2007

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Linda Eroh and Ralucca Gera, Alliance Partition Number in Graphs. Ars Combinatoria, 103 (2012), pp. 519-529 (accepted 2007).
https://hdl.handle.net/10945/41332

Alliance Partition Number in Graphs

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A (defensive) alliance in G is a subset S of $V(G)$ such that for every vertex $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V(G) - S)|$. The alliance partition number of a graph G, $\psi_a(G)$, is defined to be the maximum number of sets in a partition of $V(G)$ such that each set is a (defensive) alliance. In this paper, we give both general bounds and exact results for the alliance partition number of graphs, and in particular for regular graphs and trees.

Key Words: alliance, domination, partition, alliance partition.

AMS Subject Classification: 05C15, 05C69.

1 Introduction and motivation

Defensive and offensive alliances were introduced by Kristiansen, Hedetniemi, and Hedetniemi in [9] and [10], and numerous variants of this problem have been studied by others. The definitions were motivated by the study of alliances between different people, between different countries, and between species of plants in botany. In a graph G, a nonempty set of vertices S is a *(defensive) alliance* if for every vertex $v \in S$, $|N[v] \cap S| \geq$

*Research supported by the Research Initiation Program Grant at the Naval Postgraduate School.

$|N(v) \cap (V(G) - S)|$. Then every vertex in S is an ally of v and every vertex not in S is a (potential) enemy of v, in particular, vertices not in S adjacent to v are the enemies of v. This says that v is adjacent to at least as many allies as enemies, where v itself is counted as an ally. A (defensive) alliance S is *strong* if, for every $v \in S$, $|N[v] \cap S| > |N(v) \cap (V(G) - S)|$ (strict inequality). In this paper, we will often call a defensive alliance simply an alliance. Algorithmic complexity of alliances in graphs was first studied in [11] with more studies of complexity of different variants of alliances. For other graph theory terminology the reader should refer to [2].

The *alliance partition number of* G, $\psi_a(G)$, is defined to be the maximum number of sets in a partition of $V(G)$ such that each set is an alliance. The alliance partition number was a topic introduced in [9].

Similar concepts have been studied in which the vertex set has been partitioned into exactly two sets, each of which is some type of alliance. In [3], [4], and [5], R. D. Dutton and H. S. Khurram defined an *alliance-free partition* to be a partition of the vertex set into two nonempty sets if neither one of the two sets contains a strong defensive alliance as a subset. Also, they defined an *alliance cover set* to be a subset of the vertices of a graph that contains at least one vertex from every alliance of the graph. It turns out that the complement of an alliance cover set is an alliance free set, that is, a set that does not contain any alliance as a subset. They characterize the graphs that can be partitioned into alliance free and alliance cover sets. Gerber and Kobler in [6] introduced the *satisfactory partition problem*, which, restated in our notation, involves determining whether a particular graph has a partition into two strong defensive alliances. In [8], this idea was generalized to the *k- Satisfactory Graph Partitioning problem (k-SGP)*, which consists in determining if a graph is k-satisfiable or not, i.e., whether a given graph can be partitioned into two k-defensive alliances. An alliance A is k-defensive, if for each vertex $v \in A$, we have that $\deg_A(v) \geq \deg_{V-A} v + k$, where k is an integer. Note that if $k = 0$, then the problem reduces to finding which graphs have a partition of the vertex set into exactly two alliances. A graph G is 0-satisfiable if and only if $\psi_a(G) \geq 2$, so the alliance partition number could be viewed as a generalization of 0-satisfiability. In a similar fashion the *unfriendly graph partition problem* was introduced by Aharoni et al. [1], where the vertex set is partitioned into two sets such that each vertex has most of its neighbors in the complement of the set it belongs to.

In this paper we study the elementary properties of alliance partition number by presenting both general bounds in terms of minimum degree, order and diameter, and also exact results for the alliance partition number in graphs. We recently learned that T. W. Haynes and J. A. Lachniet [7] have independently worked on this topic, and we refer to their future papers for the study of the alliance partition number in classes of graphs.

