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Abstract. We consider the problem of identification of airborne objects from high-range-resolution radar data. We use high-frequency asymptotics to show that certain features of the object correspond to identifiable features of the radar data. We study the cases of single scattering and multiple scattering from two point-like scattering centers.

This work suggests a method for target identification that circumvents the need to create an intermediate radar image from which the object's characteristics are to be extracted. As such, this scheme may be applicable to efficient machine-based radar identification programs.

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1. Introduction

Current methods of identifying objects from radar data generally involve first forming an image, and then attempting to identify features of the image. Here we propose a different approach, namely to carry out the identification directly from examination of the raw radar data.

This approach requires determining which features of the radar data correspond to which features of the object. Our approach [7] relates the singular structure (such as edges) of the target to the singular structure of the data set. Restricting our attention to the singular structure—specifically, to a certain set in phase space called the *wavefront set*—allows us to use the tools of microlocal analysis [10, 14, 33]. This strategy was first applied to imaging problems in [1]; its uses in seismic prospecting [2, 6, 11], X-ray tomography [12, 17], and Synthetic-Aperture Radar [24] are active areas of research. An approach similar to the one we pursue here, in which we use microlocal analysis not to do imaging but instead to study the connection between features of the target and the data, was considered for the X-ray tomography problem by Quinto [26].

We begin in section 2 by examining the general properties of radar scattering and developing mathematical models for the measured data. These models involve Fourier Integral Operators with kernels that are oscillatory integrals; it is this that makes it possible to study these models with the techniques of microlocal analysis. Next we present an overview of the microlocal concepts and theorems that are relevant to our investigation (section 3.1). These two sections serve to introduce our notation, assumptions, and terminology. In particular, we assume thoughout that the target's rotational acceleration is negligible. Section 3 contains our main results: the calculation of the wavefront sets for the single-scattering and multiple-scattering cases.

2. Radar data

High-range-resolution (HRR) radar systems transmit the equivalent of a short electromagnetic pulse and measure the time delay of the corresponding waveform reflected from a target. This provides an estimate of the target's range, and, more generally, the range to individual target substructures.

To obtain different views of the target, radar systems can use multiple pulses that interrogate the target as it rotates and sequentially presents different aspects to the radar. Such systems are known as inverse synthetic-aperture radar, or ISAR, systems.

Ultimately, the behavior of radar data is determined by scattered-field solutions to the wave equation. Since radar systems transmit and receive radio waves, we should generally examine the electromagnetic (vector) wave equation. For simplicity, however, we will examine the scalar wave equation and assume that the components of the electromagnetic field each satisfy

$$\left(\nabla^2 - c^{-2}\,\partial_t^2\right) u(t, \boldsymbol{y}) = 0 \tag{1}$$

in the region exterior to the scattering object Ω. We write the total field as a sum of the incident and scattered fields $u = g + u^{\mathrm{s}}$, where the Green's function g represents the field due to a point source at $\boldsymbol{x}$, the position of the radar. Specifically, g is given by [32]

$$g(t, \boldsymbol{y} - \boldsymbol{x}) = \frac{\delta(t - |\boldsymbol{y} - \boldsymbol{x}|/c)}{4\pi|\boldsymbol{y} - \boldsymbol{x}|} = \int \frac{\mathrm{e}^{-\mathrm{i}\omega(t - |\boldsymbol{y} - \boldsymbol{x}|/c)}}{8\pi^2|\boldsymbol{y} - \boldsymbol{x}|}\,\mathrm{d}\omega \tag{2}$$

and satisfies

$$\left(\nabla^2 - c^{-2}\partial_t^2\right) g(t, \boldsymbol{y} - \boldsymbol{x}) = -\delta(t)\delta(\boldsymbol{y} - \boldsymbol{x}). \tag{3}$$

The dependence of g on the source position $\boldsymbol{x}$ induces a similar dependence in u^s, which we write as $u^s(t, \boldsymbol{y}, \boldsymbol{x})$.

In section 2.1, we develop a mathematical model for HRR radar data and explain the fundamental role played by the single scattering approximation. We examine the multiple-scattering case in section 2.2, where we construct an exact scattering solution for two isotropic point scatterers.

2.1. Kirchhoff approximation

The Kirchhoff approximation is a geometric optics approximation. We use it to obtain an expression for the scattered field as follows. First we multiply (1) by $g(t - t', \boldsymbol{x} - \boldsymbol{y})$ and (3) by $u(t', \boldsymbol{y})$, subtract the resulting equations, and apply Green's theorem to the region exterior to the scattering object Ω, and use the outgoing radiation conditions to eliminate the contributions to the integral from infinity. The result is

$$u^s(t,\boldsymbol{x},\boldsymbol{x}) = \int_{\partial\Omega} \Big(g(t-t',\boldsymbol{x},\boldsymbol{y})\partial_{\boldsymbol{\nu}} u^s(t',\boldsymbol{y},\boldsymbol{x}) - u^s(t',\boldsymbol{y},\boldsymbol{x})\partial_{\boldsymbol{\nu}} g(t-t',\boldsymbol{x},\boldsymbol{y}) \Big)\, dt' dS_{\boldsymbol{y}} \tag{4}$$

