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Adaptive Vision-Based Guidance Law with Guaranteed Performance Bounds for Tracking a Ground Target with Time-Varying Velocity

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This work extends the earlier results of authors on vision-based tracking of a ground vehicle moving with unknown time-varying velocity. The follower UAV is equipped with a single camera. The control objective is to regulate the 2D horizontal range between the UAV and the target to a constant. The extension in this paper has two distinct features.

The earlier developed guidance law used the estimates of the target’s velocity obtained from a fast estimation scheme. In this paper, we prove guaranteed performance bounds for the fast estimation scheme and explicitly derive the tracking performance bound as a function of the estimation error. The performance bounds imply that the signals of the closed-loop adaptive system remain close to the corresponding signals of a bounded closed-loop reference system both in transient and steady-state. The reference system is introduced solely for the purpose of analysis.

This paper also analyzes the stability and the performance degradation of the closed-loop adaptive system in the presence of out-of-frame events, when continuous extraction of the target’s information is not feasible due to failures in the image processing module. The feedback loop is then closed using the frozen estimates. The out-of-frame events are modelled as brief instabilities. A sufficient condition for the switching signal is derived that guarantees graceful degradation of performance during target loss. The results build upon the earlier developed fast estimation scheme of the target’s velocity, the inverse-kinematics-based guidance law and insights from switching systems theory.

I. Introduction

References 1–7 have reported theoretical and experimental results on vision-based tracking and motion estimation system for a small unmanned aerial vehicle (UAV) tasked to follow a ground target using a single camera. In that system, the UAV flies autonomously along a predefined search pattern, while a gimbal operator on the ground may select a target of interest using a joystick that steers the onboard gimballed camera. Real-time video, along with the UAV-gimbal telemetry, are transmitted to the ground station wirelessly. Once the target is selected, the UAV and the camera
automatically track the target. The system also performs real-time estimation of the target’s unknown velocity using UAV-gimbal telemetry data and the extracted target position on the image plane. For more information on the flight system, refer to Refs. 1, 2, 6 and the references therein.

Figure 1 shows the graphical illustration of the vision-based target tracking scenario. Let $\rho(t)$ denote the 2D horizontal range between the UAV and the target. The control objective is to regulate $\rho(t)$ to $\rho_d$, where $\rho_d$ is a given desired 2D horizontal range between the UAV and the target. For simplicity, we consider the case when $\rho_d$ is constant. For this system, the available measurements are listed below:

- Visual measurements $u(t)$ and $v(t)$ of the target’s center extracted by an image processing algorithm.
- Relative altitude $h(t)$ between the UAV and the target, obtained by geo-referencing the image captured by the onboard camera with a given database.
- UAV’s information from onboard sensors, such as the UAV’s velocity and orientation.

![Figure 1. Relative kinematics of UAV-target motion.](image)

In Ref. 6, we designed an inverse-kinematic-based controller to regulate $\rho(t)$ to $\rho_d$. The controller uses the estimates of the target’s time-varying velocity, obtained through a fast estimation scheme. In this paper, we analyze this controller in more details and show how the estimation performance bound affects the tracking performance.

We also analyze stability and performance degradation of the system in the presence of target loss, or out-of-frame events. Sufficient condition on the duration of the out-of-frame event is derived that guarantees stability and desired performance. The closed-loop system is viewed as a switching system that switches between one subsystem with continuous visual measurements and the other subsystem with frozen estimates. These frozen quantities are listed specifically when we discuss the out-of-frame events in Sec. V. The analysis is motivated by Ref. 4 and is cast into the framework of switching systems theory.

The paper is organized as follows. Section II describes the problem of vision-based ground target tracking. Essentials of motion estimation of the target’s unknown time-varying velocity are reviewed in Sec. III. Section IV shows that the resulting closed-loop adaptive system bears a close resemblance to a reference system, upon proper selection of the controller gains, both in transient
and steady-state. In Sec. V, we address the out-of-frame events. A sufficient condition on the out-of-frame switching signal is identified to ensure stability and tracking performance. Experimental and simulation results are given in Sec. VI. Conclusions are given in Sec. VII.

