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# The Spectrum of Generalized Petersen Graphs

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### Abstract

In this paper, we completely describe the spectrum of the generalized Petersen graph  $P(n, k)$ , thus adding to the classes of graphs whose spectrum is completely known.

## 1 Introduction and motivation

Let  $G = (V(G), E(G))$  be a simple graph. The spectrum of a graph  $G$  is the multiset of eigenvalues of the adjacency matrix. The graph spectrum is an important tool one can use to find information about the physical properties of a network, such as robustness, diameter, connectivity [3]. In this research we completely describe the spectrum for the class of graphs, defined below.

The *generalized Petersen graph* (GPG)  $P(n, k)$  has vertices, respectively, edges given by

$$\begin{aligned} V(P(n, k)) &= \{a_i, b_i, 0 \leq i \leq n-1\}, \\ E(P(n, k)) &= \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} \mid 0 \leq i \leq n-1\}, \end{aligned}$$

where the subscripts are expressed as integers modulo  $n$  ( $n \geq 5$ ), and  $k$  is the “skip”. Note that  $k \leq \lfloor \frac{n-1}{2} \rfloor$ , because of the obvious isomorphism  $P(n, k) \cong P(n, n-k)$ . Let  $A(n, k)$  (respectively,  $B(n, k)$ ) be the subgraph of  $P(n, k)$  consisting of the vertices  $\{a_i \mid 0 \leq i \leq n-1\}$  (respectively,  $\{b_i \mid 0 \leq i \leq n-1\}$ ) and edges  $\{a_i a_{i+1} \mid 0 \leq i \leq n-1\}$  (respectively,  $\{b_i b_{i+k} \mid 0 \leq i \leq n-1\}$ ). We will call  $A(n, k)$  (respectively,  $B(n, k)$ ) the *outer* (respectively, *inner*) subgraph of  $P(n, k)$ . We display in Figure 1 the graph  $P(12, 3)$ .

For other graph theoretical terminology the reader could refer to [7].

## 2 Eigenvalues of $P(n, k)$

In this section we find our description for the spectrum of generalized Petersen graphs  $P(n, k)$ . We denote the adjacency matrix of the GPG  $P(n, k)$

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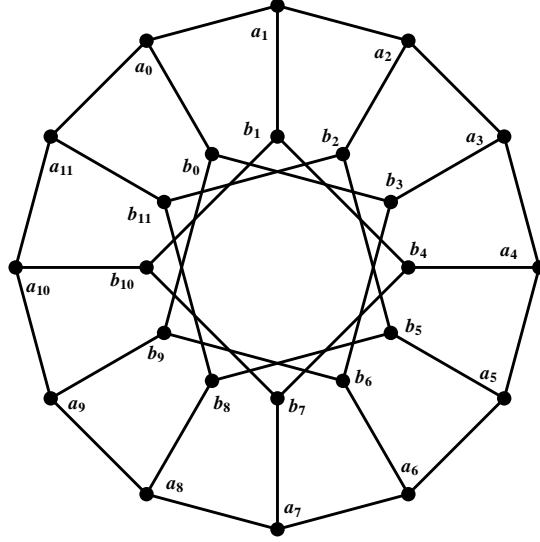


Figure 1: The Generalized Petersen Graph  $P(12, 3)$

by  $A(P(n, k))$ . Let  $\lambda_0 = 3 > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{2n-1}$  be the sequence of eigenvalues of  $P(n, k)$ .

We call an  $n \times n$  matrix *circulant*, and denote it by  $\text{circ}(a_1, a_2, \dots, a_n)$  if it is of the form

$$\text{circ}(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ \vdots & & & & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix}.$$

**Lemma 2.1** *The  $(2n) \times (2n)$  adjacency matrix of the GPG  $P(n, k)$  has the block form*

$$A(P(n, k)) = \begin{pmatrix} C_k^n & I_n \\ I_n & C^n \end{pmatrix},$$

where  $I_n$  is the  $n \times n$  identity matrix,  $C^n, C_k^n$  are circulant matrices, with  $C^n = \text{circ}(0, 1, 0, 0, \dots, 0, 1)$  and  $C_k^n = \text{circ}(\underbrace{0, \dots, 0}_{k \text{ times}}, 1, 0, \dots, 0, 1, \underbrace{0, \dots, 0}_{k-1 \text{ times}})$  being the adjacency matrix for  $A(n, k)$  and  $B(n, k)$ , respectively. Thus,  $C^n$

is the adjacency matrix of a cycle graph on  $n$  vertices  $\mathcal{C}_n$ , respectively,  $C_k^n$  is the union of  $d$  cycle graphs  $\mathcal{C}_{n/d}$  on  $n/d$  vertices, where  $d = \gcd(n, k)$ .

