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Certain improvements of Newton's method with fourth-order convergence ☆

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ABSTRACT

In this paper we present two new schemes, one is third-order and the other is fourth-order. These are improvements of second-order methods for solving nonlinear equations and are based on the method of undetermined coefficients. We show that the fourth-order method is more efficient than the fifth-order method due to Kou et al. [J. Kou, Y. Li, X. Wang, Some modifications of Newton's method with fifth-order covergence, J. Comput. Appl. Math., 209 (2007) 146–152]. Numerical examples are given to support that the methods thus obtained can compete with other iterative methods.

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1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. To solve nonlinear equations, iterative methods such as Newton's method are usually used. Throughout this paper we consider iterative methods to find a simple root ξ, i.e., $f(\xi) = 0$ and $f'(\xi) \neq 0$, of a nonlinear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D.

Newton's method for the calculation of ξ is probably the most widely used iterative scheme defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

It is well known (see e.g. Traub [1]) that this method is quadratically convergent.

Some modifications of Newton's method to achieve higher order and better efficiency have been suggested and analyzed using several different techniques such as quadrature rules [2–13], decomposition [14,15] and homotopy techniques [16,17].

A third-order variant of Newton's method appeared in Weerakoon and Fernando [2] where trapezoidal approximation to the integral in Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt \tag{2}$$

was considered to obtain the cubically convergent method

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \tag{3}$$

where from here on

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$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{4}$$

Another improvement of Newton's method was suggested in [3], where the authors considered the midpoint rule for the integral of (2) and obtained the third-order method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n+y_n}{2}\right)}. \tag{5}$$

In [4], Homeier derived the following cubically convergent iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)}\right) \tag{6}$$

by applying Newton's theorem (2) to the inverse function $x = f(y)$ instead of $y = f(x)$. It should be pointed out that this method has also been derived in [5] independently and it is now known as harmonic mean Newton method. It should also be noted that many of the known iterative methods developed in recent years including the third-order methods given above can be regarded as rediscovered methods, see [18] for more details.

To further improve the order of convergence, some fourth-order iterative methods have been proposed and analyzed. The Traub–Ostrowski method [1,19], which has fourth-order convergence, is given by

$$x_{n+1} = x_n - \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}\frac{f(x_n)}{f'(x_n)}, \tag{7}$$

where y_n is defined by (4). This method is widely used and extended in more general setting for applications. The fourth-order methods in the literatute usually require three evaluations of the given function and its first derivative per iteration, and it was shown that they can compete with Newton's method, see [1,9,13,16,20] and the references therein.

Other than the above-mentioned methods, various types of improvements of Newton's method are available in the literature [6–15] and the references therein. Among these methods it is noteworthy to mention that the method of undetermined coefficients was successfully applied in [11] to show that many methods in the literature can be derived from each other, and so proving their equivalence. The method was also used to develop new schemes. Most of the above-mentioned methods improve the order of convergence and computational efficiency of Newton's method with an additional evaluation of the function or its first derivative. To be more precise, we define informational efficiency E by

$$E = \frac{p}{d},$$

where p is the order of the method and d is the number of function- (and derivative-) evaluations per step. We also mention another measure, the efficiency index I

$$I = p^{1/d}.$$

Here we apply the method of undetermined coefficients to present two new improvements of Newton's method, one third-order and the other is fourth-order. These methods are analyzed in detail and their efficiency as well as their practical utility is compared with other methods.

2. Development of methods and convergence analysis

2.1. A new fourth-order method

Let $u_{n+1} = g_2(x_n)$ stands for any second-order iterative method. It is well known [1] that the iteration scheme of the form

$$x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{f'(u_{n+1})} \tag{8}$$

and a variant of (8)

$$x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{f'(x_n)} \tag{9}$$

are of orders four and three, respectively. The order of the method (8) is higher than that of (9), but the computation involved is more costly and thus less efficient.

The informational efficiency of the above methods is unity. The efficiency index of those methods is 1.4142 for (1) and (8) but $I = 1.442$ for (34) and (9).