2 Preliminary Results and Observations

We will start by showing an example of the alliance partition number in graphs.

Let P be the Petersen graph of order $n = 10$. We will be using labels on the vertices to denote the alliance that the particular vertex belongs to.

P:

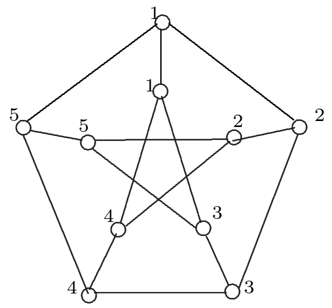

Figure 1: $ap(P) = 5$

Observe that no vertex can form an alliance by itself since its degree is 3. Thus, at least two vertices must be in each alliance, giving us that $\psi_a(P) \leq \frac{n}{2} = 5$. Also, the above labeling shows that there is an alliance partition with 5 alliances, and so $\psi_a(P) = 5$.

We consider disconnected graphs, so that we can concentrate on connected graphs for the rest of the paper. We present the relationship between the alliance partition number of a disconnected graph and its components. First, observe that each alliance is only contained in a particular component of G, and the alliances in one component form an alliance partition of that component.

Proposition 2.1 *Let G be a disconnected graph whose components are $G_1, G_2, \ldots, G_r$ $(r \geq 1)$. Then*

$$\psi_a(G) = \sum_{1 \leq i \leq r} \psi_a(G_i).$$

We also present bounds and their sharpness for disconnected graphs. Since $V(G)$ defines an alliance for any graph G, and since it is possible for each vertex of the graph to form an alliance by itself (if its degree is 0 or 1), we have the following result.

Proposition 2.2 *Let G be a disconnected graph of order $n \geq 3$. Then*

$$1 \leq \psi_a(G) \leq n.$$

To see the sharpness, note that $\psi_a(K_{2\ell+1}) = 1$ and $\psi_a(\overline{K_n}) = n$, for ℓ, n positive integers.

Since $\psi_a(G) = n$ if and only if each vertex forms an alliance by itself (i.e. each vertex can have at most one enemy), we obtain a characterization of graphs that attain the upper bound of the alliance partition number for disconnected graphs.

Proposition 2.3 *Let G be a connected graph of order n. Then $\psi_a(G) = n$ if and only if $G = kK_1 + \ell K_2$, for k and ℓ nonnegative integers.*

One may observe that most graphs of even order will have an alliance partition number of 2 or larger, since it is possible to divide the vertex set into 2 equal alliances (for example K_{2n} for some positive integer n). However, this is not always the case as we present next. We will use this result in the proof of Proposition 2.5.

Lemma 2.4 *There are classes of graphs G of even order such that $\psi_a(G) = 1$.*

Proof. Let G be the graph of order $2t \geq 8$ obtained from the complete graph $K_{2t-1} : v_1, v_2, \ldots, v_{2t-1}$, by adding a vertex v together with the edges vv_i $(1 \leq i \leq t-2)$, for $t \geq 4$. We present below the graph for $t = 4$.