II. Problem Formulation

Let \( p(t) = [p_x(t), p_y(t), p_z(t)]^T \) be the position of the target with respect to the UAV in the inertial frame and let \( h(t) \) denote the relative altitude of the UAV above the target. Let \( V_{\text{uav}}(t) \) be the UAV’s speed and let \( V_g(t) \) be the projection of \( V_{\text{uav}}(t) \) onto the horizontal plane. Denoting the UAV flight path angle by \( \gamma(t) \), one has \( V_g(t) = V_{\text{uav}}(t) \cos \gamma(t) \). Let \( V_t(t) \) and \( \psi_t(t) \) be the amplitude and the orientation of the target’s velocity in the horizontal plane and \( V_h(t) \) be the rate of change of target elevation. Let \( \eta(t) \) denote the angle between the UAV’s velocity vector and the vector perpendicular to the line-of-sight (LOS) vector, as shown in Fig. 1. The kinematic equations for a UAV tracking a target can be written as \(^1\sim\)\(^4\),\(^6\):

\[
\hat{\rho}(t) = -V_g(t) \sin \eta(t) + V_t(t) \sin(\psi_t(t) - (\psi(t) - \eta(t))), \quad \rho(0) = \rho_0, \tag{1a}
\]

\[
\hat{\eta}(t) = \frac{-V_g(t) \cos \eta(t) - V_t(t) \cos[\psi_t(t) - (\psi(t) - \eta(t))]}{\rho(t)} + \dot{\psi}(t), \quad \eta(0) = \eta_0, \tag{1b}
\]

\[
\hat{p}(t) = -\begin{bmatrix}
V_g(t) \sin \psi(t) \\
V_g(t) \cos \psi(t) \\
V_{\text{uav}}(t) \sin \gamma(t)
\end{bmatrix} + \begin{bmatrix}
V_t(t) \sin \psi_t(t) \\
V_t(t) \cos \psi_t(t) \\
V_h(t)
\end{bmatrix}, \tag{1c}
\]

where

\[
\beta_1(\omega(t)) = \text{sign}(\rho_s(t))\sqrt{\rho_s^2(t) + \rho_v^2(t)}, \quad \beta_2(\omega(t)) = \tan^{-1}\left(\frac{\rho_v(t)}{\rho_s(t)}\right),
\]

\[
\rho_s(t) = -V_g(t) + V_t(t) \cos(\psi_t(t) - \psi(t)), \quad \rho_v(t) = V_t(t) \sin(\psi_t(t) - \psi(t)),
\]

\[
V_t(t) = \sqrt{\omega_1^2(t) + \omega_2^2(t)}, \quad \psi_t(t) = \tan^{-1}\left(\frac{\omega_1(t)}{\omega_2(t)}\right).
\]

The assumptions can be summarized as follows:

• \( V_g(t) \gg V_t(t) \).

• \( V_g(t) \) and \( \psi(t) \) denote UAV’s velocity and yaw angle that are available from the onboard sensors.

• \( \dot{\psi}(t) \) denotes the UAV’s yaw rate and is the control input to be designed.

• \( V_t(t) \) and \( \psi_t(t) \) denote the target’s time-varying velocity and heading angle, which are unknown and will be estimated. These two quantities cannot be continuously measured during the out-of-frame events.

• We note that \( \rho(t) \) and \( \eta(t) \) can be computed from the visual measurements \( u(t) \) and \( v(t) \) via algebraic relationships. Recall that \( u(t) \) and \( v(t) \) denote the coordinates of the target’s center, extracted by an image processing algorithm. Specifically, \( \rho(t) \) can be computed as \( \rho(t) = h(t) \sqrt{u^2(t) + v^2(t)} \), where the camera’s focal length has been assumed to be 1, without loss of generality. From Fig. 1(b), it is obvious that the signal \( \eta(t) \) is also related to the visual measurements. Thus, the signals \( \rho(t) \) and \( \eta(t) \) cannot be continuously measured during the out-of-frame events.
The relative position $p(t)$ can be calculated as:

$$p(t) = \begin{bmatrix} p_x(t) \\ p_y(t) \\ p_z(t) \end{bmatrix} = {}^c_c R p_e(t) = \begin{pmatrix} -\frac{h(t)}{-u(t) \sin \theta(t) + v \sin \varphi(t) \cos \theta(t) + \cos \varphi(t) \cos \theta(t)} \\ 1 \end{pmatrix},$$

(3)

where ${}^c_c R$ denotes the coordinate transformation from the camera frame to the inertial frame, $\varphi(t)$ and $\theta(t)$ represent the UAV’s (known) roll and pitch Euler angles for the rotation matrix ${}^c_c R$.

