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Atmospheric inner-scale effects on normalized irradiance variance

Charles A. Davis and D. L. Walters

We have investigated five types of atmospheric optical-turbulence inner scales for their effects on normalized laser irradiance variance in the Rytov and early saturation regimes: (1) zero inner scale, (2) Gaussian inner scale, (3) Hill’s viscous-convective enhancement inner scale, (4) Frehlich’s parameterization of the viscous-convective enhancement, and (5) turbulence spectrum truncation because of the discrete grid representation. Wave-optics computer simulations yielded normalized irradiance variances within 2% of the results from numerical integrations of the Rytov-Tatarskii predictions. In the Rytov regime a Gaussian inner scale reduces the normalized irradiance variance compared with the zero-inner-scale case, and the viscous-convective inner scale first raises, then lowers the irradiance variance as the inner-scale size increases. In the saturation regime all inner-scale models increase the intensity variance for a spherical wave.

Key phrases: Atmospheric optics, laser propagation, propagation simulation, irradiance fluctuations.

Introduction

Variations in the index of refraction exist within a turbulent medium. These variations alter the phase and irradiance statistics of a wave that is propagating through the medium. For small fluctuations in the index of refraction (Rytov regime), the Rytov-Tatarskii first-order perturbation theory describes the log-amplitude and the normalized irradiance variances on the basis of the three-dimensional spatial frequency spectrum of refractive-index fluctuations. Asymptotic theories address the strong fluctuation regime. Other authors have used wave-optics computer simulations to explore the transition to the saturation regime by characterizing the behavior of the normalized irradiance variance and the effects of the Gaussian inner scale and the Hill viscous-convective enhancement inner scale. In this investigation we have examined the Rytov and the saturation regions in detail by comparing the predicted normalized irradiance variance from numerical integration of the Rytov–Tatarskii results, with the inner scale included, with the results from a full wave-optics computer simulation of the electric field (E field) that is propagated through a turbulent medium.

We have described the behavior of the normalized irradiance variance with the Gaussian and the Hill inner scales relative to the zero-inner-scale case and have compared Frehlich’s parameterization of the viscous-convective enhancement with the model of Hill.

Theory

Tatarskii derived expressions for the statistical properties of the amplitude, phase, and irradiance of a spherical wave that propagates through a turbulent medium. He neglected polarization effects and used the scalar-wave equation,

\[ \nabla^2 u + k^2 n^2(r)u = 0. \]

We write the index of refraction as

\[ n(r) = 1 + n_1(r), \]

where \( |n_1(r)| \ll 1 \) and the Rytov method of smooth perturbations is used to write the E field as \( u = A \exp(iS) \exp(P) \) and the perturbation expansions as \( u = u_0 + u_1 + \ldots \) and \( P = \Psi_0 + \Psi_1 + \ldots \), giving us

\[ \nabla^2 \Psi_1 + 2\nabla \Psi_0 \nabla \Psi_1 + 2k^2 n_1(r) = 0. \]

Equation (3) has the solution

\[ \Psi_1(r) = \frac{k^2}{2\pi u_0(r)} \int n_1(r')u_0(r') \frac{\exp(ik |r - r'|)}{|r - r'|} dV'. \]
For a spherical wave

$$u_0(r) = \frac{Q \exp(ikr)}{r}, \quad Q = \text{constant},$$  \hspace{1cm} (5)$$
and for small wavelengths $\lambda \ll l_0$, where $l_0$ is the inner scale, the Fresnel approximation reduces Eq. (4) to

$$\Psi_1(r) = \frac{k^2 z}{2\pi} \int_0^n n_1(r') \exp \left[ ik \frac{z^2(x'^2 + y'^2) + z'^2(x^2 + y^2) - 2zz'(xx' + yy')}{2zz'(z - z')} \right] dV'.$$  \hspace{1cm} (6)$$