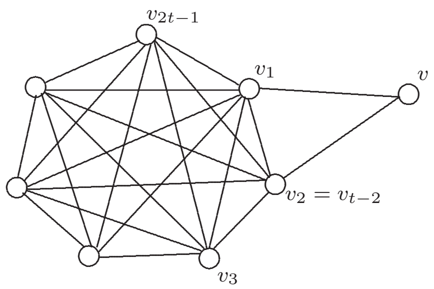

Figure 2: The graph G for $t = 4$

We now show that G has $\psi_a(G) = 1$. Assume there are at least two alliances in G, and let v belong to the alliance A. Since $\deg v \geq 2$, v must belong to the same alliance as v_i, for some i $(1 \leq i \leq t-2)$. Since for each i $(1 \leq i \leq t-2)$ we have that $\deg v_i = 2t-1$, it follows that $|A| \geq \lceil \frac{2t-1}{2} \rceil = t$. Suppose $|A| = t$. Since $\deg v \leq t-2$, it follows that there is at least one vertex $v_j \in A$ $(t-1 \leq j \leq 2t-1)$ such that $vv_j \notin E(G)$. Thus v_j has only $t-1$ allies (including itself), and t enemies, which is a contradiction. Thus A must include at least $t+1$ vertices. However then, $V(G) - A$ has at most

$t-1$ vertices with at most $t-1$ allies and at least t enemies, which is a contradiction. Thus G has only one alliance. $\square$

In the proof above, G has order at least 8. By inspection, one can find that there is no graph of order 2 or 4 with $\psi_a(G) = 1$. There are graphs of order 6 with $\psi_a(G) = 1$, such as K_5 with one edge subdivided by a new vertex.

Proposition 2.5 *Let G be a graph of order $n \geq 1$. Then all pairs (k, n) $(1 \leq k \leq n)$ except for $(1, 2)$ and $(1, 4)$ can be realized as the alliance partition number and order of some graph.*

Proof. If $k = 1$ and n is odd, let G be K_n. If $k = 1$ and n is even, $n \geq 6$, let G be the graph of Lemma 2.4. If $k = 2$ and n is even, let G be K_n. If $k = 2$ and n is odd, let G be K_{n-1} with a new vertex joined to exactly half of the $n-1$ vertices. We may assume $k \geq 3$. If k and n are of opposite parity let $G = K_{n-k} \cup P_3 \cup (k-3)K_1$, for $3 \leq k \leq n-1$. Then $\psi_a(K_{n-k}) = 1$, $\psi_a(P_3) = 2$, and $\psi_a((k-3)K_1) = k-3$. By Proposition 2.1, $\psi_a(G) = 1 + 2 + (k-3) = k$. If k and n are of the same parity let $G = K_{n-k+1} \cup (k-1)K_1$, for $1 \leq k \leq n$. Then $\psi_a(K_{n-k+1}) = 1$, and $\psi_a((k-1)K_1) = k-1$. By Proposition 2.1, $\psi_a(G) = 1 + (k-1) = k$. $\square$

3 Alliance Partition Number in Connected Graphs

We next present sharp bounds for the alliance partition number for connected graphs.

Theorem 3.1 *Let G be a connected graph of order $n \geq 3$. Then*

$$1 \leq \psi_a(G) \leq \left\lfloor n + \frac{3}{2} - \frac{\sqrt{1+4n}}{2} \right\rfloor.$$

The bounds are sharp.

Proof. Suppose G is a connected graph of order n and $\psi_a(G) = r$. Let $A_1, A_2, \ldots, A_r$ be a partition of $V(G)$ into r defensive alliances. Define the degree of each alliance $deg(A_i) = |\{uv \in E(G) | u \in A_i, v \notin A_i\}|$. Since G is connected, $\sum_{i=1}^{r} deg(A_i) = 2 \cdot |\{uv \in E(G) | u \in A_i, v \in A_j, \text{where } i \neq j\}| \geq 2(r-1)$. For arbitrary i, $|A_i| = t$ and $deg(A_i) = s$. Then some vertex $v \in A_i$ has at least $\lceil \frac{s}{t} \rceil$ enemies and at most t allies, counting itself. Hence, $t \geq \frac{s}{t}$, and so $t^2 \geq s$. Thus, for each i, $|A_i|^2 \geq deg(A_i)$, also $\sum_{i=1}^{r} |A_i|^2 \geq \sum_{i=1}^{r} deg(A_i) \geq 2r - 2$, with $n = \sum_{i=1}^{r} |A_i|$, and $|A_i| \geq 1$ for all i, $1 \leq i \leq r$.

