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# A COMPLETE SOLUTION TO A SIMPLE RENDEZVOUS PROBLEM

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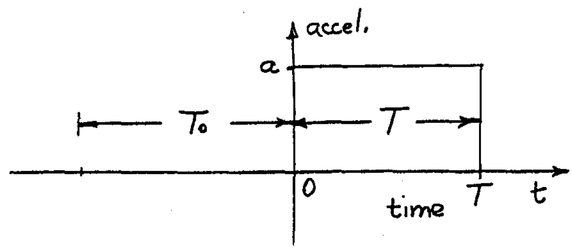
Graphs are given so that one can get all of the numbers associated with the optimum orbit-transfer problem described here. It is hoped that the ease and speed with which the solution can be obtained will make the simple case of interest. An elementary proof is given that the solution furnishes the desired optimum.

Assumptions and statement of the problem. The initial conditions giving the position and velocity of the target with respect to the missile are assumed to be given. The two are assumed to be near one another so that the difference of the gravitational acceleration on the two is negligible. The magnitude of the acceleration is assumed to be a given constant $a$ during the period of application. We wish to apply the acceleration starting at some time to be determined, for a time interval of length $T$, to be determined, so that at the end of this period the two are in orbit together; that is, rendezvous has been effected. The problem is to choose the time of starting and the direction of the acceleration vector so that the time of application $T$ is a minimum.

A reciprocal problem which can be solved with the same graphs is the one wherein the role of $T$ and $a$ is interchanged: the value of $T$ is given and we wish to find the smallest value of $a$ which will effect rendezvous.

A measure of the efficiency of the corresponding trajectory is also given.

The time at which the acceleration is first applied will be designated as $t = 0$; $T_0$ will designate the length of the interval between the time of measurement and the first application of the acceleration.



Derivation. Let us choose the coordinate set so that the relative motion is initially in the $xy$ plane; this can always be done by a simple rotation of coordinates. If the initial relative coordinates and velocity components are $X, Y, U, V$, respectively, let us further choose the axes so that $Y = Y_0 \geqq 0$, $U \geqq 0$, $V = 0$. The cases where either equality holds are trivial, so assume the strict inequalities.

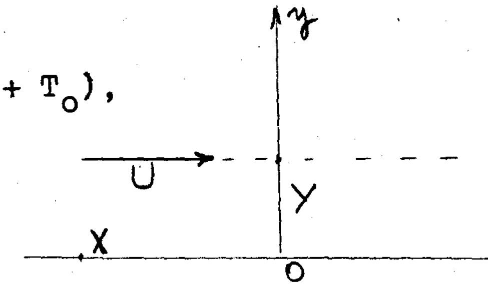
For rendezvous, there must be a time $T$ such that

$$(1) \qquad \int_0^T a \cos p \, dt = U > 0,$$

$$(2) \qquad \int_0^T a \, (T - t) \cos p \, dt = X + U(T + T_0),$$

$$(3) \qquad \int_0^T a \sin p \, dt = 0$$

$$(4) \qquad \int_0^T a \, (T - t) \sin p \, dt = Y > 0,$$




where $p$ is the angle between the acceleration vector and the $x$ axis, and is to be found. By virtue of equation (3), we may rewrite (4) as

$$(4') \qquad \int_0^T a \, t \sin p \, dt = - Y$$

Since $T_0$ can be chosen to satisfy (2) if the measurements are made early enough, we shall assume this to be the case and ignore equation (2) for the present.

Let us suppose now that we have found a path with the following properties.

($P_1$) It is admissible: that is, $p(t)$ is such that equations (1), (3), (4) are satisfied.

($P_2$) $\tan p = c_1 + c_2 t$, where $c_1$ and $c_2$ are constants (which we have been lucky or skilled enough to find, along with $T$), and

$$(5) \qquad \cos p \geqq 0$$

Theorem. For this path $T$ is a minimum.

Proof. Denote the values of $p$ associated with this path by

$\bar{p}$. Now multiply the equations (1), (3), (4') by $1, c_1, c_2$, resp., and add. We get

$$(6) \qquad \int_0^T a[\cos \bar{p} + (c_1 + c_2 t)\sin \bar{p}]\, dt = U - c_2 Y,$$

and we may rewrite the integral in the form

$$\int_0^T \vec{\lambda} \circ \vec{a}\, dt = \int_0^T |\vec{\lambda}|\, a\, dt,$$

where $\vec{\lambda} = i + (c_1 + c_2 t)j$, since $\vec{a}$ and $\vec{\lambda}$ are parallel by virtue of $(P_2)$.

