1995

# A Characterization of Graphs With Interval Two-Step Graphs

Lundgren, J. Richard; Merz, Sarah K.; Maybee, John S.; Rasmussen, Craig W.

Norrth-Holland

Linear Algebra and its Applications, v. 217, 1995, pp. 203-223
http://hdl.handle.net/10945/43174

# A Characterization of Graphs with Interval Two-Step Graphs

J. Richard Lundgren and Sarah K. Merz,
University of Colorado at Denver

John S. Maybee, University of Colorado at Boulder

Craig W. Rasmussen, Naval Postgraduate School

**Dedicated by the other authors to Professor John Maybee on the occasion of his 65th birthday.**

**Abstract.** One of the intriguing open problems on competition graphs is determining what digraphs have interval competition graphs. This problem originated in the work of Cohen [5, 6] on food webs. In this paper we consider this problem for the class of loopless symmetric digraphs. The competition graph of a symmetric digraph $D$ is the two-step graph of the underlying graph $H$ of $D$, denoted $S_2(H)$. The two-step graph is also known as the neighborhood graph, and has been studied recently by Brigham and Dutton [4] and Boland, Brigham and Dutton [1, 2]. This work was motivated by a paper of Raychaudhuri and Roberts [20] where they investigated symmetric digraphs with a loop at each vertex. Under these assumptions, the competition graph is the square of the underlying graph $H$ without loops. Here we will first consider forbidden subgraph characterizations of graphs with interval two-step graphs. Second, we will characterize a large class of graphs with interval two-step graphs using the Gilmore-Hoffman characterization of interval graphs.

**1. Introduction.** Let $G = (V, E)$ be a graph. The *two-step graph*, denoted $S_2(G)$, is a graph on the same vertex set as $G$ with an edge joining vertices $x$ and $y$ in $V$ if and only if there exists a vertex $z$ in $V$ such that $x, y \in N(z)$, the open neighborhood of $z$. The two-step graph is closely related to the *competition graph* of a digraph. Let $D = (V, A)$ be a digraph. Then the competition graph of $D$, denoted $C(D)$, is a graph on the same set of vertices with an edge between two distinct vertices $x$ and $y$ in $V$ if and only if there exists a vertex $z$ in $V$ such that there is an arc from $x$ to $z$ and from $y$ to $z$ in $A$. If $D$ is a symmetric digraph with underlying graph $H$, it is easily seen that the two-step graph of $H$ and the competition graph of $D$ are identical (see [13]). The problem of which digraphs have interval competition graphs originated in the work of Cohen [5, 6], on food webs. This problem has been studied for several special cases (see [11, 12, 22, 23]), but remains unsolved in general. Raychaudhuri and Roberts [20] were able to answer the following question: given a symmetric digraph $D$ with a loop at each vertex and underlying interval graph $H$, what conditions are necessary and sufficient for the competition graph of $D$ to be interval? Lundgren, Maybee, and Rasmussen [13] were able to solve this problem for loopless symmetric digraphs with underlying interval graph $H$. We will use ideas from [16] to characterize a large class of graphs which have interval two-step graphs.

First we will consider necessary conditions involving forbidden subgraphs. This will lead to a characterization related to the Gilmore-Hoffman characterization of interval graphs: a graph $G$ is interval if and only if the family of maximal cliques of $G$ can be ordered $C_1, C_2, \ldots, C_r$ such that if a vertex $v \in C_i$ and $v \in C_k$, then $v \in C_j$ for all $i \leq j \leq k$. Such an ordering is called a *consecutive ranking* (for a comprehensive introduction to interval graphs see Golumbic [9]). We will restrict our discussion to connected noncomplete graphs since disconnected graphs can be examined by connected component and the two-step graph of the complete graph $K_n$ is $K_n$.

**2. The Forbidden Subgraph Approach.** In earlier work, Lundgren and Rasmussen [17] take the forbidden subgraph approach to characterizing trees with an interval two-step graph. In general this approach does not work. For example, consider the forbidden subgraphs of an interval graph in Figure 1. Some of these graphs have an interval two-step graph, while others do not. Trees are one class of graphs for which a forbidden subgraph approach does work as illustrated by the following result of Lundgren and Rasmussen.

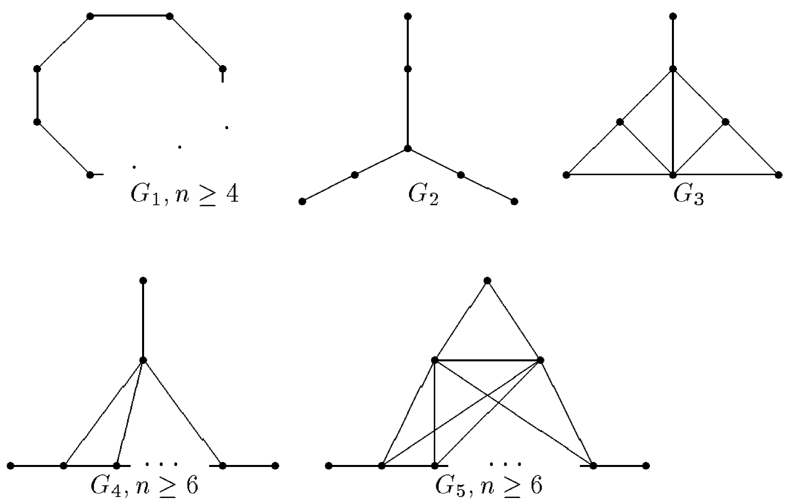


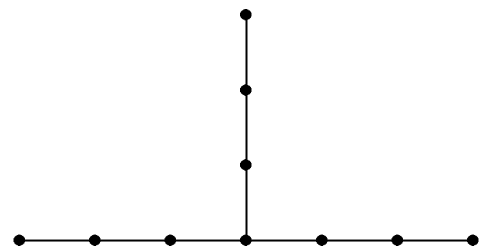

FIG. 1. *A graph is interval if and only if it contains no subgraph isomorphic to $G_1, G_2, G_3, G_4$, or $G_5$. Note the two-step graphs of $G_1 (n = 4, 6), G_2, G_3$ and $G_5$ are interval while the two-step graphs of the others are not.*

PROPOSITION 2.1. *[17] Let $T$ be a tree. Then $S_2(T)$ is interval if and only if $T$ does not contain an induced $H$, where $H$ is the graph of Figure 2.*

We provide some necessary conditions using forbidden subgraphs which establish the two-step graph as noninterval. The basic idea behind the following two theorems is that if the minimum length cycle in a graph is large enough, the two-step graph contains an induced cycle of length greater than 3.

THEOREM 2.2. *Let $G$ be a graph with girth 5. Then $S_2(G)$ is not interval.*

*Proof.* Let $C = x_1x_2x_3x_4x_5x_1$ be a cycle in $G$ of length five. Since $G$ has girth 5, $C$ is an induced subgraph of $G$. We claim $S_2(C)$ is an induced subgraph of $S_2(G)$.



$H$

FIG. 2.

