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Calculus of Variations MA 4311 Lecture Notes

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CALCULUS OF VARIATIONS
MA 4311 LECTURE NOTES

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Credits

Much of the material in these notes was taken from the following texts:

1. Bliss - Calculus of Variations, Carus monograph - Open Court Publishing Co. - 1924

2. Gelfand & Fomin - Calculus of Variations - Prentice Hall 1963

3. Forray - Variational Calculus - McGraw Hill 1968

4. Weinstock - Calculus of Variations - Dover 1974

5. J. D. Logan - Applied Mathematics, Second Edition -John Wiley 1997

The figures are plotted by Lt. Thomas A. Hamrick, USN and Lt. Gerald N. Miranda, USN using Matlab. They also revamped the numerical examples chapter to include Matlab software and problems for the reader.

CHAPTER 1

1 Functions of n Variables

The first topic is that of finding maxima or minima (optimizing) functions of n variables. Thus suppose that we have a function $f(x_1, x_2 \cdots, x_n) = f(X)$ (where X denotes the n-tuple $(x_1, x_2, \cdots, x_n)$) defined in some subset of n dimensional space R^n and that we wish to optimize f, i.e. to find a point X_0 such that

$$f(X_0) \leq f(X) \qquad \text{or} \qquad f(X_0) \geq f(X) \tag{1}$$

The first inequality states a problem in minimizing f while the latter states a problem in maximizing f.

Mathematically, there is little difference between the two problems, for maximizing f is equivalent to minimizing the function $G = -f$. Because of this, we shall tend to discuss only minimization problems, it being understood that corresponding results carry over to the other type of problem.

We shall generally (unless otherwise stated) take f to have sufficient continuous differentiability to justify our operations. The notation to discuss differentiability will be that f is of class C^i which means that f has continuous derivatives up through the i^{th} order.

1.1 Unconstrained Minimum

As a first specific optimization problem suppose that we have a function f defined on some open set in R^n. Then f is said to have an **unconstrained relative** minimum at X_0 if

$$f(X_0) \leq f(X) \tag{2}$$

for all points X in some neighborhood S of X_0. X_0 is called a relative minimizing point.

We make some comments: Firstly the word **relative** used above means that X_0 is a minimizing point for f in comparison to **nearby** points, rather than also in comparison to distant points. Our results will generally be of this "relative" nature.

Secondly, the word unconstrained means essentially that in doing the above discussed comparison we can proceed in **any** direction from the minimizing point. Thus in Figure 1, we may proceed in any direction from X_0 to any point in some neighborhood S to make this comparison.

In order for (2) to be true, then we must have that

$$\sum_{i=1}^{n} f_{x_i} h_i = 0 \Rightarrow f_{x_i} = 0 \quad i = 1, \cdots, n \tag{3a}$$

and

$$\sum_{i,j=1}^{n} f_{x_i x_j} h_i h_j \geq 0 \tag{3b}$$

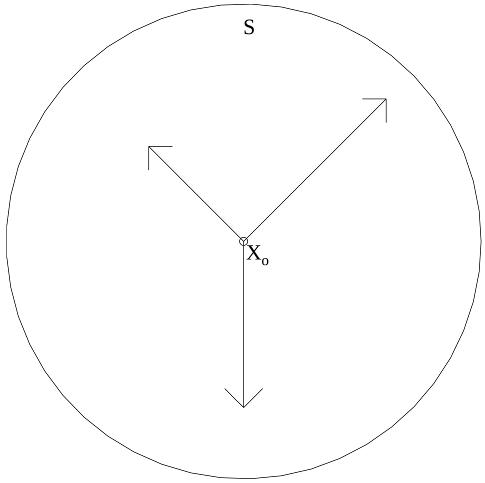

Figure 1: Neighborhood S of X_0

for all vectors H $\Big(= (h_1, h_2, \cdots, h_n)\Big)$ where f_{x_i} and $f_{x_i x_j}$ are respectively the first and second order partials at X_0.

$$f_{x_i} \equiv \frac{\partial f}{\partial x_i} \ , \quad f_{x_i x_j} \equiv \frac{\partial^2 f}{\partial x_i \partial x_j} \ ,$$

The implication in (3a), follows since the first part of (3a) holds for all vectors H.

Condition (3a) says that the first derivative in the direction specified by the vector H must be zero and (3b) says that the second derivative in that direction must be non-negative, these statements being true for all vectors H.

In order to prove these statements, consider a particular direction H and the points $X(\epsilon) = X_0 + \epsilon H$ for small numbers ϵ (so that $X(\epsilon)$ is in S). The picture is given in Figure 2.

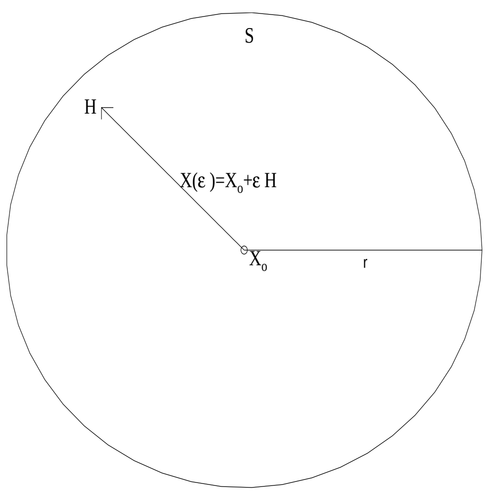

Figure 2: Neighborhood S of X_0 and a particular direction H

Define the function

$$g(\epsilon) = f(X_0 + \epsilon H) \qquad 0 \leq \epsilon \leq \delta \tag{4}$$

where δ is small enough so that $X_0 + \epsilon H$ is in S.

Since X_0 is a relative minimizing point, then

$$g(\epsilon) - g(0) = f(X_0 + \epsilon H) - f(X_0) \geq 0 \qquad 0 \leq \epsilon \leq \delta \tag{5a}$$

Since $-H$ is also a direction in which we may find points X to compare with, then we may also define g for negative ϵ and extend (5a) to read

$$g(\epsilon) - g(0) = f(X_0 + \epsilon H) - f(X_0) \geq 0 \qquad -\delta \leq \epsilon \leq \delta \tag{5b}$$

Thus $\epsilon = 0$ is a relative minimizing point for g and we know (from results for a function in one variable) that

$$\frac{dg(0)}{d\epsilon} = 0 \quad \text{and} \quad \frac{d^2 g(0)}{d\epsilon^2} \geq 0 \tag{6}$$

Now f is a function of the point $X = (x_1, \cdots, x_n)$ where the components of $X(\epsilon)$ are specified by

$$x_i(\epsilon) = x_{0,i} + \epsilon h_i \qquad -\delta \leq \epsilon \leq \delta \qquad i = 1, \cdots, n \tag{7}$$

so that differentiating by the chain rule yields

$$0 = \frac{dg(0)}{d\epsilon} = \sum_{i=1}^{n} f_{x_i} \frac{dx_i}{d\epsilon} = \sum_{i=1}^{n} f_{x_i} h_i \qquad \begin{matrix}(\text{which} \Rightarrow f_{x_i} = 0) \\ i = 1, \cdots, n\end{matrix} \tag{8a}$$

and

$$0 \leq \frac{d^2 g(0)}{d\epsilon} = \sum_{i,j=1}^{n} f_{x_i x_j} \frac{dx_i}{d\epsilon} \frac{dx_j}{d\epsilon} = \sum_{i,j=1}^{n} f_{x_i x_j} h_i h_j \tag{8b}$$

in which (8b) has used (8a). In (8) all derivatives of f are at X_0 and the derivatives of x are at $\epsilon = 0$.

This proves (3a) and (3b) which are known as the first and second order necessary conditions for a relative minimum to exist at X_0. The term necessary means that they are **required** in order that X_0 be a relative minimizing point. The terms first and second order refer to (3a) being a condition on the first derivative and (3b) being a condition on the second derivative of f.

In this course we will be primarily concerned with **necessary** conditions for minimization, however for completeness we state the following:

As a sufficient condition for X_0 to be relative minimizing point one has that if

$$\sum_{i=1}^{n} f_{x_i} h_i = 0 \quad \text{and} \quad \sum_{i,j=1}^{n} f_{x_i x_j} h_i h_j \geq 0 \tag{9}$$

for all vectors $H = (h_1, \cdots, h_n)$, with all derivatives computed at X_0, then X_0 is an unconstrained relative minimizing point for f.

Theorem 1 If $f''(x)$ exists in a neighborhood of x_0 and is continuous at x_0, then

$$f(x_0+h)-f(x_0) \;=\; f'(x_0)h \;+\; \frac{1}{2}f''(x_0)h^2 \;+\; \epsilon(h) \qquad \forall |h|<\delta \tag{10}$$

where $\lim_{h\to 0}\dfrac{\epsilon(h)}{h^2} \;=\; 0.$

Proof By Taylor's formula

$$f(x_0+h)-f(x_0) \;=\; f'(x_0)h \;+\; \frac{1}{2}f''(x_0+\Theta h)h^2$$

$$f(x_0+h)-f(x_0) \;=\; f'(x_0)h \;+\; \frac{1}{2}f''(x_0)h^2 \;+\; \frac{1}{2}\left[f''(x_0+\Theta h)-f''(x_0)\right]h^2 \tag{11}$$

The term in brackets tends to 0 as $h\to 0$ since f'' is continuous. Hence

$$\frac{\epsilon(h)}{h^2} \;=\; \frac{1}{2}\left[f''(x_0+\Theta h)-f''(x_0)\right] \;\to 0 \qquad \text{as } h\to 0. \tag{12}$$

This proves (10).

Now suppose $f \epsilon C^2[a,b]$ and f has a relative minimum at $x \;=\; x_0$. Then clearly

$$f(x_0+h)-f(x_0)\geq 0 \tag{13}$$

and

$$f'(x_0) \;=\; 0. \tag{14}$$

Using (10) and (13) we have

$$f(x_0+h)-f(x_0) \;=\; \frac{1}{2}f''(x_0)h^2 \;+\; \epsilon(h) \;\geq\; 0 \tag{15}$$

with $\lim_{h\to 0}\dfrac{\epsilon(h)}{h^2} \;=\; 0$. Now pick h_0 so that $|h_0|<\delta$, then

$$f(x_0+\lambda h_0)-f(x_0) \;=\; \frac{1}{2}f''(x_0)\lambda^2h_0^2 \;+\; \epsilon(\lambda h_0) \;\geq\; 0 \qquad \forall|\lambda|\leq 1 \tag{16}$$

Since

$$\frac{1}{2}f''(x_0)\lambda^2h_0^2 \;+\; \epsilon(\lambda h_0) \;=\; \frac{1}{2}\lambda^2h_0^2\left(f''(x_0) \;+\; 2\frac{\epsilon(\lambda h_0)}{\lambda^2h_0^2}\right)$$

and since

$$\lim_{h\to 0}\frac{\epsilon(\lambda h_0)}{\lambda^2h_0^2} \;=\; 0$$

we have by necessity

$$f''(x_0) \;\geq\; 0.$$

1.2 Constrained Minimization

As an introduction to constrained optimization problems consider the situation of seeking a minimizing point for the function $f(X)$ among points which satisfy a condition

$$\phi(X) = 0 \tag{17}$$

Such a problem is called a **constrained optimization** problem and the function ϕ is called a **constraint**.

If X_0 is a solution to this problem, then we say that X_0 is a relative minimizing point for f **subject to** the **constraint** $\phi = 0$.

In this case, because of the constraint $\phi = 0$ **all** directions **are no longer** available to get comparison points. Our comparison points must satisfy (17). Thus if $X(\epsilon)$ is a curve of comparison points in a neighborhood S of X_0 and if $X(\epsilon)$ passes through X_0 (say at $\epsilon = 0$), then since $X(\epsilon)$ must satisfy (17) we have

$$\phi(X(\epsilon)) - \phi(X(0)) = 0 \tag{18}$$

so that also

$$\frac{d}{d\epsilon}\phi(0) = \lim_{\epsilon \to 0} \frac{\phi(X(\epsilon)) - \phi(X(0))}{\epsilon} = \sum_{i=1}^{n} \phi_{x_i} \frac{dx_i(0)}{d\epsilon} = 0 \tag{19}$$

In two dimensions (i.e. for $N = 2$) the picture is

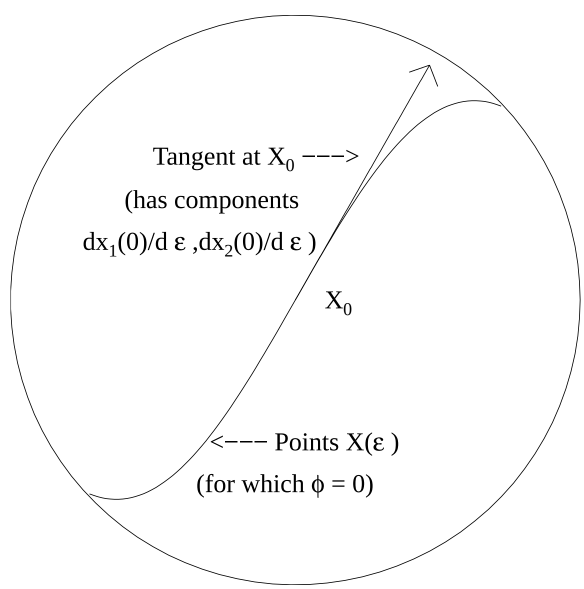

Figure 3: Two dimensional neighborhood of X_0 showing tangent at that point

Thus these tangent vectors, i.e. vectors H which satisfy (19), become (with $\dfrac{dx_i(0)}{d\epsilon}$ replaced by h_i)

$$\sum_{i=1}^{n} \phi_{x_i} h_i = 0 \tag{20}$$

and are the **only** possible directions in which we find comparison points.

Because of this, the condition here which corresponds to the first order condition (3a) in the unconstrained problem is

$$\sum_{i=1}^{n} f_{x_i} h_i = 0 \tag{21}$$

for all vectors H satisfying (19) **instead** of for **all** vectors H.

This condition is not in usable form, i.e. it does not lead to the implications in (3a) which is really the condition used in solving unconstrained problems. In order to get a usable condition for the constrained problem, we depart from the geometric approach (although one could pursue it to get a condition).

As an example of a constrained optimization problem let us consider the problem of finding the minimum distance from the origin to the surface $x^2 - z^2 = 1$. This can be stated as the problem of

$$\begin{aligned} &\text{minimize} \quad f = x^2 + y^2 + z^2 \\ &\text{subject to} \quad \phi = x^2 - z^2 - 1 = 0 \end{aligned}$$

and is the problem of finding the point(s) on the hyperbola $x^2 - z^2 = 1$ closest to the origin.

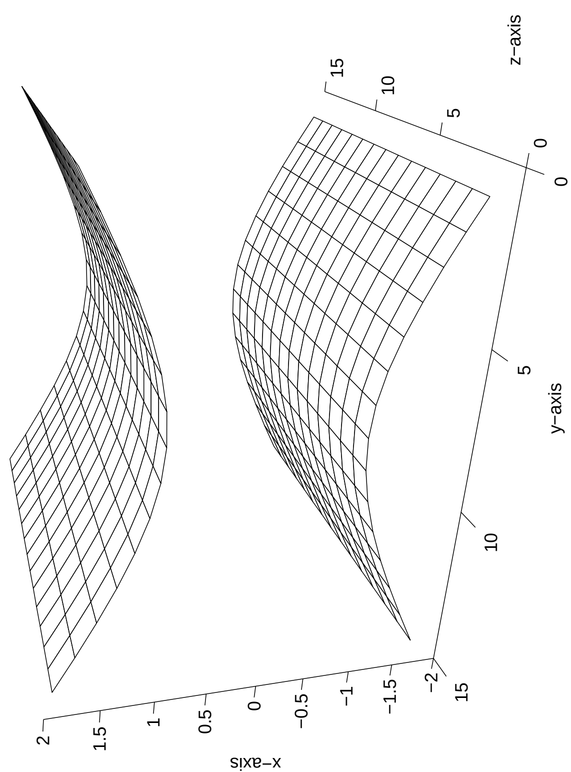

Figure 4: The constraint ϕ

A common technique to try is substitution i.e. using ϕ to solve for one variable in terms of the other(s).

Solving for z gives $z^2 = x^2 - 1$ and then

$$f = 2x^2 + y^2 - 1$$

and then solving this as the unconstrained problem

$$\min f = 2x^2 + y^2 - 1$$

gives the conditions

$$0 = f_x = 4x \quad \text{and} \quad 0 = f_y = 2y$$

which implies $x = y = 0$ at the minimizing point. But at this point $z^2 = -1$ which means that there is no real solution point. But this is nonsense as the physical picture shows.

A surer way to solve constrained optimization problems comes from the following: For the problem of

$$\begin{aligned} &\text{minimize} \quad f \\ &\text{subject to} \quad \phi = 0 \end{aligned}$$

then if X_0 is a relative minimum, then there is a constant λ such that with the function F defined by

$$F = f + \lambda\phi \tag{22}$$

then

$$\sum_{i=1}^{n} F_{x_i} h_i = 0 \qquad \text{for all vectors } H \tag{23}$$

This constitutes the first order condition for this problem and it is in usable form since it's true for all vectors H and so implies the equations

$$F_{x_i} = 0 \quad i = 1, \cdots, n \tag{24}$$

This is called the method of Lagrange Multiplers and with the n equations (24) together with the constraint equation, provides $n+1$ equations for the $n+1$ unknowns $x_1, \cdots, x_n, \lambda$.

Solving the previous problem by this method, we form the function

$$F = x^2 + y^2 + z^2 + \lambda(x^2 - z^2 - 1) \tag{25}$$

The system (24) together with the constraint give equations

$$\begin{aligned} 0 &= F_x = 2x + 2\lambda x = 2x(1+\lambda) && \text{(26a)} \\ 0 &= F_y = 2y && \text{(26b)} \\ 0 &= F_z = 2z - 2\lambda z = 2z(1-\lambda) && \text{(26c)} \\ \phi &= x^2 - z^2 - 1 = 0 && \text{(26d)} \end{aligned}$$

Now (26b) $\Rightarrow$ $y = 0$ and (26a) $\Rightarrow$ $x = 0$ or $\lambda = -1$. For the case $x = 0$ and $y = 0$ we have from (26d) that $z^2 = -1$ which gives no real solution. Trying the other possibility, $y = 0$ and $\lambda = -1$ then (26c) gives $z = 0$ and then (26d) gives $x^2 = 1$ or $x = \pm 1$. Thus the only possible points are $(\pm 1, 0, 0,)$.

The method covers the case of **more** than one constraint, say k constraints.

$$\phi_i = 0 \quad i = 1, \cdots, k < n \tag{27}$$

and in this situation there are k constants (one for each constraint) and the function

$$F = f + \sum_{i=1}^{k} \lambda_i \phi_i \tag{28}$$

satisfying (24). Thus here there are $k+n$ unknowns $\lambda_1, \cdots, \lambda_k,\ x_1, \cdots, x_n$ and $k+n$ equations to determine them, namely the n equations (24) together with the k constraints (27).

Problems

1. Use the method of Lagrange Multipliers to solve the problem

$$\text{minimize} \quad f = x^2 + y^2 + z^2$$
$$\text{subject to} \quad \phi = xy + 1 - z = 0$$

2. Show that

$$\max_{\lambda} \left| \frac{\lambda}{\cosh \lambda} \right| = \frac{\lambda_0}{\cosh \lambda_0}$$

where λ_0 is the positive root of

$$\cosh \lambda - \lambda \sinh \lambda = 0.$$

Sketch to show λ_0.

3. Of all rectangular parallelepipeds which have sides parallel to the coordinate planes, and which are inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

determine the dimensions of that one which has the largest volume.

4. Of all parabolas which pass through the points (0,0) and (1,1), determine that one which, when rotated about the x-axis, generates a solid of revolution with least possible volume between $x = 0$ and $x = 1$. [Notice that the equation may be taken in the form $y = x + cx(1-x)$, when c is to be determined.

5. a. If $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is a real vector, and $\mathbf{A}$ is a real symmetric matrix of order n, show that the requirement that

$$F \equiv \mathbf{x}^T \mathbf{A} \mathbf{x} - \lambda \mathbf{x}^T \mathbf{x}$$

be stationary, for a prescibed $\mathbf{A}$, takes the form

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$$

Deduce that the requirement that the quadratic form

$$\alpha \equiv \mathbf{x}^T \mathbf{A} \mathbf{x}$$

be stationary, subject to the constraint

$$\beta \equiv \mathbf{x}^T \mathbf{x} = \text{constant},$$

leads to the requirement

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x},$$

where λ is a constant to be determined. [Notice that the same is true of the requirement that β is stationary, subject to the constraint that $\alpha =$ constant, with a suitable definition of λ.]

b. Show that, if we write

$$\lambda = \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \equiv \frac{\alpha}{\beta},$$

the requirement that λ be stationary leads again to the matrix equation

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$$

[Notice that the requirement $d\lambda = 0$ can be written as

$$\frac{\beta d\alpha - \alpha d\beta}{\beta^2} = 0$$

or

$$\frac{d\alpha - \lambda d\beta}{\beta} = 0]$$

Deduce that stationary values of the ratio

$$\frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

are characteristic numbers of the symmetric matrix $\mathbf{A}$.

CHAPTER 2

2 Examples, Notation

In the last chapter we were concerned with problems of optimization for functions of a finite number of variables.

Thus we had to select values of n variables

$$x_1, \cdots, x_n$$

in order to solve for a minimum of the function

$$f(x_1, \cdots, x_n)\,.$$

Now we can also consider problems of an infinite number of variables such as selecting the value of y at each point x in some interval $[a, b]$ of the x axis in order to minimize (or maximize) the integral

$$\int_{x_1}^{x_2} F(x, y, y')dx\,.$$

Again as in the finite dimensional case, maximizing $\int_{x_1}^{x_2} F\,dx$ is the same as minimizing $\int_{x_1}^{x_2} -F\,dx$ so that we shall concentrate on minimization problems, it being understood that these include maximization problems.

Also as in the finite dimensional case we can speak of relative minima. An arc $\underline{y_0}$ is said to provide a relative minimum for the above integral if it provides a minimum of the integral over those arcs which (satisfy all conditions of the problem and) are in a neighborhood of $\underline{y_0}$. A neighborhood of $\underline{y_0}$ means a neighborhood of the points $(x, y_0(x), y_0'(x))$ $\quad x_1 \le x \le x_2$ so that an arc $\underline{y}$ is in this neighborhood if

$$\max_{x_1 \le x \le x_2} |y(x) - y_0(x)| \, < \, \gamma$$

and

$$\max_{x_1 \le x \le x_2} |y'(x) - y_0'(x)| \, < \, \gamma$$

for some $\gamma > 0$. *

Thus a relative minimum is in contrast to a global minimum where the integral is minimized over all arcs (which satisfy the conditions of the problem). Our results will generally be of this relative nature, of course any global minimizing arc is also a relative minimizing arc so that the necessary conditions which we prove for the relative case will also hold for the global case.

The simplest of all the problems of the calculus of variations is doubtless that of determining the shortest arc joining two given points. The co-ordinates of these points will be

*We shall later speak of a different type of relative minimum and a different type of neighborhood of $\underline{y_0}$.

denoted by (x_1, y_1) and (x_2, y_2) and we may designate the points themselves when convenient simply by the numerals 1 and 2. If the equation of an arc is taken in the form

$$\underline{y} : y(x) \quad (x_1 \leq x \leq x_2) \tag{1}$$

then the conditions that it shall pass through the two given points are

$$y(x_1) = y_1, \quad y(x_2) = y_2 \tag{2}$$

and we know from the calculus that the length of the arc is given by the integral

$$I = \int_{x_1}^{x_2} \sqrt{1 + y'^{\,2}}\, dx \,,$$

where in the evaluation of the integral, y' is to be replaced by the derivative $y'(x)$ of the function $y(x)$ defining the arc. There is an infinite number of curves $y = y(x)$ joining the points 1 and 2. The problem of finding the shortest one is equivalent analytically to that of finding in the class of functions $y(x)$ satisfying the conditions (2) one which makes the integral I a minimum.

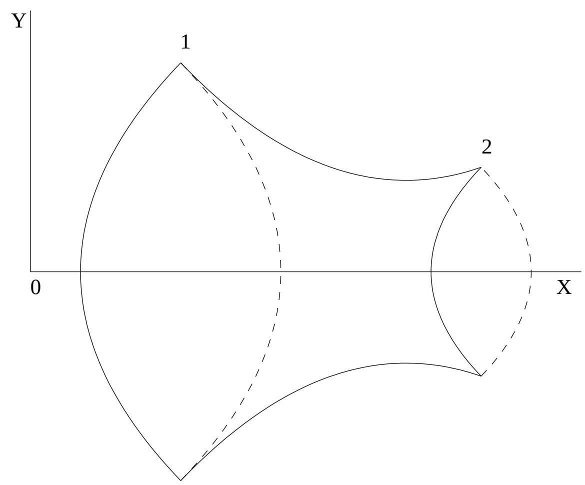

Figure 5: The surface of revolution for the soap example

There is a second problem of the calculus of variations, of a geometrical-mechanical type, which the principles of the calculus readily enable us to express also in analytic form. When a wire circle is dipped in a soap solution and withdrawn, a circular disk of soap film bounded by the circle is formed. If a second smaller circle is made to touch this disk and then moved away the two circles will be joined by a surface of film which is a surface of revolution (in the particular case when the circles are parallel and have their centers on the same axis perpendicular to their planes.) The form of this surface is shown in Figure 5. It is provable by the principles of mechanics, as one may surmise intuitively from the elastic properties of a soap film, that the surface of revolution so formed must be one of minimum area, and the problem of determining the shape of the film is equivalent therefore to that of determining such a minimum surface of revolution passing through two circles whose relative positions are supposed to be given as indicated in the figure.

In order to phrase this problem analytically let the common axis of the two circles be taken as the x-axis, and let the points where the circles intersect an xy-plane through that axis be 1 and 2. If the meridian curve of the surface in the xy-plane has an equation $y = y(x)$ then the calculus formula for the area of the surface is 2π times the value of the integral

$$I = \int_{x_1}^{x_2} y\sqrt{1 + y'^{\,2}}\, dx\,.$$

The problem of determining the form of the soap film surface between the two circles is analytically that of finding in the class of arcs $y = y(x)$ whose ends are at the points 1 and 2 one which minimizes the last-written integral I.

As a third example of problems of the calculus of variations consider the problem of the brachistochrone (shortest time) i.e. of determining a path down which a particle will fall from one given point to another in the shortest time. Let the y-axis for convenience be taken vertically downward, as in Figure 6, the two fixed points being 1 and 2.

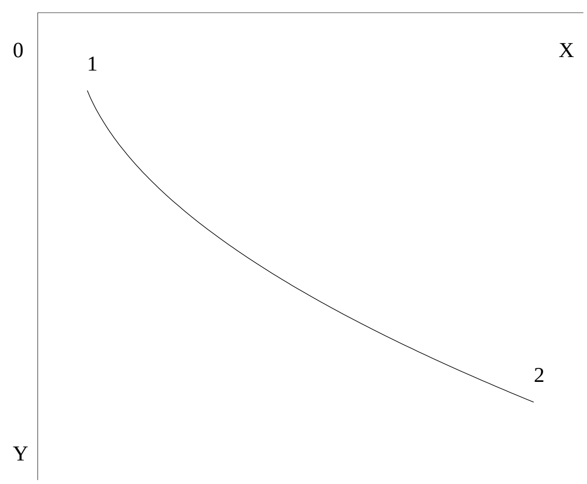

Figure 6: Brachistochrone problem

The initial velocity v_1 at the point 1 is supposed to be given. Later we shall see that for an arc defined by an equation of the form $y = y(x)$ the time of descent from 1 to 2 is $\dfrac{1}{\sqrt{2g}}$ times the value of the integral

$$I = \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^{\,2}}{y - \alpha}}\, dx\,,$$

where g is the gravitational constant and α has the constant value $\alpha = y_1 - \dfrac{v_1^2}{2g}$. The problem of the brachistochrone is then to find, among the arcs $\underline{y} : y(x)$ which pass through two points 1 and 2, one which minimizes the integral I.

As a last example, consider the boundary value problem

$$-u''(x) \;=\; r(x), \qquad 0 < x < 1$$

subject to

$$u(0) = 0, \qquad u(1) = 1.$$

The Rayleigh-Ritz method for this differential equation uses the solution of the following minimization problem:

Find u that minimizes the integral

$$I(u) \;=\; \int_0^1 \left[\frac{1}{2}(u')^2 \;-\; r(x)u\right] dx$$

where $u \epsilon V = \{v \epsilon C^2[0,1], v(0) = 0, v(1) = 0\}$. The function $r(x)$ can be viewed as force per unit mass.

2.1 Notation & Conventions

The above problems are included in the general problem of minimizing an integral of the form

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx \tag{3}$$

within the class of arcs which are **continuously differentiable** and also satisfy the end-point conditions

$$y(x_1) = y_1 \quad y(x_2) = y_2 \tag{4}$$

where y_1, y_2 are constants. In the previous three problems F was respectively $F = \sqrt{1 + y'^{\,2}}$, $F = y\sqrt{1 + y'^{\,2}}$, $F = \dfrac{\sqrt{1 + y'^{\,2}}}{\sqrt{y - \alpha}}$ and y_1, y_2 were the y coordinates associated with the points 1 and 2.

It should be noted that in (3) the symbols x, y, y' denote free variables and are not directly related to arcs. For example, we can differentiate with respect to these variables to get in the case of our last example

$$F_x = 0 \quad F_y = \frac{-1}{2}(y - \alpha)^{-3/2}(1 + y'^2)^{1/2}\,, \quad F_{y'} = y'(y - \alpha)^{-1/2}(1 + y'^{\,2})^{-1/2} \tag{5a}$$

It is when these functions are to be evaluated along an arc that we substitute $y(x)$ for y and $y'(x)$ for y'.

The above considered only the two dimensional case. In the $n + 1$ $(n > 1)$ dimensional case our arcs are represented by

$$\underline{y}: \qquad y_i(x) \quad x_1 \le x \le x_2 \quad i = 1, \cdots, n \tag{5b}$$

(the distinction between $y_i(x)$ and y_1, y_2 of (4) should be clear from the context) and the integral (3) is

$$I = \int_{x_1}^{x_2} F(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx \tag{6}$$

so that the integrals are functions of $2n+1$ variables and similar conventions to those for the two dimensional case hold for the $n+1$ dimensional case. Thus for example we will be interested in minimizing an integral of the form (6) among the class of **continuously differentiable** arcs (5b) which satisfy the end-point conditions

$$y_i(x_1) = y_{i,1} \qquad y_i(x_2) = y_{i,2} \quad i = 1, \cdots, n \tag{7}$$

where $y_{i,1},\ y_{i,2}$ are constants. For now, **continuously differentiable** arcs for which (6) is well-defined are called **admissible** arcs. Our problem in general will be to minimize the integral (6) over some sub-class of admissible arcs. In the type of problems where the end-points of the arcs are certain fixed values (as the problems thus far considered) the term fixed end point problem applies. In problems where the end points can vary, the term variable end point applies.

2.2 Shortest Distances

The shortest arc joining two points. Problems of determining shortest distances furnish a useful introduction to the theory of the calculus of variations because the properties characterizing their solutions are familiar ones which illustrate very well many of the general principles common to all of the problems suggested above. If we can for the moment eradicate from our minds all that we know about straight lines and shortest distances we shall have the pleasure of rediscovering well-known theorems by methods which will be helpful in solving more complicated problems.

Let us begin with the simplest case of all, the problem of determining the shortest arc joining two given points. The integral to be minimized, which we have already seen may be written in the form

$$I = \int_{x_1}^{x_2} F(y')dx \tag{8}$$

if we use the notation $F(y') = (1 + y'^{\,2})^{\frac{1}{2}}$, and the arcs $\underline{y} : y(x) \quad (x_1 \le x \le x_2)$ whose lengths are to be compared with each other will always be understood to be continuous with a tangent turning continuously, as indicated in Figure 7.

Analytically this means that on the interval $x_1 \le x \le x_2$ the function $y(x)$ is continuous, and has a continuous derivative. As stated before, we agree to call such functions *admissible functions* and the arcs which they define, admissible arcs. Our problem is then to find among all admissible arcs joining two given points 1 and 2 one which makes the integral I a minimum.

A first necessary condition. Let it be granted that a particular admissible arc

$$\underline{y_0} : \qquad y_0(x) \qquad (x_1 \le x \le x_2)$$

furnishes the solution of our problem, and let us then seek to find the properties which distinguish it from the other admissible arcs joining points 1 and 2. If we select arbitarily an admissible function $\eta(x)$ satisfying the conditions $\eta(x_1) = \eta(x_2) = 0$, the form

$$y_0(x) + \epsilon\eta(x) \qquad (x_1 \le x \le x_2)\,, \tag{9}$$

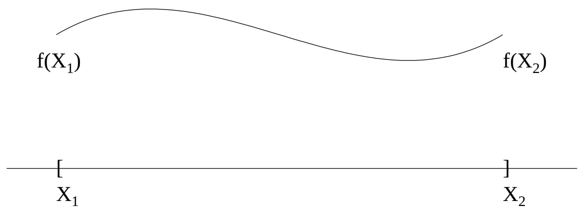

Figure 7: An arc connecting X_1 and X_2

involving the arbitrary constant a, represents a one-parameter family of arcs (see Figure 8) which includes the arc $\underline{y_0}$ for the special value $\epsilon = 0$, and all of the arcs of the family pass through the end-points 1 and 2 of $\underline{y_0}$ (since $\eta = 0$ at endpoints).

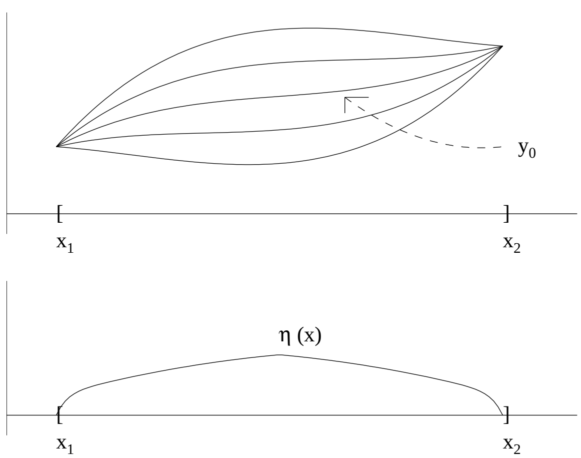

Figure 8: Admissible function η vanishing at end points (bottom) and various admissible functions (top)

The value of the integral I taken along an arc of the family depends upon the value of ϵ and may be represented by the symbol

$$I(\epsilon) = \int_{x_1}^{x_2} F(y_0' + \epsilon\eta')dx \,. \tag{10}$$

Along the initial arc $\underline{y_0}$ the integral has the value $I(0)$, and if this is to be a minimum when compared with the values of the integral along all other admissible arcs joining 1 with 2 it must, in particular, be a minimum when compared with the values $I(\epsilon)$ along the arcs of the family (9). Hence according to the criterion for a minimum of a function given previously we must have $I'(0) = 0$.

It should perhaps be emphasized here that the method of the calculus of variations, as it has been developed in the past, consists essentially of three parts; first, the deduction of necessary conditions which characterize a minimizing arc; second, the proof that these conditions, or others obtained from them by slight modifications, are sufficient to insure the minimum sought; and third, the search for an arc which satisfies the sufficient conditions. For the deduction of necessary conditions the value of the integral I along the minimizing arc can be compared with its values along any special admissible arcs which may be convenient for the purposes of the proof in question, for example along those of the family (9) described above, but the sufficiency proofs must be made with respect to all admissible arcs joining the points 1 and 2. The third part of the problem, the determination of an arc satisfying the sufficient conditions, is frequently the most difficult of all, and is the part for which fewest methods of a general character are known. For shortest-distance problems fortunately this determination is usually easy.

By differentiating the expression (10) with respect to ϵ and then setting $\epsilon = 0$ the value of $I'(0)$ is seen to be

$$I'(0) = \int_{x_1}^{x_2} F_{y'} \eta' dx \, , \tag{11}$$

where for convenience we use the notation $F_{y'}$ for the derivative of the integrand $F(y')$ with respect to y'. It will always be understood that the argument in F and its derivatives is the function $y_0'(x)$ belonging to the arc $\underline{y_0}$ unless some other is expressly indicated.

We now generalize somewhat on what we have just done for the shortest distance problem.

Recall that in the finite dimensional optimization problem, a point X_0 which is a relative (unconstrained) minimizing point for the function f has the property that

$$\sum_{i=1}^{n} f_{x_i} h_i = 0 \quad \text{and} \quad \sum_{i,j=1}^{n} f_{x_i x_j} h_i h_j \geq 0 \tag{12}$$

for all vectors $H = (h_1, \cdots, h_n)$ (where all derivatives of f are at X_0). These were called the first and second order **necessary** conditions.

We now try to establish analogous conditions for the two dimensional fixed end-point problem

$$\text{minimize} \;\; I = \int_{x_1}^{x_2} F(x, y, y') dx \tag{13}$$

among arcs which are **continuously differentiable**

$$\underline{y} : \qquad y(x) \qquad x_1 \leq x \leq x_2 \tag{14}$$

and which satisfy the end-point conditions

$$y(x_1) = y_1 \qquad y(x_2) = y_2 \tag{15}$$

with y_1, y_2 constants.