**Proof.** The outer subgraph (whose adjacency matrix is  $C^n$ ) of  $P(n, k)$  is the cycle graph  $\mathcal{C}_n$  and the inner subgraph (whose adjacency matrix is  $C_k^n$ ) has  $d$  connected components each isomorphic to  $\mathcal{C}_{n/d}$ . Also, the adjacency matrix (which depends on the labeling) has the claimed form where the labels used on the outer subgraph are consecutively  $1, 2, \dots, n$ , and on the inner subgraph the adjacent labels are  $i, i + k, i + 2k, \dots$  (where  $i + sk$  is understood as  $1 + (i - 1 + sk) \pmod{n}$ ). Note that  $b_0$  is adjacent to vertex  $b_k$  in the subgraph and to vertex labeled  $b_{n-k}$  in  $B(n, k)$ , and so  $C_k^n = \text{circ}(\overbrace{0, \dots, 0}^{k \text{ times}}, 1, 0, \dots, 0, 1, \overbrace{0, \dots, 0}^{k-1 \text{ times}})$   $\square$

We recall the Chebyshev's polynomial of the first kind [5], defined by the identity  $T_n(\cos \theta) = \cos(n\theta)$ , with the generating function  $\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-xt}{1-2xt+t^2}$ . We now present the eigenvectors and eigenvalues for  $\mathcal{C}_n$  (see [2, p. 53 and pp. 72–73]). Let  $\mathbf{v}^t$  denote the transpose of  $\mathbf{v}$ .

**Lemma 2.2** *The eigenvalues of the cycle graph  $\mathcal{C}_n$  on  $n$  vertices are*

$$\alpha_j = 2 \cos \left( \frac{2\pi j}{n} \right)$$

with a corresponding eigenvector

$$\mathbf{v}_j = (1, \zeta^j, \zeta^{2j}, \dots, \zeta^{(n-1)j})^t,$$

$0 \leq j \leq n - 1$ . The characteristic polynomial of the cycle  $\mathcal{C}_n$  is  $2T_n(x/2) - 2$  where  $T_n$  is the Chebyshev's polynomial of the first kind.

**Corollary 2.3** *The eigenvalues corresponding to the circulant  $C$  in the adjacency matrix  $A(P(n, k))$  are  $\alpha_j = 2 \cos \left( \frac{2\pi j}{n} \right)$  ( $0 \leq j \leq n - 1$ ), and the eigenvalues corresponding to  $C_k$  are  $\beta_j = 2 \cos \left( \frac{2\pi jk}{n} \right)$  ( $0 \leq j \leq n - 1$ ).*

We now state our main theorem which adds to the class of graphs whose spectrum is now known.

**Theorem 2.4** *The eigenvalues of  $P(n, k)$ , say  $\delta_{2j}, \delta_{2j+1}$ , are all roots of the quadratic equation*

$$\delta^2 - (\alpha_j + \beta_j)\delta + \alpha_j\beta_j - 1 = 0, \tag{1}$$

where  $\alpha_j = 2 \cos \left( \frac{2\pi j}{n} \right)$ ,  $\beta_j = 2 \cos \left( \frac{2\pi jk}{n} \right) = 2T_k(\alpha_j/2)$  ( $0 \leq j \leq n - 1$ ) are the eigenvalues of  $C$ , respectively  $C_k$ .

**Proof.** We first consider the case of  $d = \gcd(n, k) = 1$ . Since  $d = 1$ , then  $C_k$  is the adjacency matrix of a cycle graph isomorphic to  $C_n$ , and so it is similar to  $C$ , that is, there exists a permutation matrix  $P$ , such that  $P^{-1}C_kP = C$ . This implies that the two matrices will have the same eigenvalues and eigenvectors. Then  $\alpha_j, \beta_j$  are eigenvalues corresponding to the same eigenvector, say  $\mathbf{v}_j = (1, \zeta_n^j, \dots, \zeta_n^{(n-1)j})^t$ . We are looking for an eigenvector for  $A(P(n, k))$  of the form  $\mathbf{w}_j = (a_j\mathbf{v}_j, \mathbf{v}_j)^t$ , where  $a_j$  will be determined later. If two distinct values for  $a_j$  are to be found, for any  $0 \leq j \leq n-1$ , then we are done with our search for the eigenvectors/eigenvalues.