Now consider any other admissible path, whereon $p$ is denoted by $P = P(t)$, which effects rendezvous in the same interval of time. It must satisfy (1), (3), (4') to be admissible. Form the sum and the equation corresponding to (6) and subtract both sides from the corresponding side of (6). Let $A$ be the magnitude of the acceleration on the second path; $A \leqq a$, and it may be less than if, for example,it is zero at some points, as would be the case if a shorter interval resulted. From the difference of the integrals, we get

$$\int_0^T [\vec{\lambda} \circ \vec{a} - \vec{\lambda} \circ A(\cos P\, i + \sin P\, j)]\, dt = 0,$$

or

$$\int_0^T |\vec{\lambda}|[(a - A) + A(1 - \cos(P - \bar{p}))]\, dt = 0;$$

the quantity $|\vec{\lambda}|A$ has been added and subtracted in the integrand. Clearly the integral vanishes only if $A = a$ for all $t$ in $(0,T)$ and if $P = \bar{p} \pmod{2\pi}$. There is a contradiction in assuming that another admissible path existed.

Thus, if we can find such a trajectory, it is the best.

It should be observed that the end/conditions were written in a form which did not involve $T$.

Let us take up now the integration of equations (1)--(4'). To cut down on the writing, let $R = \sqrt{1 + (c_1+c_2 t)^2}$. Then $\cos p = 1/R$ and $\sin p = (c_1 + c_2 t)/R$. From equation (3) $c_1 + c_2 t = \tau$

$$0 = a\int_0^T \frac{(c_1+c_2 t)}{R}\, dt = \frac{a}{c_2}\int_{c_1}^{c_1+c_2 T} \frac{\tau\, d\tau}{\sqrt{(1+\tau^2)}}$$

This vanishes if and only if $-c_1 = c_1 + c_2T$, or

(7) $c_2 = -2c_1/T$;

James Retka pointed out this relation. In the remainder, let $c_1$ be designated by simply $c$; $c = (\tan p)_{t=0}$.

Now consider equation (1)

$$U = \int_0^T (a/R)dt = \frac{aT}{2c}\int_{-c}^{c} \frac{d\tau}{\sqrt{(1+\tau^2)}} = aT\frac{1}{c}\ln(\sqrt{1+c^2} + c)$$

or, in dimensionless form,

(8) $U^* = U/(aT) = (1/c)\ln(\sqrt{1+c^2} + c)$.

Finally, let us consider equation (4'),

$$Y = -a\int_0^T t \sin p\, dt = aT^2/(4c^2)\int_{-c}^{c}\frac{\tau^2 d\tau}{\sqrt{(1+\tau^2)}},$$

which may be integrated to give the dimensionless variable

(9) $Y^* = Y/(aT^2/4) = (1/c^2)[c\sqrt{1+c^2} - \ln(\sqrt{1+c^2} + c)]$.

We may consider equations (8) and (9) as the parametric form of the equation of a curve in $Y^*U^*$ space. This curve is one of those drawn on the graph of Fig. 1. We will also need $W^* = U^{*2}$ vs. $Y^*$, so this is included.

The direction of thrust is given by (5) which becomes

(10) $\tan p = c(1 - 2t/T)$:

We need another curve to get $c$ or $p_0 = \arctan c$ ; the curves for $U^*$ and $Y^*$ vs $p_0$ are given in Fig. 2.

After these are found, from equation (2) we get

$$x(T) = (aT^2/2)(1/2c)\int_{-c}^{c}\frac{d\tau}{\sqrt{(1+\tau^2)}} = aT^2/2\ U^* = UT/2$$

$$= X + U(T + T_0),$$

so that

(11) $T_0 = -X/U - T/2$.

As was pointed out, it is necessary that $T_o \geqq 0$ for the solution to pertain.

The proof includes the case where the acceleration $A$ on the admissible paths is not required to equal $a$ but is only bounded by the inequality $A(t) \leqq a$, and the solution is the same for minimum $T$.

Use of the graphs. Problem 1. Suppose $X,Y,U,V=0,a$ have all been given: we wish to find $T,T_o$ such that $T$ is a minimum. We see that

$$W^*/Y^* = \frac{U^2}{a^2T^2} \cdot \frac{aT^2/4}{Y} = \frac{U^2}{4aY} ,$$

so that

$$W^* = \frac{U^2}{4aY} Y^*$$

represents a straight line in the $Y^*W^*$ plane. The intersection of this line with the curve for $W^*$ vs $Y^*$ of Fig.1 determines the value of $Y^*, W^*$ and hence $U^*$. We get $T$ then from (8),

$$T = U/(aU^*).$$

From the curve for $Y^*$ or $U^*$ vs $p_o$, the initial direction of thrust is found, and equation (11) gives $T_o$ .