*$S_2(T)$ is interval if and only if $T$ contains no subgraph isomorphic to $H$*

Suppose $S_2(C)$ is not an induced subgraph of $S_2(G)$. Then there are two vertices $x_i$ and $x_j$ in $C$ that are adjacent in $S_2(G)$ but are not in the open neighborhood of a vertex in $C$. Therefore $x_i$ and $x_j$ are adjacent in $C$. Since $x_i$ and $x_j$ are joined by a path of length two in $G$ but not in $C$, there exists a vertex $z$ in $G$ such that $x_i, x_j \in N(z)$. Then $x_i x_j z x_i$ is a cycle in $G$ of length less than 5, a contradiction. Thus $S_2(C)$ is an induced subgraph of $S_2(G)$. It is easy to check that the two-step graph of a five cycle is also a five cycle; thus $S_2(G)$ contains an induced subgraph isomorphic to a cycle of length five which implies $S_2(G)$ is not chordal and therefore not interval, completing the proof. □

Observe that such an approach will not work for graphs with girth three, four, or six, as the two-step graph of each of these graphs is a triangle, two paths of length one, and two triangles respectively. We can eliminate graphs of all other girth.

THEOREM 2.3. *Let $G$ be a graph with girth $p \geq 7$. Then $S_2(G)$ is not interval.*

*Proof.* Let $C = x_1 x_2 \ldots x_p x_1$ be a cycle in $G$ of length $p$. Since $G$ has girth $p$, $C$ is an induced subgraph of $G$. Suppose $S_2(C)$ is not an induced subgraph of $S_2(G)$. Then there are two vertices $x_i$ and $x_j$ in $C$ that are adjacent in $S_2(G)$ but are not in the open neighborhood of a vertex in $C$. Therefore $x_i$ and $x_j$ are more than distance two apart on the cycle or they are adjacent. Since $x_i$ and $x_j$ are joined by a path of length two in $G$ but not in $C$, there exists a vertex $z$ in $G$ such that $x_i, x_j \in N(z)$. If $x_i$ and $x_j$ are adjacent then $x_i x_j z x_i$ is a cycle of length less than $p$, a contradiction. Otherwise, $x_1 x_2 x_3 \ldots x_i z x_j \ldots x_p, x_1$ is a cycle in $G$ of length less than $p$, a contradiction. Thus $S_2(C)$ is an induced subgraph of $S_2(G)$. If $p$ is odd, it is easy to check that $S_2(C)$ is a cycle of length $p$. Thus $S_2(G)$ contains an induced subgraph isomorphic to a cycle of length $p \geq 7$, i.e., $S_2(G)$ is not interval. If $p$ is even, it is easy to check that $S_2(C)$ is a graph isomorphic to two cycles of length $p/2$. Thus $S_2(G)$ contains an induced subgraph isomorphic to a cycle of length $q = p/2 \geq 4$, i.e., $S_2(G)$ is not interval, completing the proof. □

In the sections that follow, we will draw an important connection between open and/or closed neighborhoods and the maximal cliques in the two-step graph. One consequence of this approach is a result involving open neighborhoods in graphs of girth at least 7.

**3. Using Open and Closed Neighborhoods to Find Maximal Cliques.** We begin with a relatively simple class of graphs: trees. Though a characterization of trees with an interval two-step graph has already been provided, we consider that searching a graph for a forbidden subgraph is not necessarily an easy task. If we can find the maximal cliques of the two-step graph in the original graph easily, we can then use known linear-time algorithms to test for a consecutive ranking. We will disregard maximal cliques in the two-step graph of magnitude 1, since these maximal cliques can be arbitrarily added at either the beginning or end of a consecutive ranking, should one exist. Recall a pendant vertex is a vertex with precisely one neighbor.

THEOREM 3.1. *Let $T$ be a tree. Then the maximal cliques in $S_2(T)$ of magnitude at least 2 correspond to the open neighborhoods of the nonpendant vertices in $T$.*

*Proof.* Let $S = N(v)$, where $v$ is a nonpendant vertex in $T$. Clearly $N(v)$ is a clique in $S_2(T)$. Suppose it is not maximal. Then there exists a vertex $w \notin S$ that is joined to every vertex in $S$ by a path of length two. Since $|S| \geq 2$, there exist distinct vertices $x$ and $y$ in $N(v)$. Since $T$ is a tree, $x$ and $y$ are not adjacent. Then there exist vertices $t$ and $u$ such that $x, w \in N(t)$ and $y, w \in N(u)$. If $t = u$, $vxuyv$ is a cycle in $T$, a contradiction. Therefore $t \neq u$. Then $vxtwuyv$ forms a cycle in $T$, a contradiction. Thus no such $w$ can exist; therefore $N(v) = S$ is a maximal clique in $S_2(T)$. Furthermore, if $N(v) = N(z)$ for two vertices $v$ and $z$, then there exist $x$ and $y \in N(v) \cap N(z)$ and $vxzyv$ is a cycle, a contradiction.

Let $S$ be a maximal clique in $S_2(T)$. Then $|S| \geq 2$, so there exist distinct $x$ and $y$ in $S$. Since $S$ is a maximal clique in $S_2(T)$, there exists a vertex $z$ such that $x, y \in N(z)$. Suppose $S \neq N(z)$. Then there exists a vertex $w \in S$ such that $w \notin N(z)$. Since $T$ is a tree, $x$ and $y$ are not adjacent. Since $S$ is a maximal clique in $S_2(T)$ there exist vertices $t$ and $u$ such that $w, x \in N(t)$ and $w, y \in N(u)$. If $t = u$, $txzyt$ is a cycle in $T$, a contradiction. Therefore $t \neq u$. Then $wtxzyuw$ is a cycle in $T$, a contradiction. Thus no such $w$ can exist, i.e., $N(z) = S$, completing the proof. □

Using the Gilmore-Hoffman characterization of interval graphs we obtain the following corollary.

COROLLARY 3.2. *Let $T$ be a tree. Then $S_2(T)$ is interval if and only if the maximal open neighborhoods of the nonpendant vertices in $T$ have a consecutive ranking.*

We would like to take this characterization further to triangle-free graphs. Again the 6-cycle poses a problem. This is captured in the following lemma, the proof of which is easily observed.

LEMMA 3.3. *Let $G$ be a graph and let $x, y$ and $z$ be vertices contained in a maximal clique in $S_2(G)$. If there does not exist $v$ such that $x, y, z \in N[v]$ then there must exist distinct $a, b$ and $c$ such that $x, y \in N(a)$, $y, z \in N(b)$ and $x, z \in N(c)$, i.e., $xaybzcx$ is a 6-cycle.*

So in order to find classes of graphs in which the maximal cliques of the two-step graph correspond to open or closed neighborhoods in the original graph, we must exclude graphs containing 6-cycles.