Substituting spectral expansions for $\Psi_1$ and $n_1$ and assuming local homogeneity and isotropy, Tatarskii developed an expression for the log-amplitude variance of a spherical wave,

$$\overline{X^2} = 4\pi^2 k^2 \int_0^L dK \Phi_n(K) \int_0^L dz \sin^2 \left( \frac{K^2 z(L - z)}{2kL} \right),$$  \hspace{1cm} (7)$$
where $\Phi_n(K)$ represents the three-dimensional spatial frequency spectrum of the index of refraction fluctuations for spatial wave number $K$. If we assume that the irradiance follows a log-normal distribution, the normalized irradiance variance relates to the log-amplitude variance as

$$\frac{\sigma_I^2}{\overline{X^2}} = \exp(4\overline{X^2}) - 1.$$  \hspace{1cm} (8)$$

Using a Kolmogorov turbulence spectrum with zero inner scale in Eq. (7), Tatarskii found the following analytical form for the log-amplitude variance of a spherical wave when a uniform strength of turbulence $C_n(z)$ along the path is assumed:

$$\overline{X^2} = 0.124 C_n^2 k^{7/6} L^{11/6}.$$  \hspace{1cm} (9)$$
Then the normalized irradiance variance of Eq. (8) with this log-amplitude variance becomes

$$\frac{\sigma_I^2}{\overline{X^2}} = \exp(4\overline{X^2}) - 1 = \exp(0.497 C_n^2 k^{7/6} L^{11/6}) - 1.$$  \hspace{1cm} (10)$$
A small $C_n^2 k^{7/6} L^{11/6}$ makes the exponent small enough to apply the approximation $\exp(\alpha) \approx 1 + \alpha$, which gives us

$$\frac{\sigma_I^2}{\overline{X^2}} = 0.497 C_n^2 k^{7/6} L^{11/6} \approx \beta_0^2,$$  \hspace{1cm} (11)$$
where $\beta_0^2$ serves as a baseline parameter for comparing the effects of the various nonzero inner scales and facilitates the plotting of normalized irradiance variance over a broad range of turbulence strengths because of the linearity with $C_n^{10}$. The assumption of isotropic, homogeneous Kolmogorov turbulence introduces a $K^{-11/3}$ power-law form of the spectrum. The inner scale alters the high-frequency roll-off of this spectrum. Flatté et al. write the three-dimensional isotropic spectrum in the form

$$\Phi_n(K) = 0.033 C_n^2 K^{-11/3} F(Kl_0),$$  \hspace{1cm} (12)$$
where $F(Kl_0)$ represents a particular functional form of the inner scale. The parameter $Kl_0$ consists of the spatial wave number $K$ (rad/m) and the inner-scale parameter $l_0$ (m).

For zero inner scale, $F(Kl_0)$ takes the form

$$F(Kl_0) = 1, \quad 0 < K < \infty.$$  \hspace{1cm} (13)$$
For theoretical and computational convenience, a Gaussian inner scale of the form

$$F(Kl_0) = \exp \left[ -\left( \frac{Kl_0}{5.9} \right)^2 \right]$$  \hspace{1cm} (14)$$
is often used. A more realistic viscous-convective enhancement inner scale advocated by Hill and by Frehlich exhibits an enhanced spectrum for the wave numbers just before spectrum roll-off. The Hill formulation is based on experimental temperature fluctuation measurements. Frehlich presents a slightly different inner-scale model based on laser scintillation measurements and provides a four-parameter fit to describe this version of the viscous-convective enhancement inner scale. Furthermore, the discrete numerical-grid mesh used to represent the E field imposes an abrupt inner scale at some maximum spatial frequency $K_{\text{max}} = 318 \text{ rad/m}$, where $\delta x$ is the grid element size; consequently