To derive the new fourth-order scheme, we consider the expression

$$f'(u_{n+1}) = Af'(x_n) + Bf(x_n) + Cf(u_{n+1}). \tag{10}$$

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Expand the terms $f'(u_{n+1}), f'(x_n)$ and $f(u_{n+1})$ about the point x_n up to second derivatives and collect terms. Upon comparing the coefficients of the derivatives of f at x_n, we have the following system of equations for the unknowns $A, \dots, D$

$$B + C = 0, \tag{11}$$
$$A + \alpha C = 1, \tag{12}$$
$$\frac{1}{2}\alpha^2 C = \alpha, \tag{13}$$

where $\alpha = u_{n+1} - x_n$. Solving the equations (11)–(13), we get

$$A = -1, \tag{14}$$
$$B = -\frac{2}{\alpha}, \tag{15}$$
$$C = \frac{2}{\alpha}. \tag{16}$$

The method is now

$$x_{n+1} = u_{n+1} - \frac{\alpha f(u_{n+1})}{2[f(u_{n+1}) - f(x_n)] - \alpha f'(x_n)}, \tag{17}$$

where u_{n+1} is computed by any second-order method. This is a generalization of Traub–Ostrowski scheme.

For the method defined by (17), we have the following analysis of convergence.

Theorem 2.1. *Let $\xi \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \mathbb{R}$ for an open interval I. Let $u_{n+1} = g_2(x_n)$ be any second-order method and assume that it satisfies*

$$u_{n+1} - \xi = Ke_n^2 + O(e_n^3), \tag{18}$$

for some $K \neq 0$, and $e_n = x_n - \xi$. Then the new method defined by (17) is of fourth-order. The error at the nth step, e_n, satisfies the relation

$$e_{n+1} = K[Kc_2 - c_3]e_n^4 + O(e_n^5), \tag{19}$$

where

$$c_n = (1/n!)f^{(n)}(\xi)/f'(\xi). \tag{20}$$

Proof. For later use, we assume that

$$u_{n+1} - \xi = Ke_n^2 + Me_n^3 + O(e_n^4). \tag{21}$$

Using the Taylor expansion and taking into account $f(\xi) = 0$, we easily obtain

$$\alpha = u_{n+1} - x_n = -e_n + Ke_n^2 + Me_n^3 + O(e_n^4), \tag{22}$$
$$f(u_{n+1}) = f'(\xi)\left[(u_{n+1} - \xi) + c_2(u_{n+1} - \xi)^2 + O(e_n^6)\right], \tag{23}$$
$$f(x_n) = f'(\xi)\left[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)\right], \tag{24}$$
$$f'(x_n) = f'(\xi)\left[1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)\right] \tag{25}$$

from which it follows that

$$2[f(u_{n+1}) - f(x_n)] = 2f'(\xi)\left[-e_n + (u_{n+1} - \xi) - c_2e_n^2 - c_3e_n^3 + O(e_n^4)\right]. \tag{26}$$

Using (24)–(26) we find

$$\begin{aligned}\frac{\alpha f(u_{n+1})}{2[f(u_{n+1}) - f(x_n)] - \alpha f'(x_n)} &= (u_{n+1} - \xi) - 2K(u_{n+1} - \xi)e_n + \frac{2}{e_n}(u_{n+1} - \xi)^2 - (6K - c_2)(u_{n+1} - \xi)^2 \\ &\quad - \left(2M + 2Kc_2 - c_3 - 2K^2\right)(u_{n+1} - \xi)e_n^2 + \frac{4}{e_n^2}(u_{n+1} - \xi)^3 + O(e_n^5).\end{aligned} \tag{27}$$

Hence we obtain

$$\begin{aligned}e_{n+1} &= u_{n+1} - \xi - \Big[(u_{n+1} - \xi) - 2K(u_{n+1} - \xi)e_n + \frac{2}{e_n}(u_{n+1} - \xi)^2 - (6K - c_2)(u_{n+1} - \xi)^2 \\ &\quad - \left(2M + 2Kc_2 - c_3 - 2K^2\right)(u_{n+1} - \xi)e_n^2 + \frac{4}{e_n^2}(u_{n+1} - \xi)^3 + O(e_n^5)\Big] = 2K(u_{n+1} - \xi)e_n - \frac{2}{e_n}(u_{n+1} - \xi)^2 \\ &\quad + (6K - c_2)(u_{n+1} - \xi)^2 + \left(2M + 2Kc_2 - c_3 - 2K^2\right)(u_{n+1} - \xi)e_n^2 - \frac{4}{e_n^2}(u_{n+1} - \xi)^3 + O(e_n^5).\end{aligned} \tag{28}$$