THEOREM 3.4. *Let $G = (V, E)$ be a connected, noncomplete triangle- and 6-cycle-free graph. Then $C \subseteq V$ such that $|C| \geq 2$ is a maximal clique in $S_2(G)$ if and only if $C = N(z)$ for some $z$ in $G$ such that the open neighborhood of $z$ is not properly contained in the open neighborhood of any other vertex.*

*Proof.* ($\Rightarrow$) Let $C$ be a maximal clique in $S_2(G)$. If $|C| = 2$ the statement is clearly true so assume $|C| \geq 3$. Let $R \subseteq C$. We prove by induction on $|R|$ that there exists $z$ such that $R \subseteq N[z]$ in $G$. By Lemma 3.3 if $|R| = 3$ the claim is true so assume $|R| \geq 4$. Assume the claim is true for all $R$ such that $|R| < k \leq |C|$ and consider the case $|R| = k \leq |C|$. Pick arbitrary $x \in R$. Let $R' = R - \{x\}$. By induction hypothesis there exists $z_1$ such that $R' \subseteq N[z_1]$ in $G$. Pick arbitrary $y \neq x \in R$. Let $R'' = R - \{y\}$. By induction hypothesis there exists $z_2$ such that $R'' \subseteq N[z_2]$. Since $x$ and $y$ are in $R$ there exists $z$ such that $x, y \in N(z)$. If $z$ is $z_1$ or $z_2$ we are done so assume not. Observe $z_1, z_2 \notin R$ since $G$ is triangle-free (for example, if $z_1 \in R$ then $y$ and $z_1$ are adjacent and joined by a path of length two). Since $|R| \geq 4$ there exists $w \in R$ ($w \neq z, w \neq x, w \neq y, w \neq z_1, w \neq z_2$) such that $w$ is adjacent to $z_1$ and $z_2$. Then $xzyz_1wz_2x$ is a 6-cycle in $G$, a contradiction. Therefore without loss of generality we conclude $z = z_1$, i.e., $R \subseteq N[z_1]$ for all $R \subseteq C$. In particular $C \subseteq N[z_1]$ and so by maximality of $C$ we conclude $C = N(z_1)$.

($\Leftarrow$) Let $z$ be a vertex in $G$ such that the open neighborhood of $z$ is not properly contained in the open neighborhood of any other vertex in $G$. Clearly $N(z)$ is a clique in $S_2(G)$. Suppose it is not maximal. Then there is a vertex $w \notin N(z)$ such that $w$ is joined by a path of length two to every vertex in $N(z)$ in $G$. Let $x \in N(z)$. Since $w$ and $x$ are joined by a path of length two in $G$ there exists $a$ such that $x, w \in N(a)$. But $N(z)$ is not properly contained in $N(a)$ so there exists $y \in N(z)$ such that $a$ and $y$ are not adjacent. Then $w$ and $y$ are joined by a path of length two implies there exists a distinct vertex $b$ such that $y, w \in N(b)$. Since $G$ is triangle-free, $b \neq x$. Then $zxawbyz$ is a 6-cycle in $G$, a contradiction. Thus $N(z)$ forms a maximal clique in $S_2(G)$, completing the proof. □

If an open neighborhood has the property that it is not properly contained in the open neighborhood of any other vertex, we say it is *maximal*. This result does not state that there is a one-to-one correspondence between the maximal cliques in $S_2(G)$ and the maximal closed neighborhoods of $G$. For example, consider the graph in Figure 3. In this graph, $N(v_1) = N(v_2)$. Since the existence of a consecutive ranking of a family of sets is not affected by allowing a set in the family to appear more than once, we use the Gilmore-Hoffman characterization of interval graphs to conclude the following.

COROLLARY 3.5. *Let $G$ be a connected, noncomplete, triangle- and 6-cycle-free graph. Then $S_2(G)$ is interval if and only if the maximal open neighborhoods of $G$ have a consecutive ranking.*

Theorem 2.3 and Corollary 3.5 then prove:

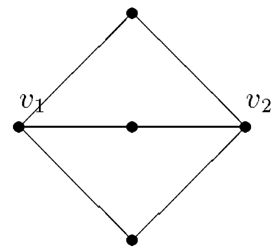



FIG. 3.

*The maximal cliques in the two-step of this graph do not correspond one-to-one with the maximal open neighborhoods of the original graph.*

COROLLARY 3.6. *Let $G$ be a graph with girth $p \geq 7$. Then the maximal open neighborhoods of $G$ do not have a consecutive ranking.*

Now consider 6-cycle-free graphs such that every edge is contained in a triangle.

THEOREM 3.7. *Let $G = (V, E)$ be a connected, noncomplete, 6-cycle-free graph such that every edge is contained in a triangle. Then $C \subseteq V$ such that $|C| \geq 2$ is a maximal clique in $S_2(G)$ if and only if $C = N[z]$ for some $z$ in $G$ such that the closed neighborhood of $z$ is not properly contained in the closed neighborhood of any other vertex.*

*Proof.* ($\Rightarrow$) Let $C$ be a maximal clique in $S_2(G)$. By an analogous argument to that in Theorem 3.4 we can show that there exists $z$ such that $C \subseteq N[z]$. Since every edge is contained in a triangle and $C$ is maximal we conclude $C = N[z]$.

($\Leftarrow$) Let $z$ be a vertex in $G$ such that $N[z]$ is not properly contained in another closed neighborhood in $G$. Since every edge of $G$ is contained in a triangle, clearly $N[z]$ forms a clique in $S_2(G)$. Suppose it is not maximal. Then there exists $w$ such that $w$ is joined to every vertex in $N[z]$ by a path of length two but $w$ and $z$ are not adjacent. Since $w$ and $z$ are joined by a path of length two there exists a vertex $v$ such that $w, z \in N(v)$. Since $N[z]$ is not properly contained in $N[v]$ there exists $y \in N[z]$ such that $y \notin N[v]$. Then $w$ and $y$ are joined by a path of length two so there exists $u$ such that $w, y \in N(u), (u \neq v)$. Then $v, z \in N(u)$ since otherwise the edge $(v, z)$ is contained in a triangle implies there exists a vertex $t$ such that $v, z \in N(t)$ and $ztvwuyz$ is a 6-cycle. But then $N[z]$ is not properly contained in $N[u]$ so there exists $x \in N[z]$ such that $x \notin N[u]$. If $x \notin N(v)$ we are done since $x$ and $w$ are joined by a path of length two implies there exists $s$ (possibly $y$) such that $w, x \in N(s)$ and then $wvuzxsw$ forms a 6-cycle in $G$. Thus $x$ and $v$ are adjacent. Then $wvxzyuw$ forms a 6-cycle in $G$. This contradiction proves no such $w$ can exist, completing the proof. □

COROLLARY 3.8. *Let $G$ be a connected, noncomplete, 6-cycle-free graph such that every edge is contained in a triangle. Then $S_2(G)$ is interval if and only if the maximal closed neighborhoods of $G$ have a consecutive ranking.*

To generalize these results we need some definitions.

**4. The Competition Cover Approach.** We begin with the following definition from Lundgren, Maybee, and Rasmussen [16]. Let $G$ be a graph. A family $S =$

$\{S_1, \ldots, S_r\}$ of sets of vertices of $G$ is called a *competition cover* of $G$ if the following conditions are satisfied:

1. $i, j \in S_m$ implies there exists a vertex $k$ such that $i, j \in N(k)$.
2. if $i, j \in N(k)$ for some $k$, then $i, j \in S_m$ for some $m$.