In the process of establishing the above analogy, we first establish the concepts of the first and second derivatives of an integral (13) about a general admissible arc. These concepts are analagous to the first and second derivatives of a function $f(X)$ about a general point X.

Let $\underline{y_0} : y_0(x), \quad x_1 \le x \le x_2$ be any **continuously differentiable** arc and let $\eta(x)$ be another such arc (nothing is required of the end-point values of $y_0(x)$ or $\eta(x)$). Form the **family of arcs**

$$y_0(x) + \epsilon\eta(x) \qquad x_1 \le x \le x_2 \tag{16}$$

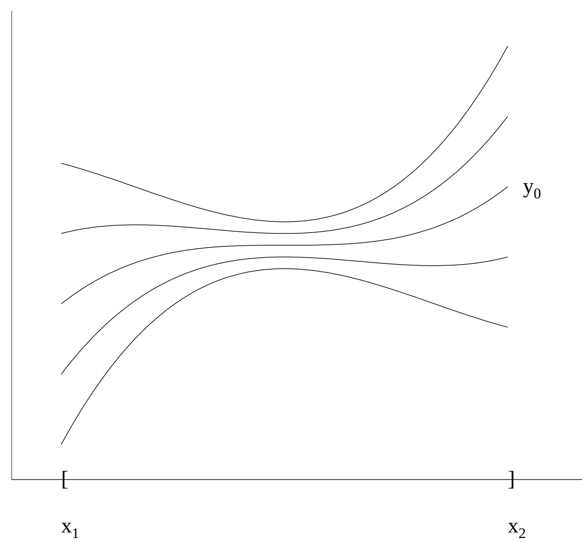

Figure 9: Families of arcs $y_0 + \epsilon\eta$

Then for sufficiently small values of ϵ say $-\delta \le \epsilon \le \delta$ with δ small, these arcs will all be in a neighborhood of $\underline{y_0}$ and will be admissible arcs for the integral (13). Form the function

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, y_0(x) + \epsilon\eta(x),\ y_0'(x) + \epsilon\eta'(x))dx, \quad -\delta < \epsilon < \delta \tag{17}$$

The derivative $I'(\epsilon)$ of this function is

$$\begin{aligned} I'(\epsilon) \quad = \quad & \int_{x_1}^{x_2} [F_y(x, y_0(x) + \epsilon\eta(x),\ y_0'(x) + \epsilon\eta'(x))\eta(x) + \\ & +F_{y'}(x, y_0(x) + \epsilon\eta(x), y_0'(x) + \epsilon\eta'(x))\eta'(x)]dx \end{aligned} \tag{18}$$

Setting $\epsilon = 0$ we obtain the first derivative of the integral I along $\underline{y_0}$

$$I'(0) = \int_{x_1}^{x_2} [F_y(x, y_0(x), y_0'(x))\eta(x) + F_{y'}(x, y_0(x), y_0'(x))\eta'(x)]dx \tag{19}$$

Remark: The first derivative of an integral I about an admissible arc $\underline{y_0}$ is given by (19).

Thus the first derivative of an integral I about an admissible arc $\underline{y_0}$ is obtained by evaluating I across a family of arcs containing $\underline{y_0}$ (see Figure 9) and differentiating that function at $\underline{y_0}$. Note how analagous this is to the first derivative of a function f at a point X_0 in the finite dimensional case. There one evaluates f across a family of points containing the point X_0 and differentiates the function.

We will often write (19) as

$$I'(0) = \int_{x_1}^{x_2} [F_y \eta + F_{y'} \eta'] dx \tag{20}$$

where it is understood that the arguments are along the arc $\underline{y_0}$.

Returning now to the function $I(\epsilon)$ we see that the second derivative of $I(\epsilon)$ is

$$\begin{aligned} I''(\epsilon) &= \int_{x_1}^{x_2} [F_{yy}(x, y_0(x) + \epsilon\eta(x), y_0'(x) + \epsilon\eta'(x))\eta^2(x) + \\ &\quad +2F_{yy'}(x, y_0(x) + \epsilon\eta(x), y_0'(x) + \epsilon\eta'(x))\eta(x)\eta'(x) + \\ &\quad +F_{y'y'}(x, y_0(x) + \epsilon\eta(x), y_0'(x) + \epsilon\eta'(x))\eta'^{\,2}(x)] dx \end{aligned} \tag{21}$$

Setting $\epsilon = 0$ we obtain the second derivative of I along $\underline{y_0}$. The second derivative of I about $\underline{y_0}$ corresponds to the second derivative of f about a point X_0 in finite dimensional problems.

$$\begin{aligned} I''(0) &= \int_{x_1}^{x_2} [F_{yy}(x, y_0(x), y_0'(x))\eta^2(x) + 2F_{yy'}(x, y_0(x), y_0'(x))\eta(x)\eta'(x) + \\ &\quad F_{y'y'}(x, y_0(x), y_0'(x))\eta'^{\,2}(x)] dx \end{aligned} \tag{22}$$

or more concisely

$$I''(0) = \int_{x_1}^{x_2} [F_{yy}\eta^2 + 2F_{yy'}\eta\eta' + F_{y'y'}\eta'^{\,2}] dx \tag{23}$$

where it is understood that all arguments are along the arc $\underline{y_0}$.

As an illustration, consider the integral

$$I = \int_{x_1}^{x_2} y(1 + y'^{\,2})^{1/2} dx \tag{24}$$

In this case we have

$$F = y(1 + y'^{\,2})^{1/2} \quad F_y = (1 + y'^{\,2})^{1/2} \quad F_{y'} = yy'(1 + y'^{\,2})^{-\frac{1}{2}} \tag{25}$$

so that the first derivative is

$$I'(0) = \int_{x_1}^{x_2} [(1 + y'^{\,2})^{1/2}\eta + yy'(1 + y'^{\,2})^{-1/2}\eta'] dx \tag{26}$$

Similarly

$$F_{yy} = 0 \quad F_{yy'} = y'(1 + y'^{\,2})^{-1/2} \quad F_{y'y'} = y(1 + y'^{\,2})^{-3/2} \tag{27}$$

and the second derivative is

$$I''(0) = \int_{x_1}^{x_2} [2y'(1 + y'^{\,2})^{-1/2}\eta\eta' + y(1 + y'^{\,2})^{-3/2}\eta'^{\,2}] dx\,. \tag{28}$$

The functions $\eta(x)$ appearing in the first and second derivatives of I along the arc $\underline{y_0}$ correspond to the directions H in which the family of points $X(\epsilon)$ was formed in chapter 1.

Suppose now that an admissible arc $\underline{y_0}$ gives a relative minimum to I in the class of admissible arcs satisfying $y(x_1) = y_1$, $y(x_2) = y_2$ where y_1, y_2, x_1, x_2 are constants defined in the problem. Denote this class of arcs by B. Then there is a neighborhood R_0 of the points $(x, y_0(x), y_0'(x))$ on the arc $\underline{y_0}$ such that

$$I_{y_0} \leq I_y \tag{29}$$

(where I_{y_0}, I_y means I evaluated along $\underline{y_0}$ and I evaluated along $\underline{y}$ respectively) for all arcs in B whose points lie in R_0. Next, select an arbitrary admissible arc $\eta(x)$ having $\eta(x_1) = 0$ and $\eta(x_2) = 0$. For all real numbers ϵ the arc $y_0(x) + \epsilon\eta(x)$ satisfies

$$y_0(x_1) + \epsilon\eta(x_1) = y_1 \ , \quad y_0(x_2) + \epsilon\eta(x_2) = y_2 \tag{30}$$

since the arc $\underline{y_0}$ satisfies (30) and $\eta(x_1) = 0$, $\eta(x_2) = 0$. Moreover, if ϵ is restricted to a sufficiently small interval $-\delta < \epsilon < \delta$, with δ small, then the arc $y_0(x) + \epsilon\eta(x)$ will be an admissible arc whose points be in R_0. Hence

$$I_{y_0+\epsilon\eta} \geq I_{y_0} \qquad -\delta < \epsilon < \delta \tag{31}$$

The function

$$I(\epsilon) = I_{y_0+\epsilon\eta}$$

therefore has a relative minimum at $\epsilon = 0$. Therefore from what we know about functions of one variable (i.e. $I(\epsilon)$), we must have that

$$I'(0) = 0 \qquad I''(0) \geq 0 \tag{32}$$

where $I'(0)$ and $I''(0)$ are respectively the first and second derivatives of I along $\underline{y_0}$. Since $\eta(x)$ was an arbitrary arc satisfying $\eta(x_1) = 0$, $\eta(x_2) = 0$, we have:

Theorem 2 If an admissible arc $\underline{y_0}$ gives a relative minimum to I in the class of admissible arcs with the same endpoints as $\underline{y_0}$ then

$$I'(0) = 0 \qquad I''(0) \geq 0 \tag{33}$$

(where $I'(0)$, $I''(0)$ are the first and second derivatives of I along $\underline{y_0}$) for all admissible arcs $\eta(x)$, with $\eta(x_1) = 0$ and $\eta(x_2) = 0$.

The above was done with all arcs $y(x)$ having just **one** component, i.e. the n dimensional case with $n = 1$. Those results extend to $n(n > 1)$ dimensional arcs
$\underline{y} : y_i(x) \ \ x_1 \leq x \leq x_2 \ \ i = 1, \cdots n)$.

In this case using our notational conventions the formula for the first and second derivatives of I take the form

$$I'(0) = \int_{x_1}^{x_2} \sum_{i=1}^{n} [F_{y_i}\eta_i + F_{y_i'}\eta_i'] dx \tag{34a}$$

$$I''(0) = \int_{x_1}^{x_2} \sum_{i,j=1}^{n} [F_{y_i y_j} \eta_i \eta_j + 2F_{y_i y'_j} \eta_i \eta'_j + F_{y'_i y'_j} \eta'_i \eta'_j] dx \tag{34b}$$

where $\eta' = \dfrac{d\eta}{dx}$.

Problems

1. For the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

with

$$f = y^{1/2} \left(1 + y'^2\right)$$

write the first and second variations $I'(0)$, and $I''(0)$.

2. Consider the functional

$$J(y) = \int_0^1 (1 + x)(y')^2 dx$$

where y is twice continuously differentiable and $y(0) = 0$ and $y(1) = 1$. Of all functions of the form

$$y(x) = x + c_1 x(1 - x) + c_2 x^2 (1 - x),$$

where c_1 and c_2 are constants, find the one that minimizes J.

CHAPTER 3

3 First Results

Fundamental Lemma. *Let $M(x)$ be a piecewise continuous function on the interval $x_1 \leq x \leq x_2$. If the integral*

$$\int_{x_1}^{x_2} M(x)\eta'(x)dx$$

vanishes for every function $\eta(x)$ with $\eta'(x)$ having at least the same order of continuity as does $M(x)$ † *and also satisfying $\eta(x_1) = \eta(x_2) = 0$, then $M(x)$ is necessarily a constant.*

To see that this is so we note first that the vanishing of the integral of the lemma implies also the equation

$$\int_{x_1}^{x_2} [M(x) - C]\eta'(x)dx = 0 \tag{1}$$

for every constant C, since all the functions $\eta(x)$ to be considered have $\eta(x_1) = \eta(x_2) = 0$.

The particular function $\eta(x)$ defined by the equation

$$\eta(x) = \int_{x_1}^{x} M(x)dx - C(x - x_1) \tag{2}$$

evidently has the value zero at $x = x_1$, and it will vanish again at $x = x_2$ if, as we shall suppose, C is the constant value satisfying the condition

$$0 = \int_{x_1}^{x_2} M(x)dx - C(x_2 - x_1)\,.$$

The function $\eta(x)$ defined by (2) with this value of C inserted is now one of those which must satisfy (1). Its derivative is $\eta'(x) = M(x) - C$ except at points where $M(x)$ is discontinuous, since the derivative of an integral with respect to its upper limit is the value of the integrand at that limit whenever the integrand is continuous at the limit. For the special function $\eta(x)$, therefore, (1) takes the form

$$\int_{x_1}^{x_2} [M(x) - C]^2 dx = 0$$

and our lemma is an immediate consequence since this equation can be true only if $M(x) \equiv C$.

With this result we return to the shortest distance problem introduced earlier. In (9) of the last chapter, $y = y_0(x) + \epsilon\eta(x)$ of the family of curves passing through the points 1 and 2, the function $\eta(x)$ was entirely arbitrary except for the restrictions that it should be admissible and satisfy the relations $\eta(x_1) = \eta(x_2) = 0$, and we have seen that the expression for (11) of that chapter for $I'(0)$ must vanish for every such family. The lemma just proven is therefore applicable and it tells us that along the minimizing arc $\underline{y_0}$ an equation

$$F_{y'} = \frac{y'}{\sqrt{1 + y'^2}} = C$$

†Thus if $M(x)$ is continuous (piecewise continuous), then $\eta'(x)$ should be continuous (at least piecewise continuous)

must hold, where C is a constant. If we solve this equation for y' we see that y' is also a constant along $\underline{y_0}$ and that the only possible minimizing arc is therefore a single straight-line joining the point 1 with the point 2.

The property just deduced for the shortest arc has so far only been proven to be necessary for a minimum. We have not yet demonstrated conclusively that the straight-line segment $\underline{y_0}$ joining 1 and 2 is actually shorter than every other admissible arc joining these points. This will be done later.

3.1 Two Important Auxiliary Formulas:

At this point we shall develop two special cases of more general formulas which are frequently applied in succeeding pages. Let $\underline{y_{34}}$ be a straight-line segment of variable length which moves so that its end-points describe simultaneously the two curves C and D shown in Figure 10, and let the equations of these curves in parametric form be

$$\begin{aligned} (C): \quad & x = x_1(t), y = y_1(t)\,, \\ (D): \quad & x = x_2(t), y = y_2(t)\,. \end{aligned}$$

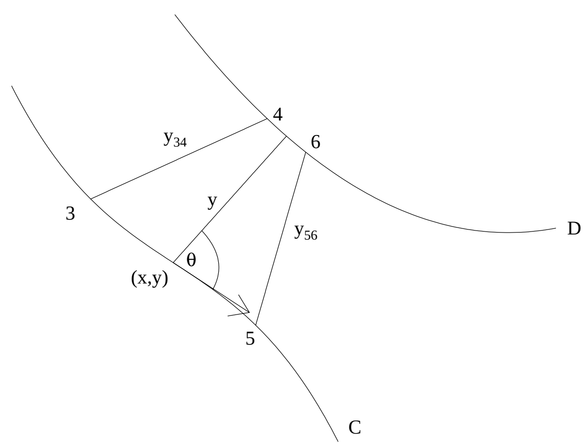

Figure 10: Line segment of variable length with endpoints on the curves C, D

For example, the point 3 in Figure 10 is described by an (x, y) pair at time t_1 as $x_3 = x_1(t_1)$, $y_3 = y_1(t_1)$. The other points are similarly given, $(x_4, y_4) = (x_2(t_1), y_2(t_1))$, $(x_5, y_5) = (x_1(t_2), y_1(t_2))$, and $(x_6, y_6) = (x_2(t_2), y_2(t_2))$.
The length ℓ

$$\ell = \sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2}$$

of the segment $\underline{y_{34}}$ has the differential

$$d\ell = \frac{(x_4 - x_3)(dx_4 - dx_3) + (y_4 - y_3)(dy_4 - dy_3)}{\sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2}}\,.$$

Note that since $\underline{y_{34}}$ is a straight line, then $(y_4 - y_3)/(x_4 - x_3)$ is the constant slope of the line . This slope is denoted by p. This result may be expressed in the convenient formula of the following theorem:

Theorem 3 *If a straight-line segment $\underline{y_{34}}$ moves so that its end-points 3 and 4 describe simultaneously two curves C and D, as shown in Figure 10, then the length ℓ of $\underline{y_{34}}$ has the differential*

$$d\ell(\underline{y_{34}}) = \frac{dx + pdy}{\sqrt{1+p^2}}\bigg|_3^4 \tag{3}$$

where the vertical bar indicates that the value of the preceding expression at the point 3 is to be subtracted from its value at the point 4. In this formula the differentials dx, dy at the points 3 and 4 are those belonging to C and D, while p is the constant slope of the segment $\underline{y_{34}}$.

We shall need frequently to integrate the right hand side of (3) along curves such as C and D. This is evidently justifiable along C, for example, since the slope $p = (y_4 - y_3)/(x_4 - x_3)$ is a function of t and since the differentials dx, dy can be calculated in terms of t and dt from the equations of C, so that the expression takes the form of a function of t. The integral I^* defined by the formula

$$I^* = \int \frac{dx + pdy}{\sqrt{1+p^2}}$$

will also be well defined along an arbitrary curve C when p is a function of x and y (and no longer a constant), provided that we agree to calculate the value of I^* by substituting for x, y, dx, dy the expressions for these variables in terms of t and dt obtained from the parametric equations of C.

It is important to note that I^* is parametrically defined, i.e. we integrate with respect to t. Before we state the next theorem, let's go back to Figure 10 to get the geometric interpretation of the integrand in I^*.

The integrand of I^* has a geometric interpretation at the points of C along which it is evaluated. At the point (x, y) on C, we can define two tangent vectors, one along the curve C (see Figure 10) and one along the line $\underline{y}$.

The tangent vector along C is given by

$$\vec{v}_1 = \frac{x'}{\sqrt{x'^2 + y'^2}}\vec{i} + \frac{y'}{\sqrt{x'^2 + y'^2}}\vec{j}$$

and the tangent vector along $\underline{y}$ is

$$\vec{v}_2 = \frac{1}{\sqrt{1+p^2}}\vec{i} + \frac{p}{\sqrt{1+p^2}}\vec{j}.$$

The angle θ between these two vectors $\vec{v}_1$ and $\vec{v}_2$ is given by the dot product (since the vectors are of unit length),

$$\cos\theta = \vec{v}_1 \cdot \vec{v}_2$$

or

$$\cos\theta = \frac{x' + py'}{\sqrt{(1+p^2)(x'^2 + y'^2)}}. \tag{4}$$

The element of arc length, ds, along C can be written as

$$ds = \sqrt{x'^2 + y'^2} dt$$

From (4) it follows that the integral I^* can also be expressed in the convenient form

$$I^* = \int \frac{dx + pdy}{\sqrt{1+p^2}} = \int \cos\theta \, ds \,. \tag{5}$$

Let t_3 and t_5 be two parameter values which define points 3 and 5 on C, and which at the same time define two corresponding points 4 and 6 on D, as in Figure 10. If we integrate the formula (3) with respect to t from t_3 to t_5 and use the notation I^* just introduced, we find as a further result:

Theorem 4 The difference of the lengths $\ell(\underline{y_{34}})$ and $\ell(\underline{y_{56}})$ of the moving segment in two positions $\underline{y_{56}}$ and $\underline{y_{34}}$ is given by the formula

$$\ell(\underline{y_{56}}) - \ell(\underline{y_{34}}) = I^*(D_{46}) - I^*(C_{35}) \,. \tag{6}$$

This and the formula (3) are the two important ones which we have been seeking. It is evident that they will still hold in even simpler form when one of the curves C or D degenerates into a point, since along such a degenerate curve the differentials dx and dy are zero.

We now do a similar investigation of a necessary condition for the general problem defined in (13) and (15) of the last chapter: Minimize an integral

$$I = \int_{x_1}^{x_2} F(x, y, y')dx \tag{7}$$

on the class β of admissible arcs joining two fixed points (x_1, y_1) and (x_2, y_2) in the xy plane (i.e. in 2 dimensional space). Suppose we are given an arc $\underline{y_0}$ that gives a relative minimum to I on the class β. Then by the previous chapter, the first derivative $I'(0)$ of I about $\underline{y_0}$ has the property that

$$I'(0) = \int_{x_1}^{x_2} [F_y \eta + F_{y'} \eta']dx = 0 \tag{8}$$

for all admissible arcs η with $\eta(x_1) = 0$ and $\eta(x_2) = 0$ where the arguments in the derivatives of F are along $\underline{y_0}$.

If we make use of the formula

$$\eta F_y(x) = \frac{d}{dx}(\eta \int_{x_1}^{x} F_y ds) - \eta' \int_{x_1}^{x} F_y ds \tag{9}$$

and the fact that $\eta(x_1) = \eta(x_2) = 0$ then (8) becomes

$$I'(0) = \int_{x_1}^{x_2} [F_{y'} - \int_{x_1}^{x} F_y ds]\eta' dx \tag{10}$$

Then by use of the fundamental lemma we find that

$$F_{y'}(x) = \int_{x_1}^{x} F_y ds + C \qquad x_1 \leq x \leq x_2 \tag{11}$$

holds at every point along $\underline{y_0}$. Since we are only thus far considering arcs on which $y'(x)$ is continuous, then we may differentiate (11) to obtain

$$\frac{d}{dx} F_{y'}(x) = F_y(x) \qquad x_1 \leq x \leq x_2 \tag{12}$$

along $\underline{y_0}$ (i.e. the arguments in $F_{y'}$ and F_y are those of the arc $\underline{y_0}$).

This is the famous Euler equation.

There is a second less well-known Euler equation, namely:

$$\frac{d}{dx}(F - y'F_{y'}) = F_x \tag{13}$$

which is true along $\underline{y_0}$.

For now, we prove this result only in the case that $\underline{y_0}$ is of class C^2 (i. e. has continuous second derivative y_0''). It is however true when $\underline{y_0}$ is of class C^1 (i.e. has continuous tangent) except at most at a finite number of points. Beginning with the left hand side of (13)

$$\frac{d}{dx}[F - y'F_{y'}] = F_x + F_y y' + \underbrace{F_{y'}y'' - y''F_{y'}}_{=0} - y'\frac{d}{dx}F_{y'} \tag{14}$$

Thus, factoring y' from last terms, we have

$$\frac{d}{dx}[F - y'F_{y'}] = F_x + y' \underbrace{\left(F_y - \frac{d}{dx}F_{y'}\right)}_{=0 \text{ by } (12)} \tag{15}$$

Thus we end up with the right hand of (13). This proves:

Theorem 5 The Euler equations (12) and (13) are satisfied by an admissible arc $\underline{y_0}$ *which provides a relative minimum to* I *in the class of admissible arcs joining its endpoints.*

Definition: An admissible arc $\underline{y_0}$ of class C^2 that satisfies the Euler equations on all of $[x_1, x_2]$ is called an extremal.

We note that the proof of (13) relied on the fact that (12) was true. Thus on arcs of class C^2, then (13) is not an independent result from (12). However (13) is valid on much more general arcs and on many of these constitutes an independent result from (12).

We call (12)-(13) the complete set of Euler equations.

Euler's equations are in general second order differential equations (when the 2^{nd} derivative y_0'' exists on the minimizing arc). There are however some special cases where these equations can be reduced to first order equations or algebraic equations. For example:

Case 1 Suppose that the integrand F does not depend on y, i. e. the integral to be minimized is

$$\int_{x_1}^{x_2} F(x, y')\, dx \tag{16}$$

where F does not contain y explicitly. In this case the first Euler's equation (12) becomes along an extremal

$$\frac{d}{dx} F_{y'} = 0 \tag{17}$$

or

$$F_{y'} = C \tag{18}$$

where C is a constant. This is a first order differential equation which does not contain y. This was the case in the shortest distance problem done before.

Case 2 If the integrand does not depend on the independent variable x, i. e. if we have to minimize

$$\int_{x_1}^{x_2} F(y, y')\, dx \tag{19}$$

then the second Euler equation (13) becomes

$$\frac{d}{dx}(F - y' F_{y'}) = 0 \tag{20}$$

or

$$F - y' F_{y'} = C \tag{21}$$

(where C is a constant) a first order equation.

Case 3 If F does not depend on y', then the first Euler equation becomes

$$0 = F_y(x, y) \tag{22}$$

which is not a differential equation, but rather an algebraic equation.

We next develop for our general problem the general version of the two auxiliary formulas (3) and (4) which were developed for the shortest distance problem.

3.2 *Two Important Auxiliary Formulas in the General Case*

For the purpose of developing our new equations let us consider a one-parameter family of extremal arcs

$$\underline{y} : y(x, b) \qquad (x_3 \le x \le x_4) \tag{23}$$

satisfying the Euler differential equation

$$\frac{\partial}{\partial x} F_{y'} = F_y\,. \tag{24}$$

The partial derivative symbol is now used because there are always the two variables x and b in our equations. If $x_3,\ x_4$ and b are all regarded as variables the value of the integral I along an arc of the family is a function of the form

$$I(x_3,\ x_4,\ b) = \int_{x_3}^{x_4} F(x, y(x,b),\ y'(x,b))dx\,.$$

With the help of Euler's equation (24), we see that along an extremal

$$\frac{\partial F}{\partial b} = \underbrace{F_y}_{\text{use (24)}} \frac{\partial y}{\partial b} + F_{y'}\frac{\partial y'}{\partial b} = \frac{\partial y}{\partial b}\frac{\partial}{\partial x}F_{y'} + \frac{\partial y'}{\partial b}F_{y'} = \frac{\partial}{\partial x}\left(F_{y'}\frac{\partial y}{\partial b}\right).$$

and the three partial derivatives of the function $I(x_3, x_4, b)$ have therefore the values

$$\frac{\partial I}{\partial x_3} = -F\Big|^3\,, \quad \frac{\partial I}{\partial x_4} = F\Big|^4\,,$$

$$\frac{\partial I}{\partial b} = \int_{x_3}^{x_4} \frac{\partial}{\partial x}\left(F_{y'}\frac{\partial y}{\partial b}\right)dx = F_{y'}\frac{\partial y}{\partial b}\Big|_3^4\,,$$

in which the arguments of F and its derivatives are understood to be the values y, y' belonging to the family (23).

Suppose now that the variables x_3, x_4, b are functions $x_3(t), x_4(t), b(t)$ of a variable t so that the end-points 3 and 4 of the extremals of the family (23) describe simultaneously two curves C and D in Figure 11 whose equations are

$$\begin{aligned} x &= x_1(t)\,, \qquad y = y(x_1(t), b(t)) = y_1(t)\,, \\ x &= x_2(t)\,, \qquad y = y(x_2(t), b(t)) = y_2(t)\,. \end{aligned} \tag{25}$$

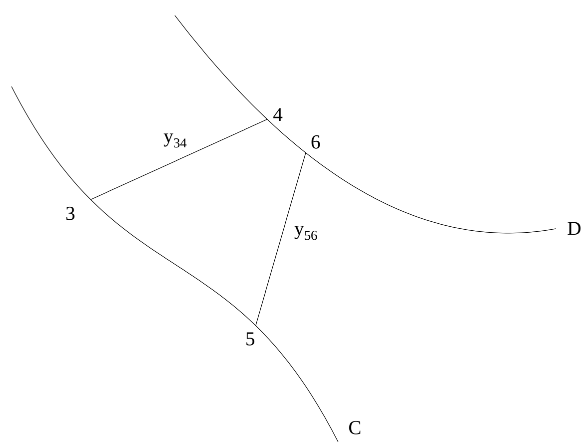

Figure 11: Curves described by endpoints of the family $y(x,b)$

The differentials $dx_3,\ dy_3$ and $dx_4,\ dy_4$ along these curves are found by attaching suitable subscripts 3 and 4 to dx, and dy in the equations

$$dx = x'(t)dt\,, \quad dy = y_x dx + y_b\,db\,. \tag{26}$$

From the formulas for the derivatives of I we now find the differential

$$dI = \frac{\partial I}{\partial x_3}dx_3 + \frac{\partial I}{\partial x_4}dx_4 + \frac{\partial I}{\partial b}db = (Fdx + F_{y'}y_b db)\Big|_3^4 = (Fdx + F_{y'}(dy - pdx))\Big|_3^4$$

where the vertical bar indicates the difference between the values at the points 4 and 3. With the help of the second of (26) this gives the following important result:

Theorem 6 The value of the integral I taken along a one-parameter family of extremal arcs $\underline{y_{34}}(x, b)$ whose end-points describe the two curves C and D shown in Figure 11 has the differential

$$dI = [F(x, y, p)dx + (dy - p\,dx)F_{y'}(x, y, p)]\Big|_3^4, \tag{27}$$

where at the points 3 and 4 the differentials dx, dy are those belonging to C and D, while y and p are the ordinate and slope of $\underline{y_{34}}(x, b)$.

We may denote by I^* the integral

$$I^* = \int \{F(x, y, p)dx + (dy - p\,dx)F_{y'}(x, y, p)\}\,.$$

If we integrate the formula (27) between the two values of t defining the points 3 and 5 in Figure 11 we find the following useful relation between values of this integral and the original integral I.

COROLLARY: For two arcs $\underline{y_{34}}(x, b)$ and $\underline{y_{56}}(x, b)$ of the family of extremals shown in Figure 11 the difference of the values of the integral I is given by the formula

$$I(\underline{y_{56}}(x, b)) - I(\underline{y_{34}}(x, b)) = I^*(D_{46}) - I^*(C_{35})\,. \tag{28}$$

Let us now use the results just obtained in order to attack the Brachistochrone problem introduced in chapter 2. That problem is to find the path joining points 1 and 2 such that a particle starting at point 1 with velocity v_1 and acted upon only by gravity will reach point 2 in minimum time.

It is natural at first sight to suppose that a straight line is the path down which a particle will fall in the shortest time from a given point 1 to a second given point 2, because a straight line is the shortest distance between the two points, but a little contemplation soon convinces one that this is not the case. John Bernoulli explicitly warned his readers against such a supposition when he formally proposed the brachistochrone problem in 1696. The surmise, suggested by Galileo's remarks on the brachistochrone problem, that the curve of quickest descent is an arc of a circle, is a more reasonable one, since there seems intuitively some justification for thinking that steepness and high velocity at the beginning of a fall will conduce to shortness in the time of descent over the whole path. It turns out, however, that this characteristic can also be overdone; the precise degree of steepness required at the start can in fact only be determined by a suitable mathematical investigation.

The first step which will be undertaken in the discussion of the problem in the following pages is the proof that a brachistochrone curve joining two given points must be a cycloid.

A cycloid is the arched locus of a point on the rim of a wheel which rolls on a horizontal line, as shown in Figure 12. It turns out that the brachistochrone must consist of a portion of one of the arches turned upside down, and the one on the underside of which the circle rolls must be located at just the proper height above the given initial point of fall.

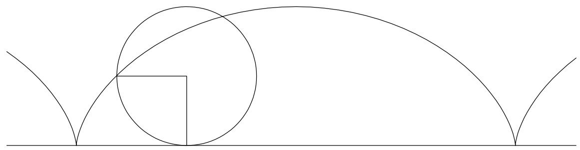

Figure 12: Cycloid

When these facts have been established we are then faced with the problem of determining whether or not such a cycloid exists joining two arbitrarily given points. Fortunately we will be able to prove that two points can always be joined by one and only one cycloid of the type desired.

The analytic formulation of the problem. In order to discuss intelligently the problem of the brachistochrone we should first obtain the integral which represents the time required by a particle to fall under the action of gravity down an arbitrarily chosen curve joining two fixed points 1 and 2. Assume that the initial velocity v_1 at the point 1 is given, and that the particle is to fall without friction on the curve and without resistance in the surrounding medium. If the effects of friction or a resisting medium are to be taken into account the brachistochrone problem becomes a much more complicated one.

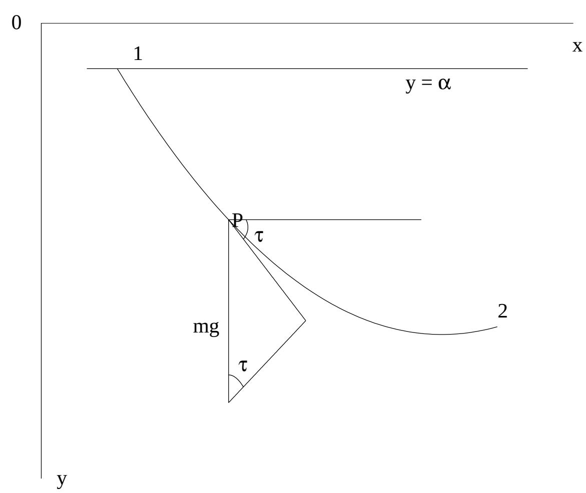

Figure 13: A particle falling from point 1 to point 2

Let m be the mass of the moving particle P in Figure 13 and s the distance through which it has fallen from the point 1 along the curve of descent C in the time t. In order to make our analysis more convenient we may take the positive y-axis vertically downward, as shown in the figure. The vertical force of gravity acting upon P is the product of the mass m by the gravitational acceleration g, and the only force acting upon P in the direction of the tangent line to the curve is the projection $mg\sin\tau$ of this vertical gravitational force upon that line. But the force along the tangent may also be computed as the product $m\frac{d^2s}{dt^2}$ of the mass of the particle by its acceleration along the curve. Equating these two values we find the equation

$$\frac{d^2s}{dt^2} = g\sin\tau = g\frac{dy}{ds}$$

in which a common factor m has been cancelled and use has been made of the formula $\sin\tau = \frac{dy}{ds}$.

To integrate this equation we multiply each side by $2\frac{ds}{dt}$. The antiderivatives of the two sides are then found, and since they can differ only by a constant we have

$$\left(\frac{ds}{dt}\right)^2 = 2gy + c\,. \tag{29}$$

The value of the constant c can be determined if we remember that the values of y and $v = \frac{ds}{dt}$ at the initial point 1 of the fall are y_1 and v_1, respectively, so that for $t = 0$ the last equation gives

$$v_1^2 = 2gy_1 + c\,.$$

With the help of the value of c from this equation, and the notation

$$\alpha = y_1 - \frac{v_1^2}{2g}\,, \tag{30}$$

equation (29) becomes

$$\left(\frac{ds}{dt}\right)^2 = 2gy + v_1^2 - 2gy_1 = 2gy + 2g\left(\frac{v_1^2}{2g} - y_1\right) = 2g(y - \alpha)\,. \tag{31}$$

An integration now gives the following result *The time T required by a particle starting with the initial velocity v_1 to fall from a point 1 to a point 2 along a curve is given by the integrals*

$$T = \frac{1}{\sqrt{2g}}\int_0^{\ell}\frac{ds}{\sqrt{y-\alpha}} = \frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\sqrt{\frac{1+y'^2}{y-\alpha}}dx \tag{32}$$

where ℓ is the length of the curve and $\alpha = y_1 - \frac{v_1^2}{2g}$.

An arc which minimizes one of the integrals (32) expressing T will also minimize that integral when the factor $\frac{1}{\sqrt{2g}}$ is omitted, and vice versa. Let us therefore use the notations

$$I = \int_{x_1}^{x_2} F(y, y')dx\,, \quad F(y, y') = \sqrt{\frac{1+y'^2}{y-\alpha}} \tag{33}$$

for our integral which we seek to minimize and its integrand. Since the value of the function $F(y, y')$ is infinite when $y = \alpha$ and imaginary when $y < \alpha$ we must confine our curves to the portion of the plane which lies below the line $y = \alpha$ in figure 13. This is not really a restriction of the problem since the equation $v^2 = \left(\frac{ds}{dt}\right)^2 = 2g(y - \alpha)$ deduced above shows that a particle started on a curve with the velocity v_1 at the point 1 will always come to rest if it reaches the altitude $y = \alpha$ on the curve, and it can never rise above that altitude. For the present we shall restrict our curves to lie in the half-plane $y > \alpha$.

In our study of the shortest distance problems the arcs to be considered were taken in the form $\underline{y} : y(x)$ $(x_1 \le x \le x_2)$ with $y(x)$ and $y'(x)$ continuous on the interval $x_1 \le x \le x_2$, *An admissible arc* for the brachistochrone problem will always be understood to have these properties besides the additional one that it lies entirely in the half-plane $y > \alpha$. The integrand $F(y, y')$ and its partial derivatives are:

$$F = \sqrt{\frac{1 + y'^2}{y - \alpha}}, \quad F_y = \frac{-1}{2}\sqrt{\frac{1 + y'^2}{(y - \alpha)^3}}, \quad F_{y'} = \frac{y'}{\sqrt{(y - \alpha)(1 + y'^2)}} \tag{34}$$

Since our integrand in (33) is independent of x we may use the case 2 special result (21) of the Euler equations.

When the values of F and its derivative $F_{y'}$ for the brachistochrone problem are substituted from (34) this equation becomes

$$F - y'F_{y'} = \frac{1}{\sqrt{(y - \alpha)(1 + y'^2)}} = \frac{1}{\sqrt{2b}}, \tag{35}$$

the value of the constant being chosen for convenience in the form $\frac{1}{\sqrt{2b}}$.