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Substituting (21) into (28), we have the error equation

$$e_{n+1} = K[Kc_2 - c_3]e_n^4 + O(e_n^5). \tag{29}$$

This means that the method defined by (17) is fourth order. This completes the proof. □

Remark 1. With iterative methods $u_{n+1} = g_2(x_n)$ that require the computation of $f(x_n)$ and $f'(x_n)$, our method requires 2 function- and one derivative-evaluation per step, the informational efficiency is $E = 4/3$ and the efficiency index is $I = 1.5874$. The fifth-order method due to Kou et al. [10] has informational efficiency $E = 5/4$ and efficiency index $I = 1.495$. Both of these measures are lower than the corresponding ones for our method (17).

If we take the Newton iteration as first step, that is,

$$u_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{30}$$

then our method (17) reduces to the well known Traub–Ostrowski fourth-order method (7).

2.2. A new third-order method

The method (6) can be rewritten

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{2f'(x_n)f'(y_n)}{f'(x_n)+f'(y_n)}}, \tag{31}$$

where y_n is given by (4).

Let us consider the application of the method of undetermined coefficients to (31) with the form

$$af'(x_n) + bf'(y_n) = \frac{2f'(x_n)f'(y_n)}{f'(x_n) + f'(y_n)} \tag{32}$$

or

$$2f'(x_n)f'(y_n) = [f'(x_n) + f'(y_n)][af'(x_n) + bf'(y_n)] \tag{33}$$

to determine the unknown constants a and b in a specific manner. By doing the same as before, we found that the resulting method will not be of order three, and therefore to improve the order, we obtain the new method

$$x_{n+1} = y_n + \frac{1}{2}\frac{f(x_n)}{f'(x_n)} - \left[\frac{f(x_n)}{f'(x_n)}\right]^2 - \frac{1}{2}\frac{(1 + y_n - x_n)^2 f(x_n)}{f'(y_n) + (y_n - x_n)^2 f'(x_n)}. \tag{34}$$

This method turns out to be third-order as we ascertain in the following theorem.

Theorem 2.2. *Let $\xi \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to ξ, then the method defined by (34) is of third-order, and satisfies the error equation*

$$e_{n+1} = \left(2c_2 + \frac{1}{2}c_3\right)e_n^3 + O(e_n^4), \tag{35}$$

where $e_n = x_n - \xi$ and c_n is given by (20).

Proof. By using (24) and (25), we obtain

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4) \tag{36}$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \xi + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + O(e_n^4), \tag{37}$$

whence

$$f'(y_n) = f'(\xi) + f''(\xi)(y_n - \xi) + O\left((y_n - \xi)^2\right) = f'(\xi)[1 + 2c_2^2e_n^2 + O(e_n^3)]. \tag{38}$$

We then easily find

$$(1 + y_n - x_n)^2 f(x_n) = f'(\xi)[e_n + (c_2 - 2)e_n^2 + (1 + c_3)e_n^3 + O(e_n^4)], \tag{39}$$

$$f'(y_n) + (y_n - x_n)^2 f'(x_n) = f'(\xi)[1 + (1 + 2c_2^2)e_n^2 + O(e_n^3)], \tag{40}$$

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so that

$$\frac{(1+y_n-x_n)^2 f(x_n)}{f'(y_n)+(y_n-x_n)^2 f'(x_n)} = e_n + (c_2-2)e_n^2 + (c_3-2c_2^2)e_n^3 + O(e_n^4). \tag{41}$$