It is on the right side of (4) that we use the geometrical optics approximation to the scattered field. In particular, the geometrical optics approximation assumes that, on the illuminated surface, the phase of the scattered field is determined by the high-frequency law of reflection (i.e., the angle of incidence is equal to the angle of reflection), and the amplitude of the scattered field is proportional to that of the incident field. This constant of proportionality is called the *reflection coefficient*; we denote it by $V(\boldsymbol{y})$. Strictly speaking, it depends also on the angle between the direction of incidence and the direction of scatter, but for typical ISAR applications these angles vary so little that we can neglect this dependence. For a rotating target, the illuminated surface also varies with angle; but again we assume that the angles vary so little that this effect can be neglected. Consequently, on the right side of (4), we use

$$\begin{aligned} u^s(t,\boldsymbol{y},\boldsymbol{x}) &\approx V(\boldsymbol{y}) \frac{\delta(t-|\boldsymbol{y}-\boldsymbol{x}|/c)}{4\pi|\boldsymbol{y}-\boldsymbol{x}|} \\ &= V(\boldsymbol{y}) \int \frac{\mathrm{e}^{-\mathrm{i}\omega(t-|\boldsymbol{y}-\boldsymbol{x}|/c)}}{8\pi^2|\boldsymbol{y}-\boldsymbol{x}|}\,\mathrm{d}\omega \\ \partial_{\boldsymbol{\nu}} u^s(t,\boldsymbol{y},\boldsymbol{x}) &\approx V(\boldsymbol{y})\partial_{\boldsymbol{\nu}}|\boldsymbol{y}-\boldsymbol{x}| \frac{\delta'(t-|\boldsymbol{y}-\boldsymbol{x}|/c)}{4\pi|\boldsymbol{y}-\boldsymbol{x}|} \\ &= V(\boldsymbol{y})\partial_{\boldsymbol{\nu}}|\boldsymbol{y}-\boldsymbol{x}| \int i\omega \frac{\mathrm{e}^{-\mathrm{i}\omega(t-|\boldsymbol{y}-\boldsymbol{x}|/c)}}{8\pi^2|\boldsymbol{y}-\boldsymbol{x}|}\,\mathrm{d}\omega \end{aligned} \tag{5}$$

where in differentiating we have retained only the leading order term for large ω. Using these expressions in (4) and simplifying results in

$$u^s(t,\boldsymbol{x},\boldsymbol{x}) = \int_{\partial\Omega}\int i\omega \frac{\mathrm{e}^{-\mathrm{i}\omega(t-2|\boldsymbol{y}-\boldsymbol{x}|/c)}}{(8\pi^2|\boldsymbol{y}-\boldsymbol{x}|)^2} V(\boldsymbol{y}) 2\partial_{\boldsymbol{\nu}}|\boldsymbol{y}-\boldsymbol{x}|\,\mathrm{d}\omega dS_{\boldsymbol{y}} \tag{6}$$

where now $\partial\Omega$ denotes the illuminated portion of the target surface.

The value of the Kirchhoff approximation is that it removes the nonlinearity in the inverse problem: it replaces the product of two unknowns (V and u) by a single unknown (V) multiplied by known quantities. This approximation is, however, a single-scattering approximation, and an important contribution to radar scattering comes from multiple bounces associated with corners. Corner scattering has the property that it can be seen from many directions; in this respect such scattering centers behave like “point” scatterers. To model corners, we simply interpret V as an effective reflection coefficient for the corner.

The model (6) applies to a stationary target and a single incident wave that starts at position $\boldsymbol{x}$ at time t. We now assume that the incident field is a series of pulses, beginning at times $t = \theta_n, n = 1, 2, \ldots$, so that

$$u_n^{\mathrm{inc}}(t',\boldsymbol{y}) = \int S_{\mathrm{inc}}(\omega') \frac{\mathrm{e}^{-\mathrm{i}\omega'(t'-\theta_n-|\boldsymbol{x}-\boldsymbol{y}|/c)}}{8\pi^2|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\omega', \tag{7}$$

where

$$S_{\rm inc}(\omega) = \mathcal{F}\{s_{\rm inc}\}(\omega) = \frac{1}{2\pi}\int s_{\rm inc}(t')\,\mathrm{e}^{\mathrm{i}\omega t'}\,\mathrm{d}t' \tag{8}$$

is the Fourier transform of the signal used to establish the interrogating field transmitted to the target. We also assume that the target is translating with velocity $\boldsymbol{v}$ and rotating, so that at time t, we have $V(t,\boldsymbol{y}) = Q_{\rm K}(\mathcal{O}^{-1}(t)(\boldsymbol{y}-\boldsymbol{v}t))$, where $\mathcal{O}(t)$ denotes a rotation operator (an orthogonal matrix).

We denote by $u_n^{\rm sc}(t,\boldsymbol{x})$ the scattered field at the radar due to the nth transmitted pulse . This field induces a system signal whose Kirchhoff-approximated value we denote by $s_{\rm sc}(\boldsymbol{x},n,t)$:

$$s_{\rm sc}(\boldsymbol{x},n,t) = \int (2\mathrm{i}\omega)\frac{\mathrm{e}^{-\mathrm{i}\omega(t-\theta_n-2|\boldsymbol{x}-\boldsymbol{y}|/c)}}{(8\pi^2|\boldsymbol{x}-\boldsymbol{y}|)^2} Q_{\rm K}(\mathcal{O}^{-1}(\theta_n)(\boldsymbol{y}-\boldsymbol{v}\theta_n))S_{\rm inc}(\omega')\partial_{\boldsymbol{\nu}}|\boldsymbol{y}-\boldsymbol{x}|\,\mathrm{d}\omega\mathrm{d}S_{\boldsymbol{y}} \tag{9}$$

Here we have made the *start-stop* approximation, i.e., the target is moving sufficiently slowly that it can be treated as stationary during the time of illumination by the radar pulse. (This depends on the length of the pulse, and the speed and size of the target.)