Problem 2. Suppose $X,Y,U,V=0,T$ have all been given. We wish to find the smallest number $a$ such that rendezvous will be effected by an acceleration which never exceeds $a$ in magnitude.

The relation

$$U^*/Y^* = \frac{U}{aT} \frac{aT^2/4}{Y} = UT/(4Y)$$

determines a straight line in the $Y^*U^*$ plane; $Y^*$ and $U^*$ are determined by the intersection of this line and the given curve for $U^*$ vs $Y^*$. Then

$$a = U/(TU^*),$$

and $p_o,T_o$ are determined as before.

Efficiency of the Orbital Transfer Path. Two measures of the efficiency of the transfer path which can be obtained quickly are the following. If at the times $t = 0$ and $t = T$, impulses of the proper size to effect rendezvous are given, the sum of the impulses corresponds to a total velocity change

$$\sqrt{U^2 + 4Y^2/T^2} = aT\sqrt{U^{*2} + (Y^*/2)^2}.$$

Then

$$\sqrt{U^{*2} + (Y^*/2)^2}$$

is a measure of the efficiency, since $aT$ corresponds to the total velocity change in the above paths.

Since the lower bound to the change in velocity is $U$ for all possible paths, for those initial conditions, $U^*$ is another measure of the absolute efficiency of the path.

Comment. The curve for $Y^*$ vs $p_0$ is an interesting one in that it is apparently a straight line for $p_0 < 75^\circ$. Various checks do not seem to verify this, nor do they reveal any error.

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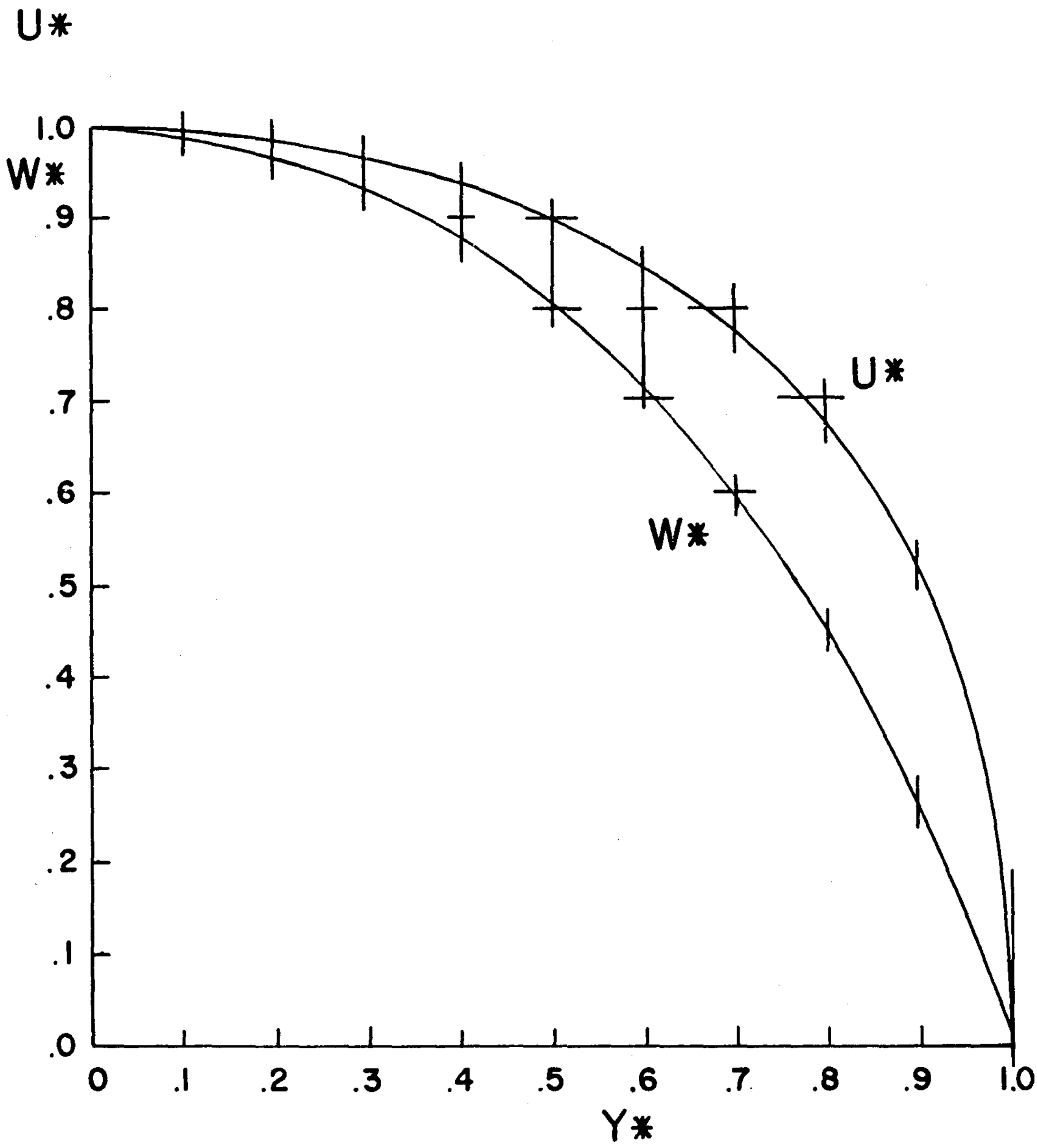