It then follows from (37) and (41) that

$$x_{n+1} = y_n + \frac{1}{2}\frac{f(x_n)}{f'(x_n)} - \left[\frac{f(x_n)}{f'(x_n)}\right]^2 - \frac{1}{2}\frac{(1+y_n-x_n)^2 f(x_n)}{f'(y_n)+(y_n-x_n)^2 f'(x_n)} = \xi + \left(2c_2+\frac{1}{2}c_3\right)e_n^3 + O(e_n^4). \tag{42}$$

This shows that the method defined by (34) has third-order convergence. This completes the proof. □

It should be mentioned that Theorems 2.1 and 2.2 can also be proven by Taylor expansions using Maple (see [15] for details). The method (34) requires one evaluation of the function and two of its first derivative per iteration, so it has the same efficiency as the third-order methods given in Weerakoon–Fernando [2], Frontini–Sormani [3], Homeier [4] and others in the literature. Note that not all third-order methods in the lietrature are as efficient. For example, Nedzhibov's third-order method (see [21] or [22]) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{1}{4}\left(f'(y_n) + 2f'\left(\frac{x_n+y_n}{2}\right) + f'(x_n)\right)} \tag{43}$$

and Hasanov's third-order method (see [23] or [22]) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{1}{6}\left(f'(y_n) + 4f'\left(\frac{x_n+y_n}{2}\right) + f'(x_n)\right)} \tag{44}$$

are both third-order but have informational efficiency $E = 3/4$ and efficiency index $I = 3^{1/4} = 1.316$.

3. Numerical examples

In this section we present some numerical experiments using our new methods and compare these results to well known third and fourth-order schemes. All computations were done using MAPLE using 128 digit floating point arithmetics (Digits := 128). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: $(i)\ |x_{n+1}-x_n| < \epsilon,\ (ii)\ |f(x_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root ξ computed. For numerical illustrations in this section we used the fixed stopping criterion $\epsilon = 10^{-25}$. We used the test functions in Weerakoon and Fernando [2] and in Neta [12]

Testfunction	x_0	x_*
$f_1(x) = x^3+4x^2-10$	1.6	1.3652300134140968457608068290
$f_2(x) = \sin^2(x) - x^2 + 1$	1.0	1.4044916482153412260350868178
$f_3(x) = (x-1)^3 - 1$	3.5	2.0
$f_4(x) = x^3 - 10$	4.0	2.1544346900318837217592935665
$f_5(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$	−1.0	−1.2076478271309189270094167584
$f_6(x) = e^{x^2+7x-30} - 1$	4.0	3.0
$f_7(x) = \sin(x) - \frac{x}{2}$	2.0	1.8954942670339809471440357381
$f_8(x) = x^5 + x - 10000$	4.0	6.3087771299726890947675717718
$f_9(x) = \sqrt{x} - \frac{1}{x} - 3$	9.0	9.6335955628326951924063127092
$f_{10}(x) = e^x + x - 20$	0.0	2.8424389537844470678165859402
$f_{11}(x) = \ln(x) + \sqrt{x} - 5$	10.0	8.3094326942315717953469556827
$f_{12}(x) = x^3 - x^2 - 1$	0.5	1.4655712318767680266567312252

We present some numerical test results for various cubically convergent iterative schemes in Table 1. Compared were Newton method(NM), the method of Weerakoon and Fernando (WF) defined by (3), Halley's method [24,25] (HalleyM) defined by

$$x_{n+1} = y_n - \frac{f(x_n)f''(x_n)}{2f'(x_n)^2 - f(x_n)f''(x_n)}\frac{f(x_n)}{f'(x_n)}, \tag{45}$$

where y_n is given by (4), Homeier's method (HM) defined by (6), and the method (34) introduced in the present contribution.

We also present some numerical test results for various fourth-order iterative schemes in Table 2. The following methods were compared: Newton method(NM), Jarratt's method [20] (JM) defined by

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$$z_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \tag{46}$$

$$x_{n+1} = x_n - \frac{1}{2}\left[\frac{3f'(z_n) + f'(x_n)}{3f'(z_n) - f'(x_n)}\right]\frac{f(x_n)}{f'(x_n)}, \tag{47}$$

King's method with $\beta = 3$ [13] (KM) defined by

$$x_{n+1} = y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \tag{48}$$

where y_n is defined by (4), Kou's method [9] (KouM) defined by

Table 1
Comparison of various third-order iterative schemes and the Newton method.