The curves which satisfy the differential equation (35) may be found by introducing a new variable u defined by the equation

$$y' = -\tan\frac{u}{2} = -\frac{\sin u}{1 + \cos u}. \tag{36}$$

From the differential equation (35) it follows then, with the help of some trigonometry, that along a minimizing arc $\underline{y_0}$ we must have

$$y - \alpha = \frac{2b}{1 + y'^2} = 2b\cos^2\frac{u}{2} = b(1 + \cos u)$$

Thus

$$\frac{dy}{du} = -b\sin u.$$

Now

$$\frac{dx}{du} = \frac{dx}{dy}\frac{dy}{du} = -\frac{1 + \cos u}{\sin u}(-b\sin u) = b(1 + \cos u)$$

Integrating, we get x

$$x = a + b(u + \sin u)$$

where a is the new constant of integration. It will soon be shown that curves which satisfy the first and third of these equations are the cycloids described in the following theorem:

Theorem 7 A curve down which a particle, started with the initial velocity v_1 at the point 1, will fall in the shortest time to a second point 2 is necessarily an arc having equations of the form

$$x - a = b(u + \sin u)\,, \qquad y - \alpha = b(1 + \cos u)\,. \tag{37}$$

These represent the locus of a point fixed on the circumference of a circle of radius b as the circle rolls on the lower side of the line $y = \alpha = y_1 - \dfrac{v_1^2}{2g}$. Such a curve is called a cycloid.

Cycloids. The fact that (37) represent a cycloid of the kind described in the theorem is proved as follows: Let a circle of radius b begin to roll on the line $y = \alpha$ at the point whose co-ordinates are (a, α), as shown in Figure 14. After a turn through an angle of u radians the point of tangency is at a distance bu from (a, α) and the point which was the lowest in the circle has rotated to the point (x, y). The values of x and y may now be calculated in terms of u from the figure, and they are found to be those given by (37).

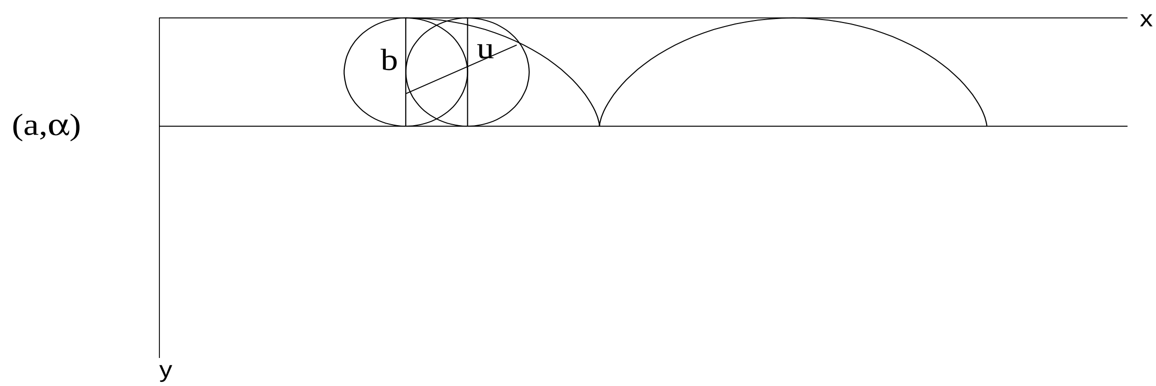

Figure 14: Cycloid

The fact that the curve of quickest descent must be a cycloid is the famous result discovered by James and John Bernoulli in 1697 and announced at approximately the same time by a number of other mathematicians.

We next continue using the general theory results to develop two auxiliary formulas for the Brachistochrone problem which are the analogues of (3), (4) for the shortest distance problem.

Two Important Auxiliary Formulas If a segment $\underline{y_{34}}$ of a cycloid varies so that its end-points describe two curves C and D, as shown in Figure 15 then it is possible to find a formula for the differential of the value of the integral I taken along the moving segment, and a formula expressing the difference of the values of I at two positions of the segment.

The equations

$$x = a(t) + b(t)(u + \sin u)\,, \qquad y = \alpha + b(t)(1 + \cos u)$$

$$(u_3(t) \leq u \leq u_4(t)) \tag{38}$$

define a one-parameter family of cycloid segments $\underline{y_{34}}$ when a, b, u_3, u_4 are functions of a parameter t as indicated in the equations. If t varies, the end-points 3 and 4 of this segment describe the two curves C and D whose equations in parametric form with t as independent variable are found by substituting $u_3(t)$ and $u_4(t)$, respectively, in (38). These curves and two of the cycloid segments joining them are shown in Figure 15.

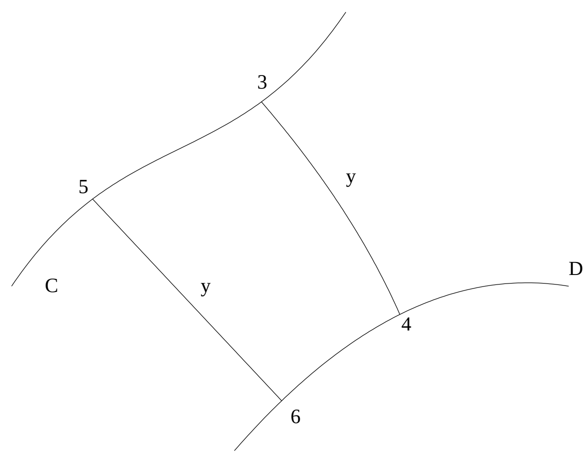

Figure 15: Curves C, D described by the endpoints of segment $\underline{y_{34}}$

Now applying (27) of the general theory to this problem, regrouping (27), then the integral in (33) has the differential

$$d\ell = (F - pF_{y'})dx + F_{y'}dy \tag{39}$$

where (recalling (27)) the differentials dx, dy in (39) are those of C and D while p is the slope of $\underline{y_{34}}$. Then by (35) and the last part of (34) substituted into (39) the following important result is obtained.

Theorem 8 If a cycloid segment $\underline{y_{34}}$ varies so that its end-points 3 and 4 describe simultaneously two curves C and D, as shown in Figure 15, then the value of the integral I taken along $\underline{y_{34}}$ has the differential

$$d\ell = \left. \frac{dx + pdy}{\sqrt{y-\alpha}\sqrt{1+p^2}} \right|_3^4 \tag{40}$$

At the points 3 and 4 the differentials dx, dy in this expression are those belonging to C and D, while p is the slope of the segment $\underline{y_{34}}$.

If the symbol I^* is now used to denote the integral

$$I^* = \int \frac{dx + p\,dy}{\sqrt{y-\alpha}\sqrt{1+p^2}} \tag{41}$$

then by an integration of the formula (39) with respect to t from t_3 to t_5 we find the further result that

Theorem 9 The difference between the values of ℓ at two different positions $\underline{y_{34}}$ and $\underline{y_{56}}$ of the variable cycloid segment, shown in Figure 15, is given by the formula

$$\ell(\underline{y_{56}}) - \ell(\underline{y_{34}}) = I^*(D_{46}) - I^*(C_{35})\,. \tag{42}$$

The formulas (40) and (42) are the analogues for cycloids of the formulas (3) and (4) for the shortest distance problems. We shall see that they have many applications in the theory of brachistochrone curves.

Problems

1. Find the extremals of

$$I = \int_{x_1}^{x_2} F(x, y, y')\,dx$$

for each case

a. $F = (y')^2 - k^2y^2 \qquad (k \text{ constant})$

b. $F = (y')^2 + 2y$

c. $F = (y')^2 + 4xy'$

d. $F = (y')^2 + yy' + y^2$

e. $F = x\,(y')^2 - yy' + y$

f. $F = a(x)\,(y')^2 - b(x)y^2$

g. $F = (y')^2 + k^2 \cos y$

2. Solve the problem minimize $I = \displaystyle\int_a^b \left[(y')^2 - y^2\right]\,dx$

with

$$y(a) = y_a, \qquad y(b) = y_b.$$

What happens if $b - a = n\pi$?

3. Show that if $F = y^2 + 2xyy'$, then I has the same value for all curves joining the endpoints.

4. A geodesic on a given surface is a curve, lying on that surface, along which distance between two points is as small as possible. On a plane, a geodesic is a straight line. Determine equations of geodesics on the following surfaces:

a. Right circular cylinder. [Take $ds^2 = a^2 d\theta^2 + dz^2$ and minimize $\displaystyle\int \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\,d\theta$ or $\displaystyle\int \sqrt{a^2 \left(\frac{d\theta}{dz}\right)^2 + 1}\,dz$]

b. Right circular cone. [Use spherical coordinates with $ds^2 = dr^2 + r^2 \sin^2 \alpha d\theta^2$.]

c. Sphere. [Use spherical coordinates with $ds^2 = a^2 \sin^2 \phi d\theta^2 + a^2 d\phi^2$.]

d. Surface of revolution. [Write $x = r\cos\theta$, $y = r\sin\theta$, $z = f(r)$. Express the desired relation between r and θ in terms of an integral.]

5. Determine the stationary function associated with the integral

$$I = \int_0^1 (y')^2 \, f(x) \, ds$$

when $y(0) = 0$ and $y(1) = 1$, where

$$f(x) = \begin{cases} -1 & 0 \leq x < \frac{1}{4} \\ 1 & \frac{1}{4} < x \leq 1 \end{cases}$$

6. Find the extremals

a. $J(y) = \int_0^1 y' dx, \qquad y(0) = 0, \, y(1) = 1.$

b. $J(y) = \int_0^1 yy' dx, \qquad y(0) = 0, \, y(1) = 1.$

c. $J(y) = \int_0^1 xyy' dx, \qquad y(0) = 0, \, y(1) = 1.$

7. Find extremals for

a. $J(y) = \int_0^1 \frac{y'^2}{x^3} dx,$

b. $J(y) = \int_0^1 \left(y^2 + (y')^2 + 2ye^x \right) dx.$

8. Obtain the necessary condition for a function y to be a local minimum of the functional

$$J(y) = \int \int_R K(s,t) y(s) y(t) ds dt + \int_a^b y^2 dt - 2 \int_a^b y(t) f(t) dt$$

where $K(s,t)$ is a given continuous function of s and t on the square R, for which $a \leq s, t \leq b$, $K(s,t)$ is symmetric and $f(t)$ is continuous.

Hint: the answer is a Fredholm integral equation.

9. Find the extremal for

$$J(y) = \int_0^1 (1+x)(y')^2 dx, \qquad y(0) = 0, \, y(1) = 1.$$

What is the extremal if the boundary condition at $x = 1$ is changed to $y'(1) = 0$?

10. Find the extremals

$$J(y) = \int_a^b \left(x^2 (y')^2 + y^2 \right) dx.$$

CHAPTER 4

4 Variable End-Point Problems

We next consider problems in which one or both end-points are not fixed.

For illustration we again consider the shortest arc problem. However now we investigate the shortest arc from a fixed point to a curve.

If a fixed point 1 and a fixed curve N are given instead of two fixed points then the shortest arc joining them must again be a straight-line segment, but this property alone is not sufficient to insure a minimum length. There are two further conditions on the shortest line from a point to a curve for which we shall find very interesting analogues in connection with the problems considered in later chapters.

Let the equations of the curve N in Figure 16 be written in terms of a parameter τ in the form

$$x = x(\tau)\,, \quad y = y(\tau)\,,$$

Let $\underline{y_{12}}$ be the solution to the problem of finding the shortest arc from point 1 to curve N.

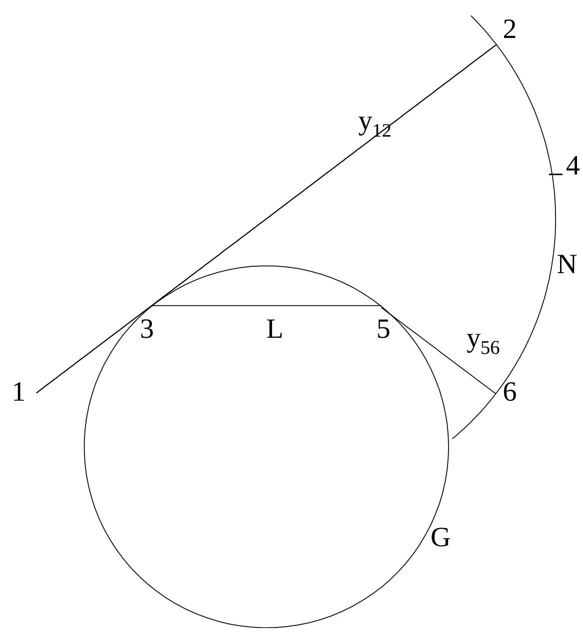

Figure 16: Shortest arc from a fixed point 1 to a curve N. G is the evolute

Let τ_2 be the parameter value defining the intersection point 2 of N. Clearly the arc $\underline{y_{12}}$ is a straight-line segment. The length of the straight-line segment joining the point 1 with an arbitrary point $(x(\tau)\,,\ y(\tau))$ of N is a function $I(\tau)$ which must have a minimum at the value τ_2 defining the particular line $\underline{y_{12}}$. The formula (3) of chapter 3 is applicable to the one-parameter family of straight lines joining 1 with N when in that formula we replace C by the point 1 and D by N. Since along C (now degenerated to a point) the differentials

dx, dy are then zero it follows that the differential of the function $I(\tau)$ along the arc $\underline{y_{12}}$ is

$$dI = \frac{dx + pdy}{\sqrt{1+p^2}}\Big|^2$$

where the bar indicates that the value of the preceding expression is to be taken at the point 2. Since for a minimum the differential dI must vanish it follows that at the point 2 the differentials dx, dy of N and the slope p of $\underline{y_{12}}$ satisfy the condition $dx + pdy = 0$, and hence that these two curves must intersect at right angles (see (5) of chapter 3).

Even a straight-line segment through 1 and intersecting N at right angles may not be a shortest arc joining 1 with N, as may be seen with the help of the familiar string property of the evolute of N‡. The segments of the straight lines perpendicular to N cut off by N and its evolute G in Figure 16 form a family to which the formula (6) of chapter 3 is applicable. If in that formula we replace the curve C by G and D by N then (note that the points 2,3,5,6 are vertices of a quadrilateral similar to figure 11)

$$\ell(\underline{y_{56}}) - \ell(\underline{y_{32}}) = I^*(N_{26}) - I^*(G_{35})\,.$$

But by using (5) of chapter 3 the integrals on the right hand side of this formula are seen to have the values

$$I^*(N_{26}) = \int_{s_1}^{s_2} \cos\theta ds = 0\,, \qquad I^*(G_{35}) = I(G_{35})$$

since $\cos\theta = 0$ along N (the straight lines of the family meet N at right angles), and $\cos\theta = 1$ along the envelope G (to which these lines are tangent). Hence from the next to last equation we have the formula

$$\ell(\underline{y_{32}}) = I(G_{35}) + \ell(\underline{y_{56}})\,.$$

This is the string property of the evolute, for it implies that the lengths of the arcs $\underline{y_{32}}(x)$ and $G_{35} + \underline{y_{56}}$ are the same, and hence that the free end 6 of the string fastened at 3 and allowed to wrap itself around the evolute G will describe the curve N.

It is evident now that the segment $\underline{y_{12}}$ cannot be a shortest line from 1 to N if it has on it a contact point 3 with the evolute G of N. For the composite arc $\underline{y_{13}} + G_{35} + \underline{y_{56}}$ would in that case have the same length as $\underline{y_{12}}$ and the arc $\underline{y_{13}} + L_{35} + \underline{y_{56}}$ formed with the straight line segment L_{35}, would be shorter than $\underline{y_{12}}$. It follows then that:

If an arc $\underline{y_{12}}$ intersecting the curve N at the point 2 is to be the shortest joining 1 with N it must be a straight line perpendicular to N at the point 2 and having on it no contact point with the evolute G of N.

Our main purpose in this section was to obtain the straight line condition and also the perpendicularity condition at N for the minimizing arc as we have done above. This last result concerning the evolute G, is a hint of something that we shall see more of later on.

‡The evolute of a curve is the locus of the centers of curvature of the given curve. The family of straight lines normal to a given curve are tangent to the evolute of this curve, and the changes in length of the radius of curvature is equal to the change in length of arc of the evolute as the point on the curve moves continuously in one direction along the curve.

4.1 The General Problem

We now consider the general problem:

Minimize the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')dx \tag{1}$$

on the class of arcs joining fixed point 1 with coordinates (x, y) with the curve N. Note that now point 2 with coordinates (x_2, y_2) is not fixed since it is as yet an undetermined point on N.

Necessary conditions when one end-point is variable. A minimizing arc $\underline{y_{12}}$ for this problem, meeting the curve N at the point 2, must evidently be a minimizing arc for the problem with end-points fixed at 1 and 2, and hence must satisfy at least the necessary conditions (12), (13) of chapter 3.

For the problem with one variable end point there is a new necessary condition for a minimum, involving the directions of the curves $\underline{y_{12}}$ and N at their intersection point 2, which is called the *transversality condition*. This condition may be proved with the help of the formula (27) of the last chapter. Let the points of N be joined to the point 1 of $\underline{y_{12}}$ by a one-parameter family of arcs containing $\underline{y_{12}}$ as one member of the family. If the curve C of the formula just cited is replaced by the fixed point 1, and the curve D by N, then this formula shows that the value of I taken along the arcs of the one-parameter family has at the particular arc $\underline{y_{12}}$ the differential

$$dI = [F(x, y, y')dx + (dy - y'dx)F_{y'}(x, y, y')]\Big|^{2},$$

where at the point 2 the differentials dx, dy are those of N and the element (x, y, y') belongs to $\underline{y_{12}}$. If the values of I along the arcs of the family are to have $I(\underline{y_{12}})$ as a minimum then the differential dI must vanish along $\underline{y_{12}}$ and we have the following result:

THE TRANSVERSALITY CONDITION. *If for an admissible arc $\underline{y_{12}}$ joining a fixed point 1 to a fixed curve N the value $I(\underline{y_{12}})$ is a minimum with respect to the values of I on neighboring admissible arcs joining 1 with N, then at the intersection point 2 of $\underline{y_{12}}$ and N the direction $dx : dy$ of N and the element (x, y, y') of $\underline{y_{12}}$ must satisfy the relation*

$$F(x, y, y')dx + (dy - y'dx)F_{y'}(x, y, y') = 0\,. \tag{2}$$

If this condition is satisfied the arc N is said to cut $\underline{y_{12}}$ transversally at the point 2. When the arc N is the vertical line $x = x_1$ or $x = x_2$, this condition is called a natural boundary condition. For many problems the transversality condition implies that $\underline{y_{12}}$ and N must meet at right angles. Indeed (2) when applied to the shortest distance problem gives the condition of perpendicularity obtained there. However (2) does not in general imply perpendicularity as one may verify in many special cases.

By a slight modification of the above reasoning we may treat the problem of minimizing the integral (1) on the class of arcs joining two given curves C and D as in Figure 11. Let

$\underline{y_{12}}$ be a minimizing arc meeting curves C and D at points 1 and 2 respectively. Then $\underline{y_{12}}$ must also be a minimizing arc for the problem with fixed endpoints 1 and 2 and hence must sastisfy the necessary conditions (12) and (13) of the last chapter. Furthermore, $\underline{y_{12}}$ is also a minimizing arc for the problem of joining point 1 with the curve D so that the transversality condition just deduced for the problem with one end-point varying must hold at point 2. By a similar argument, with point 2 fixed for arcs joining point 2 with C, we see that the transversality condition must also hold at point 1. Thus we have:

THE TRANSVERSALITY CONDITION (when both end-points vary). *If for an admissible arc* $\underline{y_{12}}$ *joining two fixed curves* C *and* D*, the value* $I(\underline{y_{12}})$ *is a minimum with respect to the values of* I *on neighboring admissible arcs joining* C *and* D*, then at the intersection points 1 and 2 of* $\underline{y_{12}}$ *with* C *and* D *respectively, the directions* $dx : dy$ *of* C *and the element* (x, y, y') *of* $\underline{y_{12}}$ *at points 1 and 2 must satisfy the separate relations*

$$[F(x,y,y')dx + (dy - y'dx)F_{y'}(x,y,y')]\Big|_i = 0 \qquad i = 1,2 \tag{3}$$

We now use the results just developed for the general theory by applying them to the brachistochrone problem.

The path of quickest descent from a point to a curve. First necessary conditions. At the conclusion of his now famous solution of the brachistochrone problem, published in 1697, James Bernoulli proposed to other mathematicians, but to his brother in particular, a number of further questions. One of them was the problem of determining the arc down which a particle, starting with a given initial velocity, will fall in the shortest time from a fixed point to a fixed vertical straight line. This is a special case of the more genreral problem of determining the brachistochrone arc joining a fixed point 1 to an arbitrarily chosen fixed curve N.

Let the point 1, the curve N, and the path $\underline{y_{12}}$ of quickest descent be those shown in Figure 17, (where α has the significance described in the previous chapter), and let the given initial velocity at the point 1 again be v_1. Since by our general theory just developed, we know that Euler's equations (12) and (13) of the previous chapter apply, then by what has been shown in chapter 3, the minimizing arc $\underline{y_{12}}$ must be a cycloid joining point 1 to some as yet undetermined point 2 on the curve N. This constitutes a first necessary condition for this problem.

Applying (2) to the present problem and using (33) of chapter 3 gives at point 2

$$\frac{\sqrt{1+y'^2}}{\sqrt{y-\alpha}}dx + (dy - y'dx)\frac{y'}{\sqrt{1+y'^2}\sqrt{y-\alpha}} = 0 \tag{4}$$

where y', y are values on the minimizing arc $\underline{y_{12}}$ at point 2 and dy, dx are values of the curve N at point 2.

After multiplying and dividing by $\sqrt{1+y'^2}$ one obtains the condition

$$dx + y'dy = 0 \tag{5}$$

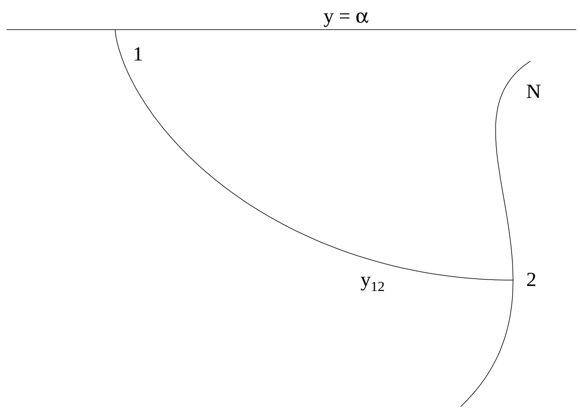

Figure 17: Path of quickest descent, $\underline{y_{12}}$, from point 1 to the curve N

which is the transversality condition for this problem. This condition means that $\underline{y_{12}}$ must be perpendicular to curve N at point 2. So the transversality condition here as in the shortest distance problem, is one of perpendicularity, but as already noted, this is not true for all problems.

Then for the brachistochrone problem from a point to a curve N, we have the result:

For a particle starting at point 1 with initial velocity v_1, the path of quickest descent from 1 to a curve N, is necessarily an arc $\underline{y_{12}}$ of a cycloid, generated by a point fixed on the circumference of a circle, rolling on the lower side of the line $y = y_1 - v_1^2/2g$. The path $\underline{y_{12}}$ must furthermore be cut at right angles by the curve N at their intersection point 2.

Example: Minimize the integral

$$I = \int_0^{\pi/4} \left[y^2 - (y')^2 \right] dx$$

with left end point fixed

$$y(0) = 1$$

and the right end point is along the curve

$$x = \frac{\pi}{4}.$$

Since $F = y^2 - (y')^2$, then the Euler equation becomes

$$y'' + y = 0.$$

The solution is

$$y(x) = A\cos x + B\sin x$$

Using the condition at $x = 0$, we get

$$y = \cos x + B\sin x$$

Now for the transversality condition

$$F + (\phi' - y')F_{y'}\Big|_{x=\frac{\pi}{4}} = 0$$

where ϕ is the curve on the right end. Since the curve is a vertical line, the slope is infinite, thus we have to rewrite the condition after dividing by ϕ'. This will become (noting that $1/\phi' = 0$)

$$F_{y'}\Big|_{x=\frac{\pi}{4}} = 0$$

In our case

$$y'(\frac{\pi}{4}) = 0$$

This implies $B = 1$, and thus the solution is

$$y = \cos x + \sin x.$$

4.2 Appendix

Let's derive the transversality condition obtained before by a different method. Thus consider the problem

$$\min I = \int_{x_1}^{x_2} F(x, y, y')dx \tag{1}$$

among arcs

$$\underline{y}: \qquad y(x) \qquad x_1 \le x \le x_2$$

(where x_1, x_2 can vary with the arc) satisfying

$$y(x_i) = Y_i(x_i) \quad i = 1, 2 \tag{2}$$

This is a variable end-point problem with $Y_1(x)$ as the left boundary curve and $Y_2(x)$ as the right boundary curve.

Assume

$$\underline{y_0}: \qquad y(x) \qquad x_{01} \le x \le x_{02}$$

is a solution to this problem. Let $\eta(x)$ be an arc and create the family of arcs

$$\underline{y(\epsilon)}: \qquad y_0(x) + \epsilon\eta(x) \qquad x_1(\epsilon) \le x \le x_2(\epsilon) \qquad -\delta < \epsilon < \delta \tag{3}$$

for some $\delta > 0$, where $\eta(x), x_1(\epsilon), x_2(\epsilon)$ are as yet arbitrary functions. In order that each arc in this family satisfies (2) we must have

$$y_0(x_i(\epsilon)) + \epsilon\eta(x_i(\epsilon)) = Y_i(x_i(\epsilon)) \qquad i = 1, 2 \tag{4}$$

Differentiating (4) with respect to ϵ at $\epsilon = 0$ gives (recall that $\eta'(x_i)$ term has a factor of ϵ)

$$y_0'(x_i(0))\frac{dx_i(0)}{d\epsilon} + \eta(x_i(0)) = \frac{dY_i}{d\epsilon}(x_i(0)) \tag{5}$$

Equation (5) gives a relation between η, $\dfrac{dx_i(0)}{d\epsilon}$ at the end-points of the solution arc and $\dfrac{dY_i(0)}{d\epsilon}$ of the boundary curves. Namely, (with $x_{0i} = x_i(0)$, the end-points of y_0)

$$\eta(x_{0i}) = \frac{dY_i(0)}{d\epsilon} - y_0'(x_{0i})\frac{dx_i(0)}{d\epsilon} \qquad i = 1, 2 \tag{6}$$

These are the only $\eta(x)$ arcs that can be used in this problem since they are the ones which create families of arcs satisfying (2). We call these, admissible $\eta(x)$. For such an admissible $\eta(x)$, evaluate I on the resultant family to get

$$I(\epsilon) = \int_{x_1(\epsilon)}^{x_2(\epsilon)} F(x, y_0 + \epsilon\eta, y_0' + \epsilon\eta')dx \tag{7}$$

Differentiating with respect to ϵ at $\epsilon = 0$ gives

$$I'(0) = \int_{x_1(0)}^{x_2(0)} [F_y\eta + F_{y'}\eta']dx + F(x_{0i})\frac{dx_i(0)}{d\epsilon}\Big|_1^2 = 0 \tag{8}$$

where $F(x_{0i})$ means $F(x_i(0), y_0(x_i(0)), y_0'(x_i(0)))$ i.e. F evaluated on the arc $\underline{y_0}$ at the i^{th} end-point and all terms in F or its derivatives are on $\underline{y_0}$ and the last term in the right side means to difference the value at the left end point from its value at the right end-point and where we have set $I'(0) = 0$ (why?). By doing the usual integration by parts we get

$$0 = I'(0) = \int_{x_{01}}^{x_{02}} [F_{y'} - \int_{x_{01}}^{x} F_y ds]\eta' dx + \int_{x_{01}}^{x_{02}} \frac{d}{dx}[\eta \int_{x_{01}}^{x} F_y ds]dx + F(x_{0i})\frac{dx_i(0)}{d\epsilon}\Big|_1^2 \tag{9}$$

Evaluating the second integral on the right side gives

$$0 = I'(0) = \int_{x_{01}}^{x_{02}} [F_{y'} - \int_{x_{01}}^{x} F_y ds]\eta' dx + [\eta(x_{0i}) \int_{x_{01}}^{x_{0i}} F_y ds + F(x_{0i})\frac{dx_i(0)}{d\epsilon}]\Big|_1^2 \tag{10}$$

and noting that $\displaystyle\int_{x_{01}}^{x_{01}} F_y ds = 0$, gives

$$0 = I'(0) = \int_{x_{01}}^{x_{02}} [F_{y'} - \int_{x_{01}}^{x} F_y ds]\eta' dx + \eta(x_{02}) \int_{x_{01}}^{x_{02}} F_y ds + F(x_{0i})\frac{dx_i(0)}{d\epsilon}\Big|_1^2 \tag{11}$$

Now a particular class of admissible $\eta(x)$ are those for which

$$\eta(x_{02}) = 0 \qquad \frac{dx_i(0)}{d\epsilon} = 0 \qquad i = 1, 2 \tag{12}$$

For such $\eta(x)$, all terms after the first integral on the right side in (11) are zero so that for such $\eta(x)$ we have

$$0 = \int_{x_{01}}^{x_{02}} [F_{y'} - \int_{x_{01}}^{x} F_y ds]\eta' dx \tag{13}$$

then by the fundamental lemma we have that

$$F_{y'}(x) - \int_{x_{01}}^{x} F_y ds = c \tag{14}$$

holds along the solution arc $\underline{y_0}$. This is the same as the Euler equation for the fixed end-point problem. Furthermore by (14)

$$c = F_{y'}(x_{01}) \tag{15}$$

Now let $\eta(x)$ be any arc satisfying (6), i.e. we are returning to the full class of admissible $\eta(x)$. Then by (14) and (15) we get that the first integral on the right side in (11) is

$$\int_{x_{01}}^{x_{02}} [F_{y'} - \int_{x_{01}}^{x} F_y ds]\eta' dx = \int_{x_{01}}^{x_{02}} c\eta' dx = \quad c(\eta(x_{02}) - \eta(x_{01})) \tag{16}$$

$$= \quad F_{y'}(x_{01})[\eta(x_{02}) - \eta(x_{01})]$$

Then by (16), (15) and (14) the equation (11) becomes

$$0 = F_{y'}(x_{01})[\eta(x_{02}) - \eta(x_{01})] + \eta(x_{02}) \cdot \underbrace{\underbrace{\int_{x_{01}}^{x_{02}} F_y ds}_{F_{y'}(x_{02})-c \text{ by } (14)}}_{F_{y'}(x_{02})-F_{y'}(x_{01}) \text{ by } (15)} + F(x_{0i})\frac{dx_i(0)}{d\epsilon}\Big|_1^2 \tag{17}$$

Simplifying gives

$$[F_{y'}(x_{0i})\eta(x_{0i}) + F(x_{0i})\frac{dx_i(0)}{d\epsilon}]\Big|_1^2 = 0 \tag{18}$$

Then by (6), for all admissible $\eta(x)$, (18) becomes

$$\left([F(x_{0i}) - y_0'(x_{0i})F_{y'}(x_{0i})]\frac{dx_i(0)}{d\epsilon} + F_{y'}(x_{0i})\frac{dY_i(0)}{d\epsilon}\right)\Big|_1^2 = 0 \tag{19}$$

When (19) is multiplied by $d\epsilon$, this is the transversality condition obtained previously.

Next, for future work, we'll need an alternate form of the fundamental lemma which we've been using.

Alternate fundamental lemma If $\alpha(x)$ is continuous on $[x_1, x_2]$ and if

$$\int_{x_1}^{x_2} \alpha(x)\eta(x)dx = 0$$

for every arc $\eta(x)$ of class C^1 satisfying $\eta(x_1) = \eta(x_2) = 0$ then $\alpha(x) \equiv 0$ for all x on $[x_1, x_2]$.

Problems

1. Solve the problem minimize $I = \int_0^{x_1} \left[y^2 - (y')^2\right] dx$

   with left end point fixed and $y(x_1)$ is along the curve

$$x_1 = \frac{\pi}{4}.$$

2. Find the extremals for

$$I = \int_0^1 \left[\frac{1}{2}(y')^2 + yy' + y' + y\right] dx$$

where end values of y are free.

3. Solve the Euler-Lagrange equation for

$$I = \int_a^b y\sqrt{1 + (y')^2}\, dx$$

where

$$y(a) = A, \qquad y(b) = B.$$

   b. Investigate the special case when

$$a = -b, \qquad A = B$$

and show that depending upon the relative size of b, B there may be none, one or two candidate curves that satisfy the requisite endpoints conditions.

4. Solve the Euler-Lagrange equation associated with

$$I = \int_a^b \left[y^2 - yy' + (y')^2\right] dx$$

5. What is the relevant Euler-Lagrange equation associated with

$$I = \int_0^1 \left[y^2 + 2xy + (y')^2\right] dx$$

6. Investigate all possibilities with regard to tranversality for the problem

$$\min \int_a^b \sqrt{1 - (y')^2}\, dx$$

7. Determine the stationary functions associated with the integral

$$I = \int_0^1 \left[(y')^2 - 2\alpha y y' - 2\beta y' \right] dx$$

where α and β are constants, in each of the following situations:

a. The end conditions $y(0) = 0$ and $y(1) = 1$ are preassigned.

b. Only the end conditions $y(0) = 0$ is preassigned.

c. Only the end conditions $y(1) = 1$ is preassigned.

d. No end conditions are preassigned.

8. Determine the natural boundary conditions associated with the determination of extremals in each of the cases considered in Problem 1 of Chapter 3.

9. Find the curves for which the functional

$$I = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y} dx$$

with $y(0) = 0$ can have extrema, if

a. The point (x_1, y_1) can vary along the line $y = x - 5$.

b. The point (x_1, y_1) can vary along the circle $(x-9)^2 + y^2 = 9$.

10. If F depends upon x_2, show that the transversality condition must be replaced by

$$\left[F + (\phi' - y') \frac{\partial F}{\partial y'} \right] \Big|_{x=x_2} + \int_{x_1}^{x_2} \frac{\partial F}{\partial x_2} dx = 0.$$

11. Find an extremal for

$$J(y) = \int_1^e \left(\frac{1}{2} x^2 (y')^2 - \frac{1}{8} y^2 \right) dx, \qquad y(1) = 1,\ y(e) \text{ is unspecified.}$$

12. Find an extremal for

$$J(y) = \int_0^1 (y')^2 dx + y(1)^2, \qquad y(0) = 1, y(1) \text{ is unspecified.}$$

CHAPTER 5

5 Higher Dimensional Problems and Another Proof of the Second Euler Equation

Up to now our problems have been two-dimensional i.e. our arcs have been described by two variables, namely x, y. A natural generalization is to consider problems involving arcs in three-space with coordinates x, y, z or in even higher dimensional space, say $N+1$ dimensional space with coordinates $x, y_1, \cdots, y_N$. The problem to be considered then involves an integral of the form

$$I = \int_{x_1}^{x_2} F(x, y_1, \cdots, y_N, y_1', \cdots, y_N')dx\,. \tag{1}$$

and a class of admissible arcs $\underline{\overline{y}}$ where superscript bar designates a vector arc, with components

$$\underline{\overline{y}}: \quad y_i(x) \qquad x_1 \le x \le x_2 \qquad i = 1, \cdots, N \tag{2}$$

on which the integral (1) has a well defined value. As a fixed end-point version of this problem one would want to minimize (1) on a subclass of arcs (2) that join two fixed points $\overline{y}_1$ and $\overline{y}_2$, i.e. arcs that satisfy

$$y_i(x_1) = y_{i1} \qquad y_i(x_2) = y_{i2} \tag{3}$$

where $\overline{y}_1$ has coordinates $(y_{11}, \cdots, y_{N1})$ and $\overline{y}_2$ has coordinates $(y_{12}, \cdots, y_{N2})$.

Analogous to the proof used in obtaining the first Euler equation in chapter 3 for the two-dimensional problem we could obtain the following condition along a minimizing arc

$$F_{y_i'}(x) = \int_{x_1}^{x} F_{y_i} dx + c_i \qquad i = 1, \cdots, N \tag{4}$$

where c_i are constants. And then by differentiation

$$\frac{d}{dx} F_{y_i'}(x) = F_{y_i}(x) \qquad x_1 \le x \le x_2 \tag{5}$$

Using this result we can now prove the second Euler equation for the **two-dimensional** problem with the same generality as for the first Euler equation. We previously stated this result in chapter 3 but only really proved it for arcs of class C^2 (i.e. having a continuous second derivative y''). Thus let us now consider the two-dimensional problem of chapter 3, defined by (7) and the remarks following it, without assuming even the existence of y'' on our arcs. We now write our arcs in parametric form. In particular our minimizing arc $\underline{y_0}$ (from chapter 3) is now written as

$$x = t \quad y = y_0(t) \quad x_1 \le t \le x_2 \tag{6}$$

where t is the parameter and $x(t) = t$ is the first component and $y(t) = y_0(t)$ is the second component. Being a minimizing arc, then this arc must minimize the integral (7) of chapter 3 on the class of parametric arcs

$$x = \epsilon(t) \qquad y = \rho(t) \qquad t_1 \le t \le t_2 \tag{7}$$

(where we have set $t_1 = x_1$ and $t_2 = x_2$), which join the fixed points (x_1, y_1) and (x_2, y_2) of that problem and have $\epsilon'(t) > 0$ on $[t_1, t_2]$. This is true since each non-parametric arc of the originally stated problem can be written (as in (6)) as a parametric vector arc of the class just described and vice versa.