This definition leads to the following result.

PROPOSITION 4.1. [16] *Let $G$ be a graph. Then $S_2(G)$ is interval if and only if $G$ has a competition cover $S$ which has a consecutive ranking.*

The difficulty with this result is finding the right competition cover. Furthermore, it is very difficult to use this characterization to prove that the two-step graph of a given graph is not interval. This leads to the following question: can we define a specific family of sets in $G$ that determines whether or not $S_2(G)$ is interval? We have already shown this family of sets is the open neighborhoods for trees and triangle- and 6-cycle-free graphs and the closed neighborhoods for 6-cycle-free graphs such that every edge is contained in a triangle. Using the competition cover approach, this problem was solved for interval graphs in [16]. The family of sets is found through categorizing the *nonsimplicial vertices* of $G$ (recall a simplicial vertex is a vertex whose neighborhood is a clique). Let $v_i$ be a nonsimplicial vertex in $G$. We say $v_i$ is Type I if every maximal clique containing $v_i$ contains three or more vertices. We say $v_i$ is Type II if every maximal clique containing $v_i$ contains exactly two vertices. Otherwise we say $v_i$ is Type III.

Let $G$ be a noncomplete connected graph with nonsimplicial vertices $\{v_1, \ldots, v_r\}$. Define $S(G) = \{S_1, \ldots, S_r\}$, where $S_i$ is

1. $N[v_i]$, the closed neighborhood of $v_i$, if $v_i$ is Type I.
2. $N(v_i)$, the open neighborhood of $v_i$, if $v_i$ is Type II.
3. actually two sets $S_{i_1}$ and $S_{i_2}$ otherwise, where

$$S_{i_1} = \mathcal{C}_{v_i} = \bigcup\{C | C \in \mathcal{C}, v_i \in C, |C| \geq 3\} \quad \text{and} \quad S_{i_2} = N(v_i),$$

   where $\mathcal{C}$ is the family of maximal cliques in $G$.

Define $S'(G)$ as the set of all sets in $S(G)$ such that no set is properly contained in any other. We note the following previous result.

PROPOSITION 4.2. [16] *Let $G$ be a connected noncomplete interval graph. $S'(G)$ is a competition cover of $G$.*

For this reason $S'(G)$ is called the *maximal nonsimplicial competition cover* of $G$. $S'(G)$ is particularly useful in characterizing interval graphs with interval two-step graphs as was proved in the following result.

PROPOSITION 4.3. [16] *Let $G$ be a connected noncomplete interval graph. $S_2(G)$ is interval if and only if $S'(G)$ has a consective ranking.*

Observe that Proposition 4.2 does not say anything about the maximal cliques in $S_2(G)$. A competition cover of a graph does not necessarily correspond precisely to the maximal cliques in the two-step graph. For example, the open neighborhoods of a

6-cycle form a competition cover, but the two-step graph of a 6-cycle is two triangles. Figure 4 gives another example in which this is not the case. We now ask the following question: when does the competition cover $S'(G)$ correspond to the maximal cliques in $S_2(G)$?

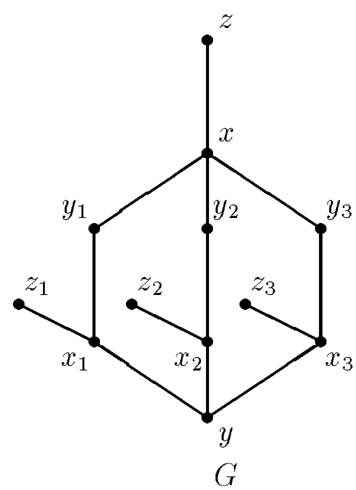



FIG. 4.

*Observe that $\{x, x_1, x_2, x_3\}$ forms a maximal clique in $S_2(G)$ but this set is not a member of $S'(G)$.*

Though Proposition 4.3 already characterizes interval graphs with interval two-step graphs, we consider whether or not for an interval graph $G$, $S'(G)$ corresponds to the maximal cliques in $S_2(G)$. The following result of Lundgren, Maybee, and Rasmussen proves the first half of the next theorem.

PROPOSITION 4.4. [16] *Let $G$ be a connected, noncomplete, interval graph. Let $S'(G) = \{S_1, \dots, S_m\}$ be the maximal nonsimplicial competition cover of $G$. Let $x \in V(G)$. If $x$ is connected by a path of length two to every vertex in some $S_i \in S'(G)$, then $x \in S_i$.*

THEOREM 4.5. *Let $G = (V, E)$ be a connected, noncomplete, interval graph. Then $C \subseteq V$ is a maximal clique in $S_2(G)$ if and only if $C \in S'(G)$.*

*Proof.* ($\Leftarrow$) Let $C \in S'(G)$. Clearly $C$ is a clique in $S_2(G)$. Suppose it is not maximal. Then there exists a vertex $w \notin C$ such that $w$ is joined to every vertex in $C$ by a path of length two. But Proposition 4.4 implies $w$ must be an element of $C$. This contradiction proves $C$ must be a maximal clique in $S_2(G)$.

($\Rightarrow$) Let $C$ be a maximal clique in $S_2(G)$. Since $G$ is interval the maximal cliques of $G$ have a consecutive ranking $\{C_1, \dots, C_l\}$. We claim there exists a nonsimplicial vertex $z$ such that $C \subseteq N[z]$. First we will show there exists a vertex $z$ such that $C \subseteq N[z]$. Suppose not. Let $i$ be the smallest integer such that there exists a vertex $x$ that is an element of both $C_i$ and $C$, but $x \notin C_{i+1}$. This must occur since $C \nsubseteq N[x]$. Let $j$ be the largest integer such that there exists a vertex $y$ that is an element of both $C_j$ and $C$, but $y \notin C_{j-1}$. This must occur since $C \nsubseteq N[y]$. Note $i$ must be less than $j$, for if not then $C \subseteq C_k$ for all $j \leq k \leq i$. Since $x$ and $y$ are joined by a path of length two, there exists a vertex $z$ such that $x$ and $z$ are contained in a maximal clique and $y$ and $z$ are contained in a maximal clique. Since this ranking is consecutive and $x \notin C_{i+1}$, $z$ must be in a clique $C_k$ such that $k \leq i$. Since $y \notin C_{j-1}$, $z$ must be in a clique $C_m$ such that $m \geq j$. This ranking of cliques is consecutive, therefore $z \in C_p$ for all $p$, $i \leq p \leq j$. Note every vertex of $C$ is contained in a clique $C_p$ such that $i \leq p \leq j$. Thus $C \subseteq N[z]$, a contradiction. Thus there must exist a vertex $z$ such that $C \subseteq N[z]$.

We now return to the proof of our claim. If $z$ is simplicial, then $C$ is a clique in $G$. Since $G$ is connected and not complete, there exists a vertex $x \notin C$ such that $x$ is adjacent to a vertex $y \in C$. If $y$ is nonsimplicial we are done since $C \subseteq N[y]$, so assume $y$ is simplicial. Then $\{x\} \cup C$ is a clique in $S_2(G)$ containing $C$, a contradiction. Therefore $z$ is nonsimplicial, completing the proof of our claim.