f		NM	WF	HM	HalleyM	(34)
f_1	IT	6	4	4	4	5
	NFE	12	12	12	12	15
	$f(x_*)$	1.29e−61	3.01e−76	1.55e−107	6.58e−83	−1.90e−126
	δ	1.26e−31	4.07e26	3.14e−36	2.81e−28	1.00e−56
f_2	IT	7	5	5	5	6
	NFE	14	15	15	15	18
	$f(x_*)$	−1.04e−50	8.90e−89	−1.0e−127	1.38e−114	1.20e−99
	δ	7.33e−26	3.79e−30	2.18e−62	1.02e−38	6.69e−34
f_3	IT	9	6	6	6	7
	NFE	18	18	18	18	21
	$f(x_*)$	1.41e−84	1.23e−109	0	0	0
	δ	6.86e−43	3.28e−37	5.22e−73	1.45e−49	3.57e−74
f_4	IT	8	6	5	5	5
	NFE	16	18	15	15	15
	$f(x_*)$	5.44e−72	0	5.90e−113	2.47e−80	0
	δ	9.17e−37	1.35e−64	4.91e−38	2.31e−27	2.18e−52
f_5	IT	7	5	5	4	5
	NFE	14	15	15	12	15
	$f(x_*)$	−2.27e−63	4.62e−98	−1.10e−129	8.57e−91	−1.01e−104
	δ	8.63e−33	8.87e−34	1.80e−60	5.50e−31	6.29e−36
f_6	IT	21	15	12	12	13
	NFE	42	45	36	36	39
	$f(x_*)$	9.09e−78	−2.0e−126	5.00e−105	0	0
	δ	3.26e−40	3.75e−73	2.98e−36	6.95e−68	1.73e−50
f_7	IT	6	4	4	12	4
	NFE	12	12	12	36	12
	$f(x_*)$	−1.54e−80	−8.21e−104	−2.0e−128	−3.64e−98	−5.71e−81
	δ	1.81e−40	6.92e−35	3.55e−49	4.81e−33	1.84e−27
f_8	IT	10	8	6	18	
	NFE	20	24	18	54	
	$f(x_*)$	1.74e−62	−4.42e−89	0	0	div
	δ	2.63e−33	3.54e−31	1.33e−55	6.13e−61	
f_9	IT	5	4	4	4	5
	NFE	10	12	12	12	15
	$f(x_*)$	−2.21e−54	−1.35e−125	0	0	0
	δ	2.05e−26	3.44e−41	5.18e−45	1.15e−44	1.95e−44
f_{10}	IT	14	89	21	15	
	NFE	28	267	63	45	
	$f(x_*)$	6.08e−54	−9.97e−79	0	2.0e−126	div
	δ	8.42e−28	5.67e−27	4.59e−70	3.36e−58	
f_{11}	IT	6	4	4	4	
	NFE	12	12	12	12	
	$f(x_*)$	−2.21e−74	3.79e−83	3.63e−97	2.89e−102	div
	δ	1.33e−36	3.39e−27	9.33e−32	1.99e−33	
f_{12}	IT	10	7	6	6	7
	NFE	20	21	18	18	21
	$f(x_*)$	8.30e−99	−1.0e−127	−1.0e−127	2.71e−88	−1.0e−127
	δ	4.94e−50	1.01e−63	1.20e−47	4.91e−30	2.72e−67

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Table 2
Comparison of various fourth-order iterative schemes and the Newton method.