In (9), we neglect the overall target velocity (set $\boldsymbol{v}=0$), let $t \to t-\theta_n$, and make the change of variables $\boldsymbol{z} = \mathcal{O}^{-1}(\theta_n)\boldsymbol{y}$. This approximation converts (9) into

$$s_{\rm sc}(\boldsymbol{x},n,t) = \int (2\mathrm{i}\omega)\frac{\mathrm{e}^{-\mathrm{i}\omega(t-\theta_n-2|\boldsymbol{x}-\mathcal{O}(\theta_n)\boldsymbol{z}|/c)}}{(8\pi^2|\boldsymbol{x}-\mathcal{O}(\theta_n)\boldsymbol{z}|)^2} Q_{\rm K}(\boldsymbol{z})S_{\rm inc}(\omega)\partial_{\boldsymbol{\nu}}|\mathcal{O}(\theta_n)\boldsymbol{z}-\boldsymbol{x}|\,\mathrm{d}\omega\mathrm{d}S_{\boldsymbol{z}}\,. \tag{10}$$

We use the far-field approximation $|\boldsymbol{x}-\boldsymbol{w}| = |\boldsymbol{x}| - \hat{\boldsymbol{x}}\cdot\boldsymbol{w} + \mathrm{O}(|\boldsymbol{x}|^{-1})$ (with the hat denoting unit vector), the orthogonality of $\mathcal{O}$, and the notation $R=|\boldsymbol{x}|$, $\hat{\boldsymbol{R}}_n = -\mathcal{O}^{\rm T}(\theta_n)\hat{\boldsymbol{x}}$ to rewrite (10) as

$$s_{\rm sc}(\boldsymbol{x},n,t) \approx \frac{2}{(8\pi^2R)^2}\int Q_{\rm K}(\boldsymbol{z})(\mathrm{i}\omega)S_{\rm inc}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega[t-\theta_n-(R+\hat{\boldsymbol{R}}_n\cdot\boldsymbol{z})/c]}\boldsymbol{\nu}\cdot\hat{\boldsymbol{R}}_n\,\mathrm{d}\omega\mathrm{d}S_{\boldsymbol{z}}\,. \tag{11}$$

ISAR systems typically use a correlation receiver. This means that our model for the radar data must include the correlation process. In particular, we correlate the signal (11) with a signal of the form $s_{\rm inc}(t'-t) = \int S(\omega')\exp(-\mathrm{i}\omega'(t'-t))\,\mathrm{d}\omega'$ to obtain the output of the correlation receiver:

$$\begin{aligned}\eta_{\rm K}(\theta_n,t) &= \int s_{\rm sc}(\boldsymbol{x},n,t')\overline{s}_{\rm inc}(t'-t)\,\mathrm{d}t' \\ &= \frac{2}{(8\pi^2R)^2}\int Q_{\rm K}(\boldsymbol{z})(\mathrm{i}\omega)S_{\rm inc}(\omega)\overline{S}_{\rm inc}(\omega')\,\mathrm{e}^{-\mathrm{i}\omega[t'-\theta_n-2(R+\hat{\boldsymbol{R}}_n\cdot\boldsymbol{z})/c]} \\ &\qquad\times\ \boldsymbol{\nu}\cdot\hat{\boldsymbol{R}}_n|\mathrm{e}^{\mathrm{i}\omega'(t'-t)}\,\mathrm{d}\omega\mathrm{d}\omega'\mathrm{d}t'\mathrm{d}S_{\boldsymbol{z}}\,,\end{aligned} \tag{12}$$

where the bar denotes complex conjugation. In (12) we carry out the integrations over ω' and t' to obtain

$$\eta_{\rm K}(\theta_n,t) = \frac{4\pi}{(8\pi^2R)^2}\int Q_{\rm K}(\boldsymbol{z})\,(\mathrm{i}\omega)|S_{\rm inc}(\omega)|^2\,\mathrm{e}^{-\mathrm{i}\omega[t-2(R+\hat{\boldsymbol{R}}(\theta_n)\cdot\boldsymbol{z})/c]}\boldsymbol{\nu}\cdot\hat{\boldsymbol{R}}_n\,\mathrm{d}\omega\mathrm{d}S_{\boldsymbol{z}}\,. \tag{13}$$

We introduce the notation $r_n(\boldsymbol{z}) = \hat{\boldsymbol{R}}_n\cdot\boldsymbol{z}$ and insert $\delta_{\partial\Omega}$ in order to convert the $\boldsymbol{z}$ integral to a three-dimensional one:

$$\eta_{\rm K}(\theta_n,t) = \frac{4\pi}{(8\pi^2R)^2}\int (\mathrm{i}\omega)|S_{\rm inc}(\omega)|^2\,\mathrm{e}^{-\mathrm{i}\omega[t-2r_n(\boldsymbol{z})/c]}Q_{\rm K}(\boldsymbol{z})\delta_{\partial\Omega}(\boldsymbol{z})\boldsymbol{\nu}\cdot\hat{\boldsymbol{R}}_n\,\mathrm{d}\omega\mathrm{d}\boldsymbol{z}\,. \tag{14}$$

Equation (14) is our model for the radar data in the single-scattering case. We note that the kernel of (14) involves an oscillatory integral, which suggests that it can be analyzed with the techniques of microlocal analysis.