If $z \in C$, since $C$ is a maximal clique and $z$ is joined to every vertex in $C$ by a path of length two, it follows that $C = \mathcal{C}_z$. If $z \notin C$, since $C$ is a maximal clique it follows that $C = N(z)$. In either case, $C \in S'(G)$. □

Proposition 4.3 is then an immediate corollary.

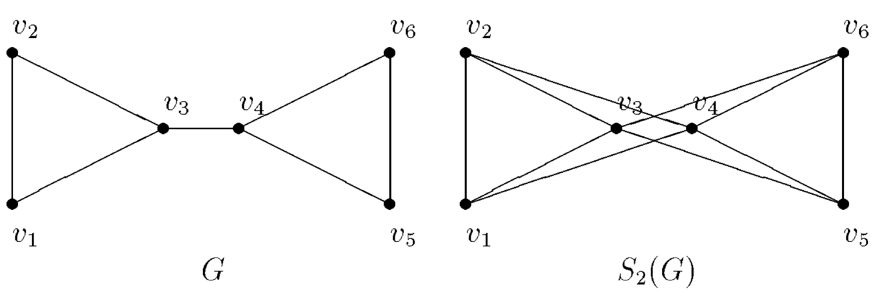



FIG. 5.
*An interval graph with a noninterval two-step graph.*

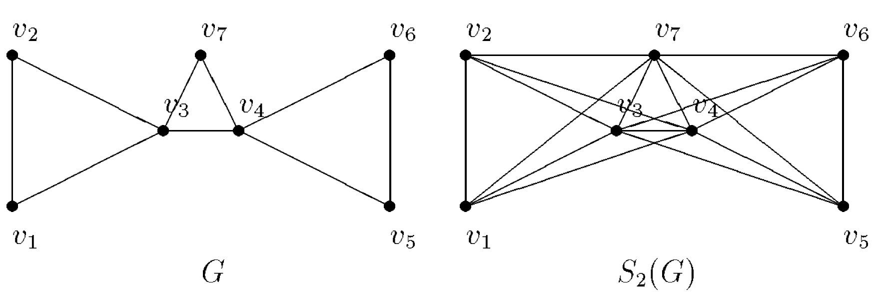



FIG. 6.
*An interval graph with an interval two-step graph.*

Observe that Theorem 4.5 characterizes some graphs which have an interval two-step graph and some which do not. For example, the graph in Figure 5 is interval while

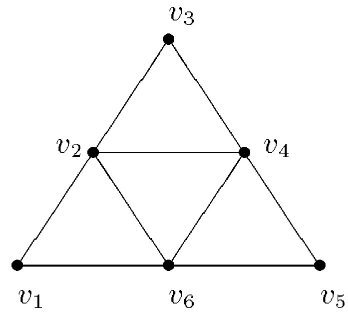



FIG. 7.
*$S'(G) = \{\{v_1, v_2, v_3, v_4, v_6\}, \{v_2, v_3, v_4, v_5, v_6\}, \{v_1, v_2, v_4, v_5, v_6\}\}$ does not have a consecutive ranking although $S_2(G) = K_6$ is interval.*

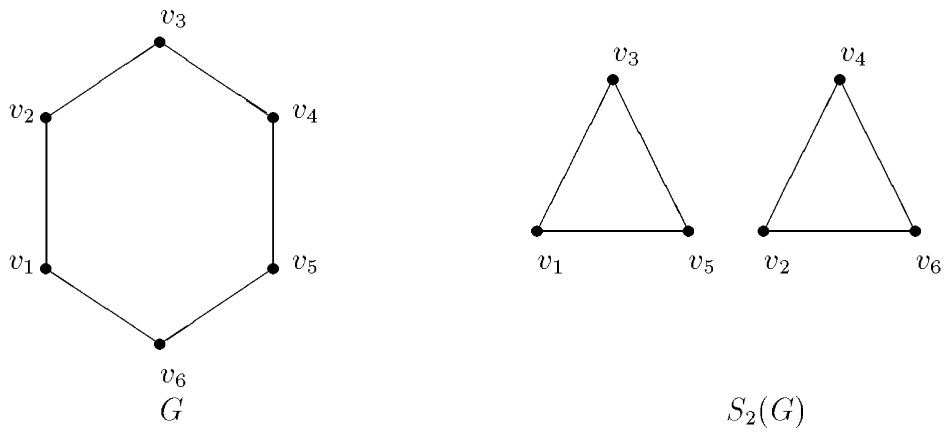



FIG. 8.
*$S'(G) = \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_4, v_6\}, \{v_1, v_5\}, \{v_2, v_6\}\}$ does not have a consecutive ranking although $S_2(G)$ is interval.*

its two-step graph is not. The graph in Figure 6 is just one example of an interval graph with an interval two-step graph. The graphs shown in Figures 7 and 8 are useful examples demonstrating that Proposition 4.3 does not necessarily hold if $G$ is not interval or connected. In both cases the sets of $S'(G)$ do not have a consecutive ranking, while $S_2(G)$ is interval. Both examples also contain 6-cycles.

**5. A Characterization for 6-cycle-free Graphs.** We now show that by considering the maximal nonsimplicial competition cover we can characterize a large class of graphs with interval two-step graphs. 6-cycles must be forbidden.

THEOREM 5.1. *Let $G = (V, E)$ be a connected, noncomplete, 6-cycle-free graph. Then $C \subseteq V$ such that $|C| \geq 2$ is a maximal clique in $S_2(G)$ if and only if $C \in S'(G)$.*

*Proof.* ($\Rightarrow$) Let $C$ be a maximal clique in $S_2(G)$. By an induction argument similar to that used in the proof of Theorem 3.4, we can show there exists a nonsimplicial vertex $z$ such that $C \subseteq N[z]$. First we show there exists $z$ such that $C \subseteq N[z]$. Clearly this is true if $|C| = 2$. By Lemma 3.3 it is true if $|C| = 3$, so assume $|C| \geq 4$. The induction is on $|R|$ where $R \subseteq C$. Assume there exists $z$ such that $R \subseteq N[z]$ for $R$ such that $|R| < k \leq |C|$ and assume $|R| = k \leq |C|$. Pick $x \in R$. Let $R' = R - \{x\}$. By induction hypothesis there exists $z_1$ such that $R' \subseteq N[z_1]$. Pick $y \in R$ such that $y \neq x$. Let $R'' = R - \{y\}$. By induction hypothesis there exists $z_2$ such that $R'' \subseteq N[z_2]$. If $z_1 = z_2$ we are done so assume not. Then $x$ and $y$ are joined by a path of length two implies there exists $z$ such that $x, y \in N(z)$. If there exists $w \in R$ such that $w \neq x, y, z_1, z_2, z$ we are done since $w \in N[z_1]$ and $w \in N[z_2]$ implies $xzyz_1wz_2x$ is a 6-cycle, so assume not. Then $R \subseteq \{x, y, z_1, z_2, z\}$. Since $|R| \geq 4$, at least one of the set $\{z_1, z_2\}$ is in $R$. Without loss of generality assume $z_1 \in R$. Then $z_1 \in N[z_2]$. If $z_1 \notin N[z]$ we are done since $z_1$ and $y$ are joined by a path of length two implies there exists $w$ such that $z_1, y \in N[w]$ implies $zwyz_1z_2xz$ is a 6-cycle. Therefore assume $z_1$ and $z$ are adjacent. If $z_2 \notin R$ we are done since $R \subseteq N[z]$ so assume $z_2 \in R$. Similarly, if $z_2 \in N[z]$ we are done so assume not. Then $x$ and $z_2$ are joined by a path of length two implies there exists $w$ such that $x, z_2 \in N[w]$ implies $xwz_2z_1yzx$ is a 6-cycle. Therefore there exists $z$ such that $R \subseteq N[z]$, completing the proof of our claim.