In terms of these parametric arcs, the integral (7) of chapter 3 takes the form

$$I = \int_{t_1}^{t_2} \underbrace{F(\epsilon, \rho, \frac{\rho'}{\epsilon'})\epsilon'\, dt}_{F \text{ in } (1)} \tag{8}$$

where the primes now mean derivatives with respect to t [§]. This is an integral like (1) (i.e. in three-dimensional t, x, y space). By (4) applied to the variable ϵ (use $i = 2$ in (4)), there results

$$F - \frac{\rho'}{\epsilon'} F_{y'} = \int_{t_1}^{t} F_x \epsilon' dt + c \tag{9}$$

When we write y' for $\dfrac{\rho'}{\epsilon'}$ and use (6) we get along $\underline{y_0}$

$$F - y'F_{y'} = \int_{x_1}^{x} F_x dx + c \tag{10}$$

and by differentiation

$$\frac{d}{dx}[F - y'F_{y'}] = F_x \tag{11}$$

which is the result listed in chapter 3.

5.1 Variational Problems with Constraints

5.1.1 Isoparametric Problems

In the general problem considered thus far, the class of admissible arcs was specified (apart from certain continuity conditions) by conditions imposed on the end-points. However many applications of the calculus of variations lead to problems in which not only boundary conditions, but also conditions of quite a different type, known as subsidiary conditions (or also side conditions or constraints) are imposed on the admissible arcs. As an example of this, we consider the isoparametric problem. This problem is one of finding an arc $\underline{y}$ passing through the points $(-x_1, 0)$ and $(x_1, 0)$ of given length L which together with the interval $[-x_1, x_1]$ on the x-axis, encloses the largest area. In general form, this problem can be stated as finding the arc $\underline{y}$ for which the integral

$$I[\underline{y}] = \int_{x_1}^{x_2} F(x, y, y')dx \tag{12}$$

[§]The argument $\dfrac{\rho'}{\epsilon'}$ replaces y' in the original integral. This follows since by calculus $\dfrac{dy}{dx} = \dfrac{dy}{dt} / \dfrac{dx}{dt}$.

is minimized (or maximized) where the admissible arcs satisfy the end-point conditions

$$y(x_i) = y_i \quad i = 1, 2 \tag{13}$$

and also are such that another integral

$$K[\underline{y}] = \int_{x_1}^{x_2} G(x, y, y')dx \tag{14}$$

has a fixed value L.

In the specific application noted above (assuming that $\underline{y}$ is in the positive half-plane), the integral I giving the area is

$$I = \int_{-x_1}^{x_1} y(x)dx \tag{15}$$

while the integral K is the length integral

$$K = \int_{-x_1}^{x_1} \sqrt{1 + y'^2}dx \tag{16}$$

and is required to have fixed length.

Returning now to the general version of this problem stated in (12)-(14), we will follow the reasoning employed originally in solving the shortest distance problem. Thus assume that $\underline{y_0}$ is a solution to this problem. Let $\eta_1(x)$ and $\eta_2(x)$ be two functions satisfying

$$\eta_1(x_i) = 0\,, \quad \eta_2(x_i) = 0 \quad i = 1, 2 \tag{17}$$

Create the two-parameter ¶ family of arcs

$$\underline{y}(\alpha_1, \alpha_2) : y_0(x) + \alpha_1\eta_1(x) + \alpha_2\eta_2(x) \tag{18}$$

By (17) this family satisfies the end-point conditions of our problem. Consider the integrals (12) and (14) evaluated on this family. For example

$$I(\underline{y}(\alpha_1, \alpha_2)) = \int_{x_1}^{x_2} F(x, y_0(x) + \alpha_1\eta_1(x) + \alpha_2\eta_2(x), y_0'(x) + \alpha_1\eta_1'(x) + \alpha_2\eta_2'(x))dx \tag{19}$$

and similarly for $K(\underline{y}(\alpha_1, \alpha_2))$. On this family of arcs our problem can be stated as

$$\text{minimize } I(\underline{y}(\alpha_1, \alpha_2)) \quad \text{subject to} \quad K(\underline{y}(\alpha_1, \alpha_2)) = L \tag{20}$$

Now noting that on this family, these integrals can be considered as functions of two variables (α_1, α_2) (instead of arcs), then, when considering this family, our problem can be stated as

$$\min I(\alpha_1, \alpha_2) \;\text{ subject to }\; K(\alpha_1, \alpha_2) = L \tag{21}$$

where in somewhat loose notation we have written $I(\alpha_1, \alpha_2)$ for $I(\underline{y}(\alpha_1, \alpha_2))$ and similarly for K. This is a finite (actually, two) dimensional problem of the type described in chapter 1.

¶Note that up to now our families of arcs have been one parameter families.

By the results there and also noting that our minimizing arc $\underline{y}_0 = \underline{y}(0,0)$ solves this problem we must have that

$$0 = \frac{dI}{d\alpha_i}(0,0) + \lambda\frac{dK}{d\alpha_i}(0,0) \qquad i = 1,2 \tag{22}$$

where λ is a Lagrange multiplier.

Writing the integrals I and K out in terms of the family (18) and differentiating separately with respect to α_1, α_2 under the integral sign gives the two equations

$$0 = \int_{x_1}^{x_2}[F_y\eta_i + F_{y'}\eta_i']dx + \lambda\int_{x_1}^{x_2}[G_y\eta_i + G_{y'}\eta_i']dx \quad i = 1,2 \tag{23}$$

where the partial derivatives of F and G are at $(\alpha_1, \alpha_2) = (0,0)$ i.e. along the arc $\underline{y_0}$.

Writing this as one integral, gives

$$0 = \int_{x_1}^{x_2}[(F+\lambda G)_y\eta_i + (F+\lambda G)_{y'}\eta_i']dx = \int_{x_1}^{x_2}[\overline{F}_y\eta_i + \overline{F}_{y'}\eta_i']dx \quad i = 1,2 \tag{24}$$

where $\overline{F} \equiv F + \lambda G$ and where this is true for all functions $\eta_i(x)$ satisfying (17).

Making use of the integration by parts, formula (9) of chapter 3, but with $\overline{F}$, we get as there

$$0 = \int_{x_1}^{x_2}[\overline{F}_{y'} - \int_{x_1}^{x}\overline{F}_y ds]\eta_i' dx \tag{25}$$

Then by the fundamental lemma we obtain

$$\overline{F}_{y'}(x) = \int_{x_1}^{x}\overline{F}_y ds + c \qquad x_1 \le x \le x_2 \tag{26}$$

(where c is a constant) which holds at every point along the arc $\underline{y_0}$ and then also by differentiating

$$\frac{d}{dx}\overline{F}_{y'}(x) = \overline{F}_y(x) \qquad x_1 \le x \le x_2 \tag{27}$$

along $\underline{y_0}$. In terms of the functions F and G, this is

$$\frac{d}{dx}(F+\lambda G)_{y'} = (F+\lambda G)_y \qquad x_1 \le x \le x_2 \tag{28}$$

This is the first Euler equation for the isoparametric problem.

In a manner similar to that used in the beginning of this chapter, it can be shown that the second Euler equation

$$(F+\lambda G) - y'(F+\lambda G)_{y'} = \int_{x_1}^{x}(F+\lambda G)_x dx \qquad x_1 \le x \le x_2 \tag{29}$$

or in differentiated form

$$\frac{d}{dx}[(F+\lambda G) - y'(F+\lambda G)_{y'}] = (F+\lambda G)_x \qquad x_1 \le x \le x_2 \tag{30}$$

also holds along $\underline{y_0}$.

These results are summarized in

Theorem 10. For the problem stated in (12) - (14) let $\underline{y_0}$ *be a solution, then there exists a constant* λ *such that (26), (28), (29), and (30) are true along* $\underline{y_0}$.

Note that if our problem did not have a fixed right end point but instead was required to intersect some curve N then $\eta(x)$ would not have to satisfy (17) for $i = 2$ and then a line of reasoning similar to that used in chapter 4, would give

$$(F + \lambda G)dx + (dy - y'dx)(F + \lambda G)_{y'} = 0 \tag{31}$$

as the transversality condition at intersection with N, where the direction $dy : dx$ comes from N, and the arguments of F, G are from $\underline{y_0}$. For this problem a corresponding condition at left end point would hold if the left end point were not prescribed.

Let's go through an application: Consider the problem of determining the curve of length L with end-points (0,0) and (1,0) which encloses the largest area between it and the x-axis. Thus we need to

$$\text{maximize } I = \int_0^1 y dx \tag{32}$$

subject to fixed end points

$$y(0) = y(1) = 0 \tag{33}$$

and fixed length constraint

$$K = \int_0^1 \sqrt{1 + y'^2} dx = L \tag{34}$$

Setting

$$\overline{F} = y + \lambda\sqrt{1 + y'^2} \tag{35}$$

the first Euler equation (27) is

$$\lambda \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) - 1 = 0 \tag{36}$$

Direct integration gives

$$\frac{\lambda y'}{\sqrt{1 + y'^2}} = x - c_1 \tag{37}$$

Now make the substitution

$$\tan\theta = y' \tag{38}$$

then (37) gives (recall that $1 + \tan^2\theta = \sec^2\theta$)

$$\sin\theta = \frac{x - c_1}{\lambda} \tag{39}$$

Now since $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\pm\sin\theta}{\sqrt{1 - \sin^2\theta}}$, then (38), (39) give

$$y' = \frac{\pm(x - c_1)}{\lambda\sqrt{1 - \frac{(x-c_1)^2}{\lambda^2}}} = \frac{\pm(x - c_1)}{\sqrt{\lambda^2 - (x - c_1)^2}} \tag{40}$$

or when using $y' = \dfrac{dy}{dx}$

$$dy = \frac{\pm(x - c_1)}{\sqrt{\lambda^2 - (x - c_1)^2}} dx. \tag{41}$$

Integration gives

$$y = \mp\sqrt{\lambda^2 - (x - c_1)^2} + c_2 \tag{42}$$

or then

$$(y - c_2)^2 + (x - c_1)^2 = \lambda^2 \tag{43}$$

This is part of a circle with center (c_1, c_2), and radius λ.

The three constants c_1, c_2, λ are determined to satisfy the two end-point conditions and the fixed length constraint this completes the problem solution. (see problem 5)

5.1.2 Point Constraints

We next discuss problems in which minimization is to be done subject to a different type of constraint than the one just discussed. Thus consider the problem

$$\text{minimize } I = \int_{x_1}^{x_2} F(x, y, z, y', z')dx \tag{44}$$

subject to fixed endpoints

$$y(x_i) = y_i \quad z(x_i) = z_i \quad i = 1, 2 \tag{45}$$

and also subject to a constraint

$$\phi(x, y, z, y', z') = 0 \tag{46}$$

Assume (as for previous problems) that $\underline{\overline{y_0}}$ is a solution to this problem. The notation $\underline{\overline{y_0}}$ denotes a vector arc with components $y_0(x), z_0(x)$. All arcs considered in this problem have two components. Also assume that $\phi_{y'}, \phi_{z'}$ do not equal zero simultaneously at any point on $\underline{\overline{y_0}}$.

Next, let $\psi(x)$, $\eta(x)$ be functions which satisfy

$$\psi(x_i) = 0 \quad \eta(x_i) = 0 \quad i = 1, 2\,. \tag{47}$$

As in previous chapters, create the one-parameter family of arcs (but note that now our arcs are vector arcs)

$$\underline{\overline{y}}(\epsilon): \quad y_0(x) + \epsilon\psi(x)\,, \quad z_0(x) + \epsilon\eta(x) \quad x_1 \le x \le x_2 \tag{48}$$

We assume also that for some $\delta > 0$, and for $|\epsilon| < \delta$, the functions $\psi(x), \eta(x)$ satisfy

$$\phi(x, y_0(x) + \epsilon\psi(x), z_0(x) + \epsilon\eta(x), y_0'(x) + \epsilon\psi'(x), z_0'(x) + \epsilon\eta'(x)) = 0 \qquad x_1 \le x \le x_2. \tag{49}$$

Again, similar to previous chapters, evaluate the integral in (44) on our family and define

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, y_0(x) + \epsilon\psi(x), z_0(x) + \epsilon\eta(x), y_0'(x) + \epsilon\psi'(x), z_0'(x) + \epsilon\eta'(x))dx \tag{50}$$

Differentiating this with respect to ϵ at $\epsilon = 0$ gives

$$0 = I'(0) = \int_{x_1}^{x_2} [F_y\psi + F_z\eta + F_{y'}\psi' + F_{z'}\eta']dx \tag{51}$$

where the partials of F are taken at points along $\underline{\overline{y_0}}$. Next, differentiate (49) with respect to ϵ at $\epsilon = 0$ to get

$$\phi_y\psi + \phi_z\eta + \phi_{y'}\psi' + \phi_{z'}\eta' = 0 \qquad x_1 \le x \le x_2 \tag{52}$$

where the partials of ϕ are at points along $\underline{\overline{y}_0}$. Equation (52) reveals that the ψ, η functions are not independent of each other but are related. Multiplying (52) by an as yet unspecified function $\lambda(x)$ and adding the result to the integrand of (51) yields.

$$\int_{x_1}^{x_2} [(F_y + \lambda\phi_y)\psi + (F_{y'} + \lambda\phi_{y'})\psi' + (F_z + \lambda\phi_z)\eta + (F_{z'} + \lambda\phi_{z'})\eta']dx = 0 \tag{53a}$$

Setting $\hat{F} = F + \lambda\phi$ gives (53a) in the form

$$\int_{x_1}^{x_2} [\hat{F}_y\psi + \hat{F}_{y'}\psi' + \hat{F}_z\eta + \hat{F}_{z'}\eta']dx = 0 \tag{53b}$$

Using the now familiar integration by parts formula on the 1^{st} and 3^{rd} terms in the integrand of (53b) gives:

$$\psi\hat{F}_y = \frac{d}{dx}(\psi\int_{x_1}^{x} \hat{F}_y dx) - \psi'\int_{x_1}^{x} \hat{F}_y dx \tag{54}$$

and similarly for $\eta\hat{F}_z$. Using these and (47) yields

$$0 = I'(0) = \int_{x_1}^{x_2} ([\hat{F}_{y'} - \int_{x_1}^{x} \hat{F}_y ds]\psi' + [\hat{F}_{z'} - \int_{x_1}^{x} \hat{F}_z ds]\eta')dx \tag{55}$$

However we cannot take the step that we essentially did in developing the Euler equation in the unconstrained case at the start of this chapter and say that the ψ, η functions, are independent since as noted above (see (52)), they are not. Now, assuming that $\phi_{y'} \neq 0$ (consistent with our assumption either $\phi_{y'}$ or $\phi_{z'} \neq 0$) we can choose λ such that the coefficient of ψ' is constant (i.e. choose λ such that $\frac{d}{dx}(F_{y'} + \lambda\phi_{y'}) - (F_y + \lambda\phi_y) = 0$ or then $\dot{\lambda} = (F_y + \lambda\phi_y - \frac{d}{dx}F_{y'} - \lambda\frac{d}{dx}\phi_{y'})/\phi_{y'}$ and integrate this result). Next choose η arbitrarily (consistent with (47)) and ψ consistent with (49) and (47). By (47) and the fundamental lemma, the coefficient of η' must also be constant. This results in

$$\hat{F}_{y'}(x) - \int_{x_1}^{x} \hat{F}_y ds = c_1 \tag{56a}$$

$$\hat{F}_{z'}(x) - \int_{x_1}^{x} \hat{F}_z ds = c_2 \tag{56b}$$

where c_1, c_2 are constants. In differentiated form this is

$$\hat{F}_y - \frac{d}{dx}\hat{F}_{y'} = 0 \tag{56c}$$

$$\hat{F}_z - \frac{d}{dx}\hat{F}_{z'} = 0 \tag{56d}$$

Substituting for $\hat{F}$, then (56c), (56d) become

$$(F_y + \lambda\phi_y) - \frac{d}{dx}(F_{y'} + \lambda\phi_{y'}) = 0 \tag{57a}$$

$$(F_z + \lambda\phi_z) - \frac{d}{dx}(F_{z'} + \lambda\phi_{z'}) = 0 \tag{57b}$$

This result is actually contained in a larger result as follows. If the constraint (46) does not depend on y', z' i.e. if the constraint is

$$\phi(x, y, z) = 0 \tag{58}$$

and if ϕ_y and ϕ_z are not simultaneously zero at any point of $\underline{\overline{y_0}}$ then the analogous equations for (57a) and (57b) are

$$F_y + \lambda\phi_y - \frac{d}{dx}F_{y'} = 0 \tag{59a}$$

$$F_z + \lambda\phi_z - \frac{d}{dx}F_{z'} = 0 \tag{59b}$$

These results are summarized in the following:

Theorem: Given the problem

$$\min I = \int_{x_1}^{x_2} F(x, y, z, y', z')dx \tag{60}$$

subject to fixed end points and the constraint

$$\phi(x, y, z, y', z') = 0 \tag{61}$$

then if $\phi_{y'}, \phi_{z'}$ (or in case ϕ does not depend on y', z', then if ϕ_y, ϕ_z) do not simultaneously equal zero at any point of a solution $\underline{\overline{y_0}}$, then there is a function $\lambda(x)$ such that with $\hat{F} \equiv F + \lambda\phi$, then (56a) and (56b) or in differentiated form, (56c) and (56d) are satisfied along $\underline{\overline{y_0}}$.

The three equations (56a,b) or (56c,d) and (61) are used to determine the three functions $y(x), z(x), \lambda(x)$ for the solution.

In more general cases if our integrand has k dependent variables

$$I = \int_{x_1}^{x_2} F(x, y_1, y_2, \cdots y_k, y_1', \cdots y_k')dx \tag{62}$$

and we have $(N < k)$ constraints

$$\phi_i(x, y_1, \cdots y_k, y_1', \cdots y_k') = 0\,,\ i = 1, \cdots, N \tag{63}$$

such that the matrix $\frac{\partial \phi_i}{\partial y_j'}$ (or in case the ϕ are independent of $y_1' \cdots y_k'$, then assume $\frac{\partial \phi_i}{\partial y_j}$) $i = 1, \cdots, N \quad j = 1, \cdots k$ has maximal rank along a solution curve, $\underline{\overline{y_0}}$ then with

$$\hat{F} = F + \sum_{i=1}^{N} \lambda_i(x)\phi_i \tag{64}$$

we have

$$\hat{F}_{y_j} - \frac{d}{dx}\left(\hat{F}_{y_j'}\right) = 0 \qquad\qquad j = 1, \cdots, k \tag{65}$$

holding on $\underline{\overline{y_0}}$ where the $\lambda_i(x)$ are N multiplier functions.

As an application, consider the problem of finding the curve of minimum length between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) on a surface

$$\phi(x, y, z) = 0 \tag{66}$$

Doing this in parametric form our curves will be written as $x = x(t),\ \ y = y(t),\ \ z = z(t)$ and with arc length as

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt \tag{67}$$

where "$\cdot$" denotes differentiation with respect to t. Then our problem is

$$\text{minimize }\ I = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\ dt \tag{68}$$

with fixed end points

$$x(t_i) = x_i \qquad y(t_i) = y_i \qquad z(t_i) = z_i \qquad i = 1, 2 \tag{69}$$

subject to (66).

For this problem, with

$$F = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{70}$$

the Euler-Lagrange equations (65) are

$$\lambda\phi_x - \frac{d}{dt}\left(\frac{\dot{x}}{F}\right) = 0 \quad \lambda\phi_y - \frac{d}{dt}\left(\frac{\dot{y}}{F}\right) = 0 \quad \lambda\phi_z - \frac{d}{dt}\left(\frac{\dot{z}}{F}\right) = 0 \tag{71}$$

Now noting that

$$F = \frac{ds}{dt} \tag{72}$$

where s is arc length then e.g.

$$\frac{d}{dt}\Big(\frac{\dot{x}}{F}\Big) = \frac{d}{dt}\Big(\frac{dx}{dt}\Big/\frac{ds}{dt}\Big) = \frac{d}{dt}\Big(\frac{dx}{dt}\cdot\frac{dt}{ds}\Big) = \frac{d}{dt}\Big(\frac{dx}{ds}\Big) \tag{73}$$

and if we multiply this by $\dfrac{dt}{ds}$ we get

$$\Big[\frac{d}{dt}\Big(\frac{\dot{x}}{F}\Big)\Big]\frac{dt}{ds} = \Big[\frac{d}{dt}\Big(\frac{dx}{ds}\Big)\Big]\frac{dt}{ds} = \frac{d^2x}{ds^2} \tag{74a}$$

and similarly

$$\frac{d}{dt}\Big[\frac{\dot{y}}{F}\Big]\frac{dt}{ds} = \frac{d^2y}{ds^2} \tag{74b}$$

$$\frac{d}{dt}\Big[\frac{\dot{z}}{F}\Big]\frac{dt}{ds} = \frac{d^2z}{ds^2} \tag{74c}$$

Thus, multiplying each of the equations of (71) by $\dfrac{dt}{ds}$ give as shown above

$$\frac{d^2x}{ds^2} = \lambda\phi_x\frac{dt}{ds} \qquad \frac{d^2y}{ds^2} = \lambda\phi_y\frac{dt}{ds} \qquad \frac{d^2z}{ds^2} = \lambda\phi_z\frac{dt}{ds} \tag{75}$$

or then

$$\frac{d^2x}{ds^2} : \ \frac{d^2y}{ds^2} : \ \frac{d^2z}{ds^2} = \phi_x : \ \phi_y : \ \phi_z \tag{76}$$

which has the geometric interpretation that the principal normal to the curve is parallel to the gradient to the surface (i.e. it's perpendicular to the surface).

If we do this in particular for geodesics on a sphere so that (66) is

$$\phi(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0 \tag{77}$$

where R is the radius of sphere, then (71) becomes (after solving for λ)

$$\frac{F\ddot{x} - \dot{x}\dot{F}}{2xF^2} = \frac{F\ddot{y} - \dot{y}\dot{F}}{2yF^2} = \frac{F\ddot{z} - \dot{z}\dot{F}}{2zF^2} \tag{78}$$

Multiplying by $\dfrac{2F^2}{F}$ gives

$$\frac{\ddot{x} - \dot{x}\frac{\dot{F}}{F}}{x} = \frac{\ddot{y} - \dot{y}\frac{\dot{F}}{F}}{y} = \frac{\ddot{z} - \dot{z}\frac{\dot{F}}{F}}{z} \tag{79}$$

which after cross multiplying gives

$$y\ddot{x} - y\dot{x}\frac{\dot{F}}{F} = x\ddot{y} - x\dot{y}\frac{\dot{F}}{F} \ \text{ and } \ y\ddot{z} - y\dot{z}\frac{\dot{F}}{F} = z\ddot{y} - z\dot{y}\frac{\dot{F}}{F} \tag{80}$$

and then

$$y\ddot{x} - x\ddot{y} = \frac{\dot{F}}{F}(y\dot{x} - x\dot{y}) \ \text{ and } \ y\ddot{z} - z\ddot{y} = \frac{\dot{F}}{F}(y\dot{z} - z\dot{y}) \tag{81}$$

or

$$\frac{y\ddot{x} - x\ddot{y}}{y\dot{x} - x\dot{y}} = \frac{y\ddot{z} - z\ddot{y}}{y\dot{z} - z\dot{y}} = \frac{\dot{F}}{F} \tag{82}$$

The first equality can be restated as

$$\frac{\frac{d}{dt}(y\dot{x} - x\dot{y})}{y\dot{x} - x\dot{y}} = \frac{\frac{d}{dt}(y\dot{z} - z\dot{y})}{y\dot{z} - z\dot{y}} \tag{83}$$

and integration using $\int \frac{du}{u} = \ln|u| + c$ gives

$$y\dot{x} - x\dot{y} = A(y\dot{z} - z\dot{y}) \tag{84}$$

where A is a constant of integration. This gives

$$y(\dot{x} - A\dot{z}) = \dot{y}(x - Az) \tag{85}$$

or then

$$\frac{\dot{x} - A\dot{z}}{x - Az} = \frac{\dot{y}}{y} \tag{86}$$

so that another integration gives

$$x - Az = By \tag{87}$$

where B is a constant. This is the equation of a plane through the center of sphere and containing the two end points of the problem. The intersection of this plane with the two points and passing through center of sphere is a great circle. This completes the problem solution.

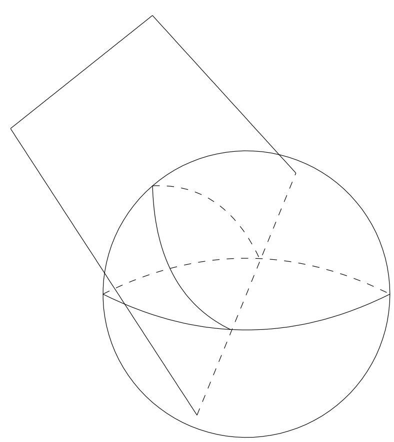

Figure 18: Intersection of a plane with a sphere

Note that to cover all possible pairs of points we really have to do this problem in parametric form since for example if we tried to express solutions in terms of x as $x, y(x), z(x)$, then any two points given in yz plane would **not have** a great circle path expressible in x.

Problem

1. A particle moves on the surface $\phi(x, y, z) = 0$ from the point (x_1, y_1, z_1) to the point (x_2, y_2, z_2) in the time T. Show that if it moves in such a way that the integral of its kinetic energy over that time is a minimum, its coordinates must also satisfy the equations

$$\frac{\ddot{x}}{\phi_x} = \frac{\ddot{y}}{\phi_y} = \frac{\ddot{z}}{\phi_z}.$$

2. Specialize problem 2 in the case when the particle moves on the unit sphere, from $(0, 0, 1)$ to $(0, 0, -1)$, in time T.

3. Determine the equation of the shortest arc in the first quadrant, which passes through the points $(0, 0)$ and $(1, 0)$ and encloses a prescribed area A with the x-axis, where $A \le \frac{\pi}{8}$.

4. Finish the example on page 51. What if $L = \frac{\pi}{2}$?

5. Solve the following variational problem by finding extremals satisfying the conditions

$$J(y_1, y_2) = \int_0^{\frac{\pi}{4}} \left(4y_1^2 + y_2^2 + y_1'y_2'\right) dx$$

$$y_1(0) = 1,\ y_1\left(\frac{\pi}{4}\right) = 0,\ \ y_2(0) = 0,\ y_2\left(\frac{\pi}{4}\right) = 1.$$

6. Solve the isoparametric problem

$$J(y) = \int_0^1 \left((y')^2 + x^2\right) dx,\ \ y(0) = y(1) = 0,$$

and

$$\int_0^1 y^2 dx = 2.$$

7. Derive a necessary condition for the isoparametric problem
   Minimize

$$I(y_1, y_2) = \int_a^b L(x, y_1, y_2, y_1', y_2')dx$$

subject to

$$\int_a^b G(x, y_1, y_2, y_1', y_2')dx = C$$

and

$$y_1(a) = A_1, \quad y_2(a) = A_2, \quad y_1(b) = B_1, \quad y_2(b) = B_2$$

where C, A_1, A_2, B_1, and B_2 are constants.

8. Use the results of the previous problem to maximize

$$I(x,y) = \int_{t_0}^{t_1} (x\dot{y} - y\dot{x})dt$$

subject to

$$\int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}dt = 1.$$

Show that I represents the area enclosed by a curve with parametric equations $x = x(t)$, $y = y(y)$ and the contraint fixes the length of the curve.

9. Find extremals of the isoparametric problem

$$I(y) = \int_0^{\pi} (y')^2 dx, \qquad y(0) = y(\pi) = 0,$$

subject to

$$\int_0^{\pi} y^2 dx = 1.$$

CHAPTER 6

6 Integrals Involving More Than One Independent Variable

Up to now our integrals have been single integrals, i.e. integrals involving only one independent variable which we have usually called x.

There are problems in the calculus of variations where the integral involves more than one independent variable. For example, given some contour C in xyz space, then find the surface $z = z(x, y)$ contained within C that has minimum surface area. In this case we'd minimize the surface area integral

$$S = \int\int_R \sqrt{1 + z_x^2 + z_y^2}\, dy\, dx \tag{1}$$

where R is the region in the xy plane enclosed by the projection of C in the xy plane. In this problem there are two independent variables, x, y and one dependent variable, z.

In order to see what conditions for a minimum hold when the integrand involves **more than one independent** variable, i.e. the Euler Lagrange equations in this more general case, let I be defined by

$$I = \int\int_R F(x, y, z, z_x, z_y) dydx \tag{2a}$$

where x, y are the independent variables and z is a continuously differentiable function of x, y and is to be determined, subject to

$$z = g(s) \tag{2b}$$

on the boundary of R where s is arc length, R is some closed region in the xy plane, and F has continuous first and second partial derivatives with respect to its arguments.

Doing the analogous steps that we did in the single integral problems, assume that $\underline{z_0}:\ z_0(x, y)$ is a solution to this problem and that $\eta(x, y)$ is a surface which is continuous with continuous first partials defined over R and satisfies

$$\eta(x, y) = 0 \qquad \text{on boundary of } R\,. \tag{3}$$

Create the family of surfaces

$$\underline{z}(\epsilon) = z_0(x, y) + \epsilon\eta(x, y) \tag{4}$$

and evaluate I on this family to obtain

$$I(\epsilon) = \int\int_R F[x, y, z_0(x, y) + \epsilon\eta(x, y),\ \ z_{0_x}(x, y) + \epsilon\eta_x(x, y),\ \ z_{0_y}(x, y) + \epsilon\eta_y(x, y)]dxdy \tag{5}$$

Differentiating $I(\epsilon)$ with respect to ϵ at $\epsilon = 0$ and setting it to zero (why?) gives

$$0 = I'(0) = \int\int_R [F_z\eta + F_{z_x}\eta_x + F_{z_y}\eta_y]dydx \tag{6}$$

At this point, let's recall (from an earlier chapter) the line of reasoning followed for the single integral case. The expression corresponding to (6) was

$$0 = I'(0) = \int_{x_1}^{x_2} [F_y \eta + F_{y'} \eta'] dx$$

We then rewrote this integrand (by using integration by parts) to involve only η' terms instead of η' and η and used the fundamental lemma to get the Euler-Lagrange equation. As an alternate to this procedure we could have used a variant of the integration by parts formula used above and then written the integrand above in terms of η, with no η' terms. Our next step would have been to use a modified form of the fundamental lemma introduced in chapter 4, involving η but not η' terms.

As a generalization to two variables of that modified form of the fundamental lemma we have

Lemma 1. If $\alpha(x, y)$ is continuous over a region R in the xy plane and if

$$\int\int_R \alpha(x, y)\eta(x, y) dy dx = 0$$

for every continuous function $\eta(x, y)$ defined over R and satisfying $\eta = 0$ on the boundary of R, then $\alpha(x, y) \equiv 0$ for all (x, y) in R.

We will not prove this lemma since it is not pertinent to the discussion.

Returning now to our double integral and equation (6), then the second term in the integrand there can be written

$$F_{z_x} \eta_x = \frac{\partial}{\partial x}[F_{z_x} \eta] - \frac{\partial F_{z_x}}{\partial x} \eta \tag{7}$$

This is analogous to the integration by parts formula used in the single integral problems.

Now recalling Green's theorem

$$\int\int_R (Q_x + P_y) dy dx = \oint_{\substack{\text{boundary} \\ \text{of R}}} (Q \cos \nu + P \sin \nu) ds \tag{8}$$

where P, Q are functions of x, y; ν is the angle between the outward normal of the boundary curve of R and the positive x-axis (see figure 19); ds is the differential of arc length and the boundary integral is taken in a direction to keep R on the left (positive).

Integrating (7) over R and using (8) with Q as $F_{z_x} \eta$ and $P \equiv 0$ gives:

$$\int\int_R F_{z_x} \eta_x dy dx = \oint_{\substack{\text{boundary} \\ \text{of R}}} F_{z_x} \eta \cos \nu ds - \int\int_R \frac{\partial}{\partial x}(F_{z_x}) \eta dy dx \tag{9}$$

By performing a similar line of reasoning on the third term in the integrand of (6), then (6) becomes

$$0 = I'(0) = \oint_{\substack{\text{boundary} \\ \text{of R}}} [F_{z_x} \cos \nu - F_{z_y} \sin \nu] \eta ds + \int\int_R [F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y}] \eta dy dx \tag{10}$$

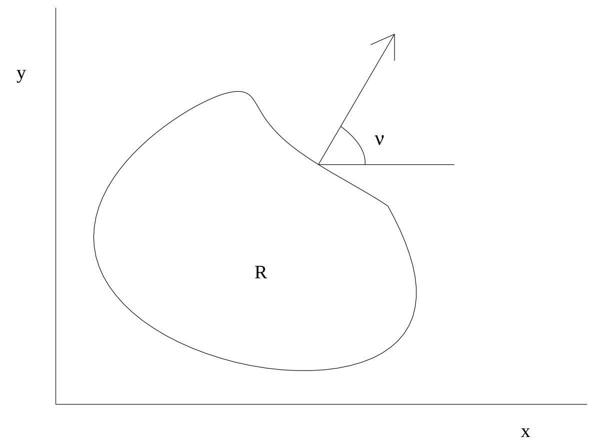

Figure 19: Domain R with outward normal making an angle ν with x axis

Thus in the expression for the derivative of I with respect to ϵ, (at $\epsilon = 0$), we have written all terms involving η and eliminated η_x and η_y. This is entirely analogous to the single integral case outlined above.

Since (10) is true for all $\eta(x, y)$ which satisfy (3) then the first integral on the right side of (10) is zero for such η and then by lemma 1, the coefficient of η in the second integral of (10) must be zero over R. That is

$$\frac{\partial}{\partial x} F_{z_x} + \frac{\partial}{\partial y} F_{z_y} - F_z = 0 \tag{11}$$

which constitutes the Euler-Lagrange equation for this problem.

As an application of the above results, consider the minimal surface problem started before. Thus minimize

$$S = \int \int_R \sqrt{1 + z_x^2 + z_y^2}\, dydx \tag{12}$$

where the surface is assumed representable in the form $z = z(x, y)$ with $z(x, y)$ specified on C, the given contour and R is the region in the xy plane, enclosed by the projection of C. Then (11) gives

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0 \tag{13}$$

which by algebra can be reduced to

$$(1 + z_y^2) z_{xx} - 2 z_x z_y z_{xy} + (1 + z_x^2) z_{yy} = 0 \tag{14}$$

Next, by setting

$$p = z_x \quad q = z_y \quad r = z_{xx} \quad u = z_{xy} \quad t = z_{yy} \tag{15}$$

then (14) becomes

$$(1 + q^2) r - 2pqu + (1 + p^2) t = 0 \tag{16}$$

Now from differential geometry the mean curvature, M, of the surface is

$$M \equiv \frac{Eg - 2Ff + Ge}{2(EG - F^2)} \tag{17}$$

where E, F, G and e, f, g are the coefficients of the first and second fundamental forms of the surface. For surfaces given by $z = z(x, y)$ then one can show that

$$E = 1 + p^2 \quad F = pq \quad G = 1 + q^2 \tag{18a}$$

and

$$e = \frac{r}{\sqrt{1 + p^2 + q^2}} \quad f = \frac{u}{\sqrt{1 + p^2 + q^2}} \quad g = \frac{t}{\sqrt{1 + p^2 + q^2}} \tag{18b}$$

so that

$$M = \frac{(1 + p^2)t - 2upq + (1 + q^2)r}{2(1 + p^2 + q^2)^{3/2}} \tag{19}$$

So the numerator is the same as the left side of Euler's equation (16). Thus (16) says that the mean curvature of the minimal surface must be zero.

Problems

1. Find all minimal surfaces whose equations have the form $z = \phi(x) + \psi(y)$.

2. Derive the Euler equation and obtain the natural boundary conditions of the problem

$$\delta \int\int_R \left[\alpha(x, y)u_x^2 + \beta(x, y)u_y^2 - \gamma(x, y)u^2\right] dxdy = 0.$$

In particular, show that if $\beta(x, y) = \alpha(x, y)$ the natural boundary condition takes the form

$$\alpha \frac{\partial u}{\partial n} \delta u = 0$$

where $\frac{\partial u}{\partial n}$ is the normal derivative of u.

3. Determine the natural boundary condition for the multiple integral problem

$$I(u) = \int\int_R L(x, y, u, u_x, u_y) dxdy, \quad u \epsilon C^2(R), \quad u \text{ unspecified on the boundary of } R$$

4. Find the Euler equations corresponding to the following functionals

   a. $I(u) = \int\int_R (x^2 u_x^2 + y^2 u_y^2) dxdy$

   b. $I(u) = \int\int_R (u_t^2 - c^2 u_x^2) dxdt, c$ is constant

CHAPTER 7

7 Examples of Numerical Techniques

Now that we've seen some of the results of the Calculus of Variations, we can study the solution of some problems by numerical techniques.

All of the numerical techniques used in variational problems are iterative in nature, that is, they do not solve the problem in one step but rather proceed from an initial estimate (usually input by the user) and generate a sequence of succeeding estimates which converges to the answer.

The iterative procedures used, are based upon a search from the present estimate to obtain a next estimate which has certain characteristics. The types of search procedures fall into two main classes called "Indirect Methods" and "Direct Methods." We will also look at a computer program for the variable end point case using indirect methods.