2.2. Multiple scattering

Multiple scattering does not fit into the model discussed in section 2.1. In the case where there are only two isotropic point scatterers, we use the exact solution derived in Appendix A for the scattered field due to the incident wave (7). We consider the case of a rotating target; i.e., we replace $\boldsymbol{z}^j$ of (A.8) by $\mathcal{O}(\theta_n)\boldsymbol{z}^j$:

$$u^{\mathrm{sc}}(t,\boldsymbol{x}) = \sum_{j=1}^{2} \frac{1}{8\pi^2}\int g(t-t', |\boldsymbol{x}-\mathcal{O}(\theta_n)\boldsymbol{z}^j|)\frac{\mu_j S_{\mathrm{inc}}(\omega')}{1-\mu_1\mu_2\,\mathrm{e}^{\mathrm{i}2\omega' L/c}/(4\pi L)^2} \times\left(\frac{\mathrm{e}^{\mathrm{i}\omega'|\boldsymbol{x}-\mathcal{O}(\theta_n)\boldsymbol{z}^j|/c}}{|\boldsymbol{x}-\mathcal{O}(\theta_n)\boldsymbol{z}^j|}+\mu_{j'}\frac{\mathrm{e}^{2\mathrm{i}\omega' L/c}}{4\pi L}\frac{\mathrm{e}^{\mathrm{i}\omega'|\boldsymbol{x}-\mathcal{O}(t')\boldsymbol{z}^{j'}|/c}}{|\boldsymbol{x}-\mathcal{O}(\theta_n')\boldsymbol{z}^{j'}|}\right)\mathrm{e}^{-\mathrm{i}\omega'(t'-\theta_n)}\mathrm{d}\omega'\mathrm{d}t'\,, \tag{15}$$

where $j'=1$ if $j=2$ and $j'=2$ if $j=1$. Equation (15) is simplified as in section 2: we use the oscillatory-integral representation (2) for g; make the far-field approximation; use the orthogonality of $\mathcal{O}$; apply the change of variables $t''=t'-\theta_n$; and use the notation defined above (11). With these substitutions we obtain

$$u^{\mathrm{sc}}(\boldsymbol{x},n,t) = \frac{1}{(8\pi^2 R)^2}\sum_{\substack{j=1,2\\ j'\neq j}}\int\frac{\mu_j S_{\mathrm{inc}}(\omega')\,\mathrm{e}^{-\mathrm{i}\omega[t-t''-\theta_n-r_n(\boldsymbol{z}^j)/c]}}{1-\mu_j\mu_{j'}\,\mathrm{e}^{\mathrm{i}2\omega' L/c}/(4\pi L)^2} \times\left[\mathrm{e}^{\mathrm{i}\omega'[t''-r_n(\boldsymbol{z}^j)/c]}+\mu_{j'}\frac{\mathrm{e}^{\mathrm{i}\omega' L/c}}{4\pi L}\,\mathrm{e}^{\mathrm{i}\omega'[t''-r_n(\boldsymbol{z}^{j'})/c]}\right]\mathrm{d}\omega'\mathrm{d}\omega\mathrm{d}t''\,, \tag{16}$$

where $L\equiv|\boldsymbol{z}^j-\boldsymbol{z}^{j'}|$. Carrying out the t'' and ω' integrations results in

$$u^{\mathrm{sc}}(\boldsymbol{x},n,t) = \frac{1}{(8\pi^2 R)^2}\sum_{\substack{j=1,2\\ j'\neq j}}\int\frac{\mu_j S_{\mathrm{inc}}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega[t-\theta_n-r_n(\boldsymbol{z}^j)/c]}}{1-\mu_j\mu_{j'}\,\mathrm{e}^{\mathrm{i}2\omega L/c}/(4\pi L)^2} \times\left[\mathrm{e}^{-\mathrm{i}\omega r_n(\boldsymbol{z}^j)/c}+\mu_{j'}\frac{\mathrm{e}^{\mathrm{i}\omega L/c}}{4\pi L}\,\mathrm{e}^{-\mathrm{i}\omega r_n(\boldsymbol{z}^{j'})/c}\right]\mathrm{d}\omega\,, \tag{17}$$

The output of the correlation receiver is

$$\eta_{\mathrm{mult}}(\theta_n,t) = \frac{1}{(4\pi R)^2}\sum_{\substack{j=1,2\\ j'\neq j}}\int\frac{\mu_j|S_{\mathrm{inc}}(\omega)|^2}{(1-\mu_j\mu_{j'}\,\mathrm{e}^{\mathrm{i}2\omega L/c}/(4\pi L)^2)}\left[\mathrm{e}^{-\mathrm{i}\omega[t_n-2r_n(\boldsymbol{z}^j)/c]} +\frac{\mu_{j'}}{4\pi L}\mathrm{e}^{-\mathrm{i}\omega[t_n-[2r_n(\boldsymbol{z}^j)+L+\hat{\boldsymbol{R}}_n\cdot(\boldsymbol{z}^{j'}-\boldsymbol{z}^j)]/c]}\right]\mathrm{d}\omega\,, \tag{18}$$

Expanding the denominator of (18), retaining only terms cubic and lower in μ_j, and simplifying, we obtain

$$\eta_{\mathrm{mult}}(\theta_n,t) \approx \frac{1}{(4\pi R)^2}\sum_{\substack{j=1,2\\ j'\neq j}}\int\mu_j|S_{\mathrm{inc}}(\omega)|^2\left[\mathrm{e}^{-\mathrm{i}\omega[t_n-2r_n(\boldsymbol{z}^j)/c]}+\frac{\mu_{j'}}{4\pi L}\mathrm{e}^{-\mathrm{i}\omega[t_n-(r_n(\boldsymbol{z}^{j'})+r_n(\boldsymbol{z}^j)+L)/c]} +\frac{\mu_j\mu_{j'}}{(4\pi L)^2}\mathrm{e}^{-\mathrm{i}\omega[t_n-2(r_n(\boldsymbol{z}^j)+L)/c]}\right]\mathrm{d}\omega\,. \tag{19}$$

Equation (19) is our model for radar data in the multiple-scattering case. The first term on the right side corresponds to single scattering from the scatterer at position $\boldsymbol{z}^j$. The second term corresponds to a wave travelling first to $\boldsymbol{z}^j$, then a distance L to the scatterer at $\boldsymbol{z}^{j'}$, and then back to the radar. The third term corresponds to a wave travelling first to $\boldsymbol{z}^j$, then travelling a distance $2L$ to the other scatterer and back, and then returning to the radar.

