacement and normal velocity of the beam at the center as a function of time.

Figure 2.
Flow between rigid and flexible walls. The periodic boundary condition was applied to the left and right side boundaries of the flow field

Figure 3.
Snapshot of fluid
velocities at two different
times for the case shown in
Figure 2 with $Re = 16.5$

Note: The bold arrows indicate the displacements of the flexible beam. The velocities and displacements were scaled independently for visual clarity

Figure 14 shows a case with contained fluid which is subjected to a uniform horizontal velocity at the top of the container with a flexible boundary at the bottom. The material properties used for this example were the same as those in the previous examples. The shear loading at the top of the container produced circular fluid motions which resulted in vibration of the flexible bottom structure. The flow motion is shown in Figure 15 while the vibration of the flexible bottom is shown in Figure 16. As shown in Figure 16, the vibration of the flexible bottom of the container does not start until the fluid at the bottom is agitated from the top

Figure 4.
Time history plot of the
transverse displacement
and velocity at the center
of the beam with
 $Re = 16.5$ for Figure 2

Figure 5.
Time history plot of the
transverse displacement
and velocity at the left
quarter point of the beam
with $Re = 16.5$ for Figure 2

Figure 6.
Time history plot of the
transverse displacement
and velocity at the right
quarter point of the beam
with $Re = 16.5$ for Figure 2

Figure 7.
Time history plot of the
transverse displacement
and velocity at the center
of the beam with $Re = 33$
for Figure 2

loading. Two different densities were considered for the fluid. As the fluid density increases, the damping of the vibrating structure becomes greater as shown in Figure 16, as expected.

Summary

Because the LBM was developed relatively recently and was proved to be very useful for fluid mechanics application, a coupling technique was introduced for FSI application using LBM for the fluid domain and FEM for the structural domain. The

Figure 8. Flow between rigid and flexible walls. The periodic boundary condition was applied to the left and right side boundaries of the flow field

Notes: Figure 9 shows vortex flow at the upper right side. The bold arrows indicate the displacements of the flexible beam. The velocities and displacements were scaled independently for visual clarity

Figure 9. Snapshot of fluid velocities for the case shown in Figure 8 with duct $Re = 33$

Figure 10.
Time history plot of the
transverse displacements
at the left and right sides
of the beam with duct
 $Re = 33$ for Figure 8

Figure 11.
Flow between two rigid
walls containing a flexible
structure inside

coupling procedure was described and some numerical examples were presented to demonstrate the technique. The example showed the usefulness of the coupled techniques. Because the LBM can be applied to multiphase problems with relative ease, this coupling technique for FSI will be beneficial to various applications including complex multiphase flows inside or outside flexible structures. Those cases will be studied and reported later in another publication. Furthermore, stability analysis will be conducted and reported later.

Notes: The bold arrows indicate the displacements of the flexible beam.
The velocities and displacements were scaled independently for visual clarity

Figure 12.
Snapshot of fluid
velocities at an instant for
the case shown in
Figure 11

Figure 13.
Plot of time history of the
transverse deflection and
normal velocity at the
center of the beam shown
in Figure 11

Figure 14.
Contained fluid subjected
to a constant horizontal
velocity at the top with a
flexible structure at the
bottom

Figure 15.
A snapshot of the fluid
flow and the structural
displacement by a bold
arrows for the case of
Figure 14

(a) Solid to fluid density ratio = 100

(b) Solid to fluid density ratio = 50

Figure 16.
Comparison of
time-displacement plot of
the flexible top at the
center shown in Figure 14
for two different fluid
densities

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