7.1 Indirect Methods

Indirect methods are those which seek a next estimate satisfying certain of the necessary conditions for a minimizing arc, established previously. Thus these methods for example seek arcs that satisfy the Euler equations. An example of an indirect method is Newton's method for variational problems. We will now discuss this method and provide a sample computer program written in Matlab for students to try on their computer. First we discuss the fixed end points case.

7.1.1 Fixed End Points

Consider the fixed endpoint problem of minimizing the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')dx \tag{1}$$

among arcs satisfying

$$y(x_1) = Y_1, \qquad y(x_2) = Y_2. \tag{2}$$

The indirect method seeks to find an arc $\underline{y_0}$ which satisfies the Euler equations and also satisfies the endpoint conditions (2).

Writing the Euler equation

$$\frac{d}{dx} f_{y'} = f_y \tag{3}$$

and then differentiating, gives

$$f_{y'x} + f_{y'y}y' + f_{y'y'}y'' = f_y \tag{4}$$

(Note that we assumed that our solution will have a second derivative y'' at each point).

In this procedure, the selection of $y(x)$ and $y'(x)$ for $x_1 < x \leq x_2$ is dictated by (4) as soon as

$$y(x_1) \quad \text{and} \quad y'(x_1) \tag{5}$$

are selected.

Thus each time we alter the initial conditions (5), we will get a different solution of (4). Since by the first part of (2), the value of $y(x_1)$ is fixed, then the only variable left to satisfy the second part of (2) is $y'(x_1)$. Calling the initial estimate of the minimizing arc $\underline{y_1}$ with value $y_1'(x_1)$ and denoting the value of left end-point slope for any other arc $y'(x_1, c) = y_1'(x_1) + c$, then the solutions to (4) are a family of arcs

$$\underline{y}(c): \qquad \underline{y}(x, c) \qquad x_1 \leq x \leq x_2 \tag{6}$$

so that

$$y'(x_1, c) = y_1'(x_1) + c \quad \text{and} \quad y'(x_1, 0) = y_1'(x_1) \tag{7}$$

Differentiating the family (6) with respect to c at $c = 0$ we obtain (since $\left.\dfrac{\partial y'(x_1, c)}{\partial c}\right|_{c=0} = 1$)

$$\eta(x) \equiv \left.\frac{\partial y(x, c)}{\partial c}\right|_{c=0} \left(= \left.\frac{\partial y(x, c)}{\partial y'(x_1, c)}\right|_{c=0} \right) \quad x_1 \leq x \leq x_2 \tag{8}$$

where we have assigned the name $\eta(x)$ to $\left.\dfrac{\partial y(x, c)}{\partial c}\right|_{c=0}$

In particular at $x = x_2$ we get $\left.\dfrac{\partial y(x_2, c)}{\partial y'(x_1, c)}\right|_{c=0} (= \eta(x_2))$ as the change in the value of $y(x_2, 0)$ to a solution to (4) with each unit change in value of its left end-point slope $y_1'(x_1)\,(= y'(x_1, 0))$. Thus knowing $\eta(x_2)$, we can form the differential correction to $y_1'(x_1)$ as

$$\Delta y_1'(x_1) = \frac{Y_2 - y_1(x_2)}{\eta(x_2)} \tag{9}$$

and use this to iterate on $y_1'(x_1)$ to satisfy the second part of (2). In order to obtain $\eta(x)$ we note that for any arc

$$\underline{y}(c): \qquad y(x, c) \qquad y'(x, c) \qquad x_1 \leq x \leq x_2 \tag{10}$$

in our family (6) then by (4) we must have

$$\begin{aligned} &f_{y'x}\,(x, y(x, c),\ y'(x, c)) + f_{y'y}\,(x, y(x, c), y'(x, c))\, y'(x, c) \\ &\quad + f_{y'y'}\,(x, y(x, c),\ y'(x, c))\, y''(x, c) = f_y\,(x, y(x, c), y'(x, c)) \end{aligned} \tag{11}$$

Differentiating (11) with respect to c at $c = 0$ and assuming that in our family, $y(x, c)$ is continuously differentiable up through third order in x, c so that order of differentiation is immaterial and

$$\eta'(x) = \left.\frac{\partial^2 y(x, c)}{\partial x \partial c}\right|_{c=0} = \left.\frac{\partial^2 y(x, c)}{\partial c \partial x}\right|_{c=0} = \left.\frac{\partial y'(x, c)}{\partial c}\right|_{c=0}$$

and

$$\eta''(x) = \left.\frac{\partial^3 y(x,c)}{\partial x \partial x \partial c}\right|_{c=0} = \left.\frac{\partial^3 y(x,c)}{\partial c \partial x \partial x}\right|_{c=0}$$

which results in,

$$\begin{aligned} f_{y'xy}\eta + f_{y'xy'}\eta' + (f_{y'yy}\eta + f_{y'yy'}\eta')y_1' + f_{y'y}\eta' \\ +(f_{y'y'y}\eta + f_{y'y'y'}\eta')y_1'' + f_{y'y'}\eta'' - f_{yy}\eta - f_{yy'}\eta' = 0 \end{aligned} \tag{12}$$

where in (12) all arguments of the derivatives of f are $x, y(x), y'(x)$ i. e. along the arc $\underline{y_1}$.

Equation (12) represents a second order linear differential equation for η. The initial conditions for solution are obtained by differentiating (10) with respect to c at $c = 0$. Thus

$$\left.\frac{\partial y(x_1,c)}{\partial c}\right|_{c=0} = \eta(x_1)\ , \quad 1 = \left.\frac{\partial y'(x_1,c)}{\partial y'(x_1,c)}\right|_{c=0} = \left.\frac{\partial y'(x_1,c)}{\partial c}\right|_{c=0} = \eta'(x_1) \tag{13}$$

where in the second equation in (13) we have recalled the definition of c. Then by the second equation of (13) we get that $\eta'(x_1) = 1$. Furthermore, by the first part of (2) we see that for any c, $y(x_1, c) = Y_1 = y_1(x_1)$ so that $\eta(x_1) = 0$.

Thus we solve for $\eta(x)$ on $x_1 \le x \le x_2$ by solving the second order differential equation (12) with initial conditions

$$\eta(x_1) = 0 \qquad \eta'(x_1) = 1.$$

For example, suppose we wish to find the minimum of

$$I = \int_0^1 \left[(y')^2 + y^2\right] dx \tag{14}$$

$$y(0) = 0$$
$$y(1) = 1.$$

The function odeinput.m supplies the user with the boundary conditions, a guess for the initial slope, tolerance for convergence. All the derivatives of f required in (4) are supplied in rhs2f.m.

```
function [fy1y1,fy1y,fy,fy1x,t0,tf,y1,y2,rhs2,sg,tol] = odeinput
% Defines the problem for solving the ode:
%                  (f_{y'y'} )y"  +  (f_{y'y})y'  =  f_y - f_{y'x}

% t0     - start time
% tf     - end time
% y1     - left hand side boundary value
% y2     - right hand side boundary value
% sg     - initial guess for the slope
% tol    - tolerance e.g. 1e-4
```

```
t0      = 0;
tf       = 1;
y1     = 0;
y2     = 1;
sg     = 1;
tol     = 1e-4;
```

```
%rhs2f.m
%
function [rhs2]=rhs2f(t,x)
%
%input
% t is the time
% x is the solution vector (y,y')
%
% fy1fy1 - fy'y' (2nd partial wrt y' y')
% fy1y   - fy'y  (2nd partial wrt y' y)
% fy       - fy  (1st partial wrt y)
% fy1x   - fy'x  (2nd partial wrt y' x)
%
fy1y1  = 2;
fy1y   = 0;
fy       = 2*x(1);
fy1x   = 0;

rhs2=[-fy1y/fy1y1,(fy-fy1x)/fy1y1];
```

The main program is ode1.m which uses a modified version of ode23 from matlab. This modified version is called ode23m.m. Since we have to solve a second order ordinary differential equation, we have to transform it to a system of first order to be able to use ode23. To solve the η equation, the ode23 is used without any modifications. We also need the right hand side of the 2 equations to be solved (one for y and one for η). These are called odef.m and feta.m, respectively. All these programs (except the original ode23.m) are given here

```
% ode1.m
% This program requires an edited version of ode23 called ode23m.m
% Also required is odef.m, feta.m & odeinput.m
% All changes to a problem should ONLY be entered in odeinput.m
```

```
[fy1y1,fy1y,fy,fy1x,t0,tf,y1,y2,rhs2,sg,tol] = odeinput;
correct = 100;
while abs(correct) > tol
   %solve the initial value with the slope guessed
   x0=[y1,sg]';
   [t,x]=ode23m('odef',t0,tf,x0,y2,'rhs2f',tol,0);
   n1=size(x,1);
   yy(1:n1)=x(1:n1,1);
   plot(t,yy)

   % check the value at tf
   % change the value of the slope to match the solution
   eta0=[0,1]';
   [tt,eta]=ode23('feta',t0,tf,eta0);
   [nn1,nn2]=size(eta);
   correct=(y2-yy(n1))/eta(nn1);
   sg=sg+correct;
end
```

```
% msode23m.m
%
% This code is a modified version of MATLAB's ODE23 to find a numerically integrated
% solution to the input system of ODEs.
%
% This code is currently defined for the variable right hand endpoint defined by the
% following boundary conditions:
% y(0) = 1, y(x1) = Y1 = x2 - 1
%
% Lines which require modification by the user when solving different problems
% (different boundary function) are identified by (user defined) at the right margin.
%
%
function [tout, yout] = msode23m(ypfun, t0, tfinal, y0, rhs2f, tol, trace)

%ODE23 Solve differential equations, low order method.
% ODE23 integrates a system of ordinary differential equations using
% 2nd and 3rd order Runge-Kutta formulas.
% [T,Y] = ODE23('yprime', T0, Tfinal, Y0, Y2, rhs2) integrates the system
% of ordinary differential equations described by the M-file YPRIME.M,
```

```
% over the interval T0 to Tfinal, with initial conditions Y0.
% [T, Y] = ODE23(F, T0, Tfinal, Y0, y2, rhs2, TOL, 1) uses tolerance TOL
% and displays status while the integration proceeds.
%
% INPUT:
% F     - String containing name of user-supplied problem description.
%         Call: yprime = fun(t,y) where F = 'fun'.
%         t      - Time (scalar).
%         y      - Solution column-vector.
%         yprime - Returned derivative column-vector;
%                        yprime(i) = dy(i)/dt.
% t0    - Initial value of t.
% tfinal- Final value of t.
% y0    - Initial value column-vector.
% tol   - The desired accuracy. (Default: tol = 1.e-3).
% trace - If nonzero, each step is printed. (Default: trace = 0).
%
% OUTPUT:
% T  - Returned integration time points (column-vector).
% Y  - Returned solution, one solution column-vector per tout-value.
%
% The result can be displayed by: plot(tout, yout).
%
% See also ODE45, ODEDEMO.
% C.B. Moler, 3-25-87, 8-26-91, 9-08-92.
% Copyright (c) 1984-93 by The MathWorks, Inc.
%
% Initialization
pow = 1/3;
if nargin < 7, tol = 1.e-3; end
if nargin < 8, trace = 0; end
t = t0;
hmax = (tfinal - t)/256; %(user defined)
   %the denominator of this expression may
                      %require adjustment to
%refine the number of subintervals over
                      %which to numerically
%integrate - consider adjustment if infinite
                      %loops are encountered
%within this routine and keep the value as
                      %a power of 2
h = hmax/8;
y = y0(:);
chunk = 128;
```

```
tout = zeros(chunk,1);
yout = zeros(chunk,length(y));
k = 1;
tout(k) = t;
yout(k,:) = y.';
if trace
   clc, t, h, y
end
% The main loop
while (t < tfinal) & (t + h > t)
   if t + h > tfinal, h = tfinal - t; end
   % Compute the slopes
   rhs2=feval(rhs2f,t,y); rhs2=rhs2(:);
   s1 = feval(ypfun, t, y,rhs2); s1 = s1(:);
   rhs2=feval(rhs2f,t+h,y+h*s1); rhs2=rhs2(:);
   s2 = feval(ypfun, t+h, y+h*s1,rhs2); s2 = s2(:);
   rhs2=feval(rhs2f,t+h/2,y+h*(s1+s2)/4); rhs2=rhs2(:);
   s3 = feval(ypfun, t+h/2, y+h*(s1+s2)/4,rhs2); s3 = s3(:);
   % Estimate the error and the acceptable error
   delta = norm(h*(s1 - 2*s3 + s2)/3,'inf');
   tau = tol*max(norm(y,'inf'),1.0);
   % Update the solution only if the error is acceptable
   if delta <= tau
      t = t + h;
      y = y + h*(s1 + 4*s3 + s2)/6;
      k = k+1;
      if k > length(tout)
         tout = [tout; zeros(chunk,1)];
         yout = [yout; zeros(chunk,length(y))];
      end
      tout(k) = t;
      yout(k,:) = y.';
   end
   if trace
      home, t, h, y
   end
   % Update the step size
   if delta ~= 0.0
      h = min(hmax, 0.9*h*(tau/delta)^pow);
   end

   varendpt = t - 1; %(user defined)
   tolbnd = 1e-2;      %(user defined)
%varendpt is the equation of the variable
```

```
                        %endpoint as defined by
%the right hand side boundary curve where
                        %t is the independent variable
%tolbnd is the desired tolerance for meeting
                        %the variable right
%endpoint condition and may require some
                        %experimentation

   if abs(y(1) - varendpt) < tolbnd
                        %this checks to see if the endpoint of the solution
      disp('hit boundary in msode23m');
          break; %curve comes within a user specified
   end %tolerance of the right hand side
                                %boundary curve

end

if (t < tfinal)
   disp('Singularity likely.')
   t
end

tout = tout(1:k);
yout = yout(1:k,:);



% feta.m

function xdot=feta(t,x)
xdot=[x(2),0]';




% odef.m
function xdot=odef(t,x,rhs2)
xdot=[x(2),rhs2(1)*x(2)+rhs2(2)]';
```

The solution obtained via matlab is plotted in figure 20.

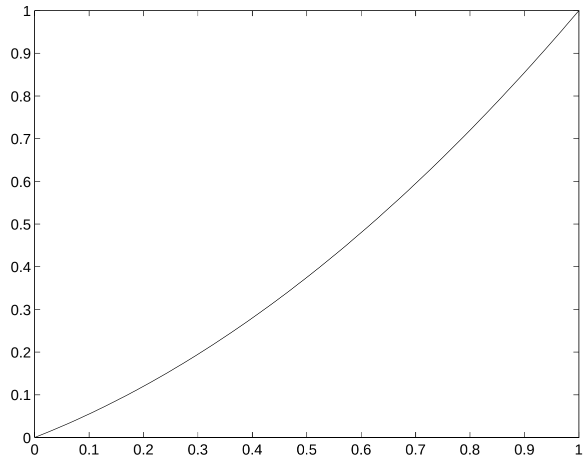

Figure 20: Solution of example given by (14)

7.1.2 Variable End Points

We have previously obtained necessary conditions that a solution arc to the variable end point problem had to satisfy.

We consider now a computer program for a particular variable end point problem. We will use Newton's method to solve the problem:

Thus consider the problem of minimizing the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')dx = \int_{x_1}^{x_2} [(y')^2 + y^2]dx \tag{15}$$

among arcs satisfying

$$y(x_1) = 1, \quad y(x_2) = Y_2 = x_2 - 1, \quad x_1 = 0 \tag{16}$$

(where we use $Y_2(x)$ for the right hand boundary, which is a straight line).

Our procedure now will be much the same as for the fixed end point problem done by Newton's method in that we'll try to find a solution to the Euler equation. Also as before, all of our estimate arcs $\underline{y}$ of solutions to this problem will have

$$y(x_1) = 1 \qquad x_1 = 0 \tag{17}$$

so that these items are **fixed**. However we note that in general this will not be the case, and in other problems we may be allowed to vary these quantities in our iterative procedure but will then be required to satisfy a transversality condition involving them.

Returning now to the problem at hand, we start with an initial estimate $\underline{y}_1$, satisfying the left end point condition

$$y_1(x_1) = 1 \qquad x_1 = 0 \tag{18}$$

and the Euler equations

$$\frac{d}{dx} f_{y'} = f_y \qquad \text{or} \qquad f_{y'x} + f_{y'y}y' + f_{y'y'}y'' = f_y. \tag{19}$$

As for the fixed endpoint case, only $y'(x_1)$ is free to iterate with, so that setting

$$y'(x_1, c) = y_1'(x_1) + c \qquad \text{with} \qquad y(x_1, c) = y_1(x_1) = 1 \tag{20a}$$

and integrating the Euler equation we get the family

$$\underline{y}(c): \qquad y(x, c) \qquad x_1 \le x \le x_2(c) \qquad -\delta \le c \le \delta \tag{20b}$$

(where only the right end value of x varies with c since the left end value is fixed and) which safisfies the Euler equation and

$$y(x_1, c) = 1 \qquad x_1 = 0 \tag{21}$$

Thus we have on this family

$$f_{y'x}(x, y(x,c), y'(x,c)) + f_{y'y}(x, y(x,c), y'(x,c))y'(x,c) + \tag{22}$$

$$f_{y'y'}(x, y(x,c), y'(x,c))y''(x,c) = f_y(x, y(x,c), y'(x,c))$$

Proceeding as we did in the fixed endpoint case we differentiate (22) with respect to c at $c = 0$. Thus

$$f_{y'xy}\eta + f_{y'xy'}\eta' + (f_{y'yy}\eta + f_{y'yy'}\eta')y_1' + f_{y'y}\eta' \tag{23}$$

$$+(f_{y'y'y}\eta + f_{y'y'y'}\eta')y_1'' + f_{y'y'}\eta'' = f_{yy}\eta + f_{yy'}\eta'$$

which is the same equation for η that we got in the fixed endpoint case.

The initial conditions for η, η', are obtained from (20a) by differentiation (at $c = 0$). In particular, differentiating the second part of (20a) yields

$$\eta(x_1) = \frac{\partial y(x_1, c)}{\partial c} = 0 \tag{24}$$

and differentiating the first part of (20a) gives

$$\eta'(x_1) = \frac{\partial y'(x_1, c)}{\partial c} = 1 \tag{25}$$

We have two conditions that our estimates have to satisfy at the right hand end, namely, (with subscript F denoting final values, e.g. $y_F(c) \equiv y(x_2(c), c)$).

$$y_F = Y_2 = x_2 - 1 \tag{26a}$$

and the transversality condition (3) of chapter 4 which applied to this problem yields

$$2y_F' - (y_F')^2 + y_F^2 = 0 \tag{26b}$$

Since x_2 is unrestricted we choose to stop integration for each estimate when (26a) is satisfied and there to evaluate the expression (26b) which we call TERM

$$\text{TERM} = \quad 2y'_F - (y'_F)^2 + y_F^2 \tag{27}$$

Then if TERM differs from 0 we compute as before how much to change c by in order to reduce this value

$$c = \frac{-TERM}{\frac{d(TERM)}{dc}} \tag{28}$$

Next, differentiating (27) yields

$$\frac{d(TERM)}{dc} = 2\frac{dy'_F}{dc} - 2y'_F\frac{dy'_F}{dc} + 2y_F\frac{dy_F}{dc} = 2[(1-y'_F)\frac{dy'_F}{dc} + y_F\frac{dy_F}{dc}] \tag{29a}$$

where all arguments are along the arc $\underline{y}_1$.

Now concentrating on y_F which is a function of c

$$y_F(c) \equiv y(x_2(c), c) \tag{29b}$$

and differentiating with respect to c at $c = 0$ (i.e. at $\underline{y}_1$) yields

$$\frac{dy_F}{dc} = \underbrace{\frac{\partial y(x_2, c)}{\partial c}}_{\eta_F} + y'_F\frac{dx_2}{dc} \tag{30a}$$

Doing analogous operations for $y'_F(c)$ yields after differentiation with respect to c at $c = 0$.

$$\frac{dy'_F}{dc} = \eta'_F + y''_F\frac{dx_2}{dc} \tag{30b}$$

Also by differentiating the middle constraint in (16) i.e. the equation $y_F(c) = Y_2 = x_2(c) - 1$ yields

$$\frac{dy_F}{dc} = \frac{dx_2}{dc} \tag{30c}$$

so that putting together (30a) and (30c) gives

$$\frac{dx_2}{dc} = \frac{dy_F}{dc} = \eta_F + y'_F\frac{dx_2}{dc} \tag{30d}$$

or then

$$\frac{dx_2}{dc}(1 - y'_F) = \eta_F \tag{30e}$$

(compare to equation (6) of the appendix 4.2 but with $\epsilon = c$ and with $x_{0i} = x_2$ and $Y_i = x_2 - 1$ so that $\frac{dY_i}{dc} = \frac{dx_2}{dc}$) or then

$$\frac{dx_2}{dc} = \frac{\eta_F}{1 - y'_F} \tag{30f}$$

and then by (29a), (30b), (30f), (30a) we get

$$\frac{d(TERM)}{dc} = 2[(1-y'_F)(\eta'_F + y''_F\frac{\eta_F}{1-y'_F}) + y_F(\eta_F + y'_F\frac{\eta_F}{1-y'_F})] \tag{31}$$

From the Euler equation we get $y''_F = y_F$ so that after collecting terms

$$\begin{aligned}\frac{d(TERM)}{dc} &= 2[(1-y'_F)(\eta'_F + \frac{y_F\eta_F}{1-y'_F}) + y_F\eta_F + \frac{y_F y'_F \eta_F}{1-y'_F}] = \\ &= 2[\eta'_F - y'_F\eta'_F + y_F\eta_F + \frac{y_F\eta_F}{1-y'_F}]\end{aligned} \tag{32}$$

We have thus obtained all of the quantities necessary to compute the correction to c.

The program for the present problem is then:

a) start with an initial estimate $\underline{y}_1$ with $y_1(x_1) = 1, \quad y'_1(x_1) = y'(x_1), \;\; x_1 = 0$

b) integrate the Euler equation for y and the equation for $\eta = \frac{\partial y}{\partial c}$ stopping the integration when the end point condition $y(x_2) = Y_2$ is met

c) determine the error in the transversality condition and the correction in $y'(x_1)$ needed to correct it $\left(\frac{-TERM}{\frac{d(TERM)}{dc}}\right) = c$, where $\frac{d(TERM)}{dc}$ is computed using η_F.

d) re-enter (b) with initial conditions $y(x_1) = 1, x_1 = 0, y'(x_1) = y'_1(x_1) + c$ and continue through the steps (b) and (c)

e) stop when the error is smaller than some arbitrary number ϵ.

7.2 Direct Methods

Direct methods are those which seek a next estimate by working directly to reduce the functional value of the problem. Thus these methods search in directions which most quickly tend to reduce I. This is done by representing I to various terms in its Taylor series and reducing the functions represented.

The direct method we will work with is the gradient method (also called method of steepest descent). This method is based on representing the integral to be minimized as a linear functional of the arcs $\underline{y}$ over which it is evaluated. The gradient method has an analogue for finite dimensional optimization problems and we will first describe this method in the finite dimensional case.

Thus suppose that we wish to minimize a function of the two dimensional vector $\overline{y} = (y_1, y_2)$

$$f(\overline{y}) \qquad (= (y_1)^2 + (y_2)^2 \;\text{ as an example}) \tag{33}$$

subject to the constraint

$$\ell(\overline{y}) = 0 \qquad (y_1 + y_2 - 1 = 0 \text{ for our example}) . \tag{34}$$

The gradient method says that starting with an initial estimate $\overline{y}_1 = (y_{1,1}, y_{1,2})$, we first linearize f as a function of the change vector $\overline{\eta} = (\eta_1, \eta_2)$.

Expanding f to first order at the point $\overline{y}_1$, gives

$$f(\overline{y}_1 + \epsilon\overline{\eta}) \sim f(\overline{y}_1) + \epsilon f_{y_1}\eta_1 + \epsilon f_{y_2}\eta_2 \quad (= y_{1,1}^2 + y_{1,2}^2 + 2\epsilon(y_{1,1}\eta_1 + y_{1,2}\eta_2)) \tag{35}$$

if $|\epsilon\eta|$ is small. Since $f(\overline{y}_1)$ is constant, then this allows us to consider f as a function F of only the change vector $\overline{\eta} = (\eta_1, \eta_2)$

$$F(\overline{\eta}) \equiv f(\overline{y}_1) + \epsilon f_{y_1}\eta_1 + \epsilon f_{y_2}\eta_2 \tag{36}$$

where we don't list ϵ as an argument of F since it will be determined independently of $\overline{\eta}$ and we wish to concentrate on the determination of $\overline{\eta}$ first.

We can similarly linearize the constraint ℓ and approximate ℓ by the function L which depends on $\overline{\eta}$

$$L(\overline{\eta}) \tag{37}$$

Now we wish to choose $\overline{\eta}$ in order that $F(\overline{\eta})$ is as small as possible and also $L(\overline{\eta}) = 0$ for a given step size length, ($|\epsilon\overline{\eta}| = ST$). Recall from calculus, that the maximum negative (positive) change in a function occurs if the change vector is opposite (in the same direction) to the gradient of the function. Now, the gradient of the function (36) considered as a function of $\overline{\eta}$ is:

$$\nabla F = (F_{\eta_1}, F_{\eta_2}) = \epsilon(f_{y_1}, f_{y_2}) \quad (= \epsilon(2y_{1,1}, 2y_{1,2}) \text{ for our example}) \tag{38}$$

so that fastest way to reduce F requires that $\overline{\eta}$ be oriented in the direction

$$\overline{\eta} = (\eta_1, \eta_2) = -(f_{y_1}, f_{y_2}) \quad (= (-2y_{1,1}, -2y_{1,2}) \text{ for our example}) \tag{39}$$

(note that this choice is independent of ϵ). However since we have a constrained problem, then our change should be restricted so that our new point $\overline{y}_1 + \epsilon\overline{\eta}$ satisfies

$$\ell(\overline{y}_1 + \epsilon\overline{\eta}) = 0 \tag{40a}$$

or by our approximation

$$L(\overline{\eta}) = 0 \tag{40b}$$

according to the way we defined L. Thus we modify $\overline{\eta}$ from (39) so that it satisfies (40b). These conditions establish the direction of $\overline{\eta}$ and then the value of ϵ is established by the requirement that $|\epsilon\overline{\eta}| = ST$, i. e. the change is equal to the step size.

The gradient procedure then computes the function $f(\overline{y}_1 + \epsilon\overline{\eta})$ which should be smaller than $f(\overline{y}_1)$ and repeats the above procedure at the point $\overline{y}_2 = \overline{y}_1 + \epsilon\overline{\eta}$.

In the infinite dimensional case, the idea is the same, except that we are now dealing with a function of an infinite number of variables, namely arcs

$$\underline{y}: \qquad y(x) \qquad x_1 \leq x \leq x_2$$

and our change vector will have direction defined by the arc

$$\underline{\eta}: \qquad \eta(x) \qquad x_1 \leq x \leq x_2$$

Thus consider the case of minimizing the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')dx \tag{41}$$

subject to the fixed endpoint conditions (the constraint on the problem)

$$y(x_1) = a \qquad y(x_2) = b \tag{42}$$

Following the procedure used in the finite dimensional case, we start with an initial arc $\underline{y}_1$ and first linearize the integral I by computing the first variation of I [‖].

$$I' = \int_{x_1}^{x_2} [f_y \eta + f_{y'} \eta']dx \tag{43}$$

Integrating (by parts) the first term in the integrand gives,

$$\int_{x_1}^{x_2} f_y \eta(x)dx = [\eta(x) \int_{x_1}^{x} f_y ds]_{x_1}^{x_2} - \int_{x_1}^{x_2} [\eta'(x) \int_{x_1}^{x} f_y ds]dx \tag{44}$$

Since the variations $\eta(x)$ must (why?) satisfy

$$\eta(x_1) = \eta(x_2) = 0 \tag{45}$$

then the first term on the right in (44) vanishes and plugging back into (43) gives

$$I' = \int_{x_1}^{x_2} [f_{y'} - \int_{x_1}^{x} f_y ds]\eta'(x)dx\,. \tag{46}$$

Corresponding to (36) we then approximate $I(\overline{y}_1 + \epsilon\eta)$ by

$$\hat{I} = I(\overline{y}_1) + \epsilon I' \tag{47}$$

where the second term is represented by (46). Analogous to the finite dimensional case, we desire to select η or equivalently $\eta(x_1)$ and $\eta'(x)$, $x_1 \leq x \leq x_2$ so that subject to a step size constraint, we have that $\hat{I}$ (and also approximately I) has minimum value at $y_1 + \epsilon\eta$. The stepsize constraint in this case looks like

‖Alternatively we can think of this as the derivative at $\epsilon = 0$ of I evaluated on the family $\underline{y}(\epsilon)$: created with the arc $\eta(x)$ but we don't set it $= 0$ (why?)

$$\begin{array}{c} \max|\epsilon\eta'(x)| \\ x_1 \le x \le x_2 \end{array} \tag{48}$$

(which represents the maximum change from $y_1'(x)$ along our arcs) and where ϵ will be selected according to the stepsize we wish. It can be shown formally that the best selection of $\eta'(x)$ at each x is

$$\eta'(x) = -[f_{y'} - \int_{x_1}^{x} f_y ds] \qquad x_1 \le x \le x_2 \tag{49}$$

This hueristically can be considered the direction opposite to the gradient of $\hat{I}$ with respect to $\eta'(x)$ for each x. However, as in the finite dimensional case, we must modify this change in order to satisfy the constraint (45). Defining the integral of $\eta'(x)$ of (49) from x_1 to x as

$$M(x) = -\int_{x_1}^{x}[f_{y'} - \int_{x_1}^{\xi} f_y ds]d\xi \tag{50}$$

and defining the average of this as

$$M_{avg} = \frac{M(x_2)}{x_2 - x_1} = -\frac{1}{x_2 - x_1}\int_{x_1}^{x_2}[f_{y'} - \int_{x_1}^{x} f_y ds]dx \tag{51}$$

(note that $M(x_1) = 0$) then with $\eta'(x)$ defined as

$$\begin{aligned} \eta'(x) = \; - \; [f_{y'} - \int_{x_1}^{x} f_y ds] - \; M_{avg} = \\ -[f_{y'} - \int_{x_1}^{x} f_y ds - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2}[f_{y'} - \int_{x_1}^{x} f_y ds]dx] \end{aligned} \tag{52}$$

we get

$$\eta(x_2) = \int_{x_1}^{x_2} \eta'(x)dx + \eta(x_1) = \eta(x_1) \tag{53}$$

which together with $\eta(x_1) = 0$ (which we can easily choose) yields η which satisfies our constraint (45). Integrate (52) from x_1 to x

$$\eta(x) = M(x) - (x - x_1)M_{avg}.$$

While this is not the only way to create η satisfying (45), it can be formally shown that subject to (45), this $\eta(x)$ is the best selection to reduce I.

We now give a matlab program that uses direct method to minimize the integral I. This program requires the user to supply the functions $f, f_y, f_{y'}$. These functions are supplied in the finput.m file that follows.

```
% This program solves problems of the form
%                 ___x2
%                |
% Minimize I =  |    f(x,y,y') dx
```

```
%            ___|x1
%
% using the direct method. The user must supply the  functions
% F(x,y,y'), Fy(x,y,y')
% and Fy'(x,y,y') in a file called finput.m
%
% See finput.m

% By Jerry Miranda, 12/10/96

% WARNING: Early termination may occur if N is TOO large or if epsilon is
%          TOO small.  The count parameter is set at 50 and can be adjusted
%          below.  Count is required to prevent runaway in the while loop or
%          excessive computation until this version is modified.
clear
C = 50;
% set the count paramater

% Here we solve the problem min [ int(0->1) {2y' + y^2} dx]
%                            s.t. y(0)=0, y(1)=1

% setup boundary conditions                                       (User define)
x1 = 0; y1 = 0;  % y(x1) = y1
x2 = 1; y2 = 1;  % y(x2) = y2

% choose an epsilon and the number of points to iterate          (User define)
epsilon = .01;  N = 25;


if x2-x1 == 0, error('x2 and x1 are the same'), break, end

deltax = (x2-x1)/N;  x = [x1:deltax:x2]'; % x is a col vector

% make an initial guess for the solution arc:                    (User define)
% this is a function satisfying the boundary conditions
ybar = (y2-y1)/(x2-x1)*(x-x1)+y1;

% this is the derivative of a linear function ybar
% if ybar is NOT linear,
% we should use finite difference to approximate yprime

yprime = ones(size(x))*(y2-y1)/(x2-x1);

% calculate M(x2) and Mavg
```

```
sum1=0;
MM(1)=0;
for i = 2:N+1
 sum2=0;
     for jj=1:i-1
        sum2= deltax*finput(x(jj),ybar(jj),yprime(jj),2)+sum2;
     end

   sum1 = deltax*(finput(x(i),ybar(i),yprime(i),3)-sum2)+sum1;
   MM(i)= - sum1;
end
Mx2 = - sum1;
Mavg = Mx2/(x2-x1);
% Calculate eta(x) for each x(i)
for i = 1:N+1
  eta(i,1) = MM(i) - Mavg*(x(i)-x1);
end
% Calculate eta'(x) for each x(i)
for i = 1:N+1
     sum2=0;
     for jj=1:i-1
        sum2= deltax*finput(x(jj),ybar(jj),yprime(jj),2)+sum2;
     end
 etaprm(i,1)= - finput(x(i),ybar(i),yprime(i),3)-sum2 -Mavg;
end
% The main loop
% We now compute Ihat = I(ybar1) + epsilon*I' and check to minimize Ihat

% First Ihat
sum1=0;

for i = 1:N+1
   F = finput(x(i),ybar(i),yprime(i),1);
   sum1 = deltax*F+sum1;
end
Ihatnew = sum1; Ihatold = Ihatnew+1;
count = 0; %set counter to prevent runaway

while (Ihatnew <= Ihatold) & (count <= C)

  count = count + 1;

  % Integrate to get I'
  sum1=0;
```

```
  for i = 1:N+1
     sum2 =0;
     for j = 1:i-1
        Fy = finput(x(j),ybar(j),yprime(i),2);
        sum2 = deltax*Fy+sum2;                   % what delta is used
     end
     Fyp = finput(x(i),ybar(i),yprime(i),3);
     sum1 = deltax*(Fyp+sum1-sum2)*etaprm(i);
  end
  Iprm = sum1;

  % Integrate to get I

  sum1=0;

  for i = 1:N+1
     F = finput(x(i),ybar(i),yprime(i),1);
     sum1 = deltax*F+sum1;
  end
  I = sum1;
  Ihatnew = I + epsilon*Iprm;
  if Ihatnew < Ihatold

    ybar = ybar + epsilon*eta;
    Ihatold = Ihatnew;
      for ij=2:N+1
        yprime(ij)=(ybar(ij)-ybar(ij-1))/(x(ij)-x(ij-1));
      end
  end
end

% we now have our solution arc ybar

plot(x,ybar), grid, xlabel('x'), ylabel('y')
title('Solution y(x) using the direct method')




function value = finput(x,y,yp,num)
% function VALUE = FINPUT(x,y,yprime,num) returns the value of the
% functions F(x,y,y'), Fy(x,y,y'), Fy'(x,y,y')  at a given
% x,y,yprime
% for a given num,
```

```
% num defines which function you want to evaluate.
%       1 for F, 2 for Fy, 3 for Fy'.

if nargin < 4, error('Four arguments are required'), break, end
if (num < 1) | (num > 3)
  error('num must be between 1 and 3'), break
end

if num == 1, value = yp^2 + y^2; end      % F
if num == 2, value = 2*y; end             % Fy
if num == 3, value = 2*yp; end            % Fy'
```

Problems

1. Find the minimal arc $y(x)$ that solves, minimize $I = \int_0^{x_1} \left[y^2 - (y')^2\right] dx$

   a. Using the indirect (fixed end point) method when $x_1 = 1$.
   b. Using the indirect (variable end point) method with $y(0)$=1 and $y(x_1) = Y_1 = x_2 - \frac{\pi}{4}$.

2. Find the minimal arc $y(x)$ that solves, minimize $I = \int_0^1 \left[\frac{1}{2}(y')^2 + yy' + y' + y\right] dx$ where $y(0) = 1$ and $y(1) = 2$.

3. Solve the problem, minimze $I = \int_0^{x_1} \left[y^2 - yy' + (y')^2\right] dx$
   a. Using the indirect (fixed end point) method when $x_1 = 1$.
   b. Using the indirect (variable end point) method with $y(0)$=1 and $y(x_1) = Y_1 = x_2 - 1$.

4. Solve for the minimal arc $y(x)$:

$$I = \int_0^1 \left[y^2 + 2xy + 2y'\right] dx$$

   where $y(0) = 0$ and $y(1) = 1$.

CHAPTER 8

8 The Rayleigh-Ritz Method

We now discuss another numerical method. In this technique, we approximate the variational problem and end up with a finite dimensional problem. So let us start with the problem of seeking a function $y = y_0(x)$ that extremizes an integral $I(y)$. Assume that we are able to approximate $y(x)$ by a linear combination of certain linearly independent functions of the type:

$$y(x) \approx \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_N\phi_N(x) \tag{1}$$

where we will need to determine the constant coefficients $c_1, \cdots c_N$.