If $z$ is simplicial then $C$ is a clique in $G$. Since $G$ is connected and not complete, there exists a vertex $x \notin C$ such that $x$ is adjacent to a vertex $y \in C$. If $y$ is nonsimplicial we are done since $C \subseteq N[y]$ so assume $y$ is simplicial. Then $\{x\} \cup C$ is a clique in $S_2(G)$ containing $C$, a contradiction. Therefore there exists nonsimplicial $z$ such that $C \subseteq N[z]$. If $z \in C$, since $C$ is a maximal clique in $S_2(G)$ and $z$ is joined to every vertex in $C$ by a path of length two, $C = \mathcal{C}_z$. If $z \notin C$, since $C$ is a maximal clique in $S_2(G)$, $C = N(z)$.

($\Leftarrow$) Let $C \in S'(G)$. By definition there exists a nonsimplicial vertex $z$ such that $C \subseteq N[z]$. We then have two cases. <u>Case 1:</u> There exists nonsimplicial $z$ such that $C = \mathcal{C}_z$. Observe that $z$ may be either Type I or Type III. Clearly $C$ is a clique in $S_2(G)$. Suppose it is not maximal. Then there exists $w \notin C$ such that $w$ is joined to every $s \in C$ by a path of length two. So there exists $x$ such that $z, w \in N(x)$. Observe that $w \notin C$, $C = \mathcal{C}_z$, and $w$ and $x$ adjacent implies $w$ and $z$ are not adjacent. Since $C \not\subseteq N[x]$, there exists a vertex $y \in C$ such that $x$ and $y$ are not adjacent. Then $y$ and $w$ are joined by a path of length two so there exists a vertex $u$ such that $w, y \in N(u), (u \neq z, u \neq x)$. If $z \notin N(u)$ we are done since $y$ and $z$ are contained in a triangle implies there exists a vertex $t (t \neq x, t \neq w)$ such that $y, z \in N(t)$. Then $ztyuwxz$ is a 6-cycle, a contradiction, so $z \in N(u)$. Then $u \in C$. Suppose $x \in C$. If $x \notin N(u)$, we are done since $x$ and $z$ contained in a triangle implies there exists $t (t \neq u, t \neq w, t \neq y)$ such that $x, z \in N(t)$. Then $ztxwuyz$ is a 6-cycle. Therefore $x \in C$ implies $x \in N(u)$. But $C \not\subseteq N[u]$ so there exists $v \in C$ such that $u$ and $v$ are not adjacent. If $v$ and $y$ are adjacent we are done since $zvyuwxz$ is a 6-cycle. So $v$ and $y$ are not adjacent. Then there exists $s$ such that $w, v \in N(s)$ where $s$ is possibly $x$ but $s \neq u, s \neq y$ and $s \neq z$. Then $zvswuyz$ is a 6-cycle. Therefore $x \notin C$. This implies $x$ and $u$ are not adjacent.

But $C \not\subseteq N[u]$ so there exists a vertex $v \in C$ such that $u$ and $v$ are not adjacent. If $v$ and $y$ are adjacent we are done since $zvyuwxz$ is a 6-cycle, so assume not. Then there exists a vertex $s$ (possibly $x$, but $s \neq y, s \neq u$) such that $w, v \in N(s)$. Then $zvswuyz$ is a 6-cycle, a contradiction. Therefore $C$ is a maximal clique in $S_2(G)$.

Case 2: There exists nonsimplicial $z$ such that $C = N(z)$. Observe that $z$ may be Type II or Type III. Clearly $C$ is a clique in $S_2(G)$. Suppose it is not maximal. Then there exists a vertex $w \notin C$ joined to every $s \in C$ by a path of length two. Let $x \in C$. Then there exists $y$ such that $x, w \in N(y)$. Since $C \not\subseteq N(y)$ there exists $v \in N(z)$ such that $y$ and $v$ are not adjacent. Then $w$ and $v$ must be adjacent to $x$ since otherwise there exists a distinct vertex $t$ such that $w, v \in N(t)$ and $wtvzxyw$ is a 6-cycle. If $z$ is Type II we are done because $z$ is contained in a triangle, namely $vxzv$, a contradiction. So assume $z$ is Type III. Then $C \not\subseteq N[x]$ implies there exists $u \in C$ such that $x$ and $u$ are not adjacent. If $u$ and $y$ are adjacent we are done since $wxvzuyw$ is a 6-cycle. So assume $u$ and $y$ are not adjacent. Then $w$ and $u$ must be adjacent to $v$ since otherwise there exists a distinct vertex $t$ such that $u$ and $w$ are adjacent to $t$ and $utwyxzu$ is a 6-cycle. Then $wvuzxyw$ is a 6-cycle, a contradiction. Therefore $C$ is a maximal clique in $S_2(G)$. □

COROLLARY 5.2. *Let $G$ be a connected, noncomplete 6-cycle-free graph. Then $S_2(G)$ is interval if and only if the maximal nonsimplicial competition cover of $G$ has a consecutive ranking.*

**6. Graphs with Sparse 6-cycles.** Since the maximal cliques in the two-step graph of a 6-cycle are easily found, it may be possible to find the maximal cliques of the two-step graph in the original graph if we require that the 6-cycles be sparsely arranged. First, a definition. Let $H = abcdefa$ denote a 6-cycle. We then say *the alternating triples* of $H$ are $\{a, c, e\}$ and $\{b, d, f\}$. Figure 8 illustrates that the family of maximal cliques in the two-step graph of a 6-cycle is precisely the set of alternating triples. We can apply this idea to the following large class of graphs. The graph in Figure 4 illustrates the difficulty when 6-cycles overlap by more than a single edge: the set $\{x, x_1, x_2, x_3\}$ forms a maximal clique in $S_2(G)$, but is neither a set in $S'(G)$ nor an alternating triple.

THEOREM 6.1. *Let $G$ be a connected, noncomplete, triangle-free graph such that no two 6-cycles in $G$ have more than a single edge in common. Let $C$ such that $|C| \geq 2$ be a maximal clique in $S_2(G)$. Then either $C = N(z)$ for some nonsimplicial vertex $z$ in $G$ or $C$ is an alternating triple from a 6-cycle in $G$.*

*Proof.* If $|C| = 2$ clearly $C$ must be the open neighborhood of a nonsimplicial vertex with precisely two neighbors so the statement is true. If $|C| = 3$ by Lemma 3.3 and maximality of $C$, we observe the statement is true. So assume $|C| \geq 4$. Let $R$ denote a subset of $C$. We will prove by induction on $|R|$ that there exists a vertex $z$ such that $C \subseteq N[z]$.