We note that (19) is a sum of oscillatory integrals, to which the techniques of microlocal analysis can be applied.

Our multiple-scattering model (19) differs significantly from that of the single-scattering case in that additional bookkeeping must be performed to account for target substructure position relative to other scatterers. In addition, the multiple-scattering expression depends on the overall target orientation and involves multiplicative terms of the form $\exp(\mathrm{i}\omega mL/c)$ (for some integer m).

3. Wavefront sets for radar data

The target features that interest us are the boundary of the scattering object and localized scattering centers such as corners. These target features we characterize by the singular structure of Q, which we describe in terms of its wavefront set.

3.1. Wavefront sets

Mathematically the singular structure of a function can be characterized by its *wavefront set*, which involves both the location $\boldsymbol{x}$ and corresponding directions $\boldsymbol{\xi}$ of singularities [10, 14, 31, 33].

Definition. The point $(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$ is *not* in the wavefront set $\mathrm{WF}(f)$ of the function f if there is a smooth cutoff function ψ with $\psi(\boldsymbol{x}_0) \neq 0$, for which the Fourier transform $\mathcal{F}(f\psi)(\lambda\boldsymbol{\xi})$ decays rapidly (i.e., faster than any polynomial in $1/\lambda$) as $\lambda \to \infty$ for $\boldsymbol{\xi}$ uniformly in a neighborhood of $\boldsymbol{\xi}_0$.

This definition says that to determine whether $(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$ is in the wavefront set of f, one should 1) localize around $\boldsymbol{x}_0$ by multiplying by a smooth function ψ supported in the neighborhood of $\boldsymbol{x}_0$, 2) Fourier transform $f\psi$, and 3) examine the decay of the Fourier transform in the direction $\boldsymbol{\xi}_0$. Rapid decay of the Fourier transform in direction $\boldsymbol{\xi}_0$ corresponds to smoothness of the function f in the direction $\boldsymbol{\xi}_0$ [17].

Example: a point scatterer. If $Q(\boldsymbol{x}) = \delta(\boldsymbol{x})$, then $\mathrm{WF}(Q) = \{(\boldsymbol{0}, \boldsymbol{\xi}) : \boldsymbol{\xi} \neq \boldsymbol{0}\}$.

Example: a specular flash. Suppose $Q(\boldsymbol{x}) = H(\boldsymbol{x} \cdot \boldsymbol{\nu})$, where H denotes the Heaviside function. Then $\mathrm{WF}(Q) = \{(\boldsymbol{x}, \alpha\boldsymbol{\nu}) : \boldsymbol{x} \cdot \boldsymbol{\nu} = 0, \alpha \neq 0\}$.

Our strategy is to work out explicitly how the wavefront set of Q corresponds (via (13)) to the wavefront set of η. We denote the wavefront set of Q by

$$\mathrm{WF}(Q) = \{(\boldsymbol{z}, \boldsymbol{\zeta}) : \boldsymbol{\zeta} \neq \boldsymbol{0}\} \ . \tag{20}$$

For calculating the wavefront set of η, the basic tool is the method of stationary phase; the results we need are the following theorems [10, 14, 33].

Theorem 1 *(Wavefront set of an oscillatory integral) Suppose K is defined by*

$$K(\boldsymbol{x}) = \int \mathrm{e}^{\mathrm{i}\phi(\boldsymbol{\omega},\boldsymbol{x})} a(\boldsymbol{x},\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}, \tag{21}$$

where ϕ and a satisfy the following conditions:

(i) ϕ *is real-valued.*

(ii) $\phi(\lambda\boldsymbol{\omega},\boldsymbol{x}) = \lambda\phi(\boldsymbol{\omega},\boldsymbol{x})$.

(iii) At every point $(\boldsymbol{\omega},\boldsymbol{x})$, at least one of the derivatives $\partial_{x_j}\phi$ or $\partial_{\omega_j}\phi$ is nonzero.

(iv) There is some μ and M for which, on any compact set X, the estimate

$$|\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}\cdots\partial_{x_N}^{n_N}\partial_{\omega_1}^{m_1}\cdots\partial_{\omega_J}^{m_J} a(\boldsymbol{x},\boldsymbol{\omega})| \le C_{X,\boldsymbol{n},\boldsymbol{m}}(1+|\boldsymbol{\omega}|)^{\mu - M|\boldsymbol{m}|+(1-M)|\boldsymbol{n}|} \tag{22}$$

holds, with $|\boldsymbol{n}| = \sum n_j$.

Then the wavefront set of K satisfies

$$\mathrm{WF}(K) \subseteq \{(\boldsymbol{x},\nabla_{\boldsymbol{x}}\phi) : \nabla_{\boldsymbol{\omega}}\phi(x) = \mathbf{0}\}. \tag{23}$$

Theorem 2 *A Fourier integral operator $f(\boldsymbol{x}) = \int K(\boldsymbol{x},\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$ maps the wavefront set of g to the wavefront set of f according to the (twisted) canonical relation*

$$\Lambda' = \{[(\boldsymbol{x};\boldsymbol{\xi}),(\boldsymbol{y},\boldsymbol{\eta})] : (\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},-\boldsymbol{\eta}) \in \mathrm{WF}(K)\} \tag{24}$$

In other words, $\mathrm{WF}(f)$ is the set of $(\boldsymbol{x};\boldsymbol{\xi})$ for which $[(\boldsymbol{x};\boldsymbol{\xi}),(\boldsymbol{y},\boldsymbol{\eta})]$ is in Λ' for some $(\boldsymbol{y};\boldsymbol{\eta}) \in \mathrm{WF}(g)$.