$$F(Kl_0) = \begin{cases} 1, & 0 < K < K_{\text{max}} \\ 0, & K > K_{\text{max}} \end{cases}.$$  \hspace{1cm} (15)$$
Figure 1 illustrates these inner scales for various inner-scale sizes. The abscissa gives us the spatial wave number (rad/m). These plots were generated with grid size $N = 1024$ and values appropriate to stratospheric propagation: path length $L = 200$ km, $\lambda = 0.5 \mu$m, and inner scales $l_0 = 2, 3, 4, 5, 10, \text{ and } 15$ cm. The numerical-grid spatial frequency cutoff was $K_{\text{max}} = 318 \text{ rad/m}$. 20 December 1994 / Vol. 33, No. 36 / APPLIED OPTICS 8407
tion algorithm propagates the complex E field $u(p, 0)$ into five different inner scales. The simulations from Cochrane, and is conceptually similar to the investigation was developed at the U.S. Air Force Phillips Laboratory by Ellerbroek. The wave-optics computer simulation used in this paper is a filtered Gaussian random array with amplitude $\phi = \text{Re}(0.0084k(C_n^2 \Delta z)^{1/2}/(N\Delta x)^{5/6})$ 

$$\times FT[(n_x^2 + n_y^2)^{-11/12}[F(Kl_0)]^{1/2}G(n_x, n_y)], \quad (18)$$

where $\Delta z$ is the optical path segment length, $C_n^2$ is the optical-turbulence parameter within the segment, $\Delta x$ is the grid element size, and FT implies a two-dimensional discrete Fourier transform. The $n_x$ and $n_y$ are the grid coordinate numbers, and $G$ represents an $N \times N$ array of complex unit-variance Gaussian random numbers. The refractive spatial-frequency spectrum $F_\nu(K)$ for the turbulent medium determined the magnitude and the spatial distribution of the random phases, and in the simulations we incorporated five inner scales, Eqs. (12)–(15), by altering the form of this spectrum.

To simulate a spherical wave, the initial field must approximate a point source. Some investigators have chosen a narrow Gaussian-like source. The simulations presented here used the discrete Fourier transform of a uniform circular aperture. This yielded the numerical equivalent of an Airy distribution as the source. After propagation with zero turbulence, this source distribution produced a uniform, circular E field with all energy confined within the computational grid. The uniformity of the E field was helpful for irradiance and coherence calculations compared with a Gaussian or other field.

The high spatial frequencies required for production of the abrupt circular representation created Fresnel fringes at intermediate ranges. We suppressed the energy in these high spatial frequencies by rounding the edges of the cylinder before the transform. However, identical runs that differed only in the shape of the cylinder edge showed less than 0.5% difference in the normalized irradiance variance. Thus, even modest amounts of turbulence introduce enough energy at high spatial frequencies to dominate the details of the high spatial frequency of the source edges.

For each propagation run we used 32 phase screens that were distributed uniformly along the optical path. The normalized irradiance variance came from an average of 30 simulation runs. For each simulation run we computed the normalized irradiance variance from the central 256 $\times$ 256 portion of the 1024 $\times$ 1024 grid. With turbulence present, the far-field intensity was not constant. We compensated the radial change in average intensity by computing the E-field intensity variance in concentric
rings and normalizing by the average intensity of each ring over all 30 runs.

The propagation parameters that were chosen relate to stratospheric propagation: propagation distance \( L = 200 \text{ km} \); wavelength \( \lambda = 0.5 \mu \text{m} \); index of refraction fluctuation inner scales \( l_0 = 5, 10, \) and \( 15 \text{ cm} \); and turbulence strength \( C_n^2 = 10^{-21}, 10^{-20}, 10^{-19}, \) and \( 10^{-17} \text{ m}^{-2/3} \). The numerical-grid spatial frequency cutoff was \( K_{\text{max}} = 318 \text{ rad/m} \).