The control objective is to regulate $p(t)$ to $\rho_d$, where $\rho_d$ is a given desired 2D horizontal range between the UAV and the target. For simplicity, we consider a constant $\rho_d$. Notice that the relative altitude $h(t)$ is not regulated in this paper. The UAV altitude can be straightforwardly controlled by the on-board autopilot.

### III. Review of Target Motion Estimation

In Ref. 6 (Sec. III), we designed an inverse-kinematics-based controller using estimates of the target’s velocity. For the sake of completeness, some essential details on target motion estimation are reviewed in this section.

Let $x(t) = [p_x(t), p_y(t)]^T$, where $p_x(t)$ and $p_y(t)$ denote the $x-$ and $y-$ component of the relative position between the UAV and the ground target in the inertial frame. Notice that $x(t)$ can be computed from UAV’s onboard sensors and visual measurements of the target as given in (3). From equation (1c), we have

$$\begin{bmatrix} \dot{p}_x(t) \\ \dot{p}_y(t) \end{bmatrix} = -V_g(t) \begin{bmatrix} \sin \psi(t) \\ \cos \psi(t) \end{bmatrix} + V_I(t) \begin{bmatrix} \sin \psi_I(t) \\ \cos \psi_I(t) \end{bmatrix}. \tag{4}$$

Let

$$\omega(t) = V_I(t) \begin{bmatrix} \sin \psi_I(t) \\ \cos \psi_I(t) \end{bmatrix}, \quad \omega(0) = \omega_0. \tag{5}$$

Since the moving ground target is a mechanical system, subject to Newton’s second law, its velocity and acceleration are bounded. Therefore there exist constants $\mu_\omega$ and $d_\omega$ such that

$$\|\omega(t)\|_\infty \leq \mu_\omega < \infty, \quad \|\dot{\omega}(t)\|_\infty \leq d_\omega < \infty, \quad \forall \ t \geq 0. \tag{6}$$

The estimates of target’s velocity $V_I(t)$ and heading angle $\psi_I(t)$ (denoted by $\hat{V}_I(t)$ and $\hat{\psi}_I(t)$, respectively) can be obtained through the following steps $6, 8, 10$:

- **State Predictor:**

  $$\dot{x}(t) = A_m \ddot{x}(t) - V_g(t) \begin{bmatrix} \sin \psi(t) \\ \cos \psi(t) \end{bmatrix} + \hat{\omega}(t), \quad \ddot{x}(t) = \tilde{x}(t) - x(t), \quad \dot{x}(0) = x_0, \tag{7}$$

  where $A_m$ is a known $n \times n$ Hurwitz matrix chosen to satisfy performance requirements.

- **Adaptive Law:**

  $$\hat{\omega}(t) = \Gamma_c \text{Proj} (\hat{\omega}(t), -P \ddot{x}(t)), \quad \hat{\omega}(0) = \hat{\omega}_0, \tag{8}$$

  where $\Gamma_c \in \mathbb{R}^+$ determines the adaptation rate, chosen sufficiently large to ensure fast convergence, and $P$ is the solution of the algebraic equation $A_m^T P + P A_m = -Q$ for some choice of matrix $Q > 0$. 

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• **Low-Pass Filter:** Let

\[
\omega_r(s) = C(s) \omega(s), \quad \omega_r(0) = \dot{\omega}_0, \tag{9a}
\]

\[
\omega_c(s) = C(s) \dot{\omega}(s), \quad \omega_c(0) = \dot{\omega}_0, \tag{9b}
\]

where \(C(s)\) is a diagonal matrix, whose \(i^{th}\) diagonal element \(C_i(s)\) is a strictly proper, stable transfer function with low-pass gain \(C_i(0) = 1\) for \(i = 1, 2\), with \(s\) being the Laplace variable. Let

\[
C_i(s) = \frac{c}{s + c}, \quad i = 1, 2, \quad \text{with } c > 0. \tag{10}
\]