3.2. Wavefront set for the Kirchhoff model

In (14), we let $t_n \equiv t - \theta_n$ denote the *fast time* (similarly, θ_n is the *slow time*). Then we can write (14) as

$$\eta_{\mathrm{K}}(\theta_n, t) = \int K_{\mathrm{K}}(\theta_n,\theta',t,\boldsymbol{z})[\delta_\Omega(\boldsymbol{z})Q_{\mathrm{K}}(\boldsymbol{z})\boldsymbol{\nu}_{\boldsymbol{z}}\cdot\hat{\boldsymbol{R}}(\theta')]\,\mathrm{d}\boldsymbol{z}\mathrm{d}\theta' , \tag{25}$$

where $\hat{\boldsymbol{R}}(\theta') = -\mathcal{O}^{\mathcal{T}}(\theta')\cdot\boldsymbol{x}$ and

$$K_{\mathrm{K}}(\theta_n,\theta',t,\boldsymbol{z}) = \frac{1}{(4\pi R)^2}\int (\mathrm{i}\omega)|S_{\mathrm{inc}}(\omega)|^2 \mathrm{e}^{-\mathrm{i}\omega[t_n - 2r_n(\boldsymbol{z})/c]} e^{i\omega'(\theta_n-\theta')}\,\mathrm{d}\omega\mathrm{d}\omega' . \tag{26}$$

Under the assumptions on S_{inc} of Theorem 1, equation (25) expresses η_{w} in terms of a Fourier Integral Operator applied to Q_{w}, and therefore the wavefront set of η_{w} can be calculated in terms of that of Q_{w} by Theorem 2.

First we calculate the canonical relation Λ' for the kernel K_{K}:

The canonical relation. We assume that $(\mathrm{i}\omega')^2|S_{\mathrm{inc}}(\omega)|^2$ satisfies the hypothesis of theorem 1. The phase of K_{w} is

$$\phi = -\omega[t_n - 2r_n(\boldsymbol{z})/c] + \omega'(\theta_n - \theta') \tag{27}$$

and so

$$\begin{aligned}\Lambda' = \Big\{ &(\theta_n, t:\sigma,\tau)(\boldsymbol{z},\theta';\boldsymbol{\zeta},\sigma') : t_n - 2r_n(\boldsymbol{z})/c = 0, \qquad \theta_n = \theta' \\ &\sigma = \partial_{\theta_n}\phi = -\frac{2\omega}{c}\frac{\partial\hat{\boldsymbol{R}}(\theta_n)}{\partial\theta_n}\cdot\boldsymbol{z} + \omega' \\ &\tau = \partial_t\phi = -\omega, \\ &\boldsymbol{\zeta} = -\nabla_z\phi = \frac{2\omega}{c}\hat{\boldsymbol{R}}(\theta_n) \\ &\sigma' = -\partial_{\theta'}\phi = \omega' \Big\}, \end{aligned} \tag{28}$$

The wavefront set of $[\delta_\Omega(\boldsymbol{z})Q_{\mathrm{K}}(\boldsymbol{z})\boldsymbol{\nu}_{\boldsymbol{z}}\cdot\hat{\boldsymbol{R}}(\theta')]$. We assume that Q_{K} and $\partial\Omega$ are smooth over the (typically small) data acquisition interval. Then the only singular part of $[\delta_\Omega(\boldsymbol{z})Q_{\mathrm{K}}(\boldsymbol{z})\boldsymbol{\nu}_{\boldsymbol{z}}\cdot\hat{\boldsymbol{R}}(\theta')]$ is the delta function supported on $\partial\Omega$. Thus the wavefront set of $[\delta_\Omega(\boldsymbol{z})Q_{\mathrm{K}}(\boldsymbol{z})\boldsymbol{\nu}_{\boldsymbol{z}}\cdot\hat{\boldsymbol{R}}(\theta')]$ is simply

$$\mathrm{WF}([\delta_\Omega(\boldsymbol{z})Q_{\mathrm{K}}(\boldsymbol{z})\boldsymbol{\nu}_{\boldsymbol{z}}\cdot\hat{\boldsymbol{R}}(\theta')]) = \{(\boldsymbol{z},\theta';\boldsymbol{\zeta},\sigma') : \ \boldsymbol{z}\in\partial\Omega, \ \theta' \text{ arbitrary}, \boldsymbol{\zeta}\propto\boldsymbol{\nu}_{\boldsymbol{z}}, \ \sigma'=0\} \tag{29}$$

The wavefront set of η_{K}. the wavefront set of η_{w} is contained in the set

$$\begin{aligned}\Big\{&(\theta_n, t;\sigma,\tau,) : \ t_n - 2r_n(\boldsymbol{z})/c = 0, \boldsymbol{z}\in\partial\Omega, \ \hat{\boldsymbol{R}}(\theta_n)\propto\boldsymbol{\nu}_{\boldsymbol{z}}, \\ &(\sigma,\tau)\propto\Big(-(2/c)\partial_{\theta_n}\hat{\boldsymbol{R}}(\theta_n)\cdot\boldsymbol{z}, 1\Big)\Big\}.\end{aligned} \tag{30}$$

In particular, the wavefront set corresponding to a single point scatterer at $\boldsymbol{z}^0$ will be the curve $t_n - 2r_n(\boldsymbol{z}^0)/c = 0$ whose normal vector is $(\sigma,\tau)\propto\Big((2/c)\partial_{\theta_n}\hat{\boldsymbol{R}}(\theta_n)\cdot\boldsymbol{z}^0, 1\Big)$.