For simplicity, let $a_i = |A_i|$ for $1 \leq i \leq r$. It can be shown that the minimum value for $n = a_1 + a_2 + \ldots + a_r$ under the conditions $a_i \geq 1$ for $1 \leq i \leq r$ and $a_1^2 + a_2^2 + \ldots + a_r^2 \geq 2r - 2$ occurs at $a_1 = a_2 = \ldots = a_{r-1} = 1$ and $a_r = \sqrt{2r - 2 - (r-1)} = \sqrt{r-1}$. Thus, $n \geq r - 1 + \sqrt{r-1}$, which simplifies to $r \leq n + \frac{3}{2} - \frac{\sqrt{1+4n}}{2}$.

To see that the bounds are sharp, consider $K_{2\ell+1}$ for the lower bound. For the upper bound, let G be the graph obtained from K_t by attaching t pendants to each vertex of K_t.

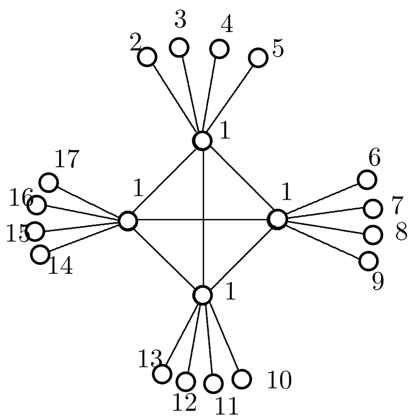

Figure 3: The graph G for $t = 4$

Then $n = |V(G)| = t(t+1)$ and $\psi_a(G) = t^2 + 1$ (where the alliances are indicated by the vertex labels), which verifies the bound. $\Box$

For the lower bound, we find relations between the size of G and the alliance partition number.

Proposition 3.2 *Let G be a graph with $2t + 1$ vertices and at least $\binom{2t+1}{2}$ $-(t-1)$ edges $(t \geq 1)$. Then $\psi_a(G) = 1$.*

Proof. Assume, to the contrary, that there is a graph G with $2t + 1$ vertices, at least $\binom{2t+1}{2}$ $-(t-1)$ edges, and $\psi_a(G) \geq 2$. Then $V(G)$ can be partitioned into i sets A_i $(i \geq 2)$, so that each A_i is an alliance.

Since the size of G is at least $\binom{2t+1}{2}$ $-(t-1)$, there is some vertex $v \in V(G)$ such that $\deg v = 2t$. Let v belong to the alliance A. Then $|A| \geq t + 1$. Then $V(G) - A$ is an alliance or a union of alliances, with $|V(G) - A| = s \leq t$.

Let $u \in V(G) - A$. Then u has at most $s \leq t$ allies (including itself), so u can have at most s enemies. Thus u is adjacent to at most s of the

$2t+1-s$ vertices in A. So there are at least $2t+1-2s \geq 1$ vertices in A not adjacent to u. Since this is true for each $u \in V(G)-A$, there are at least $s(2t+1-2s)$ potential edges not present for some s with $1 \leq s \leq t$.

We next claim that there are at least t potential edges not in G, i.e. $|E(G)| \leq \binom{2t+1}{2} - t$. The critical points of $f(s) = s(2t+1-2s)$ are $s = \frac{1}{4}(2t+1)$, and the endpoints $s=1$ and $s=t$. Now, for $t \geq 1$, the minimum value occurs when $s=t$. Thus $|E(G)| \leq \binom{2t+1}{2} - t$. This contradicts the fact that G has at least $\binom{2t+1}{2} - (t-1)$ edges. □

The next two upper bounds involve the minimum degree and the girth.