• **Extraction of \(\dot{V}_i(t)\) and \(\dot{\psi}_i(t)\) from \(\omega_c(t)\):**

\[
\dot{V}_i(t) = \sqrt{\omega_c^2(t) + \omega_c^2(t)}, \quad \dot{\psi}_i(t) = \tan^{-1}\left(\frac{\omega_c(t)}{\omega_c(t)}\right). \tag{11}
\]

The fast adaptive estimator ensures that \(\omega_c(t)\) estimates the unknown signal \(\omega(t)\) with the final precision:

\[
\lim_{t \to \infty} ||\omega_c - \omega||_{L_\infty} \leq ||\omega_c - \omega_r||_{L_\infty} + ||\omega_r - \omega||_{L_\infty} \leq \frac{\gamma_c}{\sqrt{\Gamma_c}} + \frac{1 - C(s)||L_1||\omega||L_\infty}}{\gamma_c}, \tag{12}
\]

where \(\cdot||L_\infty\) denotes the \(L_\infty\)-norm of a signal, and \(\cdot||L_1\) denotes the \(L_1\) gain of the system, while

\[
\gamma_c = \sqrt{\frac{\omega_m}{\lambda_{\min}(P)}} \||C(s)H^{-1}(s)||_{L_1}, \quad H(s) = (s \mathbb{I} - A_m)^{-1}, \quad \omega_m = 4\mu_0^2 + 2\mu_0 d^2 \lambda_{\max}(P) \lambda_{\min}(Q). \tag{13}
\]

Definitions of \(\cdot||L_\infty\) and \(\cdot||L_1\) are reviewed in Appendix VIII.A.

When the transients of \(C(s)\) due to the initial condition \(\dot{\omega}(0) - \omega(0)\) die out, \(\omega_c(t)\) estimates \(\omega(t)\) with the final precision given in (12). It is obvious that both the final estimation precision and the transient time can be arbitrarily reduced by increasing the bandwidth of \(C(s)\), which leads to smaller \(L_1\) gain for \(1 - C(s)||L_1\). However, the large bandwidth of \(C(s)\) leads to further increase of \(\gamma_c\) in (13), which requires large \(\Gamma_c\) to keep the term \(\frac{\gamma_c}{\sqrt{\Gamma_c}}\) small. We note that larger \(\Gamma_c\) implies faster computation and requires smaller integration step. The transient of \(\omega(t) - \omega_r(t)\) can also be quantified as

\[
||\omega(t) - \omega_r(t)||_{\infty} \leq ||\omega_0 - \dot{\omega}_0||_{\infty} e^{-ct} + \gamma_c, \quad \forall t \geq 0. \tag{14}
\]

### IV. Guaranteed Transient and Steady State Performance of the Vision-Based Guidance Law

Consider equations (1a) and (1b). The following guidance law was designed in our early work to regulate \(\rho(t)\) to \(\rho_d\) in the presence of continuous visual measurements.

**Controller with continuous visual measurements:**

\[
\begin{cases}
\psi(t) = \frac{V_\theta(t) \cos \eta(t) - \dot{V}_\theta(t) \cos(\dot{\psi}(t) - (\psi(t) - \eta(t))]}{\rho(t)} - k_2(\eta(t) - \eta_d(\omega_c(t), \rho(t))), \tag{15}
\end{cases}
\]

\[
\eta_d(\omega_c(t), \rho(t)) = \sin^{-1}\left(\frac{-k_1(\rho(t) - \rho_d)}{\beta_1(\omega_c(t))}\right) - \beta_2(\omega_c(t)),
\]

where \(k_i > 0\) for \(i = 1, 2\) are the design gains. The signal \(\omega_c(t)\) estimates the estimation of \(\omega(t)\), obtained through (7)–(11). The signals \(\dot{V}_i(t)\) and \(\dot{\psi}_i(t)\) are the estimated amplitude and heading angle of the target velocity.
IV.A. Closed-loop Reference System