**Results**

In Fig. 2 we plot the normalized irradiance variance versus \( \beta_0^2 \) in the Rytov regime for both numerical integration of the analytical theory, Eq. (7) (dotted lines), and Huygens–Fresnel simulation (solid lines) for Gaussian inner-scale values that include the numerical-grid cutoff and 5, 10, and 15 cm. All values were normalized by \( \beta_0^3 \). Numerical integration values for the numerical-grid cutoff inner scale agreed to within 1% of the theoretical zero-inner-scale values. The small difference in numerical integration values came from the energy neglected by the numerical integration for frequencies above the numerical grid cutoff \( K_{\text{max}} \). Larger \( K_{\text{max}} \) values yielded closer agreement. The abrupt inner scale that was inherent in the discrete numerical-grid representation had a negligible effect on the normalized irradiance variance in the Rytov regime for a sufficiently large grid (e.g., \( 1024 \times 1024 \)).

The computer-simulated normalized irradiance variances agreed to within 2% of the numerically integrated analytical values. A nonzero Gaussian inner scale reduced the normalized irradiance variance below the zero-inner-scale value (by 10, 25, and 40% for the 5-, 10-, and 15-cm cases, respectively). Effectively, the nonzero Gaussian inner scale suppressed the higher spatial frequency index of refraction fluctuations, reducing the variance.

Figure 3 shows the normalized irradiance variance (divided by \( \beta_0^2 \)) for \( \beta_0^2 = 5.0 \times 10^{-4} \) and Gaussian inner-scale sizes of numerical-grid cutoff and 5, 10, and 15 cm. Both the numerical integration values (dotted curve) and computer-simulation values (solid curve) were nearly identical and showed an almost-linear decrease of the normalized irradiance variance in the Rytov regime as the inner scale increased.

Figure 4 shows the normalized irradiance variance in the Rytov regime for Hill viscous-convective inner-scale sizes of numerical-grid cutoff and 5, 10, and 15 cm. Numerical integration of Eq. (7) (dotted lines) and computer simulation (solid lines) agreed to within 2%. For the smaller inner scales the normalized irradiance variance exceeded the zero-inner-scale values (by 30% for the 5-cm case). The slight spectral enhancement near the inner scale increased the optical variance. Yet, for a large enough inner scale, the overall reduction in high-frequency components reduced the variance below the zero-inner-scale variance (by 30% for the 15-cm case). The crossover occurred at 10 cm.

In Fig. 5 we plot the normalized irradiance variance (divided by \( \beta_0^2 \)) for \( \beta_0^2 = 5.0 \times 10^{-4} \) and Hill viscous-convective inner-scale sizes of 2, 3, 4, 5, 6, 7, 10, and 15 cm. The numerical integration (dotted
1.5 cm .................................
1.0 cm ....................................
0.5 cm ....................................
0.0 cm ....................................
0.0001 0.0010 0.0100
0.0001 0.0010 0.0100

Fig. 4. Normalized irradiance variance in the Rytov regime for the Hill viscous-convective enhancement inner scale: thick dashed line, Rytov-Tatarskii theory with a zero inner scale; dotted lines, numerical integration of the Rytov-Tatarskii results, Eq. (7), with Hill inner scales; solid lines, computer simulation with Hill inner scales.

Fig. 5. Normalized irradiance variance for \( \beta_0^2 = 5 \times 10^{-4} \) and viscous-convective enhancement inner scales of numerical-grid cutoff and 2, 3, 4, 5, 6, 7, 10, and 15 cm: dotted curve, numerical integration of Eq. (7) with the Hill inner scale; solid curve, computer simulation with the Hill inner scale; dashed curve, computer simulation with the Frehlich inner scale. The normalized irradiance variance reached a maximum for \( l_0 = 4 \) cm. The Hill and Frehlich versions agreed within 3%.

Fig. 6. Normalized irradiance variances from computer simulation for a spherical wave with zero inner scale (solid curve) and Gaussian inner scales of 5 cm (dashed curve), 10 cm (dashed–dotted curve), and 15 cm (dotted curve). In the Rytov regime the normalized irradiance variance monotonically decreased with increasing inner-scale size. In the saturation regime the normalized irradiance variance monotonically increased with increasing inner-scale size. A transition in the behavior occurred around \( \beta_0^2 = 1–3 \).