Fig. 1

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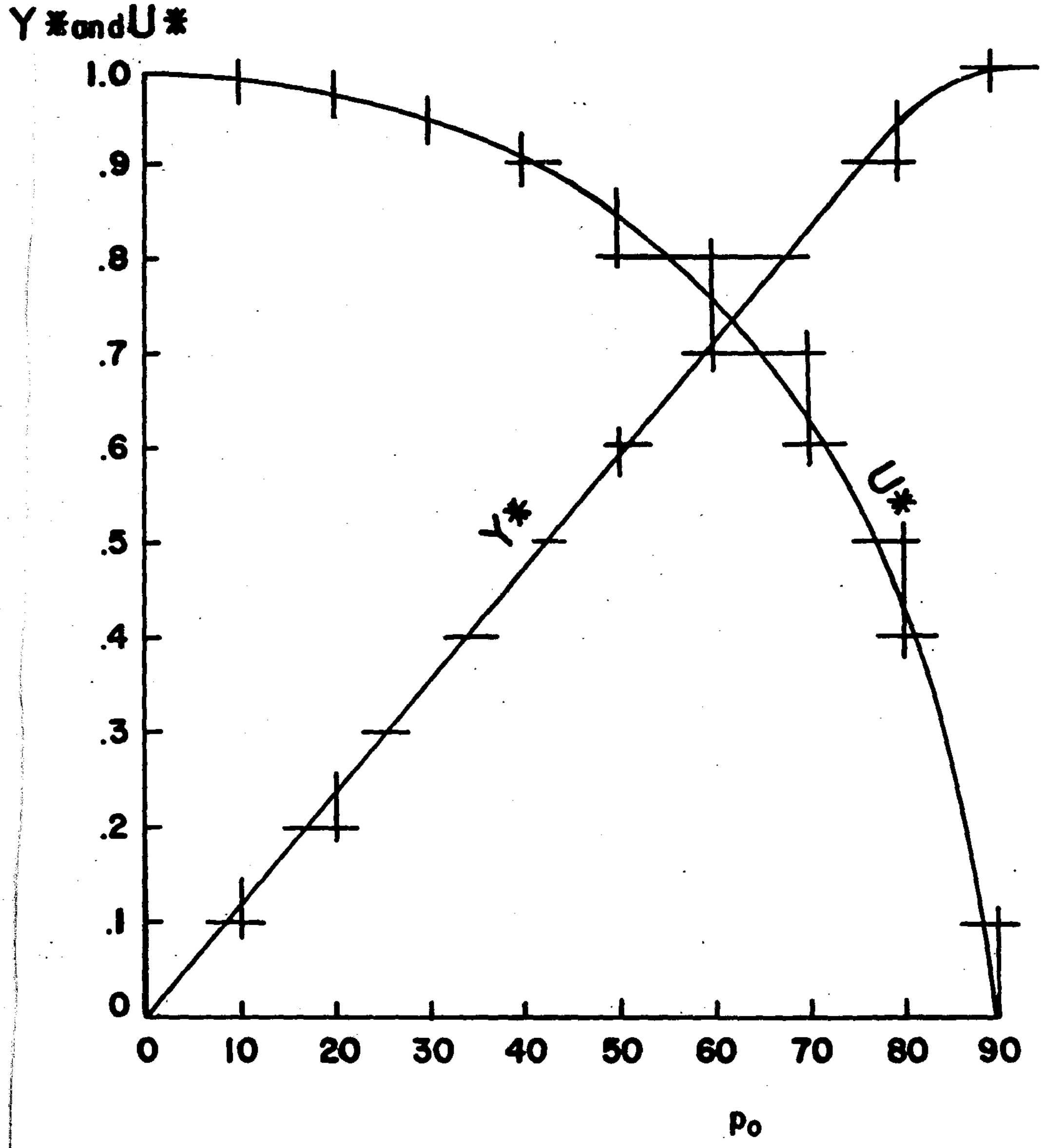


Fig. 2

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