Theorem 3.3 *Let G be a graph with minimum degree δ. Then*

$$\psi_a(G) \leq \left\lfloor \frac{n}{\lceil \frac{\delta+1}{2} \rceil} \right\rfloor.$$

Proof. Since δ is the minimum degree in G, it follows that every vertex must have at least $\lceil \frac{\delta+1}{2} \rceil$ allies, including itself. Therefore, $\psi_a(G) \leq \left\lfloor \frac{n}{\lceil \frac{\delta+1}{2} \rceil} \right\rfloor$. To see the sharpness, consider C_n $(n \geq 3)$. □

Proposition 3.4 *Let G be a graph with girth $g \geq 3$ and minimum degree $\delta(G) \geq 4$. Then $\psi_a(G) \leq \lfloor \frac{n}{g} \rfloor$.*

Proof. Let A be a defensive alliance in G. Any vertex $v \in A$ has at least 4 neighbors and hence at least 2 neighbors in A. Thus, the subgraph induced by A has minimum degree at least 2, and so it cannot be a tree; it must contain a cycle. Since G has girth g, it follows that $|A| \geq g$. Since every alliance of G has order at least g, $\psi_a(G) \leq \lfloor \frac{n}{g} \rfloor$. In the case that $g=3$, one can observe that $C_n \times C_3$ gives sharpness of the bound. □

Notice that the result does not hold if $\delta(G) \leq 3$. For instance, if P is the Petersen graph, then $\delta(P) = 3$, $g(P) = 5$, and $\psi_a(P) = 5 > \lfloor \frac{10}{5} \rfloor$, contradiction.

4 Alliance Partition Number in Regular Graphs

As a consequence of Theorem 3.3, we have the following corollary.

Corollary 4.1 *If G is r-regular, then*

$$\psi_a(G) \leq \left\lfloor \frac{n}{\lceil \frac{r+1}{2} \rceil} \right\rfloor.$$

The bound is sharp for C_n. If $r = 2$, then C_n is the only 2-regular graph ($n \geq 3$) and $\psi_a(C) = \left\lfloor \frac{n}{2} \right\rfloor$. We now consider $r \geq 3$.

Theorem 4.2 *Let G be a connected 3-regular graph. If M is a maximum matching, then $\psi_a(G) = |M|$.*

Proof. Let G be a 3-regular graph with a maximum matching M of size $|M| = k$. Since the two end vertices of each edge of a maximum matching form an alliance, and each of the leftover vertices can merge with one of the neighboring alliances already formed, we have that $\psi_a(G) \geq |M|$. Assume that $\psi_a(G) > |M| = k$, with the alliance partition $\{A_1, A_2, \ldots, A_{\psi_a(G)}\}$. Then there are at least $k+1$ alliances. Since G is 3-regular, it follows that no vertex can form an alliance by itself. Thus $|A_i| \geq 2$, $\forall i$ and each graph $\langle A_i \rangle$ induced by A_i contains at least one edge, and so there is a matching of size $\psi_a(G) > k$, contradiction. The result is sharp for $C_n \times K_2$ ($n \geq 3$). □

We next consider the n-regular hypercube Q_n.

Observation For $n \geq 1$, $\psi_a(Q_n) \geq 2^{\lceil \frac{n}{2} \rceil}$.

To see this, represent the vertices of Q_n by bit strings of length n, so that two bit strings are adjacent if and only if they differ in exactly one bit. We can define $2^{\lceil \frac{n}{2} \rceil}$ vertex sets $A_1, A_2, \ldots, A_{2^{\lceil \frac{n}{2} \rceil}}$ as follows. Let A_i be the set of vertices for which the first $\left\lceil \frac{n}{2} \right\rceil$ bits are the integer $i-1$ written in binary. (Thus, A_1 is the set of bit strings that begin with $\left\lceil \frac{n}{2} \right\rceil$ zeros, A_2 is the set of bit strings that begin with $\left\lceil \frac{n}{2} \right\rceil - 1$ zeros followed by a 1, etc.) We claim that each A_i is an alliance. Let $v \in A_i$. Then v has $n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$ neighbors in A_i, each formed by changing one of the last $\left\lfloor \frac{n}{2} \right\rfloor$ bits in v. Since $\deg(v) = n$, v is defended.