3.3. Wavefront sets for multiple scattering

In the case of the two isotropic point scatters that we modeled in section 2.2, the target is simply a sum of two delta functions $Q_{\mathrm{mult}}(\boldsymbol{z}) = \delta(\boldsymbol{z}-\boldsymbol{z}^1) + \delta(\boldsymbol{z}-\boldsymbol{z}^2)$. The corresponding wavefront set is

$$\mathrm{WF}(Q_{\mathrm{mult}}) = \left\{(\boldsymbol{z}^1,\boldsymbol{\zeta}) : \ \text{all } \boldsymbol{\zeta}\neq\boldsymbol{0}\right\}\cup\left\{(\boldsymbol{z}^2,\boldsymbol{\zeta}) : \ \text{all } \boldsymbol{\zeta}\neq\boldsymbol{0}\right\} . \tag{31}$$

We see from (19) that multiple-scattering data can be expressed as a sum of oscillatory integrals $\eta_{\mathrm{mult}} \approx \eta_1 + \eta_2 + \eta_3$; to each we can simply apply Theorem 1. The corresponding phases are

$$\begin{aligned}\phi_1 &= -\omega(t_n - 2r_n(\boldsymbol{z}^j)/c) \\ \phi_2 &= -\omega(t_n - (r_n(\boldsymbol{z}^{j'}) + r_n(\boldsymbol{z}^j) + L)/c) \\ \phi_3 &= -\omega(t_n - 2(r_n(\boldsymbol{z}^j) + L)/c) .\end{aligned} \tag{32}$$

The wavefront set of η_1 is calculated from Theorem 1:

$$\mathrm{WF}(\eta_1) \subseteq \bigcup_{j=1,2} \{(\theta_n, t; \sigma'', \tau'') : \; t_n - 2r_n(\boldsymbol{z}^j)/c = 0, \\ (\sigma'', \tau'') = \omega((2/c)\partial_{\theta_n} \hat{\boldsymbol{R}}(\theta_n) \cdot \boldsymbol{z}^j, 1)\} \,. \tag{33}$$

The wavefront set of η_2 is

$$\mathrm{WF}(\eta_2) \subseteq \bigcup_{j=1,2} \{(\theta_n, t; \sigma'', \tau'') : \; t_n - (r_n(\boldsymbol{z}^{j'}) + r_n(\boldsymbol{z}^j) + L)/c = 0, \\ (\sigma'', \tau'') \propto \Big((2/c)\partial_{\theta_n} \hat{\boldsymbol{R}}(\theta_n) \cdot (\boldsymbol{z}^{j'} + \boldsymbol{z}^j), 1\Big)\} \,. \tag{34}$$

The wavefront set of η_3 is

$$\mathrm{WF}(\eta_3) \subseteq \bigcup_{j=1,2} \{(\theta_n, t; \sigma'', \tau'') : \; t_n - 2(r_n(\boldsymbol{z}^j) + L)/c = 0, \\ (\sigma'', \tau'') \propto \Big((2/c)\partial_{\theta_n} \hat{\boldsymbol{R}}(\theta_n) \cdot \boldsymbol{z}^j, 1\Big)\} \,. \tag{35}$$

Finally, the wavefront set of our three-term approximation to η_{mult} is the union $\mathrm{WF}(\eta_1) \cup \mathrm{WF}(\eta_2) \cup \mathrm{WF}(\eta_3)$.

We note that the critical curves in the θ_n–t_n plane are somewhat different for the single-, double-, and triple-scattering contributions. In particular, single-scattering curves are

$$t_n = \frac{2}{c}\left(R - \mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \cdot \boldsymbol{z}\right), \tag{36}$$

double-scattering curves are described by

$$t_n = \frac{2}{c}\left[R + \tfrac{1}{2}L - \mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \cdot \frac{\boldsymbol{z} + \boldsymbol{z}'}{2}\right] \tag{37}$$

and triple-scattering curves obey

$$t_n = \frac{2}{c}\left[R + L - \mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \cdot \boldsymbol{z}\right] \tag{38}$$

Multiple scattering from pairs of scattering centers can potentially be recognized in the data by the occurrence of collections of such curves. We note that the double- and triple-scattering curves are the same as single-scattering curves for scatterers rotating about more distant center points, and, in the double-scattering case, the apparent position of the scatterer relative to the center of rotation is midway between the two scatterers at $\boldsymbol{z}$ and $\boldsymbol{z}'$.

4. Conclusions and Future Work

Our discussion has not actually been about radar imaging. Instead, it has focused on the structure imposed upon measured radar data by a class of image features associated with the singular set of the radar target. Standard radar imaging schemes attempt to estimate precisely this class of features, however, and so our approach has "imaging" at its heart. In particular, we have shown that when the single-scattering approximation is valid, the location of the target's scattering centers can be estimated directly from the data wavefront set. We have also shown that the wavefront set for multiple-scattering events can be distinguished from single-scattering data.

We leave for the future the question of how knowledge of the singular structure of the radar data can best be exploited for target imaging and identification. There are a number of issues here. For image formation, the wavefront-set analysis suggests that reconstruction methods related to local tomography [12, 17] may be useful. In particular, analysis of wavefront sets can determine whether backprojection will provide an image free of certain artifacts [23, 25]. In addition, wavefront-set analysis suggests an approach for producing artifact-free, superresolved images: remove all components of the data set except those that correspond to well-understood target features, and form an image from those components only.

Practical implementation of the analysis in this paper requires that we be able to extract the wavefront set from radar data under conditions in which the data are noisy, have limited bandwidth, and are discretely sampled. The problem of extracting wavefront sets under such conditions is closely related to image processing problems such as edge detection, and these are active areas of current research. We explore one possible approach in [5], where we provide numerical examples of synthetic radar data and show how the wavefront set analysis enables us to estimate target parameters from very noisy data.