With this value for  $\mathbf{w}_j$ , we need  $\delta$  (dependent on  $j$ ) such that

$$\begin{pmatrix} C_k & I_n \\ I_n & C \end{pmatrix} \begin{pmatrix} a_j\mathbf{v}_j \\ \mathbf{v}_j \end{pmatrix} = \delta \begin{pmatrix} a_j\mathbf{v}_j \\ \mathbf{v}_j \end{pmatrix}$$

and so, we get the system

$$\begin{cases} a_j C_k \mathbf{v}_j + \mathbf{v}_j = \delta a_j \mathbf{v}_j \\ a_j \mathbf{v}_j + C \mathbf{v}_j = \delta \mathbf{v}_j \end{cases} \iff \begin{cases} a_j \beta_j \mathbf{v}_j + \mathbf{v}_j = \delta a_j \mathbf{v}_j \\ a_j \mathbf{v}_j + \alpha_j \mathbf{v}_j = \delta \mathbf{v}_j \end{cases}$$

which implies

$$\begin{cases} a_j(\delta - \beta_j)\mathbf{v}_j = \mathbf{v}_j \\ (\delta - \alpha_j)\mathbf{v}_j = a_j\mathbf{v}_j \end{cases}$$

and so,  $(\delta - \beta_j)(\delta - \alpha_j) = 1$ , which renders the claim for this case, that is,  $\delta$  must satisfy the equation  $\delta^2 - (\alpha_j + \beta_j)\delta + \alpha_j\beta_j - 1 = 0$ .

The case of  $d > 1$  is treated similarly. The eigenvectors  $\mathbf{w}_j$  must have the form  $\mathbf{w}_j = (a_1\mathbf{v}'_j, a_2\mathbf{v}'_j, \dots, a_d\mathbf{v}'_j, \mathbf{v}_j)$ , with  $\mathbf{v}_j$  as before and  $\mathbf{v}'_j = (1, \zeta_n^j, \dots, \zeta_n^{(n'-1)j})^t$ ,  $n' = n/d$ , for some appropriate multipliers  $a_i$ . A similar system to the one for  $d = 1$  case will be obtained and, interestingly enough, the same polynomial whose roots are the eigenvalues  $\lambda_i$  will be found. The theorem is proved.  $\square$

Using the quadratic formula in (1) and simplifying we get the following corollary.

**Corollary 2.5** *The eigenvalues of  $P(n, k)$  are given by*

$$\cos\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{2\pi jk}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi j}{n}\right) - \cos\left(\frac{2\pi jk}{n}\right)\right)^2 + 1}, \quad 0 \leq j \leq n-1.$$

The largest eigenvalue of  $P(n, k)$ ,  $\lambda_0 = 3$ , is one of the two values obtained for  $j = 0$  in the previous corollary. It is known (see [2, Thm. 3.11]) that if a graph is bipartite, then its spectrum is symmetric with respect to 0. In our case, we have the following result.

**Corollary 2.6** *If  $n$  is even and  $k$  is odd, then the eigenvalues of the bipartite graph  $P(n, k)$  are given by  $\pm 3$  and*

$$\begin{aligned} & \cos(2j\pi/n) + \cos(2jk\pi/n) \pm \sqrt{(\cos(2j\pi/n) - \cos(2jk\pi/n))^2 + 1} \\ & - \cos(2j\pi/n) + (-1)^k \cos(2jk\pi/n) \mp \sqrt{(\cos(2j\pi/n) + (-1)^k \cos(2jk\pi/n))^2 + 1}, \end{aligned}$$

for  $0 \leq j < n/2$ .

### 3 Bounds on the eigenvalues of $P(n, 2)$

In the previous section we found the complete set of eigenvalues of  $P(n, k)$  under no restrictions on  $n$  and  $k$ . Here, we would like to find some bounds on some eigenvalues. Eigenvalue interlacing techniques (see the great survey by Haemers [4] on the topic) will not work easily since there is no visible connection between the various  $P(n, k)$ , and moreover, the technique is not sensitive enough for our purpose. We shall use a different method.

Here, we will take  $k = 2$  and consider  $P(n, 2)$  (this includes the case of the classical Petersen graph  $P(5, 2)$ ). Since the second Chebyshev polynomial of the first kind is  $T_2(x) = 2x^2 - 1$ , we immediately obtain the following:

**Theorem 3.1** *The eigenvalues of  $P(n, 2)$  are (for  $0 \leq j \leq n - 1$ )*

$$2 \cos^2(2j\pi/n) + \cos(2j\pi/n) - 1 \pm \sqrt{(2 \cos^2(2j\pi/n) - \cos(2j\pi/n) - 1)^2 + 1}.$$