f		NM	JM	KM	KouM	OM
f_1	IT	6	4	4	4	4
	NFE	12	12	12	12	12
	$f(x_*)$	1.29e−61	−6.0e−127	−6.0e−127	−6.0e−127	−6.0e−127
	δ	1.26e−31	2.42e−65	4.94e−48	7.83e−55	1.64e−45
f_2	IT	7	4	9	5	6
	NFE	14	12	27	15	18
	$f(x_*)$	−1.04e−50	−1.34e−110	−1.0e−127	2.10e−127	−1.0e−127
	δ	7.33e−26	3.41e−28	5.27e−76	1.71e−42	1.15e−94
f_3	IT	9	5	6	5	6
	NFE	18	15	18	15	18
	$f(x_*)$	1.41e−84	0	0	1.11e−120	0
	δ	6.86e−43	2.21e−49	4.28e−85	6.10e−31	1.10e−88
f_4	IT	8	5	5	5	5
	NFE	16	15	15	15	15
	$f(x_*)$	5.44e−72	0	0	0	0
	δ	9.17e−37	5.82e−82	3.78e−42	7.40e−56	1.23e−32
f_5	IT	7	4	5	5	4
	NFE	14	12	15	15	12
	$f(x_*)$	−2.27e−63	−1.10e−126	−1.94e−101	1.20e−126	−1.10e−126
	δ	8.63e−33	2.40e−50	1.46e−26	9.01e−90	1.04e−55
f_6	IT	21	10	13	12	10
	NFE	42	30	52	36	30
	$f(x_*)$	9.09e−78	0	9.22e−118	0	0
	δ	3.26e−40	1.75e−51	4.46e−31	7.87e−46	2.63e−33
f_7	IT	6	4	4	4	4
	NFE	12	12	12	12	12
	$f(x_*)$	−1.54e−80	−2.0e−128	−2.0e−128	6.0e−128	−2.0e−128
	δ	1.81e−40	7.49e−79	4.59e−64	1.40e−70	3.84e−62
f_8	IT	10	5	48	12	14
	NFE	20	15	144	36	42
	$f(x_*)$	1.74e−62	−7.0e−124	0	5.93e−102	0
	δ	2.63e−33	2.46e−35	1.12e−63	9.85e−27	2.12e−40
f_9	IT	5	3	4	3	4
	NFE	10	9	12	9	12
	$f(x_*)$	−2.22e−54	1.96e−115	0	−3.98e−109	−3.10e−126
	δ	2.05e−26	5.39e−28	1.28e−93	1.69e−26	1.55e−31
f_{10}	IT	14	6			14
	NFE	28	18			42
	$f(x_*)$	6.08e−54	0	div	div	0
	δ	8.42e−28	1.56e−69			2.72e−57
f_{11}	IT	6	4	4	4	4
	NFE	12	12	12	12	12
	$f(x_*)$	−2.21e−74	1.0e−127	1.0e−127	−1.0e−127	−7.17e−116
	δ	1.33e−36	2.62e−85	1.23e−57	2.62e−71	4.92e−29
f_{12}	IT	10	5	6	6	6
	NFE	20	15	18	18	18
	$f(x_*)$	8.30e−99	2.09e−116	−1.0e−127	−1.0e−127	−1.0e−127
	δ	4.94e−50	9.86e−30	4.78e−44	9.80e−67	3.75e−40

$$x_{n+1} = x_n - \frac{f(x_n)^2 + f(y_n)^2}{f'(x_n)(f(x_n) - f(y_n))}, \tag{49}$$

where y_n is defined by (4), and our new method (17) with u_{n+1} given by

$$u_{n+1} = x_n - \frac{f(x_n)}{f(x_n) + f'(x_n)}, \tag{50}$$

which is of order two [17]. It is well-known that Newton's method may fail to converge in case the initial guess is far from zero or the derivative is small in the vicinity of the required root. In cases that Newton's method is not successful, several second-order alternative methods were developed and tested to be robust and reliable. Some of these as well as Wu's method (50) are Stiring's method [26] given by

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$$u_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n))}, \tag{51}$$

Steffensen's method [27] given by

$$u_{n+1} = x_n - \frac{f^2(x_n)}{f'(x_n + f(x_n)) - f(x_n)} \tag{52}$$

and Mamta's method [28] given by

$$u_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'^2(x_n + f^2(x_n)) - f(x_n)}. \tag{53}$$

Also displayed are the number of iterations to approximate the zero (IT), the number of functional evaluations (NFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative, the value $f(x_*)$ and the distance δ of two consecutive approximations for the zero.

The test results in Table 2 show that for most of the functions we tested, the method (17) introduced in the present work have equal or better performance as compared to the other methods of the same order. However, it is observed that the other third order methods in comparison outperformed the proposed method (34).

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