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Appendix A. Multiple Scattering

For a time-harmonic incident wave $U^{\rm inc}(\boldsymbol{x})$, the frequency-domain field $U^{\rm sc}$ scattered from N "point" scatterers can be obtained from the Foldy-Lax [34] equations together with the assumption that the scattered field from a single "point" scatterer is proportional to the Green's function G [28]:

$$U^{\rm sc}(\boldsymbol{x}) = \sum_{j=1}^{N} G(|\boldsymbol{x} - \boldsymbol{z}^j|)\mu_j U_j(\boldsymbol{z}^j) \tag{A.1}$$

$$U_j(\boldsymbol{x}) = U^{\rm inc}(\boldsymbol{x}) + \sum_{i\neq j} G(|\boldsymbol{x} - \boldsymbol{z}^i|)\mu_i U_i(\boldsymbol{z}^i), \quad j = 1, 2, \ldots, N, \tag{A.2}$$

where $G(r) = (4\pi r)^{-1}\exp(\mathrm{i}\omega r/c)$. Equation (A.1) says that the scattered field is the sum of the fields scattered from each scatterer; moreover, the field scattered from the jth scatterer is proportional to the field U_j that is incident upon the jth scatterer. Equations (A.2) say that the jth local incident field is the overall incident field plus the field scattered from all the other scatterers. If the scattering strengths $\mu_1, \mu_2, \ldots, \mu_N$ are known, the equations (A.2) can be solved for the U_j; then the total field can be found from (A.1).

In the case of two "point" scatterers, equations (A.2) are

$$U_1(\boldsymbol{x}) = U^{\rm inc}(\boldsymbol{x}) + G(|\boldsymbol{x} - \boldsymbol{z}^1|)\mu_2 U_2(\boldsymbol{z}^2) \tag{A.3}$$

$$U_2(\boldsymbol{x}) = U^{\rm inc}(\boldsymbol{x}) + G(|\boldsymbol{x} - \boldsymbol{z}^2|)\mu_1 U_1(\boldsymbol{z}^1) \tag{A.4}$$

Evaluating (A.3) at $\boldsymbol{z}^1$ and (A.4) at $\boldsymbol{z}^2$ gives rise to the system of equations

$$\begin{pmatrix} 1 & -\mu_2 G(L) \\ -\mu_1 G(L) & 1 \end{pmatrix} \begin{pmatrix} U_1(\boldsymbol{z}^1) \\ U_2(\boldsymbol{z}^2) \end{pmatrix} = \begin{pmatrix} U^{\rm inc}(\boldsymbol{z}^1) \\ U^{\rm inc}(\boldsymbol{z}^2) \end{pmatrix}, \tag{A.5}$$

where $L = |\boldsymbol{z}^2 - \boldsymbol{z}^1|$. These equations have the solutions

$$U_j(\boldsymbol{z}^j) = \frac{U^{\rm inc}(\boldsymbol{z}^j) + \mu_{j'} G(L) U^{\rm inc}(\boldsymbol{z}^{j'})}{1 - \mu_1\mu_2 G^2(L)}, \qquad j = 1, 2, \tag{A.6}$$

where $j' = 2$ if $j = 1$ and $j' = 1$ if $j = 2$. Using (A.6) in (A.1) yields

$$U^{\rm sc}(\boldsymbol{x}) = \sum_{j=1}^{2} G(|\boldsymbol{x} - \boldsymbol{z}^j|)\mu_j \frac{U^{\rm inc}(\boldsymbol{z}^j) + \mu_{j'} G(L) U^{\rm inc}(\boldsymbol{z}^{j'})}{1 - \mu_1\mu_2 G^2(L)}. \tag{A.7}$$

The time-domain scattered field due to the incident field (7) can be found by taking $G(|\boldsymbol{x} - \boldsymbol{z}^j|) = \int \exp(\mathrm{i}\omega'(t - t')g(t - t', |\boldsymbol{x} - \boldsymbol{z}^j|)\,\mathrm{d}t'$ and $U^{\rm inc}(\boldsymbol{z}^j) = S_{\rm inc}(\omega')(8\pi^2|\boldsymbol{x} - \boldsymbol{z}^j|)^{-1}\exp(\mathrm{i}\omega'|\boldsymbol{x} - \boldsymbol{z}^j|/c)\exp(\mathrm{i}\omega'\theta_n)$ in (A.7) and Fourier transforming from ω' to t. The exponentials involving t cancel, and we obtain

$$\begin{aligned} u^{\rm sc}(t, \boldsymbol{x}) = \sum_{j=1}^{2} \int g(t - t', |\boldsymbol{x} - \boldsymbol{z}^j|) \frac{\mu_j S_{\rm inc}(\omega')}{1 - \mu_1\mu_2\,\mathrm{e}^{\mathrm{i}2\omega' L}/(4\pi L)^2} \\ \times \left(\frac{\mathrm{e}^{\mathrm{i}\omega'|\boldsymbol{x}-\boldsymbol{z}^j|/c}}{8\pi^2|\boldsymbol{x} - \boldsymbol{z}^j|} + \mu_{j'} \frac{\mathrm{e}^{\mathrm{i}2\omega' L}}{4\pi L} \frac{\mathrm{e}^{\mathrm{i}\omega'|\boldsymbol{x}-\boldsymbol{z}^{j'}|/c}}{8\pi^2|\boldsymbol{x} - \boldsymbol{z}^{j'}|} \right) \mathrm{e}^{-\mathrm{i}\omega'(t'-\theta_n)}\,\mathrm{d}\omega'\mathrm{d}t'. \end{aligned} \tag{A.8}$$

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