The selection of which approximating function $\phi(x)$ to use is arbitrary except for the following considerations:

a) If the problem has boundary conditions such as fixed end points, then $\phi_0(x)$ is chosen to satisfy the problem's boundary conditions, and all other $\phi_i$vanish at the boundary. (This should remind the reader of the method of eigenfunction expansion for inhomogeneous partial differential equation. The functions ϕ_i are not necessarily eigenfunctions though.)

b) In those problems where one knows something about the form of the solution then the functions $\phi_i(x)$ can be chosen so that the expression (1) will have that form.

By using (1) we essentially replace the variational problem of finding an arc $y(x)$ that extremizes I to finding a set of constants $c_1, \cdots, c_N$ that extremizes $I(c_1, c_2, \cdots, c_N)$. We solve this problem as a finite dimensional one i. e. by solving

$$\frac{\partial I}{\partial c_i} = 0, \quad i = 1, \cdots, N \tag{2}$$

The procedure is to first determine an initial estimate of c_1 by the approximation $y \approx \phi_0 + c_1\phi_1$. Next, the approximation $y \approx \phi_0 + c_1\phi_1 + c_2\phi_2$ is used (with c_1 being redetermined). The process continues with $y \approx \phi_0 + c_1\phi_1 + c_2\phi_2 + c_3\phi_3$ as the third approximation and so on. At each stage the following two items are true:

a) At the i^{th} stage, the terms $c_1 \cdots, c_{i-1}$ that have been previously determined are re-determined

b) The approximation at the i^{th} stage

$$y \approx \phi_0 + c_1\phi_1 + \cdots + c_i\phi_i \tag{3}$$

will be better or at least no worse than the approximation at the $i - 1^{st}$ stage

$$y \approx \phi_0 + c_1\phi_1 + \cdots + c_{i-1}\phi_{i-1} \tag{4}$$

Convergence of the procedure means that

$$\lim_{N\to\infty} \left(\phi_0 + \sum_{i=1}^{N} c_i\phi_i \right) = y_0(x) \tag{5}$$

where $y_0(x)$ is the extremizing function. In many cases one uses a complete set of functions e. g. polynomials or sines and cosines. A set of functions $\phi_i(x) \quad (i = 1, 2, \cdots)$ is called complete over $[a, b]$ if for each Riemann integrable** function $f(x)$, there is a number N (depending on $\epsilon, c_1, \cdots, c_N$) such that

$$\max_{[a,b]}[f - \sum_{i=1}^{N} c_i\phi_i]^2 < \epsilon \tag{6}$$

The above outlined procedure can be extended in a number of ways. For example, more than one independent variable may be involved. So for the problem of

$$\min I = \int\int_R F(x, y, w, w_x, w_y)dydx \tag{7}$$

subject to

$$w = h(s) \quad \text{on the boundary} \quad \Gamma \text{ of } R \tag{8}$$

where $h(s)$ is some prescribed function and s is the arc length along Γ. Analogous to (1) we write

$$w(x, y) = \phi_0(x, y) + c_1\phi_1(x, y) + \cdots + c_N\phi_N(x, y) \tag{9}$$

and ϕ_0 satisfies (8) and $\phi_i(x) \;\; i = 1, 2, 3\cdots$ are zero on Γ. We could also extend the procedure to functions involving higher derivatives, more independent variables, etc.

Example 1: Apply the procedure to the problem of

$$\min I = \int_0^1 [(y')^2 - y^2 - 2xy]dx \tag{10}$$

with the boundary conditions

$$y(0) = 1 \qquad y(1) = 2 \tag{11}$$

Solution: Since the boundary conditions are NOT homogeneous, we have to take ϕ_0 to satisfy the boundary conditions, i.e. $\phi_0 = 1 + x$. We choose $\phi_1(x) = x(1 - x)$ since it should satisfy zero boundary conditions. Setting

$$y_1(x) = 1 + x + c_1x(1 - x)\,. \tag{12}$$

Substituting (12) into (10) and performing the integration gives

$$I = 1 + \frac{3}{10}c_1^2 - \frac{2}{3}c_1 \tag{13}$$

Solving $\dfrac{\partial I}{\partial c_1} = 0$ implies that $c_1 = \dfrac{10}{9}$ or

$$y_1(x) = 1 + x + \frac{10}{9}x(1 - x) \tag{14}$$

**A function f is Riemann integrable over $[a, b]$ if all points of discontinuity of f can be enclosed in a set of subintervals, the sum of whose lengths can be made as small as desired as the first approximate solution. We note that $\frac{d^2 I}{dc_1^2}$ is positive at $c_1 = \frac{1}{9}$; thus we have minimized I on the class of functions defined by (12).

Continuing we try

$$y_n(x) = 1 + x + x(1 - x)[c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}] \tag{15}$$

where $n = 2, 3, 4, \cdots$. The boundary conditions are satisfied by y_n for all values of c_i that is for $n = 1, 2, 3, \cdots$

$$y_n(0) = 1 \quad y_n(1) = 2\,. \tag{16}$$

For $n = 2$, when

$$y_2(x) = 1 + x + x(1 - x)[c_1 + c_2 x] \tag{17}$$

is used and the integration carried out, we get 2 equations for the two parameters c_1 and c_2 when solving $\frac{\partial I}{\partial c_1} = \frac{\partial I}{\partial c_2} = 0$. This gives

$$c_1 = 0.9404 \qquad c_2 = 0.3415 \tag{18}$$

so that

$$y_2(x) = 1 + x + x(1 - x)[0.9404 + 0.3415x] \tag{19}$$

Comparing the two approximations $y_1(x)$ and $y_2(x)$ with the exact solution[††]

$$y = \cos x + \frac{3 - \cos 1}{\sin 1} \sin x - x \tag{20}$$

In the next figure we plot the exact solution and $y_1(x)$ and $y_2(x)$.

It can be seen that y_1 is already reasonably close.

8.1 Euler's Method of Finite Differences

Euler solved many variational problems by the method of finite differences. Suppose we want to extremize the integral

$$I(y) = \int_{x_0}^{x_{n+1}} F(x, y, y')dx \tag{21}$$

where x_0 and x_{n+1} are given and the function y is subject to the boundary conditions $y(x_0) = y_0$ and $y(x_{n+1}) = y_{n+1}$. Dividing the interval $[x_0, x_{n+1}]$ into $n + 1$ *equal* parts, the width of each piece is

$$\Delta x = \frac{x_{n+1} - x_0}{n + 1}$$

[††]The exact solution is obtained in this problem, by noting that the variational problem (10) and (11) has the same solution as the boundary-value problem

$$y'' + y = -x \quad y(0) = 1 \quad y(1) = 2$$

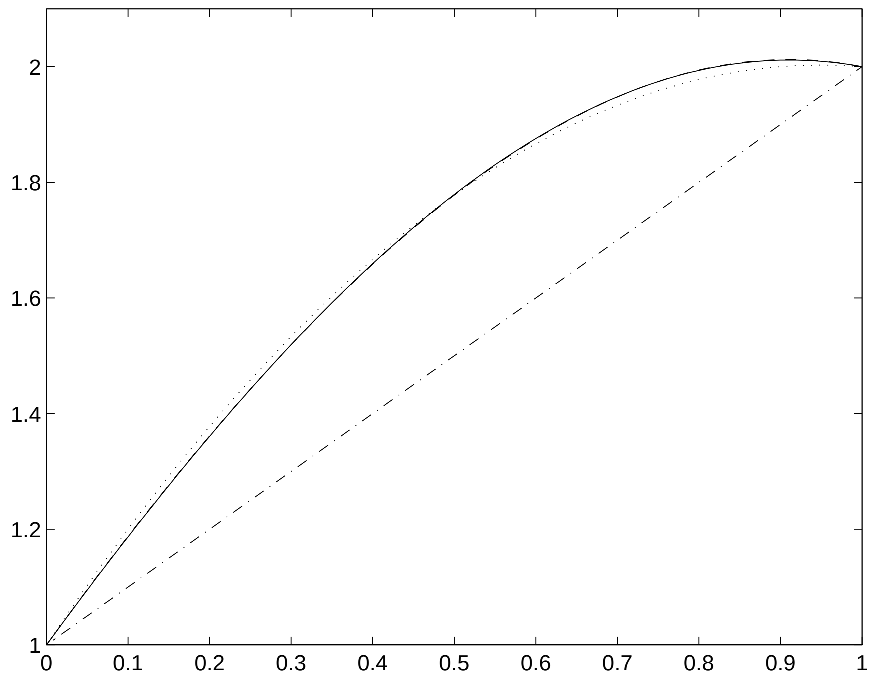

Figure 21: The exact solution (solid line) is compared with ϕ_0 (dash dot), y_1 (dot) and y_2 (dash)

(See Figure 22.) Next, let $y_1, y_2, \cdots, y_n$ be the values of y corresponding to

$$x_1 = x_0 + \Delta x, \;\; x_2 = x_0 + 2\Delta x, \cdots, x_n = x_0 + n\Delta x$$

respectively. The associated values $y_1, y_2, \cdots, y_n$ are unknowns because the function which solves the problem is unknown as yet. The integral (21) (by definition) is the limit of a summation, and thus we may approximate the integral by a function of n variables $\phi(y_1, y_2, \cdots, y_n)$.

$$\phi(y_1, y_2, \cdots, y_n) = \sum_{i=0}^{n} F\Big(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\Big)\Delta x \tag{22}$$

In this way the derivative is replaced by a difference quotient and the integral by a finite sum. The quantities $y_1, y_2, \cdots, y_n$ are determined so that ϕ solves

$$\frac{\partial \phi}{\partial y_i} = 0 \qquad i = 1, 2, \cdots, n \tag{23}$$

The terms of (22) which involve y_i are

$$F\Big(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\Big)\Delta x \quad \text{and} \quad F\Big(x_{i-1}, y_{i-1}, \frac{y_i - y_{i-1}}{\Delta x}\Big)\Delta x \tag{24}$$

so that

$$\frac{\partial \phi}{\partial y_i} = F_{y_i}\Big(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\Big)\Delta x - F_{y'_i}\Big(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\Big) \tag{25}$$
$$+F_{y'_{i-1}}\Big(x_{i-1}, y_{i-1}, \frac{y_i - y_{i-1}}{\Delta x}\Big) = 0$$

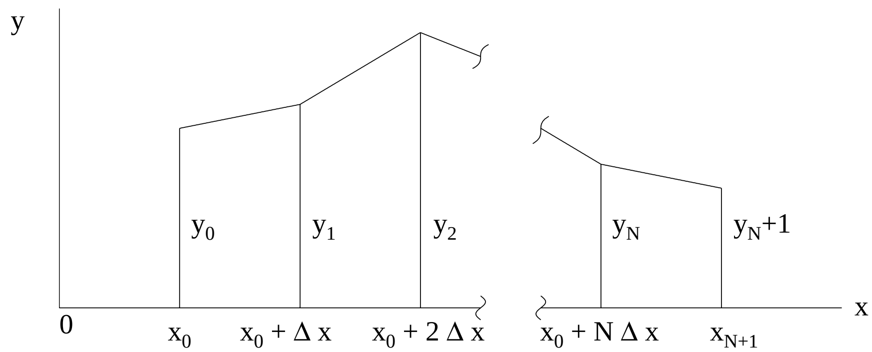

Figure 22: Piecewise linear function

where in (25)

$$y_i' = \frac{y_{i+1} - y_i}{\Delta x}\,.$$

With $\Delta y_i = y_{i+1} - y_i$, (25) is

$$F_{y_i}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{[F_{y_i'}(x_i, y_i, \frac{\Delta y_i}{\Delta x}) - F_{y_{i-1}'}(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x})]}{\Delta x} = 0\,. \tag{26}$$

This yields the following system of n equations in n unknowns:

$$F_{y_i}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\Delta F_{y_{i-1}'}}{\Delta x} = 0 \qquad i = 1, 2, \cdots, n \tag{27}$$

Equation (27) is the finite difference version of the Euler equation. As $n \to \infty$, $\Delta x \to 0$ and (27) becomes the Euler equation.

Example: Find a polygonal line which approximates the extremizing curve for

$$\int_0^2 [(y')^2 + 6x^2 y]dx \qquad y(0) = 0,\ y(2) = 4 \tag{28}$$

Solution: With $n = 1, \Delta x = 1,\ x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ y_0 = 0,\ y_2 = 4$, and $y_1 = y(x_1) = y(1)$ is unknown.

We form (in accordance with (22))

$$\begin{aligned}\phi(y_1) = \sum_{i=0}^{1}\left[6x_i^2 y_i + \left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2\right]\Delta x &= 6x_0^2 y_0 + 6x_1^2 y_1 + \left(\frac{y_1 - y_0}{\Delta x}\right)^2 + \left(\frac{y_2 - y_1}{\Delta x}\right)^2 \\ &= 0 + 6y_1 + y_1^2 + (4 - y_1)^2 = 2y_1^2 - 2y_1 + 16\end{aligned} \tag{29}$$

Now

$$\frac{d\phi}{dy_1} = 4y_1 - 2 = 0 \Rightarrow y_1 = 1/2\,. \tag{30}$$

With $n = 2$, $\Delta x = \frac{2}{3}$, $x_0 = 0$, $x_1 = \frac{2}{3}$, $x_2 = \frac{4}{3}$, $x_3 = 2$, $y_0 = 0$, and $y_3 = 4$. The variables are $y_1 = y(\frac{2}{3})$ and $y_2 = y(\frac{4}{3})$. And then

$$\phi(y_1, y_2) = [\frac{9}{2}y_1^2 + \frac{8}{3}y_1 + \frac{9}{2}y_2^2 - \frac{9}{2}y_1y_2 - \frac{22}{3}y_2 + 36]\frac{2}{3} \tag{31}$$

So then

$$\begin{aligned} \partial\phi/\partial y_1 &= 9y_1 - \frac{9}{2}y_2 + \frac{8}{3} = 0 \\ \partial\phi/\partial y_2 &= -\frac{9}{2}y_1 + 9y_2 - \frac{22}{3} = 0 \end{aligned} \tag{32}$$

giving $y_1 = \frac{4}{27}$ and $y_2 = \frac{24}{27}$.

With $n = 3$ and $\Delta x = \frac{1}{2}$, we have

$$\phi(y_1, y_2, y_3) = \frac{1}{2}[8(y_1^2 + y_2^2 + y_3^2 - y_1y_2 - y_2y_3) + \frac{3}{2}y_1 + 6y_2 - \frac{37}{2}y_3 + 64]$$

and the partial derivatives with respect to y_i give

$$\begin{aligned} 16y_1 - 8y_2 + \frac{3}{2} &= 0 \\ 16y_2 - 8(y_1 + y_3) + 6 &= 0 \\ 16y_3 - 8y_2 - \frac{37}{2} &= 0\,. \end{aligned}$$

Solving gives

$$y_1 = \frac{1}{16} \qquad y_2 = \frac{5}{16} \qquad y_3 = \frac{21}{16}.$$

With $n = 4$ and $\Delta x = 0.4$, we get

$$\begin{aligned} y_1 = y(0.4) = 0.032 &\qquad y_2 = y(0.8) = 0.1408 \\ y_3 = y(1.2) = 0.5568 &\qquad y_4 = y(1.6) = 1.664 \end{aligned}$$

The Euler equation for this problem is

$$3x^2 - y'' = 0$$

which when solved with the boundary conditions gives $y = x^4/4$. If we compare the approximate values for $n = 1, 2, 3, 4$ with the exact result, the results are consistently more accurate for the larger values of the independent variable (i.e., closer to $x = 2$). But the relative errors are large for x close to zero. These results are summarized in the table below and the figure.

x	y for $n=1$	y for $n=2$	y for $n=3$	y for $n=4$	y_{exact}
0.0	0.0	0.0000	0.0000	0	0.0000
0.4				0.0320	0.0064
0.5			0.0625		0.0156
2/3		0.1481			0.0494
0.8				0.1408	0.1024
1.0	0.5		0.3125		0.2500
1.2				0.5568	0.5184
4/3		0.8889			0.7901
1.5			1.3125		1.2656
1.6				1.6640	1.6384
2.0	4.0	4.0000	4.0000	4.0000	4.0000

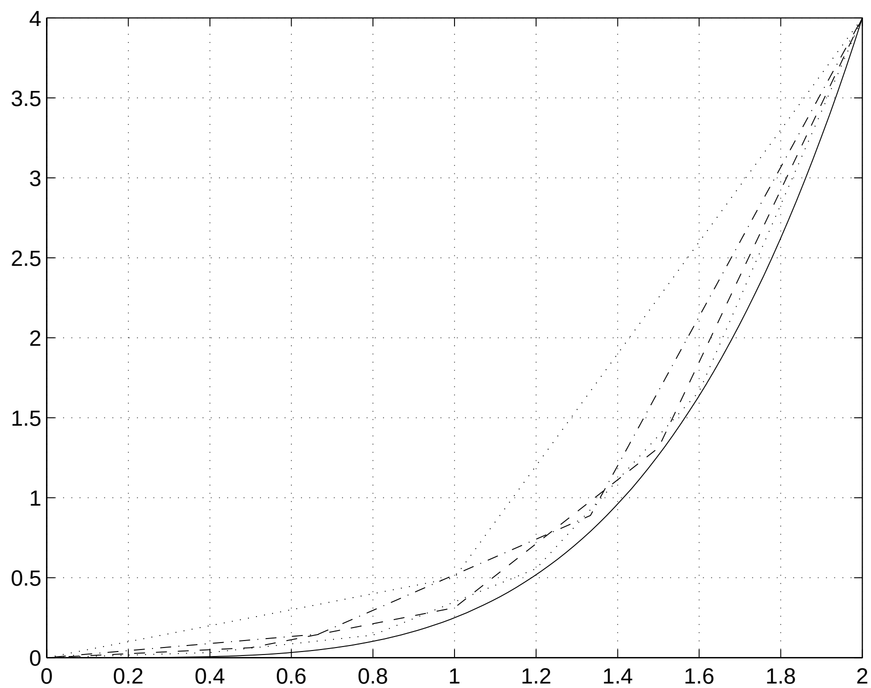

Figure 23: The exact solution (solid line) is compared with y_1 (dot), y_2 (dash dot), y_3 (dash) and y_4 (dot)

Problems

1. Write a MAPLE program for the Rayleigh-Ritz approximation to minimize the integral

$$I = \int_0^1 \left[(y')^2 - y^2 - 2xy \right] dx$$

$$y(0) = 1$$
$$y(1) = 2.$$

Plot the graph of y_0, y_1, y_2 and the exact solution.

2. Solve the same problem using finite differences.

CHAPTER 9

9 Hamilton's Principle

Variational principles enter into many physical real world problems and can be shown in certain systems to derive equations which are equivalent to Newton's equations of motion. Such a case is Hamilton's principle, the development is as follows: First let's assume Newton's equations of motion hold for a particle of mass m with vector position R acted on by force F. Thus

$$m\ddot{R} - F = 0 \tag{1}$$

(where "·" denotes time differentiation) is the differential equation which defines the motion of the particle. Consider the resulting path in time $R(t) \qquad t_1 \leq t \leq t_2$ and let

$$\delta R(t) \qquad t_1 \leq t \leq t_2 \tag{2}$$

be a curve satisfying

$$\delta R(t_1) = 0,\ \delta R(t_2) = 0 \tag{3}$$

(this is η in our previous chapters) and consider (see Figure 24) the varied path $R(t)+\delta R(t)$ *. When using the δ notation, it's often called the variation. Thus δR is the variation in R. The variation is likened to the differential. So e.g. for a function $g(x, y)$ then $\delta g = g_x \delta x + g_y \delta y$, is the variation in g due to variations δx in x and δy in y.

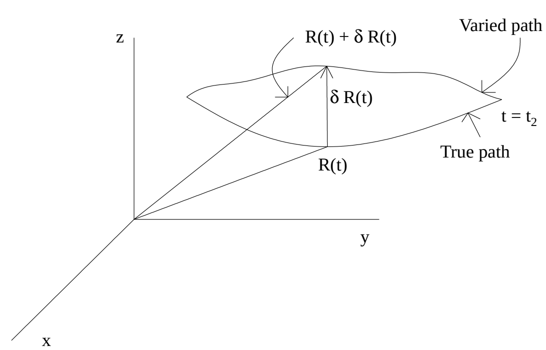

Figure 24: Paths made by the vectors R and $R + \delta R$

Now take the dot product between (1) and δR and integrate from t_1 to t_2

$$\int_{t_1}^{t_2} (m\ddot{R} \cdot \delta R - F \cdot \delta R) dt = 0 \tag{4}$$

If the first term is integrated by parts using

$$\frac{d}{dt}(\dot{R} \cdot \delta R) = \ddot{R} \cdot \delta R + \dot{R} \cdot \delta \dot{R} \tag{5}$$

*In the notation of previous chapters, δR would be called η, (and η would have three components, one each for x, y, z) however the δ notation is widely used in presenting Hamilton's principle and has some advantage over the other.

where we've used $\frac{d}{dt}\delta R = \delta\dot{R}$. Then by (3) this gives

$$\int_{t_1}^{t_2} m\ddot{R}\cdot\delta R dt = [m\dot{R}\cdot \underbrace{\delta R}_{=0 \text{ at both ends}}]_{t_1}^{t_2} - \int_{t_1}^{t_2}(m\dot{R}\cdot\delta\dot{R})dt \tag{6}$$

$$= -\int_{t_1}^{t_2}(m\dot{R}\cdot\delta\dot{R})dt$$

Now consider the variation (change) in the term $\dot{R}^2$, due to δR

$$\delta\dot{R}^2 = \delta(\dot{R}\cdot\dot{R}) = 2\dot{R}\cdot\delta\dot{R} \tag{7}$$

so that (6) becomes

$$\int_{t_1}^{t_2} m\ddot{R}\cdot\delta R dt = -\int_{t_1}^{t_2}\delta\Big(\frac{1}{2}m\dot{R}^2\Big)dt = -\int_{t_1}^{t_2}\delta T dt \tag{8}$$

where $T = \frac{1}{2}m\dot{R}^2$ is the kinetic energy of the particle. Thus using (8) in (4) gives

$$\int_{t_1}^{t_2}(\delta T + F\cdot\delta R)dt = 0 \tag{9}$$

This is the most general form of Hamilton's Principle for a single particle under a general force field and says that the path of motion is such that along it, the integral of the variation δT of the kinetic energy T plus $F\cdot\delta R$ must be zero for variations in the path satisfying $\delta R(t_1) = 0,\ \ \delta R(t_2) = 0$.

Conversely, from Hamilton's Principle we may deduce Newton's law as follows: From (9), the definition of T and (7) comes

$$\int_{t_1}^{t_2}(m\dot{R}\cdot\delta\dot{R} + F\cdot\delta R)dt = 0 \tag{10}$$

Now by (5) and integration by parts, using (3) we get

$$\int_{t_1}^{t_2}(-m\ddot{R} + F)\cdot\delta R dt = 0 \tag{11}$$

And since this holds for all $\delta R(t)$ satisfying (3) we get (by a modified form of the fundamental lemma presented in chapter 4)

$$m\ddot{R} - F = 0 \tag{12}$$

which is Newton's law of motion.

If the force field is conservative, then there is a function ϕ of position say $\phi(x, y, z)$ for motion in 3-space such that

$$\phi_x = F_1, \quad \phi_y = F_2, \quad \phi_z = F_3 \quad \text{or} \quad \text{then} \ \ F = \nabla\phi \tag{13}$$

where F_1, F_2, F_3 are the components of force F along x, y, z axes respectively and $\nabla\phi$ is the gradient of ϕ. Then

$$\delta\phi = \phi_x \delta x + \phi_y \delta y + \phi_z \delta z = F_1 \delta x + F_2 \delta y + F_3 \delta z = F \cdot \delta R \tag{14}$$

where $\delta x, \delta y, \delta z$ are the components of δR. The function ϕ is called the force potential. The function V defined by

$$V \equiv -\phi \tag{15a}$$

satisfies (by (14))

$$F \cdot \delta R = -\delta V \cdot \tag{15b}$$

This function is called the potential energy. For example in gravitational motion in a spherically symmetric field centered at the origin, the force on a particle is

$$F = \frac{-\mu}{|R|^2} \frac{R}{|R|} \quad = \frac{-\mu R}{|R|^3} \tag{16}$$

(where μ is the gravitational constant). In this case

$$V = \frac{-\mu}{|R|} \tag{17}$$

and one can check that (13) and (15) hold.

For conservative fields, by (15b), then (9) becomes

$$\int_{t_1}^{t_2} \delta(T - V) dt = 0 \tag{18}$$

and this is Hamilton's principle for a conservative force field. Thus, **Hamilton's principle for a conservative system states that the motion is such that the integral of the difference between kinetic and potential energies has zero variation**.

The difference $T - V$ is often called the Lagrangian L

$$L \equiv T - V \tag{19}$$

and in these terms, Hamilton's principle for a conservative system says that the motion is such that

$$\int_{t_1}^{t_2} \delta L dt = \delta \int_{t_1}^{t_2} L dt = 0 \tag{20}$$

(where $\delta \int_{t_1}^{t_2} L dt$ means the variation in the integral). Then in the usual way we can show that this means that the motion is such that

$$L_x - \frac{dL_{\dot{x}}}{dt} = 0 \quad L_y - \frac{dL_{\dot{y}}}{dt} = 0 \quad L_z - \frac{dL_{\dot{z}}}{dt} = 0 \tag{21}$$

i.e. the Euler equations hold for L.

For one dimensional $x(t)$, Euler's equation is

$$L_x - \frac{d}{dt} L_{\dot{x}} \;=\; 0.$$

Let's define a canonical momentum, p, by $L_{\dot{x}}$, then if $L_{\dot{x}\dot{x}} \neq 0$, then we can solve for $\dot{x}$ in terms of t, x, p,

$$\dot{x} \;=\; \phi(t, x, p).$$

Define the Hamiltonian H by

$$H(t, x, p) \;=\; -L(t, x, \phi(t, x, p)) + p\phi(t, x, p).$$

In many systems H is the total energy.

Note that

$$\frac{\partial}{\partial p} H \;=\; \phi \;=\; \dot{x}$$

$$\frac{\partial}{\partial x} H \;=\; -\dot{p}$$

These are known as Hamilton's equations.

Let's continue with our example of motion of a particle in a spherically symmetric gravitational force field. This is the situation for a satellite travelling about a spherical earth. We only assume that the force is along a line directed through the center of the Earth, where we put our origin. We don't assume any special form for F, such that as we know it depends only on distance to center of the Earth. Let t_0 be some instant and let P be the plane containing the position and velocity vectors at t_0. For ease of presentation let P be the horizontal x, y plane pictured below (we can always orient our coordinates so that this is true). Form spherical coordinates (r, θ, λ) with λ measured in P and θ, measured perpendicular to P ($\theta = 0$ for the current case). Then in these spherical coordinates

$$\delta R = e_r \delta r + e_\theta r \delta\theta + e_\lambda r \cos\theta \delta\lambda \tag{22}$$

where $r = |R|$ is distance from origin to particle and e_r, e_θ, e_λ are the unit direction vectors (see Figure 25) that R changes due to respective changes in the spherical coordinates (r, θ, λ). Then by (22)

$$F \cdot \delta R = \quad F \cdot e_r \delta r + F \cdot e_\theta r \delta\theta + F \cdot e_\lambda r \cos\theta \delta\lambda = F \cdot e_r \delta r \tag{23}$$

where the last equality follows since F is along e_r according to our assumption. Now using (15), (23) and second part of (13) results in

$$F \cdot e_r \delta r = F \cdot \delta R = -\delta V = -[\frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial \theta}\delta\theta + \frac{\partial V}{\partial \lambda}\delta\lambda] \tag{24}$$

and since $\delta r, \delta\theta, \delta\lambda$ are independent, then this gives

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial \lambda} = 0 \tag{25}$$

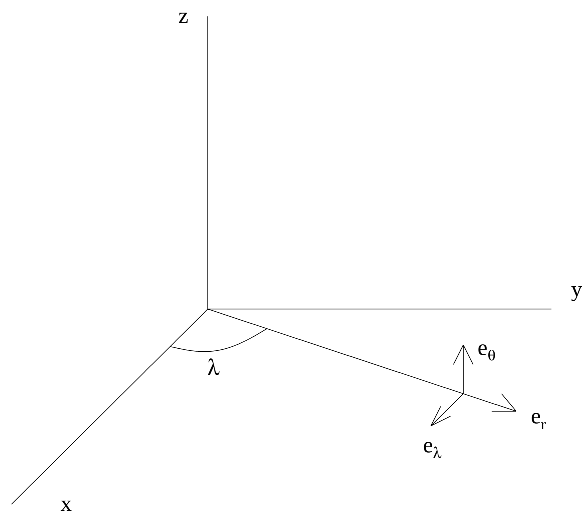

Figure 25: Unit vectors e_r, e_θ, and e_λ

i.e. V is only a functions of r (actually we know $V = \dfrac{-\mu}{r}$ where μ is a constant),

$$V = V(r) \tag{26}$$

Now for our particle in the xy plane in spherical coordinates, the velocity, of our particle at "t_0" has components along e_r, e_θ, e_λ respectively of

$$\dot{r},\ 0,\ r\dot{\lambda} \tag{27}$$

the second value being due to the velocity vector being in P and e_θ being perpendicular to P. Then the kinetic energy T is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\lambda}^2) \tag{28}$$

So that

$$L = T - V = \ \frac{1}{2}m(\dot{r}^2 + r^2\dot{\lambda}^2) - V(r) \tag{29}$$

and the Euler equations (21) given in spherical coordinates:
For r

$$L_r - \frac{d}{dt}L_{\dot{r}} = 0 \Rightarrow -\frac{dV}{dr} + mr\dot{\lambda}^2 \ - \frac{d}{dt}[m\dot{r}] = 0 \tag{30a}$$

or

$$m\ddot{r} = -\frac{dV}{dr} + mr\dot{\lambda}^2$$

While for θ we see that since θ does not enter in the problem at any time then the motion stays in same plane.
For λ

$$L_\lambda - \frac{d}{dt}L_{\dot{\lambda}} = 0 \Rightarrow \frac{d}{dt}(mr^2\dot{\lambda}) = 0 \tag{30b}$$

Equation (30a) says that (since $\frac{-dV}{dr}$ is the force in the r direction and that is the total force here then) the acceleration in that direction is the sum of that due to the force and the centrifugal acceleration. Equation (30b) gives a first integral of the motion saying that

$$mr^2\dot{\lambda} = \text{ constant} \tag{31}$$

which says that the angular momentum is constant. This is actually a first integral of the motion resulting in a first order differential equation instead of a second order differential equation as in (30a).

Problems

1. If ℓ is not preassigned, show that the stationary functions corresponding to the problem

$$\delta \int_0^1 y'^2\, dx = 0$$

subject to

$$y(0) = 2, \qquad y(\ell) = \sin \ell$$

are of the form $y = 2 + 2x \cos \ell$, where ℓ satisfies the transcendental equation

$$2 + 2\ell \cos \ell - \sin \ell = 0.$$

Also verify that the smallest positive value of ℓ is between $\frac{\pi}{2}$ and $\frac{3\pi}{4}$.

2. If ℓ is not preassigned, show that the stationary functions corresponding to the problem

$$\delta \int_0^1 \left[y'^2 + 4(y - \ell)\right] dx = 0$$

subject to

$$y(0) = 2, \qquad y(\ell) = \ell^2$$

are of the form $y = x^2 - 2\frac{x}{\ell} + 2$, where ℓ is one of the two real roots of the quartic equation $2\ell^4 - \ell^3 - 1 = 0$.

3. A particle of mass m is falling vertically, under the action of gravity. If y is distance measured downward and no resistive forces are present.

a. Show that the Lagrangian function is

$$L = T - V = m\left(\frac{1}{2}\dot{y}^2 + gy\right) + \text{ constant}$$

and verify that the Euler equation of the problem

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

is the proper equation of motion of the particle.

b. Use the momentum $p = m\dot{y}$ to write the Hamiltonian of the system.

c. Show that

$$\frac{\partial}{\partial p} H = \phi = \dot{y}$$

$$\frac{\partial}{\partial y} H = -\dot{p}$$

4. A particle of mass m is moving vertically, under the action of gravity and a resistive force numerically equal to k times the displacement y from an equilibrium position. Show that the equation of Hamilton's principle is of the form

$$\delta \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{y}^2 + mgy - \frac{1}{2} ky^2 \right) dt = 0,$$

and obtain the Euler equation.

5. A particle of mass m is moving vertically, under the action of gravity and a resistive force numerically equal to c times its velocity $\dot{y}$. Show that the equation of Hamilton's principle is of the form

$$\delta \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{y}^2 + mgy \right) dt - \int_{t_1}^{t_2} c\dot{y}\delta y \, dt = 0.$$

6. Three masses are connected in series to a fixed support, by linear springs. Assuming that only the spring forces are present, show that the Lagrangian function of the system is

$$L = \frac{1}{2} \left[m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2 - k_1x_1^2 - k_2(x_2 - x_1)^2 - k_3(x_3 - x_2)^2 \right] + \text{constant},$$

where the x_i represent displacements from equilibrium and k_i are the spring constants.

CHAPTER 10

10 Degrees of Freedom - Generalized Coordinates

If we have a system of particles whose configuration we're trying to describe, then usually, owing to the presence of constraints on the system, it is not required to give the actual coordinates of every particle. Suppose, for instance, that a rigid rod is moving in a plane, then it's sufficient to specify the (x, y) coordinates of mass center and the angle that the rod makes with the x-axis. From these, the position of all points of the rod may be found.

In order to describe the configuration of a system, we choose the smallest possible number of variables. For example, the configuration of a flywheel is specified by a single variable, namely the angle through which the wheel has rotated from its initial position. The independent variables needed to completely specify the configuration of a system are called generalized coordinates. The generalized coordinates are such that they can be varied arbitrarily and independently without violating the constraints. The number of generalized coordinates is called the number of degrees of freedom of a system. In the case of the flywheel the number of degrees of freedom is one while the rigid bar in the plane has three generalized coordinates. A deformable body does not possess a finite number of generalized coordinates.

Consider a system of N particles with masses $m_1, \cdots, m_N$ and position vectors $R_1, \cdots, R_N$ and with F_i the resultant force on the i^{th} particle. Then for each particle we have

$$m_i \ddot{R}_i - F_i = 0 \qquad i = 1, \cdots, N \tag{1}$$

For this system Hamilton's principle gives

$$\int_{t_1}^{t_2} [\delta T + \sum_{i=1}^{N} F_i \cdot \delta R_i] dt = 0 \tag{2}$$

where T is the kinetic energy of the system of N particles, and

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{R}_i^2 \, . \tag{3a}$$

As before, if there is a potential energy $V(R_1, \cdots R_N)$ then (2) becomes (see (15b) of Chapter 9)

$$\delta \int_{t_1}^{t_2} (T - V) dt = 0 \tag{3b}$$

Now each position vector consists of a triple of numbers so that the system configuration is determined by $3N$ numbers. Generally, the system is subject to constraints which implies that not all of the $3N$ coordinates are independent. Suppose that there are K constraints of the type

$$\phi_i(R_1, \cdots, R_N, t) = 0 \qquad i = 1, \cdots k \tag{4}$$

which must be satisfied by the coordinates. Then there are only $3N - k = p$ independent coordinates so that we can select a set of p independent "generalized" coordinates $q_1, \cdots q_p$

which define the configuration of the system. Therefore, the position vectors R_i can be written

$$R_i = R_i(q_1, \cdots, q_p, t) \qquad i = 1, \cdots, N \tag{5}$$

and similarly for the velocities

$$\dot{R}_i = \dot{R}_i(q_1, \cdots q_p, \dot{q}_1, \cdots \dot{q}_p, t) \tag{6}$$

so that the kinetic energy is a function of $q_1 \cdots q_p, \dot{q}_1, \cdots \dot{q}_p, t$

$$T = T(q_1 \cdots q_p, \dot{q}_1 \cdots \dot{q}_p, t) \tag{7}$$

and also if there is a potential energy $V = V(R_1 \cdots R_N)$ then

$$V = V(q_1, \cdots q_p, t) \tag{8}$$

so that

$$L = T - V = L(q_1 \cdots q_p, \dot{q}_1, \cdots \dot{q}_p, t) \tag{9}$$

Then when using (3b), the independent variations are the δq_i and not the δR_i and the resultant Euler equations are

$$\frac{d}{dt}\Big(\frac{\partial}{\partial \dot{q}_i}(T - V)\Big) - \frac{\partial}{\partial q_i}(T - V) = 0 \tag{10}$$

Before we do examples let's review some material on the potential energy V of a conservative system. We know from a previous chapter that with F as the force then $F \cdot \delta R$ is the infinitesimal amount of work done by moving through the displacement δR and also that

$$F \cdot \delta R = -\delta V \tag{11}$$

i.e. this infinitesimal work is equal to the negative of the infinitesimal change in the potential energy. For non-infinitesimal changes, then we integrate (thinking of δR as dR and similarly for δV) and

$$\int_{R_1}^{R} F \cdot \delta R = -[V(R) - V(R_1)] \tag{12}$$

and get the change in the potential energy between $R = R_1$ and R. For example if a particle moves along the y-axis (y positive down) in a constant gravity field from $y = y_1$ ($R = y$ here as the variable defining the system configuration) to y then the change in potential energy is

$$\int_{R_1}^{R} F \cdot \delta R = \int_{y_1}^{y} mg\delta y = \quad mgy - mgy_1 \quad = -[V(y) - V(y_1)] = -V(y) + c \tag{13}$$

(thinking of $V(y_1)$ as a fixed reference value) giving the potential energy

$$V(y) = -mgy \tag{14}$$

(often the constant is included in $V(y)$).