Moreover, we conjecture the following.

Conjecture 4.3 *For any integer $n \geq 1$, $\psi_a(Q_n) = 2^{\lceil \frac{n}{2} \rceil}$.*

Theorem 4.4 *Let G be an r-regular graph with girth g, where $r \geq 3$ and $g \geq 5$. Then every alliance in G has at least $1 + (g-2) \cdot \left\lceil \frac{r-3}{2} \right\rceil$ vertices.*

Proof. The bound is trivial for $r = 3$, so we assume that $r \geq 4$. For any alliance A and any vertex $v \in A$, v must have at least two neighbors in A. Thus, the graph induced by A has minimum degree at least 2, and so it must contain a cycle of length at least g. Let $u_1, u_2, \ldots, u_{g-1}$ be $g-1$ consecutive vertices on this cycle. Each u_i has $r-2$ neighbors besides u_{i-1} and u_{i+1} and at least $\left\lceil \frac{r-5}{2} \right\rceil$ of them must be in A. The only pair of vertices in $u_1, u_2, \ldots, u_{g-1}$ which could have common neighbors (other than the common neighbors on the path $u_1, u_2, \ldots, u_{g-1}$) are u_1 and u_{g-1}. Thus, there are at least $(g-2)\left(\left\lceil\frac{r-5}{2}\right\rceil\right)$ vertices in A besides $u_1, u_2, \ldots, u_{g-1}$, for a total of at least $g-1+(g-2)\left(\left\lceil\frac{r-5}{2}\right\rceil\right)=(g-2)\left(\left\lceil\frac{r-3}{2}\right\rceil\right)+1$ vertices in A. $\square$

Corollary 4.5 *Let G be an r-regular graph of order n with girth g, where $r \geq 3$ and $g \geq 5$. Then*

$$\psi_a(G) \leq \frac{n}{1+(g-2)\cdot\left\lceil\frac{r-3}{2}\right\rceil}.$$

5 Alliance Partition Number in Trees

We next find sharp bounds for the alliance partition number for trees in terms of its diameter first.

Theorem 5.1 *Let T be a tree of order $n \geq 3$ and diameter $d \geq 2$. Then*

$$\psi_a(G) \geq \left\lceil\frac{d}{2}\right\rceil+1.$$

The bound is sharp.

Proof. Let the path $P_{d+1}: u_1, u_2, \ldots, u_{d+1}$ be a path of maximum length in T, where $diam(T)=d$. Note that the vertices u_1 and u_{d+1} form alliances by themselves. Also, since $\deg(v_i) \geq 2$, for all i $(2 \leq i \leq d)$, it follows that each of these vertices need at least one ally, so each alliance containing v_i has order at least 2. Thus we have at least $\lceil\frac{d-2}{2}\rceil+2$ alliances, namely $A_i=\left\{u_{2i}, u_{2i+1}: 1 \leq i \leq \left\lceil\frac{d-2}{2}\right\rceil\right\}, A=\{u_1\}, B=\{u_d\}$. Note that leftover vertices can be grouped with their parent vertex on the diameter. For the sharpness of the lower bound, consider the path P_{2k} for some positive integer k. $\square$

For the sharpness of the next bound, we will need a definition. Recall that for a connected graph G, the *Corona of G* is $Cor(G)$, obtained by adding an end vertex to each vertex of G. We then have the following. Then a graph G is a corona graph if each vertex of G is a leaf or a stem adjacent to exactly one leaf. We now find sharp bounds for the alliance partition number in a tree in terms of its order.

Theorem 5.2 *Let T be a tree of order $n \geq 3$. Then*

$$\psi_a(T) \leq \left\lfloor\frac{3n}{4}+\frac{1}{2}\right\rfloor.$$

The bound is sharp.