The dashed curve in Fig. 5 shows simulation values derived by the use of the Frehlich parameterization of the viscous-convective enhancement inner scale. The Frehlich inner scale shifted the plot slightly to smaller inner-scale sizes, had a maximum that was 2% less than the Hill maximum, and matched the Hill values within 3% over the range of the inner scale that was plotted. Additionally, simulation runs in which 4-cm Hill and Frehlich inner scales were used agreed within 3% over the range of turbulence strengths \( 5.0 \times 10^{-4} < \beta_0^2 < 50 \), which extends well into the saturation regime.

Previous investigation have illustrated the dramatic monotonic rise in normalized irradiance variance in the saturation regime as the inner-scale size increases.\(^2,4,5\) Figure 6 shows normalized irradiance variance for 200-km computer simulations, with a Gaussian inner scale that had inner scales of numerical-grid cutoff and 5, 10, and 15 cm. Figure 7 shows a corresponding plot for Hill viscous-convective inner scales of numerical-grid cutoff and 5, 10, and 15 cm. The turbulence values range from the Rytov regime (low turbulence with \( \beta_0^2 \leq 1.0 \)) to the saturation regime (high turbulence with \( \beta_0^2 \geq 1.0 \)). The Rytov
Fig. 7. Normalized irradiance variances from computer simulation for zero-inner-scale (solid curve) and Hill viscous-convective inner scales of 5 cm (dashed curve), 10 cm (dashed-dotted curve), and 15 cm (dotted curve).

The regime shows the behaviors illustrated in Figs. 2–5: increasing Gaussian inner-scale size forced a monotonically decreasing normalized irradiance variance. Martin and Flatté\textsuperscript{2,4} provided similar plots of normalized irradiance variance with a Gaussian inner scale. The Hill viscous-convective enhancement caused the normalized irradiance variance to rise and then fall as the inner-scale size increased. However, in the saturation regime, the normalized irradiance variance increased monotonically with increasing inner-scale size for both the Gaussian and the Hill inner scales. The transition in behavior occurred with the onset of saturation around $\beta_0^3 \sim 1$. This crossing behavior was plotted for the log-intensity variance with the viscous-convective inner scale by Hill and Clifford.\textsuperscript{8}

Figures 2–5 illustrate the close agreement between numerical integration and computer-simulation values for the normalized irradiance variance at low strengths of turbulence. This agreement provided a validity check on computer simulations that incorporated an inner scale.

Conclusions

The atmospheric optical-turbulence inner scale alters the irradiance statistics of a spherical wave that is propagating through a turbulent medium. In this paper we investigated four non-Kolomogorov inner scales, including a numerical-grid cutoff inner scale, a Gaussian inner scale, and Hill and Frehlich viscous-convective inner scales that used numerical integration of the analytical Rytov–Tatarskii predictions and wave-optics computer simulation. For low turbulence strengths (Rytov regime), the variances obtained from the simulations agreed within 2% of the predictions from the analytical theory. The numerical-grid cutoff inner scale that is implicit in discrete grid wave-optics computer simulations had a negligible effect on the irradiance variance compared with a true zero inner scale at low turbulence strengths with a large grid (e.g., $1024 \times 1024$). Gaussian inner scales reduced the normalized irradiance variance by as much as 40% compared with the zero-inner-scale case. The more realistic Hill and Frehlich viscous-convective enhancement inner scale raised the normalized irradiance variance by up to 30% for 4–5-cm inner-scale values; however, values larger than 10 cm reduced the variance below the zero-inner-scale value by as much as 30% for a 15-cm inner scale. These behaviors contrasted with the behavior in the saturation regime, where larger inner scales monotonically enhanced the normalized irradiance variance, regardless of the inner-scale form. A transition occurred around a normalized irradiance variance of $\beta_0^3 \sim 1$. The Hill and Frehlich versions of the viscous-convective enhancement inner scale exhibited the same behavior for the normalized irradiance variance in the Rytov and the saturation regimes, differing by only $\sim 3\%$.

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