We consider the following closed-loop reference system with its control signal and system response being defined as:

\[
\dot{\rho}_r(t) = \beta_1(\omega_r(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t))), \quad \rho_r(0) = \rho_0, \\
\dot{\eta}_r(t) = -\frac{V_g(t) \cos(\eta_r(t) - V_c(t) \cos(\psi_r(t) - (\psi(t) - \eta_r(t))}}{\rho_r(t)} + \psi_r(t), \quad \eta_r(0) = \eta_0, \\
\dot{\psi}_r(t) = \frac{V_g(t) \cos(\eta_r(t) - V_c(t) \cos(\psi_r(t) - (\psi(t) - \eta_r(t)))}{\rho_r(t)} - k_2(\eta_r(t) - \eta_d(\omega_r(t), \rho_r(t))), \\
\eta_d(\omega_r(t), \rho_r(t)) = \sin^{-1}\left(-\frac{k_1(\rho_r(t) - \rho_d)}{\beta_1(\omega_r(t))}\right) - \beta_2(\omega_r(t)), \\
\Omega_r(t) = \sin(\eta_r(t))
\]

where \(\omega_r(t)\) is given in (9a) and

\[
V_r(t) = \sqrt{\omega_{r1}^2(t) + \omega_{r2}^2(t)}, \quad \psi_r(t) = \tan^{-1}\left(\frac{\omega_{r1}(t)}{\omega_{r2}(t)}\right).
\]

We note that the reference system in (16) is not implementable since it uses the unknown signal \(\omega_r(t)\). This closed-loop reference system is only used for analysis purpose and it does not affect the implementation of the adaptive controller in (15). The purpose of introducing this reference system is to characterize both the transient and steady-state performance of the closed-loop adaptive system, defined by application of (15) to (1). This is achieved by first characterizing the transient and steady-state performance of the closed-loop reference system in (16) (to be shown next), and then demonstrating that the signals of the adaptive system can be designed to stay arbitrarily close to the corresponding signals of this reference system, upon proper selection of the controller gains within an appropriate domain of attraction (to be shown in Sec. IV.D).

We first consider boundedness of the signals that will be used in establishing the stability of the closed-loop reference system in (16). Let

\[
f(t) = -\frac{k_1(\rho_r(t) - \rho_d)}{\beta_1(\omega_r(t))}
\]

and compute \(\hat{\eta}_r(\omega_r(t), \rho_r(t))\) from (16)

\[
\hat{\eta}_d(\omega_r(t), \rho_r(t)) = \frac{1}{\sqrt{1 - f^2(t)}} \left(\frac{-k_1\dot{\rho}_r(t)\beta_1(\omega_r(t)) + k_1(\rho_r(t) - \rho_d)\beta_1(\omega_r(t))}{\beta_1^2(\omega_r(t))}\right) - \dot{\beta}_2(\omega_r(t)), \\
= \frac{1}{\sqrt{1 - f^2(t)}} \left(\frac{-k_1\dot{\rho}_r(t)}{\beta_1(\omega_r(t))} + f(t) \frac{\dot{\beta}_1(\omega_r(t))}{\beta_1(\omega_r(t))}\right) - \dot{\beta}_2(\omega_r(t)).
\]

It can be seen from (16) that \(\dot{\rho}_r(t)\) is bounded. Bounded \(\omega_r(t)\) implies that \(\beta_1(\omega_r(t))\), \(\dot{\beta}_1(\omega_r(t))\) and \(\dot{\beta}_2(\omega_r(t))\) are also bounded. It follows from equation (19) that if \(|f(t)| \leq 1 - \epsilon\) for \(\epsilon \in (0, 1]\), there exist finite numbers \(M_{d1}, M_{d2}\) and \(M_{d}(k_1)\) such that

\[
|\hat{\eta}_d(\omega_r(t), \rho_r(t))| < M_{d1}k_1 + M_{d2} \equiv M_{d}(k_1).
\]

Besides, when \(\rho_r(t) \geq \gamma_b\), where \(\gamma_b\) is a positive constant, there exists finite constant \(M_{\rho_r}\) such that

\[
\left|\frac{V_r(t) \cos(\psi_r(t) - (\psi(t) - \eta_r(t))) - V_c(t) \cos(\psi_r(t) - (\psi(t) - \eta_r(t)))}{\rho_r(t)}\right| < M_{\rho_r}.
\]
Further, notice that \( \omega(t), \omega_r(t), V_t(t), \psi(t), V_r(t) \) and \( \psi_r(t) \) are bounded signals. Since the functions \( \beta_1(t) \) and \( \beta_2(t) \) are continuous, there exist finite constants \( \mu_{\beta_1}, \mu_{\beta_2}, M_t, M_{\beta_1}, M_{\beta_2}, \) and \( L_{\beta_1} \) such that for all \( t \geq 0 \)