Let $|R| = 4$. Pick arbitrary $x \in R$. Let $R' = R - \{x\}$. Then by Lemma 3.3 there exists $y$ such that $R' \subseteq N[y]$ or $R'$ is the set of alternating triples from a 6-cycle in $G$. Assume there exists $y$ such that $R' \subseteq N[y]$. Since $G$ is triangle-free, $y \notin C$ (otherwise $y$ would be joined by a path of length two to one of its neighbors) and hence $y \notin R'$. If

$x \in N[y]$ we are done so assume not. Further assume there does not exist $z$ such that $R \subseteq N[z]$. Since $|R| = 4$ there exists a vertex $a \in R'$. Then there exists $t$ such that $x, a \in N(t)$. Since there does not exist $z$ such that $R \subseteq N[z]$ and $|R| = 4$ there exists $b \in R'$ such that $b \neq a$ and there exists $u \neq t$ such that $x, b \in N(u)$. Since there are no triangles in $G$, $t$ and $u$ are not adjacent to $y$. Furthermore, $y \notin R$. Let $c$ denote the remaining vertex in $R'$. If there exists a distinct vertex $s$ such that $x, c \in N(s)$ we are done since $xtaybux$ and $xscybux$ are two 6-cycles with more than a singe common edge. So assume no such $s$ exists. Then $c$ must be adjacent to $t$ or $u$. WLOG, assume $c$ and $u$ are adjacent. Then $xtaybux$ and $xtaycux$ are two 6-cycles with more than a single common edge, a contradiction. Thus there must exist $z$ such that $R \subseteq N[z]$.

We now assume $R'$ is an alternating triple from a 6-cycle in $G$. Once again, let $a, b, c$ denote the vertices of $R'$. Then there exist vertices $p, q, r$ such that $bpaqcrb$ forms a 6-cycle in $G$. If there exists a vertex $z$ such that three elements of $R$ are in the open neighborhood of $z$ we can let $R'$ be the set of these vertices and we are in the former case. So assume no such $z$ exists. Then $x$ is not adjacent to $p, q$ nor $r$. So there exist distinct vertices $w$ and $y$ such that $x, b \in N(w)$ and $x, a \in N(y)$. Then $wxyapbw$ and $apbrcqa$ are two 6-cycles with more than a common edge, a contradiction. So there must exist a vertex $z$ such that $R \subseteq N[z]$.

This verifies the statement for $|R| = 4$. Assume the statement is true for all $R$ such that $|R| < k \leq |C|$ and let $|R| = k \leq |C|$. By assumption $|R| > 4$. Pick arbitrary $x \in R$ and let $R' = R - \{x\}$. By induction hypothesis there exists $z_0$ such that $R' \subseteq N[z_0]$. Suppose there does not exist $z$ such that $R \subseteq N[z]$. Then $x$ and $z_0$ are not adjacent and there must exist distinct vertices $y, z \in R'$ such that there exist distinct vertices $a$ and $b$ such that $x, z \in N(a)$ and $x, y \in N(b)$. Since there are no triangles in $G$ no two elements of $R$ are adjacent and $a, b \notin R' (a, b \notin N(z_0))$. Furthermore, $z_0 \notin R$. Since $|R| > 4$ there exists another distinct vertex $w \in R'$. If there exists a distinct vertex $c$ such that $w, x \in N(c)$ we are done since $xazz_0ybx$ and $xcwz_0ybx$ are two 6-cycles with more than a single common edge. Thus $w$ must be adjacent to $a$ or $b$. WLOG assume $w$ and $b$ are adjacent. Then $xazz_0ybx$ and $xbwz_0zax$ are two 6-cycles with more than a single common edge, a contradiction. This proves for all subsets $R$ of $C$, there must exist $z$ such that $R \subseteq N[z]$; in particular there exists $z$ such that $C \subseteq N[z]$. Since there are no triangles in $G$, $z \notin C$. Since $C$ is a maximal clique in $S_2(G)$, $C = N(z)$, completing the proof. □

Let $T(G)$ denote the set of alternating triples for all 6-cycles found in the graph $G$. Define $R(G)$ as $S'(G) \cup T(G)$. Define $R'(G)$ as the set of all sets in $R(G)$ such that no set is properly contained in any other. Figure 9 illustrates that an element of $T(G)$ may be properly contained in an element of $S'(G)$ and vice versa.

THEOREM 6.2. *Let $G$ be a connected, noncomplete, triangle-free graph such that no two 6-cycles have more than a single edge in common. Let $C \in R'(G)$. Then $C$ is a maximal clique in $S_2(G)$.*

*Proof.* Since $G$ is triangle-free, every nonsimplicial vertex is of Type II. Thus we need only consider two cases: $C$ is the open neighborhood of a nonsimplicial vertex or

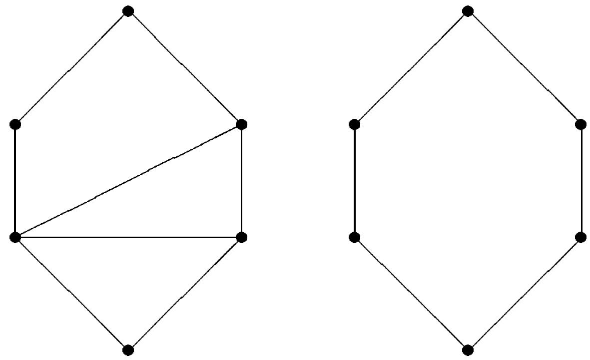


Fig. 9.

*The graph on the left contains an element of $T(G)$ which is properly contained in a set of $S'(G)$. The graph on the right contains an element of $S'(G)$ which is properly contained in a set of $T(G)$.*

$C$ is an alternating triple.

Assume $C$ is the open neighborhood of a nonsimplicial vertex $z$. Then $|C| \geq 2$. Clearly $C$ forms a clique in $S_2(G)$. Suppose it is not maximal. Then there exists $w \notin N(z)$ such that $w$ is joined to every vertex in $N(z)$ by a path of length two. Observe there does not exist a vertex $p$ such that $\{w\} \cup N(z) \subseteq N(p)$ since $N(z)$ is not properly contained in $N(p)$. Thus there exist $x, y \in N(z)$ such that there exist distinct $a$ and $b$ (not in $N(z)$ since $G$ is triangle- free) such that $x, w \in N(a)$ and $y, w \in N(b)$. There must exist another distinct vertex $u \in N(z)$ since $N(z)$ is not properly contained in an alternating triple. If there exists a distinct vertex $c$ such that $w, u \in N(c)$ we are done, as we have two 6-cycles in $G$ with more than a single edge in common. Since $G$ is triangle-free $x, y \notin N(w)$. Thus $u$ must be adjacent to $a$ or $b$, in either case creating two 6-cycles with more than a single edge in common, a contradiction. Thus no such $w$ can exist, i.e. $N(z)$ is a maximal clique in $S_2(G)$.