Proof. Suppose $V(T)$ is partitioned into a maximum number of alliances $A_1, A_2, \ldots, A_r$. Let $v \in A_i$, for some i $(1 \leq i \leq r)$. Then at least $\left\lceil \frac{1}{2}(\deg v - 1) \right\rceil$ of the neighbors of v must be in the same alliance A_i and hence $< A_i >$ contains at least $\left\lceil \frac{1}{2}(\deg v - 1) \right\rceil$ edges incident with v. Notice that the subgraph induced by each alliance is a tree, and so the number of edges in each alliance equals the number of vertices minus one, that is, $|A_i| - 1 \geq \left\lceil \frac{1}{2} \sum_{v \in A_i} \left\lceil \frac{1}{2}(\deg(v) - 1) \right\rceil \right\rceil \geq \left\lceil \frac{1}{4} \sum_{v \in A_i} (\deg v - 1) \right\rceil$. We will count one vertex from each alliance in order to count the number of alliances. There are n vertices total, and we subtract all except one vertex from each alliance. Hence, the total number of alliances is $r = n - \sum_{i=1}^{r}(|A_i| - 1) \leq n - \sum_{A_i} \left\lceil \frac{1}{4} \sum_{v \in A_i} (\deg v - 1) \right\rceil \leq n - \left\lceil \frac{1}{4} \sum_{v \in V(G)} (\deg v - 1) \right\rceil = \left\lfloor n - \frac{1}{4} \sum_{v \in V(G)} \deg v + \frac{n}{4} \right\rfloor = \left\lfloor \frac{5n}{4} - \frac{m}{2} \right\rfloor = \left\lfloor \frac{3n}{4} + \frac{1}{2} \right\rfloor$. For the sharpness, observe that $\psi_a(Cor(P_{2k})) = 3k = \frac{3n}{4} = \left\lfloor \frac{3n}{4} + \frac{1}{2} \right\rfloor$. □

Recall that a *binary tree* is a tree of maximum degree 3. For binary trees we present sharp lower bounds for the alliance partition number in terms of order and the size of a maximum matching. We let $\langle e \rangle$ be the subgraph induced by the edge e.

Proposition 5.3 *Let T be a binary tree with a maximum matching M. Then $\psi_a(T) \geq n - |M|$. The bound is sharp.*

Proof. Let $M = \{e_1, e_2, \ldots, e_{\beta'}\}$ be a maximum matching in T of size β'. Note that $G - \langle \{e_1, e_2, \ldots, e_{\beta'}\} \rangle$ consists of isolated vertices (otherwise M is not maximum). If some vertex in $G - \langle \{e_1, e_2, \ldots, e_{\beta'}\} \rangle$ is not an end-vertex in G, then there is a matching $M' = \{e'_1, e'_2, \ldots, e'_{\beta'}\}$ that overlaps part of M, such that $G - \langle \{e'_1, e'_2, \ldots, e'_{\beta'}\} \rangle$ is a union of isolated vertices which are all end-vertices in G.

Since $\Delta(T) = 3$, it follows that the size of any alliance is either 1 or 2, where each vertex needs at most one ally other than itself. Define an alliance partition of T, where alliance A_i consists of the two end-vertices of edge e_i (or e'_i if M' is used) $(1 \leq i \leq \beta')$, and each A_j consists of an isolated vertex $(1 \leq j \leq n - |M|)$. The result follows. To see the sharpness, consider the path $P_{2k+1} : v_1, v_2, \ldots, v_{2k+1}$ (for natural number k). Then the alliance partition $A_1 = \{v_1\}$ and $A_{i+1} = \{v_{2i}, v_{2i+1}\}$ $(1 \leq i \leq k)\}$ is a maximum one, giving the sharpness. □

As one can see, finding the alliance partition number in graphs is not trivial. Thus an open problem on the topic is to find the computational complexity of alliance partition number in graphs.

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