Of course, if the components of R are not all independent, and instead the q variables are the independent ones we could express everything in terms of those variables and have

$$\int_{\overline{q}_1}^{\overline{q}} F(\overline{q}) \cdot \delta\overline{q} = -[V(\overline{q}) - V(\overline{q}_1)] = -V(\overline{q}) + c \tag{15}$$

(where $\overline{q}$ is the vector with components as the independent variables q).

Example:

A simple pendulum consisting of a point mass is suspended by an inextensible string of length ℓ. The configuration of this system is completely specified by the single angle θ between the deflected position and some reference position, say equilibrium position where it's hanging vertically (Figure 26).

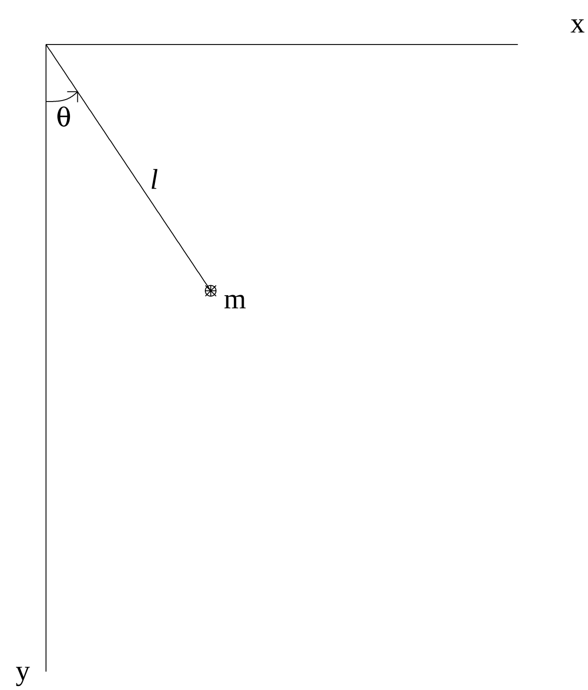

Figure 26: A simple pendulum

Using (14) we see that the potential energy $V = -mgy$. Here y is determined by θ as

$$y = \ell \cos\theta \tag{16}$$

so that

$$V = -mg\ell \cos\theta \tag{17}$$

Since the velocity is $\ell\dot{\theta}$, then the kinetic energy T is

$$T = \frac{1}{2}m(\ell\dot{\theta})^2 \tag{18}$$

so that with $q = \theta$, then (10) becomes here

$$\frac{d}{dt}\Big[\frac{\partial}{\partial\dot\theta}(\frac{1}{2}m(\ell\dot\theta)^2 + mg\ell\cos\theta)\Big] - \frac{\partial}{\partial\theta}(\frac{1}{2}m(\ell\dot\theta)^2 + mg\ell\cos\theta) = 0 \tag{19}$$

or then

$$m\ell\ddot\theta + mg\ell\sin\theta = 0 \tag{20}$$

the equation of motion for the pendulum.

Example:
Consider a compound pendulum as pictured in Figure 27.

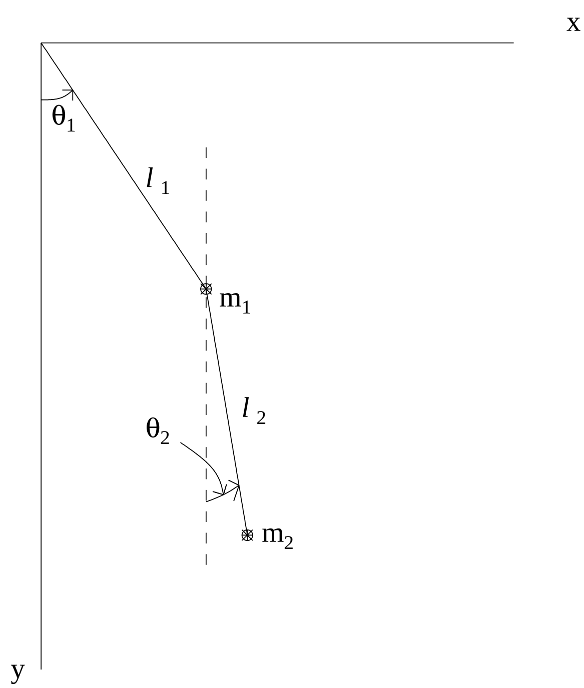

Figure 27: A compound pendulum

In this problem we can't go directly to circular coordinates since there are a different set of these for motions about the two pivot points and the motion of m_2 is the sum of these motions so that we must add these two vectorially. We use rectangular coordinates with (x_1, y_1) and (x_2, y_2) as the coordinates of the two masses m_1 and m_2 respectively. Then in terms of the independent (generalized) coordinates θ_1, θ_2 we have (choosing y negative down)

$$\begin{aligned} x_1 &= \ell_1\sin\theta_1 \qquad & y_1 = -\ell_1\cos\theta_1 \\ x_2 &= \ell_2\sin\theta_2 + \ell_1\sin\theta_1 \qquad & y_2 = -\ell_2\cos\theta_2 - \ell_1\cos\theta_1 \end{aligned} \tag{21}$$

The potential and kinetic energies analogously as done before are

$$T_1 = \frac{1}{2}m_1(\dot x_1^2 + \dot y_1^2) \qquad T_2 = \frac{1}{2}m_2(\dot x_2^2 + \dot y_2^2) \tag{22a}$$

$$V_1 = m_1 g y_1 \qquad V_2 = m_2 g y_2 \tag{22b}$$

Plugging (21) into (22) and writing the Lagrangian gives

$$\begin{aligned} L = T - V = \frac{1}{2}(m_1 + m_2)\ell_1^2\dot\theta_1^2 + m_2\ell_1\ell_2\dot\theta_1\dot\theta_2\cos(\theta_1 - \theta_2) + \frac{1}{2}m_2\ell_2^2\dot\theta_2^2 + \\ g(m_1 + m_2)\ell_1\cos\theta_1 + gm_2\ell_2\cos\theta_2 \end{aligned} \tag{23}$$

Then

$$\begin{aligned} 0 = \frac{d}{dt}[L_{\dot\theta_1}] - L_{\theta_1} = \frac{d}{dt}[(m_1 + m_2)\ell_1^2\dot\theta_1 + m_2\ell_1\ell_2\dot\theta_2\cos(\theta_1 - \theta_2)] + m_2\ell_1\ell_2\dot\theta_1\dot\theta_2\sin(\theta_1 - \theta_2) \\ +g(m_1 + m_2)\ell_1\sin\theta_1 \end{aligned} \tag{24}$$

and

$$\begin{aligned} 0 &= \frac{d}{dt}[L_{\dot\theta_2}] - L_{\theta_2} = \frac{d}{dt}[m_2\ell_1\ell_2\dot\theta_1\cos(\theta_1 - \theta_2) + m_2\ell_2^2\dot\theta_2] + gm_2\ell_2\sin\theta_2 \\ &- m_2\ell_1\ell_2\dot\theta_1\dot\theta_2\sin(\theta_1 - \theta_2) \end{aligned} \tag{25}$$

Example:

Consider the harmonic oscillator whose Lagrangian is given by

$$L(t, y, \dot y) \;=\; \frac{1}{2}m\dot y^2 - \frac{1}{2}ky^2.$$

The canonical momentum is

$$p \;=\; L_{\dot y} \;=\; m\dot y.$$

Solving for $\dot y$ gives

$$\dot y \;=\; \frac{p}{m},$$

i.e. $\phi(t, y, p) = \dfrac{p}{m}$. Therefore the Hamiltonian is

$$H \;=\; -L + \dot y L_{\dot y} \;=\; \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}ky^2$$

which is the sum of the kinetic and potential energy of the system. Differentiating

$$\frac{\partial H}{\partial y} \;=\; ky$$

$$\frac{\partial H}{\partial p} \;=\; \frac{p}{m}$$

so Hamilton's equations are

$$\dot y \;=\; \frac{p}{m}, \qquad \dot p \;=\; -ky.$$

To solve these equations in the yp plane (the so called phase plane) we divide them to get

$$\frac{\partial p}{\partial y} = \frac{-ky}{p/m}$$

or

$$pdy + kmydy = 0$$

After integration, we have

$$p^2 + kmy^2 = c, \qquad c \text{ is constant}$$

which is a family of ellipses in the py plane. These represent trajectories that the system evolves along in the position-momentum space. Fixing an initial value at time t_0 selects out the particular trajectory that the system takes.

Problems

1. Consider the functional

$$I(y) = \int_a^b \left[r(t)\dot{y}^2 + q(t)y^2\right] dt.$$

Find the Hamiltonian and write the canonical equations for the problem.

2. Give Hamilton's equations for

$$I(y) = \int_a^b \sqrt{(t^2 + y^2)(1 + \dot{y}^2)}dt.$$

Solve these equations and plot the solution curves in the yp plane.

3. A particle of unit mass moves along the y axis under the influence of a potential

$$f(y) = -\omega^2 y + ay^2$$

where ω and a are positive constants.

a. What is the potential energy $V(y)$? Determine the Lagrangian and write down the equations of motion.

b. Find the Hamiltonian $H(y, p)$ and show it coincides with the total energy. Write down Hamilton's equations. Is energy conserved? Is momentum conserved?

c. If the total energy E is $\frac{\omega^2}{10}$, and $y(0) = 0$, what is the initial velocity?

d. Sketch the possible phase trajectories in phase space when the total energy in the system is given by $E = \frac{\omega^6}{12a^2}$.

Hint: Note that $p = \pm\sqrt{2}\sqrt{E - V(y)}$.

What is the value of E above which oscillatory solution is not possible?

4. A particle of mass m moves in one dimension under the influence of the force $F(y,t) = ky^{-2}e^t$, where $y(t)$ is the position at time t, and k is a constant. Formulate Hamilton's principle for this system, and derive the equations of motion. Determine the Hamiltonian and compare it with the total energy.

5. A Lagrangian has the form

$$L(x, y, y') = \frac{a^2}{12}(y')^4 + a(y')^2 G(y) - G(y)^2,$$

where G is a given differentaible function. Find Euler's equation and a first integral.

6. If the Lagrangian L does not depend explicitly on time t, prove that $H =$ constant, and if L doesn't depend explicitly on a generalized coordinate y, prove that $p =$ constant.

7. Consider the differential equations

$$r^2\dot{\theta} = C, \qquad \ddot{r} - r\dot{\theta}^2 + \frac{k}{m}r^{-2} = 0$$

governing the motion of a mass in an inversely square central force field.

a. Show by the chain rule that

$$\dot{r} = Cr^{-2}\frac{dr}{d\theta}, \qquad \ddot{r} = C^2 r^{-4}\frac{d^2r}{d\theta^2} - 2C^2 r^{-5}\left(\frac{dr}{d\theta}\right)^2$$

and therefore the differential equations may be written

$$\frac{d^2r}{d\theta^2} - 2r^{-1}\left(\frac{dr}{d\theta}\right)^2 - r + \frac{k}{C^2 m}r^2 = 0$$

b. Let $r = u^{-1}$ and show that

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{C^2 m}.$$

c. Solve the differential equation in part b to obtain

$$u = r^{-1} = \frac{k}{C^2 m}\left(1 + \epsilon\cos(\theta - \theta_0)\right)$$

where ϵ and θ_0 are constants of integration.

d. Show that elliptical orbits are obtained when $\epsilon < 1$.

CHAPTER 11

11 Integrals Involving Higher Derivatives

Consider the problem

$$\min I = \int_{x_1}^{x_2} F(x, y, y', y'')dx \tag{1}$$

among arcs

$$y : \quad y(x) \qquad\qquad x_1 \le x \le x_2$$

with continuous derivatives up to and including the second derivative, that satisfy the boundary conditions

$$y(x_1) = A_0, \quad y'(x_1) = A_1, \quad y(x_2) = B_0, \quad y'(x_2) = B_1. \tag{2}$$

(Notice now that we also have conditions on y' at the end-points.)

Let $y_0(x)$ be a solution to this problem. Let $\eta(x)$ be an arc on the interval $[x_1, x_2]$ with continuous derivatives up through the second order and satisfying

$$\eta(x_1) = 0, \;\; \eta(x_2) = 0, \;\; \eta'(x_1) = 0, \;\; \eta'(x_2) = 0 \tag{3}$$

Create the family of arcs

$$\underline{y}(\epsilon) : \qquad y_0(x) + \epsilon\eta(x) \qquad\qquad x_1 \le x \le x_2 \qquad\qquad -\delta < \epsilon < \delta \tag{4}$$

for some $\delta > 0$.

Evaluate I on this family to get

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, y_0 + \epsilon\eta, y_0' + \epsilon\eta', y_0'' + \epsilon\eta'']dx \tag{5}$$

Differentiating at $\epsilon = 0$ and setting the derivative equal to zero gives

$$0 = I'(0) = \int_{x_1}^{x_2} [F_y\eta + F_{y'}\eta' + F_{y''}\eta'']dx \tag{6}$$

As done previously we can write the second term in the integrand of (6) as

$$F_{y'}\eta' = \frac{d}{dx}[F_{y'}\eta] - \eta\frac{d}{dx}[F_{y'}] \tag{7}$$

and the third term in the integrand can similarly be written

$$F_{y''}\eta'' = \frac{d}{dx}[F_{y''}\eta'] - \eta'\frac{d}{dx}[F_{y''}] \tag{8a}$$

and then also

$$\eta'\frac{d}{dx}[F_{y''}] = \frac{d}{dx}[\eta\frac{d}{dx}F_{y''}] - \eta\frac{d^2}{dx^2}F_{y''} \tag{8b}$$

Then using (7) and (8) in (6) gives

$$0 = I'(0) = \int_{x_1}^{x_2} [F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''}]\eta dx + \int_{x_1}^{x_2} \frac{d}{dx}[F_{y'}\eta + F_{y''}\eta' + \eta\frac{d}{dx}F_{y''}]dx \tag{9}$$

Evaluating the last integral of (9) gives

$$0 = I'(0) = \int_{x_1}^{x_2} [F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''}]\eta dx + [\eta(F_{y'} + \frac{d}{dx}F_{y''}) + \eta' F_{y''}]\Big|_{x_1}^{x_2} \tag{10}$$

By (3), the last integral is zero, leaving

$$0 = I'(0) = \int_{x_1}^{x_2} [F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''}]\eta dx \tag{11}$$

which must be true for the full class of η arcs described above. Then by a slight extension† of the modified form of the fundamental lemma we get that the integrand of (11) must be zero giving

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0 \tag{12}$$

as the Euler equation for this problem.

Note that the integrated form of the Euler equation (12) is

$$\int_{x_1}^{x} [\int_{x_1}^{s} F_y ds - F_{y'}]dx + c_1 x + c_2 = F_{y''} \tag{13}$$

where c_1, c_2 are constants.

As a generalization of this, it can be shown by a directly analagous argument that if the integrand involves the first N derivatives of y so that the corresponding problem would be

$$\min I = \int_{x_1}^{x_2} F(x, y, y', y'', \cdots, y^{(N)})dx \tag{14}$$

among arcs with continuous N^{th} derivatives on $[x_1, x_2]$ and satisfying

$$\begin{aligned} y(x_1) = A_0,\ y'(x_1) = A_1, \cdots, y^{(N-1)}(x_1) = A_{N-1} \\ y(x_2) = B_0,\ y'(x_2) = B_1, \cdots, y^{(N-1)}(x_2) = B_{N-1} \end{aligned} \tag{15}$$

then the Euler equation is

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \cdots + (-1)^N \frac{d^N}{dx^N}F_{y^{(N)}} = 0 \tag{16}$$

a differential equation of order $2N$. This result is summarized in

Theorem 1 Consider the problem defined by (14), (15) and the accompanying remarks. Then a solution arc must satisfy (16).

†To take into consideration that η'' exists and is continuous

As an application consider the problem

$$\min I = \int_0^{\pi/4} [16y^2 - (y'')^2 + \phi(x)]dx \tag{17}$$

(where ϕ is an arbitrary continuous function of x), among arcs possessing continuous derivatives through second order and satisfying

$$y(0) = 0, \quad y(\pi/4) = 1, \quad y'(0) = 1 \quad y'(\pi/4) = 0 \tag{18}$$

Applying the Euler equation gives

$$0 = F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 32y - 2y^{(4)} = 0 \tag{19}$$

or

$$y^{(4)} - 16y = 0 \tag{20}$$

The roots of the characteristic equation

$$D^4 - 16 = 0 \tag{21}$$

are

$$D = \pm 2, \pm 2i \tag{22}$$

so that the solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x \tag{23}$$

and then

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x} - 2c_3 \sin 2x + 2c_4 \cos 2x \tag{24}$$

Applying the conditions (18) gives

$$0 = \quad y(0) = c_1 + c_2 + c_3 \tag{25a}$$

$$1 = \quad y(\pi/4) = c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4 \tag{25b}$$

$$1 = \quad y'(0) = 2c_1 - 2c_2 + 2c_4 \tag{25c}$$

$$0 = \quad y'(\pi/4) = 2c_1 e^{\pi/2} - 2c_2 e^{-\pi/2} - 2c_3 \tag{25d}$$

Solving this system will yield solution of the problem.

We discuss now the Newton method to solve this problem. Analagous to our procedure for Newton method applications to problems involving terms in x, y, y' we start with an initial estimate y_1 satisfying the left hand conditions of (2) and the Euler equation

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0 \tag{26}$$

which after differentiating out (with respect to x) gives a fourth order equation in y.

Noting that

$$\frac{d}{dx}F_{y'} = F_{y'x} + F_{y'y}y' + F_{y'y'}y'' + F_{y'y''}y''' \tag{27a}$$

and

$$\frac{d}{dx}F_{y''} = F_{y''x} + F_{y''y}y' + F_{y''y'}y'' + F_{y''y''}y''' \tag{27b}$$

then the Euler equation for this problem is

$$\begin{aligned}
&F_y - F_{y'x} - F_{y'y}y' - F_{y'y'}y'' - F_{y'y''}y''' \\
&+F_{y''xx} + F_{y''xy}y' + F_{y''xy'}y'' + F_{y''xy''}y''' \\
&+\Big(F_{y''yx} + F_{y''yy}y' + F_{y''yy'}y'' + F_{y''yy''}y'''\Big)y' + F_{y''y}y'' + \\
&+\Big(F_{y''y'x} + F_{y''y'y}y' + F_{y''y'y'}y'' + F_{y''y'y''}y'''\Big)y'' + F_{y''y'}y''' + \\
&+\Big(F_{y''y''x} + F_{y''y''y}y' + F_{y''y''y'}y'' + F_{y''y''y''}y'''\Big)y''' + F_{y''y''}y^{(4)} = 0
\end{aligned} \tag{28}$$

where all but the first five terms come from differentiating (27b) with respect to x and where these have been grouped so that the second line of (28) comes from differentiating the first term on the right in (27b) and each succeeding line comes from differentiating another term on the right of (27b).

Calling this equation E_4 (fourth order Euler equation) we define a two parameter family of curves $\underline{y}(c_1, c_2)$ which satisfies E_4 and the left hand conditions of (2) and also has initial values of $\underline{y}''(x_0)$ and $y'''(x_0)$ as

$$y''(x_0) = y_1''(x_0) + c_1 \qquad\qquad y'''(x_0) = y_1'''(x_0) + c_2 \tag{29}$$

Notice that we have two conditions to satisfy (namely the two right hand conditions of (2)) and we have two parameters to do it with.

As before, we set

$$\eta_i(x) \equiv \frac{\partial y(x)}{\partial c_i} \qquad\qquad x_0 \le x \le x_1 \tag{30}$$

which now this means

$$\eta_1(x) = \frac{\partial y(x)}{\partial c_1} = \frac{\partial y(x)}{\partial y''(x_0)} \quad \text{and} \quad \eta_2(x) = \frac{\partial y(x)}{\partial c_2} = \frac{\partial y(x)}{\partial y'''(x_0)} \tag{31}$$

We obtain the differential equation that $\eta_i(x)$ has to satisfy by differentiating (28) (evaluated on the curve $\underline{y}(c)$) with respect to c_i. By examining (28) we see that the differential equation for η_i will be fourth order (since we only have terms up through fourth order in y in (28) and differentiating term by term and using

$$\eta_i = \frac{\partial y}{\partial c_i} \qquad \eta_i' = \frac{\partial y'}{\partial c_i} \qquad \eta_i'' = \frac{\partial y''}{\partial c_i} \qquad \eta_i''' = \frac{\partial y'''}{\partial c_i}, \qquad \eta_i^{(4)} = \frac{\partial y^{(4)}}{\partial c_i} \tag{32}$$

we get a fourth order equation for $\eta_i, \quad i = 1, 2)$.

The remainder of the program consists in determining initial conditions for $\eta_i,\ \eta_i',\ \eta_i'', \quad i = 1, 2$ and setting up an iteration scheme to achieve the right–hand end point conditions of (2).

Problems

1. Derive the Euler equation of the problem

$$\delta \int_{x_1}^{x_2} F(x, y, y', y'')\, dx \ = \ 0$$

in the form

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y} \ = \ 0,$$

and show that the associated natural boundary conditions are

$$\left[\left(\frac{d}{dx}\frac{\partial F}{\partial y''} - \frac{\partial F}{\partial y'}\right)\delta y\right]\Bigg|_{x_1}^{x_2} \ = \ 0$$

and

$$\left[\frac{\partial F}{\partial y''}\,\delta y'\right]\Bigg|_{x_1}^{x_2} \ = \ 0.$$

2. Derive the Euler equation of the problem

$$\delta \int_{x_1}^{x_2}\int_{y_1}^{y_2} F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\, dxdy \ = \ 0,$$

where $x_1, x_2, y_1,$ and y_2 are constants, in the form

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial u_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u_{yy}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) + \frac{\partial F}{\partial u} \ = \ 0,$$

and show that the associated natural boundary conditions are then

$$\left[\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial u_{xx}} + \frac{\partial}{\partial y}\frac{\partial F}{\partial u_{xy}} - \frac{\partial F}{\partial u_x}\right)\delta u\right]\Bigg|_{x_1}^{x_2} \ = \ 0$$

$$\left[\frac{\partial F}{\partial u_{xx}}\delta u_x\right]\Bigg|_{x_1}^{x_2} \ = \ 0,$$

and

$$\left[\left(\frac{\partial}{\partial y}\frac{\partial F}{\partial u_{yy}} + \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{xy}} - \frac{\partial F}{\partial u_y}\right)\delta u\right]\Bigg|_{y_1}^{y_2} \ = \ 0$$

$$\left[\frac{\partial F}{\partial u_{yy}}\delta u_y\right]\Bigg|_{y_1}^{y_2} \ = \ 0.$$

3. Specialize the results of problem 2 in the case of the problem

$$\delta \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[\frac{1}{2} u_{xx}^2 + \frac{1}{2} u_{yy}^2 + \alpha u_{xx} u_{yy} + (1 - \alpha) u_{xy}^2 \right] dx dy = 0,$$

where α is a constant.

Hint: Show that the Euler equation is $\nabla^4 u = 0$, regardless of the value of α, but the natural boundary conditions depend on α.

4. Specialize the results of problem 1 in the case

$$F = a(x)(y'')^2 - b(x)(y')^2 + c(x)y^2.$$

5. Find the extremals

a. $I(y) = \int_0^1 (yy' + (y'')^2) dx, \qquad y(0) = 0,\ y'(0) = 1,\ y(1) = 2,\ y'(1) = 4$

b. $I(y) = \int_0^\infty (y^2 + (y')^2 + (y'' + y')^2) dx, \qquad y(0) = 1,\ y'(0) = 2,\ y(\infty) = 0,\ y'(\infty) = 0.$

6. Find the extremals for the functional

$$I(y) = \int_a^b (y^2 + 2\dot{y}^2 + \ddot{y}^2) dt.$$

7. Solve the following variational problem by finding extremals satisfying the given conditions

$$I(y) = \int_0^1 (1 + (y'')^2) dx, \qquad y(0) = 0,\ y'(0) = 1,\ y(1) = 1,\ y'(1) = 1.$$

CHAPTER 12

12 Piecewise Smooth Arcs and Additional Results

Thus far we've only considered problems defined over a class of smooth arcs and hence have only permitted smooth solutions, i.e. solutions with a continuous derivative $y'(x)$. However one can find examples of variational problems which have no solution in the class of smooth arcs, but which do have solutions if we extend the class of admissible arcs to include piecewise smooth arcs[‡]. For example, consider the problem

$$\text{minimize} \int_{-1}^{1} y^2(1-y')^2 dx \qquad y(-1)=0 \quad y(1)=1\,. \tag{1}$$

The greatest lower bound of this integral is clearly (non negative integrand) zero, but it does not achieve this value for any smooth arc. The minimum is achieved for the arc

$$y(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 < x \le 1 \end{cases} \tag{2}$$

which is piecewise smooth and thus has a discontinuity in y' (i.e. a "corner") at the point $x = 0$.

In order to include such problems into our theory and to discuss results to follow we consider again the term admissible arcs

$$\underline{y}: \qquad y(x) \qquad x_1 \le x \le x_2 \tag{3}$$

We will need to refer to the general integral (defined many times before) which we seek to minimize

$$I = \int_{x_1}^{x_2} F(x,y,y')dx \tag{4}$$

The definition of the particular class of admissible arcs may be made in many ways, each of which gives rise to a distinct problem of the calculus of variations. For a special problem the properties defining the class will in general be in part necessitated by the geometrical or mechanical character of the problem itself, and in part free to be chosen with a large degree of arbitrariness. An example of a property of the former type is the restriction for the brachistochrone problem that the curves considered shall all lie below the line $y = \alpha$, since on arcs above that line the integral expressing the time of descent has no meaning. On the other hand we frequently find it convenient to make the arbitrary restriction that our curves shall all lie in a small neighborhood of a particular one whose minimizing properties we are investigating, always remembering that on each of the arcs of our class the integral I must have a well-defined value.

[‡]An arc $\underline{y}: y(x) \quad x_1 \le x \le x_2$ is piecewise smooth if there are at most a finite number of points $x = \overline{x}_i \;\; i = 1, \cdots, L$ in the interval $[x_1, x_2]$ where $y'(x)$ is discontinuous. The points $\overline{x}_i$ at which $y'(x)$ is discontinuous are called corners.

In order to make a definition of a class of admissible arcs, which will be generally applicable, let us first assume that there is a region R of sets of values (x, y, y') in which the integrand $F(x, y, y')$ is continuous and has continuous derivatives of as many orders as may be needed in our theory. The sets of values (x, y, y') interior to the region R may be designated as *admissible sets*. An arc (3) will now be called an *admissible arc* if it is continuous and has a continuously turning tangent except possibly at a finite number of corners, and if the sets of values $(x, y(x),\ y'(x))$ on it are all admissible according to the definition just given. For an admissible arc the interval $[x_1, x_2]$ can always be subdivided into a number of subintervals on each of which $y(x)$ is continuous and has a continuous derivative. At a value x where the curve has a corner the derivative $y'(x)$ has two values which we may denote by $y'(x-0)$ and $y'(x+0)$, corresponding to the backward and forward slopes of the curve, respectively.

With the above considerations in mind, then the problem with which we are concerned is to minimize the integral (4) on the class of admissible arcs (3) joining two fixed points.

The Euler equations which we've seen in differentiated and integrated forms, e.g. the first Euler equation

$$\frac{d}{dx}F_{y'} - F_y = 0 \qquad F_{y'} - \int^x F_y ds = c \tag{5}$$

(where c is a constant) were proven by explicitly considering only smooth arcs (i.e. arcs without corners).

Recall our proof of the integrated form of the first Euler equation (the second equation of (5)) which we originally did for the shortest distance problem. There we used the fundamental lemma involving the integral

$$\int M(x)\eta'(x)dx \tag{6}$$

where in that lemma $M(x)$ was allowed to be piecewise continuous [§] and $\eta'(x)$ was required to have at least the continuity properties of $M(x)$. The term $M(x)$ turned out to be

$$M(x) = F_{y'}(x) - \int^x F_y ds\,. \tag{7}$$

When we allowed only smooth arcs, then $F_{y'}(x)$ (i.e. $F_{y'}(x, y(x), y'(x))$) and $F_y(x)$ (i.e. $F_y(x, y(x), y'(x))$) were continuous (since $y(x)$ and $y'(x)$ were so) and the piecewise continuity provision of the fundamental lemma was not used. This is the procedure that was followed in chapter 3 and proved that when considering only smooth arcs, the first Euler equation held in integrated and in differentiated form on a minimizing arc. However, now in allowing piecewise smooth arcs, then $F_{y'}(x)$, and $F_y(x)$ may be discontinuous at the corners of the arc and then by (7) this will also be true for $M(x)$.

Since the fundamental lemma allowed for this, then the proof of the integrated form is still valid when permitting piecewise smooth arcs. The differentiated form also still holds in between the corners of the arc but may not hold at the corners themselves. A similar

[§]A function $M(x)$ is said to be piecewise continuous on an interval $[x_1, x_2]$ if there at most a finite number of points $\overline{x}_i\ \ i = 1, \cdots, L$ in $[x_1, x_2]$ where $M(x)$ is discontinuous statement is true for the other Euler equation. With this in mind the theorem concerning the Euler equations for the general problem stated above, is: *For the problem*

$$\text{minimize } I = \int_{x_1}^{x_2} F(x, y, y')dx \tag{8}$$

on the class of admissible (where the term admissible is consistent with our above discussions) arcs (3) joining two fixed points, let the arc

$$\underline{y}_0 : \qquad y_0(x) \qquad x_1 \leq x \leq x_2 \tag{9}$$

be a solution. Then the Euler equations hold in integrated form

$$F_{y'}(x) - \int^x F_y ds = c_1 \qquad F(x) - y'(x)F_{y'}(x) - \int^x F_y ds = c_2 \tag{10a}$$

(where c_1 and c_2 are constants) along $\underline{y}_0$ while the differentiated forms

$$\frac{d}{dx}F_{y'} - F_y = 0 \qquad \frac{d}{dx}(F - y'F_{y'}) = F_x \tag{10b}$$

hold between the corners of $\underline{y}_0$

Remark: These same modification to the Euler equations hold for the other types of problems considered. All other results such as the transversality condition remain unchanged when allowing piecewise smooth arcs.

Because we allow piecewise smooth arcs, then there are two additional necessary conditions to be established and one of these will imply a third additional necessary condition. Finally we will present one other necessary condition that has nothing to do with corners.

For our discussions to follow, assume that the arc $\underline{y}_0$ of (9) is a solution to our problem.

The necessary conditions of Weierstrass and Legendre. In order to prove Weierstrass' necessary condition, let us select arbitrarily a point 3 on our minimizing arc $\underline{y}_0$, and a second point 4 of this arc so near to 3 that there is no corner of $\underline{y}_0$ between them. Through the point 3 we may pass an arbitrary curve C with an equation $y = Y(x)$, and the fixed point 4 can be joined to a movable point 5 on C by a one parameter family of arcs $\underline{y}_{54}$ containing the arc $\underline{y}_{34}$ as a member when the point 5 is in the position 3.

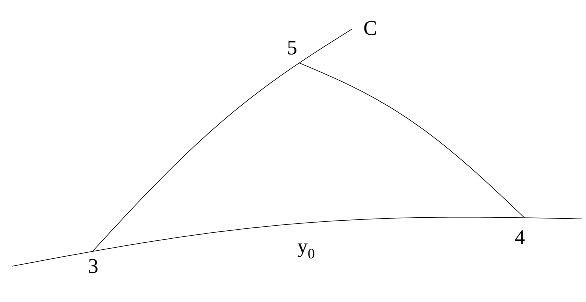

Figure 28: Two nearby points 3,4 on the minimizing arc

We shall soon see that such a family can easily be constructed. If the integral $I(\underline{y}_0)$ is to be a minimum then it is clear that as the point 5 moves along C from the point 3 the integral

$$I(C_{35} + \underline{y}_{54}) = \int_{x_3}^{x_5} F(x, Y, Y')dx + I(\underline{y}_{54}) \tag{11}$$

must not decrease from the initial value $I(\underline{y}_{34})$ which it has when 5 is at the point 3. Then at the point 3 the differential of this integral with respect to x_5 must not be negative.

The differential of the term $I(\underline{y}_{54})$ in the expression (11), at the position $\underline{y}_{34}$, is given by the expression derived in chapter 4 which we now repeat here

$$dI(\underline{y}_{54}) = F(x, y, y')dx + (dy - y'dx)F_{y'}(x, y, y')|_3^4$$

where the point 4 is fixed so that $dx_4 = dy_4 = 0$. For that formula holds along every arc of the family in question which satisfies the Euler equations and we know that our minimizing arc must satisfy these equations. Since the differential of the first integral in the expression (11) with respect to its upper limit is the value of its integrand at that limit, it follows that when 5 is at 3 we have for the differential of $I(C_{35} + \underline{y}_{54})$ the value at the point 3 of the quantity

$$F(x, Y, Y')dx - F(x, y, y')dx - (dy - y'dx)F_{y'}(x, y, y')\,.$$

The differentials in this expression belong to the arc C and satisfy the equation $dy = Y'dx$, and at the point 3 the ordinates of C and $\underline{y}$ are equal, so that the differential of (11) is also expressible in the form

$$[F(x, y, Y') - F(x, y, y') - (Y' - y')F_{y'}(x, y, y')]dx|^3 \tag{12}$$

Since this differential must be positive or zero for an arbitrarily selected point 3 and arc C through it, i.e., for every element (x, y, y') on $\underline{y}_0$ and every admissible element (x, y, Y'), we have justified the necessary condition of Weierstrass.

Theorem The Necessary Condition of Weierstrass.

At every element (x, y, y') of a minimizing arc $\underline{y}_0$, the condition

$$F(x, y, Y') - F(x, y, y') - (Y' - y')F_{y'}(x, y, y') \geq 0 \tag{13}$$

is satisfied for every admissible point (x, y, Y') different from (x, y, y').

The expression on the left side of (13) is usually called the Weierstrass E-function

$$E(x, y, y', Y') \equiv F(x, y, Y') - F(x, y, y') - (Y' - y')F_{y'}(x, y, y'). \tag{14}$$

Thus in terms of this quantity, the necessary condition of Weierstrass may be stated as

$$E(x, y, y', Y') \geq 0 \tag{15}$$

where (x, y, y') and (x, y, Y') are as noted above.

With the help of Taylor's formula, the Weierstrass E-function may be expressed in the form

$$E(x, y, y', Y') = \frac{1}{2}(Y' - y')^2 F_{y'y'}(x, y, y' + \theta(Y' - y')) \tag{16}$$

where $0 < \theta < 1$. If we let Y' approach y' we find from this formula the necessary condition of Legendre, as an immediate corollary of the condition of Weierstrass.

Theorem The Necessary Condition of Legendre

At every element (x, y, y') *of a minimizing arc* $\underline{y}_0$*, the condition*

$$F_{y'y'}(x, y, y') \geq 0 \tag{17}$$

must be satisfied.

In order now to demonstrate the consturction of a family of arcs $\underline{y}_{54}$ of the type used in the foregoing proof of Weierstrass' condition, consider the equation

$$y = y(x) + \frac{Y(a) - y(a)}{x_4 - a}(x_4 - x) = y(x, a)\,. \tag{18}$$

For $x = x_4$ these arcs all pass through the point 4, and for $x = a$ they intersect the curve C. For $a = x_3$ the family contains the arc $\underline{y}_{34}$ since at the intersection point 3 of $\underline{y}_{34}$ and C we have $Y(x_3) - y(x_3) = 0$ and the equation of the family reduces to the equation $y = y(x)$ of the arc $\underline{y}_{34}$.

For an element $(x, y, y'(x - 0))$ at a corner of a minimizing arc the proof just given for Weierstrass' necessary condition does not apply, since there is always a corner between this element and a point 4 following it on $\underline{y}_0$. But one can readily modify the proof so that it makes use of a point 4 preceding the corner and attains the result stated in the condition for the element in question.

There are two other necessary conditions that result from satisfaction of the Euler equations. One condition involves corners.

Consider the equation

$$F_{y'} = \int_{x_1}^{x} F_y dx + c\,. \tag{19}$$

The right hand side of this equation is a continuous function of x at every point of the arc $\underline{y}_0$ and the left hand side must therefore also be continuous, so that we have

Corollary 1. The Weierstrass-Erdmann Corner Condition. *At a corner* (x, y) *of a minimizing arc* $\underline{y}_0$ *the condition*

$$F_{y'}(x, y, y'(x - 0)) = F_{y'}(x, y, y'(x + 0)) \tag{20}$$

must hold.