To find good bounds on the eigenvalues in this case, we look for the extreme points of the two functions

$$f_{\pm}(x) = 2x^2 + x - 1 \pm \sqrt{(2x^2 - x - 1)^2 + 1}, \quad (2)$$

in the interval  $-1 \leq x \leq 1$ . Certainly, we cannot expect exact or even tight results, in general, since the sequence  $\frac{2j\pi}{n}$ ,  $0 \leq j < n - 1$ , is finite and therefore,  $\cos\left(\frac{2j\pi}{n}\right)$  is not dense in this interval. However, we will have lower



and upper bounds, which is what we are interested in. Since any differentiable function in a compact domain attains its extreme points at either the critical points or on the boundary, we proceed by studying first the functions' critical points:

$$f'_{\pm}(x) = 4x + 1 \pm \frac{(2x^2 - x - 1)(4x - 1)}{\sqrt{(2x^2 - x - 1)^2 + 1}} = 0,$$

has solutions (computed by Mathematica) at  $x_1 \sim -0.41100$  (for  $f_+$ ) and  $x_2 \sim -0.65041, x_3 \sim -0.04610$  (for  $f_-$ ). The values of the corresponding  $f_{\pm}$  at these critical points are

$$\begin{aligned} f_+(-0.41100) &= -0.04210\dots \\ f_-(-0.65041) &= -1.92081\dots \\ f_-(-0.04610) &= -2.42092\dots \end{aligned}$$

Further, we look at the values of  $f_{\pm}$  at  $|x| = 1$ . Thus,  $f_+(1) = 3, f_+(-1) = \sqrt{5}$ , and  $f_-(1) = 1, f_-(-1) = -\sqrt{5}$ . Certainly, the maximum value is 3, and the minimum value is approximately -2.42092. We sketch in Figure 2 the two functions  $f_{\pm}$ , to visualize our analysis from above:

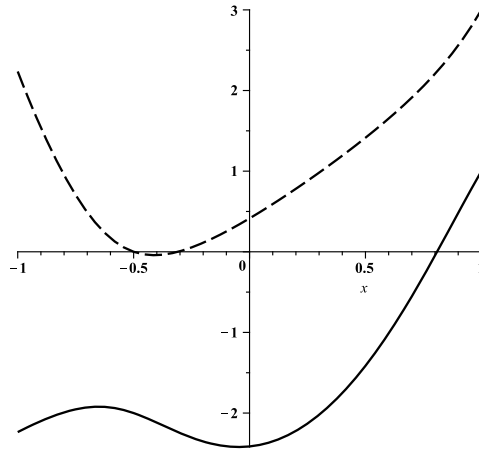


Figure 2: The top function is  $f_+$  and the bottom function is  $f_-$

Every value of  $f_+$  is above every value of  $f_-$ , and so the minimum is attained by  $f_-$  and the maximum is attained by  $f_+$ . Furthermore, we see

that the second largest eigenvalue of  $P(n, 2)$  is

$$\begin{aligned} \lambda_1 &= f_+ \left( \cos \left( \frac{2\pi}{n} \right) \right) \\ &= \cos \left( \frac{2\pi}{n} \right) + \cos \left( \frac{4\pi}{n} \right) + \sqrt{4 \left( 2 \cos \left( \frac{2\pi}{n} \right) + 1 \right)^2 \sin^4 \left( \frac{\pi}{n} \right) + 1}, \end{aligned} \quad (3)$$

which increases as  $n$  increases (shown simply by using Calculus techniques). For instance, for  $3 \leq n \leq 20$  the sequence  $\lambda_1 = \lambda_1(n)$  is

$$0, 0.41421, 1., 1.41421, 1.71083, 1.93185, 2.10199, 2.23607, 2.34356, 2.43091, \\ 2.50268, 2.56224, 2.61211, 2.65421, 2.69002, 2.7207, 2.74716, 2.77011.$$

Since  $\lim_{n \rightarrow \infty} \cos \left( \frac{2\pi}{n} \right) = 1$ , we obtain the next result.

**Theorem 3.2** *The eigenvalues of  $P(n, 2)$  are*

$$\lambda_0 = 3 > \lambda_1 \geq \dots \geq \lambda_{2n-1} \geq -2.42092.$$

Moreover, the second largest eigenvalue satisfies  $\lim_{n \rightarrow \infty} \lambda_1(n) = 3$ .

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## 4 Further comments

All of our results for  $P(n, 2)$  can be certainly extended to  $P(n, 3)$ ,  $P(n, 4)$ , etc., but to find sensitive bounds on eigenvalues for arbitrary GPG  $P(n, k)$  does not seem to be easy, since the sequence of the involved Chebyshev's polynomials of the first kind does not have a "controllable" behavior in  $|x| \leq 1$ .

Also, it would be interesting to investigate the number of and distinct values among the eigenvalues of  $P(n, k)$ , and that is presumably doable. We suspect that the methods of this paper can be also applied to the  $I$ -graphs of [1] or the *supergeneralized* Petersen graphs of [6].

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