Alternatively, assume $C$ is an alternating triple $\{x, y, z\}$. Then there exist vertices $a, b, c$ such that $xaybzcx$ is a 6-cycle in $G$. Clearly $C$ is a clique in $S_2(G)$. Suppose it is not maximal. Then there exists a vertex $w$ joined to $x, y$ and $z$ by a path of length two. Since $G$ is triangle-free $x, y, z \notin N(w)$. Suppose $w$ is adjacent to more than one element of the set $\{a, b, c\}$. One can easily show we have two 6-cycles with more than a single edge in common. Suppose $w$ is adjacent to one element of the set $\{a, b, c\}$. WLOG, assume $w$ and $c$ are adjacent. Then $w$ and $y$ are joined by a path of length two implies there exists a new vertex $d$ such that $w, y \in N(d)$ and we have two 6-cycles with more than a single edge in common. Thus $w$ is not adjacent to $a, b$ or $c$. Then there must exist new vertices $s$ and $t$ such that $w, x \in N(s)$ and $w, z \in N(t)$. If $s = t$ then $xaybzsx$ and $xaybzcx$ are two 6-cycles with more than a single common edge. Therefore $s \neq t$. But then $xczbyax$ and $xcztwsx$ are two 6-cycles with more than a single common edge. Thus $C$ is a maximal clique in $S_2(G)$, completing the proof. □

Then by Theorems 6.1, 6.2 and the Gilmore-Hoffman characterization of interval graphs we conclude:

COROLLARY 6.3. *Let $G$ be an connected, noncomplete, triangle-free graph such that no two 6-cycles in $G$ share more than one edge. Then $S_2(G)$ is interval iff $R'(G)$ has a consecutive ranking.*

**7. Conclusions and Directions for Further Research.** The following open questions may be of interest in characterizing graphs with interval two-step graphs.

1. Which graphs have complete two-step graphs or two-step graphs consisting of complete components? For example, the two-step graph of the complete bipartite graph $K_{1,m}$ is $K_1 \cup K_m$.
2. Which graphs have chordal two-step graphs? This is related to characterizing graphs with chordal squares. These problems have been considered by Phelps [18], Harary and McKee [10], and Lundgren and Merz [14]. Also related is the problem of characterizing graphs with interval squares (see [15, 14]).

Results in these areas are potentially useful with regard to the channel assignment problem. Lundgren, Maybee, and Rasmussen [13] discuss this application in greater detail. Optimal colorings or T-colorings are desired in making frequency assignments. Raychaudhuri [19] extended a result of Cozzens and Roberts [7] to give an $O(n^2)$ algorithm for finding a T-coloring of an interval graph. Rose, Tarjan, and Leuker [21] showed that a chordal graph can be recognized in linear time. A linear time algorithm developed by Fulkerson and Gross [8] can then be used to find the maximal cliques of a chordal graph. Booth and Leuker [3] showed that a family of sets, the maximal cliques in this case, can be tested for a consecutive ranking in linear time, thus proving that interval testing can be done in linear time. $S'(G)$ can be found in $O(|V|^2)$ time. The algorithm due to Booth and Leuker can then be used to test $S'(G)$ for a consecutive ranking. Thus given an incomplete connected graph with no 6-cycle, we can perform interval testing in time proportional to $|V|^2$.

## REFERENCES


[1] J. W. BOLAND, R. C. BRIGHAM, AND R. D. DUTTON, *The difference between a neighborhood graph and a wheel.* Congressus Numerantium 58 (1987) 151-156.

[2] ——, *Embedding arbitrary graphs in neighborhood graphs.* to appear.

[3] K. S. BOOTH AND G. S. LEUKER, *Testing for the consecutive ones property, interval graphs, and graph planarity using pq-tree algorithms.* J. Comput. Syst. Sci. 13 (1976) 335-379.

[4] R. D. BRIGHAM AND R. D. DUTTON, *On neighborhood graphs.* Journal of Combinatorics, Information and System Sciences 12(1-2) (1987) 75-85.

[5] J. COHEN, *Food webs and niche spaces.* (Princeton University Press, Princeton, NJ, 1978).

[6] ——, *Interval graphs and food webs: a finding and a problem.* RAND Corporation Document 17696-PR, Santa Monica, CA, 1968.

[7] M. B. COZZENS AND F. S. ROBERTS, *T-colorings of graphs and the channel assignment problem.* Congressus Numerantium 35 (1982) 191-208.

[8] D. FULKERSON AND O. GROSS, *Incidence matrices and interval graphs.* Pacific Journal of Mathematics 15 (1965) 835-855.

[9] M. GOLUMBIC, *Algorithmic graph theory and perfect graphs.* (Academic Press, New York, 1980).

[10] F. HARARY AND T. MCKEE, *The square of a chordal graph.* to appear.

[11] K. Hefner, K. Jones, S. Kim, J. Lundgren, and F. Roberts, *$(i, j)$ competition graphs.* Discrete Applied Mathematics 32 (1991) 241-262.

[12] J. Lundgren and J. Maybee, *Food webs with interval competition graphs.* in: F. Harary and J.S. Maybee, eds., Graphs and Applications: Proceedings of the First Colorado Symposium on Graph Theory (Wiley, New York, 1984) 231-244.

[13] J. Lundgren, J. S. Maybee, and C. W. Rasmussen, *An application of generalized competitons graphs to the channel assignment problem.* Congressus Numerantium 71 (1990) 217-224.

[14] J. Lundgren and S. Merz, *Elimination ordering characterizations of digraphs with chordal and interval competition graphs.* submitted to Congressus Numerantium.

[15] J. Lundgren, S. Merz, and C. Rasmussen, *A characterization of graphs with interval squares.* Congressus Numerantium 98 (1993) 132-142.

[16] J. R. Lundgren, J. S. Maybee, and C. W. Rasmussen, *Interval competition graphs of symmetric digraphs.* Discrete Mathematics, 119 (1993) 113-122.

[17] J. R. Lundgren and C. W. Rasmussen, *Two-step graphs of trees.* Discrete Mathematics 119 (1993) 123-140.

[18] E. B. Phelps, *Characterizing chordal graphs with chordal two-step graphs*, master's thesis, University of Colorado at Denver, 1992.

[19] A. Raychaudhuri, *Intersection Assignments, T-colorings, and Powers of Graphs*, PhD thesis, Rutgers University, 1985.

[20] A. Raychaudhuri and F. Roberts, *Generalized competition graphs and their applications.* Methods of Operations Research 49 (1985) 295-311.

[21] D. J. Rose, R. E. Tarjan, and G. S. Leuker, *Algorithmic aspects of vertex elimination on graphs.* SIAM Journal of Applied Mathematics 5 (1976) 266-268.

[22] G. Sugihara, *Niche Hierarchy: Structure, Organization, and Assembly in Natural Communities*, PhD thesis, Princeton University, 1982.

[23] C. Wang, *Competition Graphs, Threshold Graphs, and Threshold Boolean Functions*, PhD thesis, Rutgers Center for Operations Research, Rutgers University, New Brunswick, NJ, 1991.