This condition at a point (x, y) frequently requires $y'(x - 0)$ and $y'(x + 0)$ to be identical so that at such a point a minimizing arc can have no corners. It will always require this identity if the sets (x, y, y') with y' between $y'(x - 0)$ and $y'(x + 0)$ are all admissible and the derivative $F_{y'y'}$ is everywhere different from zero, since then the first derivative $F_{y'}$ varies monotonically with y' and cannot take the same value twice. The criterion of the corollary has an interesting application in a second proof of Jacobi's condition which will be given later.

We have so far made no assumption concerning the existence of a second derivative $y''(x)$ along our minimizing arc. If an arc has a continuous second derivative then Euler's equation along it can be expressed in the form

$$F_{y'x} + F_{y'y}y' + F_{y'y'}y'' - F_y = 0\,. \tag{21}$$

The following corollary contains a criterion which for many problems enables us to prove that a minimizing arc must have a continuous second derivative and hence satisfy the last equation.

Corolary 2. Hilbert's Differentiability condition. *Near a point on a minimizing arc $\underline{y}_0$ where $F_{y'y'}$ is different from zero, the arc always has a continuous second derivative $y''(x)$.*

To prove this let (x, y, y') be a set of values on $\underline{y}_0$ at which $F_{y'y'}$ is different from zero, and suppose further that $(x+\Delta x, y+\Delta y, y'+\Delta y')$ is also on $\underline{y}_0$ and with no corner between it and the former set. If we denote the values of $F_{y'}$ corresponding to these two sets by $F_{y'}$ and $F_{y'} + \Delta F_{y'}$ then with the help of Taylor's formula we find

$$\begin{aligned} \frac{\Delta F_{y'}}{\Delta x} &= \frac{1}{\Delta x}\{F_{y'}(x+\Delta x, y+\Delta y, y'+\Delta y') - F_{y'}(x,y,y')\} \\ &= F_{y'x}(x+\theta\Delta x, y+\theta\Delta y, y'+\theta\Delta y') \\ &+ F_{y'y}(x+\theta\Delta x, y+\theta\Delta y, y'+\theta\Delta y')\frac{\Delta y}{\Delta x} \\ &+ F_{y'y'}(x+\theta\Delta x, y+\theta\Delta y, y'+\theta\Delta y')\frac{\Delta y'}{\Delta x} \end{aligned} \tag{22}$$

where $0 < \theta < 1$. In this expression the left hand side $\Delta F_{y'}/\Delta x$ has the definite limit F_y as Δx approaches zero, because of the definition of the derivative and the differentiated form of the first Euler equation which holds on intervals that have no corners. Also, the first two terms on the right hand side of (22) have well-defined limits. It follows that the last term must have a unique limiting value, and since $F_{y'y'} \neq 0$ this can be true only if $y'' = \lim\ \Delta y'/\Delta x$ exists. The derivative $F_{y'y'}$ remains different from zero near the element (x, y, y') on the sub-arc of $\underline{y}_0$ on which this element lies. Consequently Euler's equation in the form given in (21) can be solved for y'', and it follows that y'' must be continuous near every element (x, y, y') of the kind described in the corollary.

CHAPTER 13

13 Field Theory Jacobi's Neccesary Condition and Sufficiency

In this chapter, we discuss the notion of a field and a sufficiency proof for the shortest distance from a point to a curve.

Recall figure 10 of chapter 3 in which a straight line segement of variable length moved so that its ends described two curves C and D. These curves written in parametric form are

$$C: \quad x = x_3(t) \quad y = y_3(t)$$
$$D: \quad x = x_4(t) \quad y = y_4(t)\,. \tag{1}$$

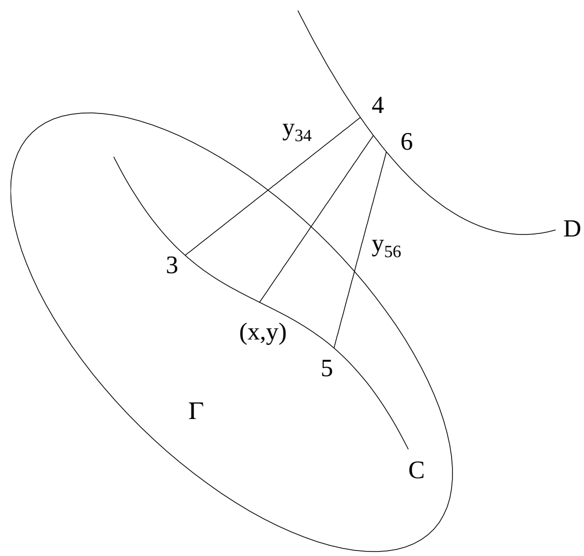

Figure 29: Line segment of variable length with endpoints on the curves C, D

Points 5, 6 are two other points on these curves.

We have seen in chapter 3 that necessary conditions on the shortest arc problem may be deduced by comparing it with other admissible arcs of special types. It is also true that for sufficiency in that problem, then a particular arc can be proved to be actually the shortest, only by comparing it with *all* of the admissible arcs satisfying the conditions of the problem. The sufficiency proof of this chapter is valid not only for the arcs which we have named admissible but also for arcs with equations in the parametric form

$$x = x(t), \quad y = y(t) \quad (t_3 \le t \le t_5)\,. \tag{2}$$

We suppose always that the functions $x(t)$ and $y(t)$ are continuous, and that the interval $[t_3, t_5]$ can be subdivided into one or more parts on each of which $x(t)$ and $y(t)$ have continuous derivatives such that $x'^2 + y'^2 \neq 0$. The curve represented is then continuous and has a continuously turning tangent except possibly at a finite number of corners. A much larger variety of curves can be represented by such parametric equations than by an equation of the form $y = y(x)$ because the parametric representation lays no restriction upon the slope of the curve or the number of points of the curve which may lie upon a single ordinate. On the other hand for an admissible arc of the form $y = y(x)$ the slope must always be finite and the number of points on each ordinate must at most be one.

The mathematician who first made satisfactory sufficiency proofs in the calculus of variations was Weierstrass, and the ingenious device which he used in his proofs is called a field. We describe first a generic field for shortest distance problems in general and after giving some other examples of fields, we introduce the particular field which will be used for the shortest distance problem from a point to a curve.

For the shortest distance problems, a field Γ is a region of the xy-plane with which there is associated a one-parameter family of straight-line segments all of which intersect a fixed curve D, and which have the further property that through each point (x, y) of Γ there passes one and only one of the segments. The curve D may be either inside the field, or outside as illustrated in Figure 29, and as a special case it may degenerate into a single fixed point.

The whole plane is a field when covered by a system of parallel lines, the curve D being in this case any straight line or curve which intersects all of the parallels. The plane with the exception of a single point 0 is a field when covered by the rays through 0, and 0 is a degenerate curve D. The tangents to a circle do not cover a field since through each point outside of the circle there pass two tangents, and through a point inside the circle there is none. If, however, we cut off half of each tangent at its contact point with the circle, leaving only a one parameter family of half-rays all pointing in the same direction around the circle, then the exterior of the circle is a field simply covered by the family of half-rays.

At every point (x, y) of a field Γ the straight line of the field has a slope $p(x, y)$, the function so defined being called the slope-function of the field. The integral I^* introduced in chapter 4

$$I^* = \int \frac{dx + pdy}{\sqrt{1 + p^2}} \tag{3}$$

with this slope function used for p and with dx, dy coming from the arc C of figure 29, has a definite value along every arc C_{35} in the field having equations of the form (2), as we have seen before. We can prove with the help of the formulas of chapter 3 that the integral I^* associated in this way with a field has the two following useful properties:

The values of I^* are the same along all curves C_{35} in the field Γ having the same end-points 3 and 5. Furthermore along each segment of one of the straight lines of the field the value of I^* is equal to the length of the segment.

To prove the first of these statements we may consider the curve C_{35} shown in the field Γ of Figure 29. Through every point (x, y) of this curve there passes, by hypothesis, a straight line of the field Γ intersecting D, and (4) of chapter 3, applied to the one parameter family of straight-line segments so determined by the points of C_{35}, gives

$$I^*(C_{35}) = I^*(D_{46}) - I(\underline{y}_{56}) + I(\underline{y}_{34})\,. \tag{4}$$

The values of the terms on the right are completely determined when the points 3 and 5 in the field are given, and are entirely independent of the form of the curve C_{35} joining these two points. This shows that the value $I^*(C_{35})$ is the same for all arcs (C_{35}) in the field joining the same two end-points, as stated in the theorem.

The second property of the theorem follows from the fact that along a straight-line segment of the field the differentials dx and dy satisfy the equation $dy = p\ dx$, and the integrand of I^* reduces to $\sqrt{1+p^2}dx$ which is the integrand of the length integral.

We now have the mechanism necessary for the sufficiency proof for the problem of shortest distance from a fixed point 1 to a fixed curve N (introduced in chapter 4).

We recall figure 16 (chapter 4) which is repeated below in which the curve N, its evolute G and two positions of normals to N, one of them containing the point 1, are shown.

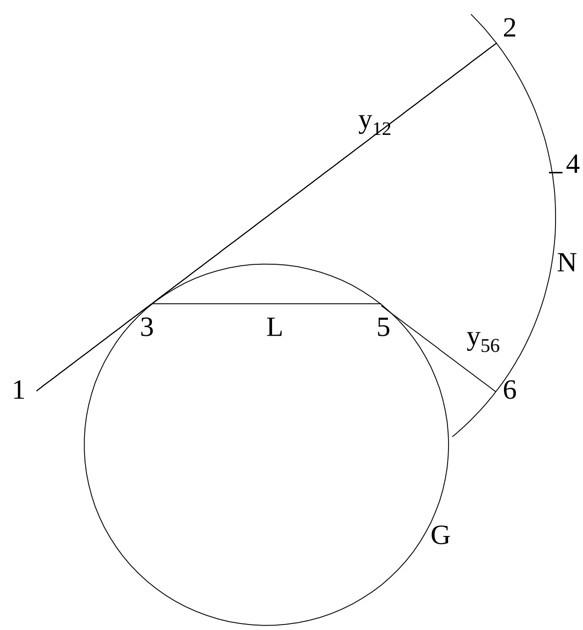

Figure 30: Shortest arc from a fixed point 1 to a curve N. G is the evolute

We infer by inspection from the Figure, that when the end-point 1 lies between 3 and 2, there is adjoining $\underline{y}_{12}$ a region Γ of the plane which is simply covered by the normals to N which are near to $\underline{y}_{12}$. An analytic proof of this statement for a more general case will be given if time permits. For the present we shall rely on our inference of it from the figure. The region Γ so covered by the normals to N forms a field such as was described above. The integral I^* formed with the slope function $p(x, y)$ of the field in its integrand, is independent of the path and has the same value as I along the straight-line segment $\underline{y}_{12}$ of the field. It also has the value zero on every arc of N since the straight lines of the field are all perpendicular to N and its integrand therefore vanishes identically along that curve. Hence for an arbitrarily selected arc C_{14} in F joining 1 with N, as shown in Figure 30, we have

$$I(\underline{y}_{12}) = I^*(\underline{y}_{12}) = I^*(C_{14} + N_{42}) = I^*(C_{14})\ , \tag{5a}$$

and the difference between the lengths of C_{14} and $\underline{y}_{12}$ is

$$I(C_{14}) - I(\underline{y}_{12}) = I(C_{14}) - I^*(C_{14}) = \int_{s_1}^{s_2} (1 - \cos\theta) ds \geq 0 \tag{5b}$$

with the last equality following from (6) of chapter 3.

The difference between the values of I along C_{14} and $\underline{y}_{12}$ is therefore

$$I(C_{14}) - I(\underline{y}_{12}) = \int_{s_1}^{s_2} (1 - \cos\theta) ds \geq 0\,. \tag{6}$$

The equality sign can hold only if C_{14} coincides with $\underline{y}_{12}$. For when the integral in the last equation is zero we must have $\cos\theta = 1$ at every point of C_{14}, from which it follows that C_{14} is tangent at every point to a straight line of the field and satisfies the equation $dy = p\,dx$. Such a differential equation can have but one solution through the initial point 1 and that solution is $\underline{y}_{12}$. We have proved therefore that the length $I(C_{14})$ of C_{14} is always greater than that of $\underline{y}_{12}$ unless C_{14} is coincident with $\underline{y}_{12}$.

For a straight-line segment $\underline{y}_{12}$ perpendicular to the curve N at the point 2 and not touching the evolute G of N there exists a neighborhood Γ in which $\underline{y}_{12}$ is shorter than every other arc joining 1 with N.

We now prove a sufficiency theorem for the general problem of chapters 12 and 3 which we repeat here for completeness.

We wish to minimize the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')dx \tag{7}$$

on a class of admissible arcs

$$\underline{y}: \quad y(x) \quad x_1 \leq x \leq x_2 \tag{8}$$

joining two given points and lying in some region R of admissible points.

We will often refer to a class of extremals. Recalling the definition from previous chapters an extremal $\underline{y}$ is an arc which is a solution to the Euler equations on $[x_1, x_2]$ and which has continuous first and second derivatives ($y'(x)$ and $y''(x)$).

We also define a field for this general problem. A region Γ of the plane is called a *field* if it has associated with it a one-parameter family of extremals all intersecting a curve D and furthermore such that through each point (x, y) of Γ there passes one and but one extremal of the family. Figure 31 is a picture suggesting such a field.

The function $p(x, y)$ defining the slope of the extremal of the field at a point (x, y) is called the *slope-function* of the field. With this slope-function substituted, then the integrand of the integral I^* of chapter 3

$$I^* = \int [F(x, y, p)dx + (dy - pdx)F_{y'}(x, y, p)] \tag{9}$$

depends only upon x, y, dx, dy, and the integral itself will have a well-defined value on every arc C_{35} in Γ having equations

$$x = x(t)\,, \quad y = y(t) \quad (t_3 \leq t \leq t_5) \tag{10}$$

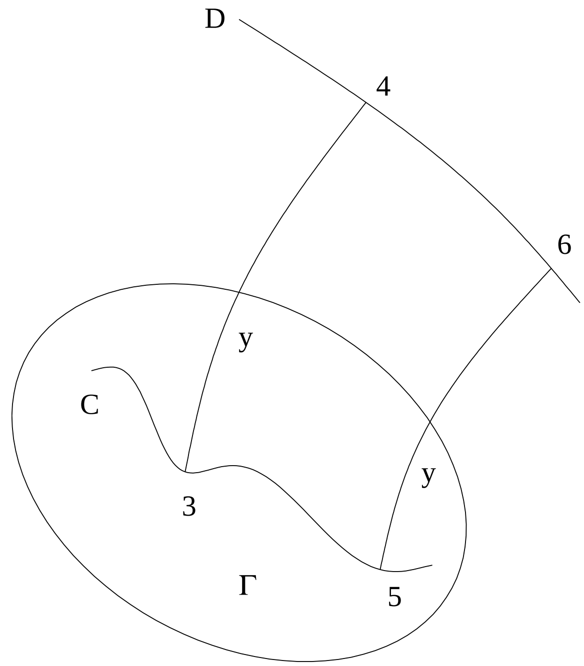

Figure 31: Line segment of variable length with endpoints on the curves C, D

of the type described in (2). Furthermore the endpoints of C_{35} determine two extremal arcs $\underline{y}_{34}$ and $\underline{y}_{56}$ of the field, and a corresponding arc D_{46}, which are related to it like those in equation (28) of chapter 3, which we repeat here

$$I(\underline{y}_{56}) - I(\underline{y}_{34}) = I^*(D_{46}) - I^*(C_{35})\,. \tag{11}$$

It is clear then that the value $I^*(C_{35})$ depends only upon the points 3 and 5, and not at all upon the form of the arc C_{35} joining them, since the other three terms in equation (11) have this property.

The importance of the integral I^* in the calculus of variations was first emphasized by Hilbert and it is usually called Hilbert's invariant integral. Its two most useful properties are described in the following corollary:

Corollary: For a field Γ *simply covered by a one parameter family of extremals all of which intersect a fixed curve D, the Hilbert integral* I^* *formed with the slope-function p(x,y) of the field has the same value on all arcs* C_{35} *in* Γ *with the same end-points 3 and 5. Furthermore on an extremal arc of the field,* I^* *has the same value as I.*

The last statement follows, since along an extremal of the field we have $dy = p\;dx$ and the integrand of I^* reduces to $F(x, y, p)dx$.

The formula (27) of chapter 3 which we also repeat

$$dI = F(x, y, p) + (dy - pdx)F_{y'}(x, y, p)\Big|_3^4 \tag{12}$$

and (11) of this chapter are the two important auxiliary formulas developed in chapter 3. They remain valid in simpler forms if one of the curves C_{35} or D_{46} degenerates into a point, since then the differentials dx, dy along that curve are zero.

We shall see that through a fixed point 1 there passes in general a one-parameter family of extremals. If such a family has an envelope G as shown in figure 32, then the contact point 3 of an extremal arc $\underline{y}_{12}$ of the family with the envelope, is called conjugate to point 1 on $\underline{y}_{12}$.

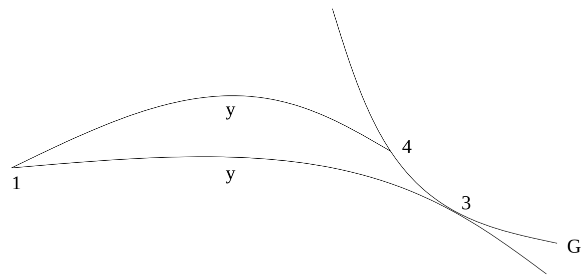

Figure 32: Conjugate point at the right end of an extremal arc

We next prove two results which are required for the sufficiency theorem.

The envelope theorem and Jacobi's condition. The formula (11) enables us to prove the envelope theorem which is a generalization of the string property of the evolute noted in the shortest distance problem of chapter 4. Let $\underline{y}_{14}$ and $\underline{y}_{13}$ be two extremals of a one-parameter family through the point 1, touching an envelope G of the family at their end-points 4 and 3, as shown in Figure 32. When we replace the arc C_{35} of the formula (11) above by the fixed point 1, and the arc D_{46} by G_{43}, we find the equation

$$I(\underline{y}_{13}) - I(\underline{y}_{14}) = I^*(G_{43}) \ . \tag{13}$$

Furthermore the differentials dx, dy at a point of the envelope G satisfy the equation $dy = p\ dx$ with the slope p of the extremal tangent to G at that point, and it follows that the value of the (Hilbert) integral (9) along G_{43} is the same as that of I. Hence we have:

The Envelope Theorem. Let $\underline{y}_{14}$ and $\underline{y}_{13}$ be two members of a one-parameter family of extremals through the point 1, touching an envelope G of the family at their end-points 4 and 3, as shown in Figure 32. Then the values of the integral I along the arcs $\underline{y}_{14}$, $\underline{y}_{13}$, G_{43} satisfy the relation

$$I(\underline{y}_{13}) + I(G_{43}) = I(\underline{y}_{14}) \tag{14}$$

for every position of the point 4 preceding 3 on G.

We next prove a condition which was hinted at in chapter 4. This is Jacobi's condition.

Theorem (Jacobi). On a minimizing arc $\underline{y}_{12}$ which is an extremal with $F_{y'y'} \neq 0$ everywhere on $\underline{y}_{12}$, there can be no point 3 conjugate to 1 between 1 and 2. We notice that according to the envelope theorem, the value of I along the composite arc $\underline{y}_{14} + G_{43} + \underline{y}_{32}$ in Figure 32 is always the same as its value along $\underline{y}_{12}$. But G_{43} is not an extremal and can be replaced therefore by an arc C_{43} giving I a smaller value. In every neighborhood of $\underline{y}_{12}$ there is consequently an arc $\underline{y}_{14} + C_{43} + \underline{y}_{32}$ giving I a smaller value than $\underline{y}_{12}$ and $I(\underline{y}_{12})$ cannot be a minimum.

To insure that G_{43} is not an extremal arc we make use of a well-known property of (Euler's) second order differential equation expanded out:

$$\frac{d}{dx}F_{y'} - F_y = F_{y'x} + F_{y'y}y' + F_{y'y'}y'' - F_y = 0 \tag{15}$$

which is satisfied by all extremals. That property states that when such an equation can be solved for the derivative y'' there is one and only one solution of it through an arbitrarily selected initial point and direction (x_3, y_3, y_3'). But we know that equation (15) is solvable for y'' near the arc $\underline{y}_{12}$ since the hypothesis of Jacobi's condition requires $F_{y'y'}$ to be different from zero along that arc. Hence if G_{43} were an extremal it would necessarily coincide with $\underline{y}_{13}$, in which case all of the extremal arcs of the family through the point 1 would by the same property be tangent to and coincide with $\underline{y}_{13}$. There would then be no one-parameter family such as the theorem supposes.

The fundamental sufficiency theorem. The conditions for a minimum which have so far been deduced for our problem have been only necessary conditions, but we shall see in the following that they can be made over with moderate changes into conditions which are also sufficient to insure an extreme value for our integral. Since the comparison of necessary with sufficient conditions is one of the more delicate parts of the theory of the calculus of variations, it's a good idea before undertaking it to consider a sufficiency theorem which in special cases frequently gives information so complete that after using it one does not need to use farther the general theory.

Using the general field described above, we as usual designate the function $p(x, y)$ defining the slope of the extremal of the field at a point (x, y) as the *slope-function* of the field. With E as the Weierstrass $E-$ function of chapter 14

$$E(x, y, y', Y') = F(x, y, Y') - F(x, y, y') - (Y' - y')F_{y'}(x, y, y') \tag{16}$$

we have the following theorem, which is fundamental for all of the sufficiency proofs:

The Fundamental Sufficiency Theorem. Let $\underline{y}_{12}$ be an extremal arc of a field Γ such that at each point (x, y) of Γ the inequality

$$E(x, y, p(x, y), y') \geq 0 \tag{17}$$

holds for every admissible set (x, y, y') different from (x, y, p). Then $I(\underline{y}_{12})$ is a minimum in Γ, or, more explicitly, the inequality $I(\underline{y}_{12}) \leq I(C_{12})$ is satisfied for every admissible arc C_{12} in Γ joining the points 1 and 2. If the equality sign is excluded in the hypothesis (17) then $I(\underline{y}_{12}) < I(C_{12})$ unless C_{12} coincides with $\underline{y}_{12}$, and the minimum is a so-called proper one.

In order to accomplish the analysis involved in the proof of the above sufficiency theorem we now list the properties of the family of extremal arcs covering the field Γ. It is supposed that the family has an equation of the form

$$y = y(x, a) \quad (a_1 \leq a \leq a_2; \;\; x_1(a) \leq x \leq x_2(a)) \tag{18}$$

in which the functions $y(x, a)$, $y'(x, a)$ and their partial derivatives up to and including those of the second order, as well as the functions $x_1(a)$ and $x_2(a)$ defining the end-points of the extremal arcs, are continuous. It is understood that the point of the curve D on each extremal is defined by a function $x = \xi(a)$ which with its first derivative is continuous on the interval $[a_1, a_2]$, and furthermore that the derivative y'_a is everywhere different from zero on the extremal arcs. To each point (x, y) in Γ there corresponds a value $a(x, y)$ which defines the unique extremal of the field through that point, and as a result of the hypothesis that y_a is different from zero we can prove that $a(x, y)$ and its first partial derivatives are continuous in Γ. The same is then true of the slope-function $p(x, y) = y'(x, a(x, y))$ of the field. These properties form the analytical basis of the theory of the field, and we assume them always.

The Hilbert integral (9) formed with the slope-function $p(x, y)$ in place of p has now a definite value on every admissible arc C_{12} in the field. Furthermore as shown above its values are the same on all such arcs C_{12} which have the same end-points, and if the points 1 and 2 are the end-points of an extremal arc $\underline{y}_{12}$ of the field, this value is that of the original integral I. Hence we find for the pair of arcs C_{12} and $\underline{y}_{12}$ shown in figure 33,

$$I(C_{12}) - I(\underline{y}_{12}) = I(C_{12}) - I^*(\underline{y}_{12}) = I(C_{12}) - I^*(C_{12})\,, \tag{19}$$

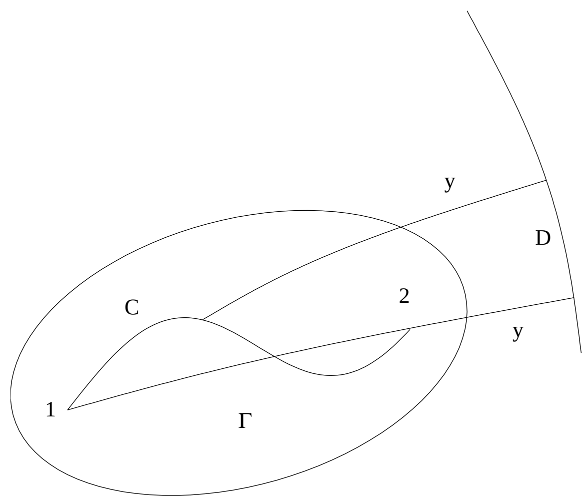

Figure 33: Line segment of variable length with endpoints on the curves C, D

and when we substitute for I and I^* their values as integrals, it follows that

$$I(C_{12}) - I(\underline{y}_{12}) = \int_{x_1}^{x_2} E(x, y, p(x, y), y')dx\,. \tag{20}$$

In the integral on the right, y and its derivative y' are functions of x obtained from the equation $y = y(x)$ of the admissible arc C_{12}.

The sufficiency theorem is an immediate consequence of this formula. For the hypothesis (17) that the $E-$function is greater than or equal to zero in the field implies at once that $I(\underline{y}_{12}) \le I(C_{12})$. If the $E-$function vanishes in the field only when $y' = p$ then the equality $I(\underline{y}_{12}) = I(C_{12})$ can hold only if the equation $y' = p(x, y)$ is satisfied at every point of C_{12}. But in that case C_{12} must coincide with $\underline{y}_{12}$ since the differential equation $y' = p(x, y)$ has one and but one solution through the initial point 1, and that one is $\underline{y}_{12}$.

The sufficiency proof of the shortest distance problem was an application of a special case of the formula (20) and this theorem. For that special problem the second derivative $F_{y'y'}$ is positive for all admissible sets (x, y, y') and the formula (16) of chapter 12 which we repeat here

$$E(x,y,p,y') = \frac{1}{2}(y'-p)^2 F_{y'y'}(x,y,p+\theta(y'-p)) \quad (0<\theta<1) \tag{21}$$

shows that the $E-$function is positive whenever $y' \neq p$, as presupposed in the last sentence of the sufficiency theorem.

In order to efficiently discuss further sufficiency results it is convenient now to collect together all of the necessary conditions which have been obtained thus far for our general problem.

I. For every minimizing arc $\underline{y}_{12}$ *there exists a constant c such that the equation*

$$F_{y'}(x,y(x),y'(x)) = \int_{x_1}^{x} F_y(x,y(x),y'(x))dx + c \tag{22}$$

holds identically on $\underline{y}_{12}$*. An immediate consequence of this equation is that on each arc of* $\underline{y}_{12}$ *having a continuously turning tangent Euler's differential equation*

$$\frac{d}{dx}F_{y'} - F_y = 0 \tag{23}$$

must also be satisied.

II. (Weierstrass). At every element (x, y, y') *of a minimizing arc* $\underline{y}_{12}$ *the condition*

$$E(x,y,y',Y') \geq 0 \tag{24}$$

must be satisfied foe every admissible set (x, y, Y') *different from* (x, y, y')*.*

III. (Legendre). At every element (x, y, y') *of a minimizing arc* $\underline{y}_{12}$ *the condition*

$$F_{y'y'}(x,y,y') \geq 0 \tag{25}$$

must be satisfied.

IV. (Jacobi). On a minimizing arc $\underline{y}_{12}$ *which is an extremal with* $F_{y'y'} \neq 0$ *everywhere on it, there can be no point 3 conjugate to 1 between 1 and 2.*

The fundamental sufficiency theorem (f.s.t.) proven above refers to a set of admissible points (of which the admissible arcs are made up) which according to our discussion in chapter 12 will be contained in some region R. The results to follow will each be closely associated with a specific R. Also the selection of R will depend in part on the field Γ (also referred to in the f.s.t) that we are able to construct.

Next, using a notation introduced by Bolza let us designate by II', III' the conditions II, III with the equality sign excluded, and by IV' the condition IV when strengthened to exclude the possibility of a conjugate point at the end-point 2 as well as between 1 and 2 on $\underline{y}_{12}$. If time permits it will be proven later that for an extremal arc $\underline{y}_{12}$ which satisfies the conditions I, III', IV' there is always some neighborhood Γ which is a field simply covered by a one-parameter family of extremals having $\underline{y}_{12}$ as a member of the family.

The value $I(\underline{y}_{12})$ is said to be a *weak relative minimum* if there is a neighborhood R' of the values (x, y, y') on $\underline{y}_{12}$ such that the inequality $I(\underline{y}_{12}) \leq I(C_{12})$ is true, not necessarily for all admissible arcs C_{12}, but at least for all those whose elements (x, y, y') lie in R'. With the help of the sufficiency theorem stated above and the field described in the last paragraph we shall be able to prove that an arc $\underline{y}_{12}$ which satisfies the conditions I, III', IV' will make the value $I(\underline{y}_{12})$ at least a weak relative minimum. This result will be established by replacing the original region R by R' and choosing R' so small that every admissible arc with respect to it is necessarily in the field Γ, and furthermore so small that the condition 17) of the theorem holds in Γ in its stronger form with respect to all of the sets (x, y, y') in R'.

Following Bolza again let us denote by II_b the condition II strengthened to hold not only for elements (x, y, y') on $\underline{y}_{12}$ but also for all such elements in a neighborhood of those on $\underline{y}_{12}$. It will be proved that for an arc which satisfies the conditions I, II_b', III', IV' the field Γ about $\underline{y}_{12}$, existent as a result of the conditions I, III', IV', can be so constructed that the stronger condition 17) holds in it with respect to the sets (x, y, y') in the region R itself. The value $I(\underline{y}_{12})$ will therefore again be a minimum in Γ, and it is called a *strong relative minimum* because it is effective with respect to all admissible comparison curves C whose elements (x, y, y') have their points (x, y) in a small neighborhood Γ of those on $\underline{y}_{12}$. No restrictions are in this case imposed upon the slopes y' except those due to the definition of the original region R.

Sufficient Condition for Relative Minima

For our immediate purposes we state now and will prove if time permits a result referred to above.

Lemma: *Every extremal arc $\underline{y}_{12}$ having $F_{y'y'} \neq 0$ along it and containing no point conjugate to 1 is interior to a field Γ of which it itself is an extremal arc.*

We now discuss the important sets of sufficient conditions which insure for an arc $\underline{y}_{12}$ the property of furnishing a relative minimum. We have seen in chapter 12 that there is a considerable degree of arbitrariness in the choice of the region R in which the minimum problem may be studied. Relative minima are really minima in certain types of sub-regions of the region R originally selected, and their existence is assured by the conditions described in the following two theorems.

Sufficient conditions for a weak relative minimum. *Let $\underline{y}_{12}$ be an arc having the properties:*

1) it is an extremal,

2) $F_{y'y'} > 0$ at every set of values (x, y, y') on it,

3) it contains no point 3 conjugate to 1.

Then $I(\underline{y}_{12})$ is a weak relative minimum, or, in other words, the inequality $I(\underline{y}_{12}) < I(C_{12})$ holds for every admissible arc C_{12} distinct from $\underline{y}_{12}$, joining 1 with 2, and having its elements (x, y, y') all in a sufficiently small neighborhood R' of those on $\underline{y}_{12}$.

To prove this we note in the first place that the conditions 1, 2, 3 of the theorem imply the conditions I, III', IV'. Furthermore the same three properties insure the existence of a field Γ having the arc $\underline{y}_{12}$ as one of its extremals, as indicated in the lemma just stated above. Let us now choose a neighborhood R' of the values (x, y, y') on $\underline{y}_{12}$ so small that all elements (x, y, y') in R' have their points (x, y) in Γ, and so small that for the slope-function

$p = p(x, y)$ of Γ, the elements $x, y, p + \theta(y' - p)$ having $0 \le \theta \le 1$ are all admissible and make $F_{y'y'} \ne 0$. Then the function

$$E(x, y, p(x, y), y') = \frac{1}{2}(y' - p)^2 F_{y'y'}(x, y, p + \theta(y' - p))\,, \tag{26}$$

is positive for all elements (x, y, y') in R' with $y' \ne p$, and the fundamental sufficiency theorem proven earlier in this chapter, with R replaced by R' in the definition of admissible sets, justifies the theorem stated above for a weak relative minimum.

Sufficient Conditions for a Strong Relative Minimum. *Let $\underline{y}_{12}$ be an arc having the properties of the preceding theorem and the further property*

4) at every element (x, y, y') in a neighborhood R' of those on $\underline{y}_{12}$ the condition

$$E(x, y, y', Y') > 0$$

is satisfied for every admissible set (x, y, Y') with $Y' \ne y'$.

Then $\underline{y}_{12}$ satisfies the conditions I, II'_b, III', IV' and $I(\underline{y}_{12})$ is a strong relative minimum. In other words, the inequality $I(\underline{y}_{12}) < I(C_{12})$ holds for every admissible arc C_{12} distinct from $\underline{y}_{12}$, joining 1 with 2, and having its points (x, y) all in a sufficiently small neighborhood Γ of those on $\underline{y}_{12}$.

The properties 1), 2), 3) insure again in this case the existence of a field Γ having $\underline{y}_{12}$ as one of its extremal arcs, and we may denote the slope-function of the field as usual by $p(x, y)$. If we take the field so small that all of the elements $(x, y, p(x, y))$ belonging to it are in the neighborhood R' of the property 4), then according to that property the inequality

$$E(x, y, p(x, y), y') > 0$$

holds for every admissible element (x, y, y') in Γ distinct from $(x, y, p(x, y))$, and the fundamental sufficiency theorem gives the desired conclusion of the theorem.

We now use the results just developed for the general theory by applying them to the brachistochrone problem of finding the curve of quickest descent for a particle to slide from a given point 1 with coordinates (x_1, y_1) to a given curve N with a given initial velocity v_1. This is the same problem we saw in chapter 4 where first necessary conditions were obtained.

Let the point 1, the curve N and the path $\underline{y}_{12}$ of quickest descent be those shown in figure 34. The constant α has the same meaning as in chapter 4, namely $\alpha = y_1 - v_1^2/2g$ where y_1 is the value of y at point 1, and g is the gravitational acceleration.

We recall the integral I to be minimized from chapter 4

$$I = \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^2}{y - \alpha}}\,dx\,. \tag{27}$$

By chapters 3 and 4 we already know that a minimizing arc $\underline{y}_{12}$ for this problem must consist of a cycloid lying below the line $y = \alpha$. We also know by Jacobi's condition that $\underline{y}_{12}$ cannot contain a conjugate point between its end-points. We now prove that with the

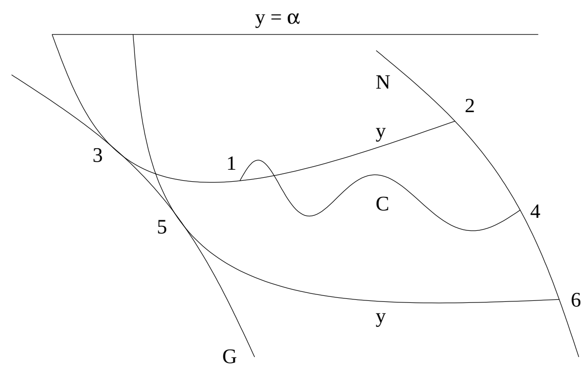

Figure 34: The path of quickest descent from point 1 to a cuve N

assumption of a slight strengthening of Jacobi's condition, this cycloid provides a strong minimizing arc for the problem at hand, (i.e. it satisfies the conditions of the strong sufficiency theorem).

With F as the integrand of (27) we first compute

$$F_{y'} = \frac{y'}{\sqrt{y-\alpha}\sqrt{1+y'^2}} \tag{28}$$

Next, by the Weierstrass-Erdmann Corner Condition (chapter 12) one sees that the expression on the right-hand side of (28) is continuous on $\underline{y}_{12}$. We now show that this implies that y' must also be continuous on $\underline{y}_{12}$. With the substitution $y' = \tan\alpha$, then $\dfrac{y'}{\sqrt{1+y'^2}} = \sin\alpha$ and the continuity of (28) implies that $\sin\alpha$ and hence also α and $\tan\alpha = y'$ must be continuous along $\underline{y}_{12}$. Thus $\underline{y}_{12}$ contains no coners. Next, note that $F_{y'y'} = \dfrac{1}{\sqrt{y-\alpha}(1-y'^2)^{3/2}} > 0$ for all admissible (see chapter 3) points (x, y, y') with $y > \alpha$, and so certainly also on $\underline{y}_{12}$ then Hilbert's Differentiability Condition (chapter 12) shows that y'' is continuous on $\underline{y}_{12}$.

Now let R' be any neighborhood of $\underline{y}_{12}$ that is contained within the admissible set of points. Let x, y, y' and x, y, Y' be any points in R' (with the same x, y). Then by (26) and the positivity of $F_{y'y'}$ for all admissible points, we have condition 4) of the strong sufficiency theorem. Finally, if we assume that $\underline{y}_{12}$ does not contain a conjugate point at its right endpoint, then all of the conditions of the strong sufficiency theorem are met and $\underline{y}_{12}$ provides a strong relative minimum for our problem as stated in that theorem.

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