\[
|V_t(t)| < M_t, \quad |\beta_1(\omega(t))| < M_{\beta_1}, \quad L_{\beta_1} < |\beta_1(\omega_r(t))| < M_{\beta_1},
\]

\[
|\beta_1(\omega(t)) - \beta_1(\omega_r(t))| < \mu_{\beta_1} \|\omega(t) - \omega_r(t)\|_{\infty} < \mu_{\beta_1} \|\omega(t) - \omega_r(t)\|_{\infty} e^{-ct} + \gamma_{\omega}, \quad (20c)
\]

\[
|\beta_2(\omega(t)) - \beta_2(\omega_r(t))| < \mu_{\beta_2} \|\omega(t) - \omega_r(t)\|_{\infty} < \mu_{\beta_2} \|\omega(t) - \omega_r(t)\|_{\infty} e^{-ct} + \gamma_{\omega},
\]

where \( \gamma_{\omega} \) is given in (12).

The next theorem establishes the stability of the closed-loop reference system in (16).

**Theorem 1** If the controller gains \( k_1, k_2 \) and the low-pass filter in the estimator are chosen to verify

\[
k_2 > \frac{8(M_{\rho_r} + M_d(k_1))M_{\beta_1}}{L_{\beta_1}(1 - \epsilon)}, \quad k_2 > k_1,
\]

(21a)

\[
\|1 - C(s)\|_{L_1} < \frac{L_{\beta_1}(1 - \epsilon)}{8M_t(M_{\beta_1} + \mu_{\beta_1})},
\]

(21b)

the initial conditions \( \omega_r(0), \rho_r(0), \eta_r(0) \) and the reference \( \rho_d \) comply with

\[
\|\omega_r(t) - \omega_r(0)\|_{\infty} < \frac{L_{\beta_1}(1 - \epsilon)}{4(M_{\beta_1} + \mu_{\beta_1})},
\]

(22a)

\[
|\rho_r(0) - \rho_d| < \frac{L_{\beta_1} \epsilon}{k_1} + \left[ \frac{(M_{\rho_r} + M_d(k_1))M_{\beta_1}}{k_1} \right] + \left[ \frac{(M_{\beta_1} + \mu_{\beta_1})(\gamma_{\omega} + \|\omega_0 - \omega_r\|_{\infty})}{k_1} \right],
\]

(22b)

\[
|\eta_r(0) - \eta_d(\omega_r(0), \rho_r(0))| < \frac{k_1}{M_{\beta_1}} |\rho_r(0) - \rho_d|,
\]

(22c)

\[
B_0 + \gamma_6 < \rho_r(0),
\]

(22d)

\[
\rho_d \geq |\rho_r(0) - \rho_d| + B_0 + \gamma_6,
\]

(23)

the closed-loop reference system (16) is uniformly ultimately bounded. Moreover,

\[
|\rho_r(t) - \rho_d| < \gamma_{r_1}(t) < \frac{L_{\beta_1} \epsilon}{k_1} (1 - \epsilon),
\]

(24a)

\[
|\eta_r(t) - \eta_d(\omega_r(t), \rho_r(t))| < \gamma_{r_2}(t),
\]

(24b)

with

\[
\gamma_{r_1}(t) = e^{-k_1 t} |\rho_r(0) - \rho_d| + \frac{M_{\beta_1} |\eta_r(0) - \eta_d(\omega_r(0), \rho_r(0))|}{k_1} (e^{-k_2 t} - e^{-k_1 t})
\]

\[
+ \frac{M_{\beta_1} \mu_{\beta_2} + \mu_{\beta_1}}{k_1} \|\omega_0 - \omega_0\|_{\infty} e^{-ct} + \frac{(M_{\rho_r} + M_d(k_1))M_{\beta_1}}{k_1} + \frac{(M_{\beta_1} + \mu_{\beta_1})(\gamma_{\omega})}{k_1},
\]

(25a)

\[
\gamma_{r_2}(t) = e^{-k_2 t} (\eta_r(0) - \eta_d(\omega_r(0), \rho_r(0))) + \frac{M_{\rho_r} + M_d(k_1)}{k_2}.
\]

(25b)

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Therefore, before continuing with the proof, let us clarify the relationship between 

If (21) is not true for all \( r \), there exists \( \tau_u \) such that

If (29) is not true for all \( t \geq 0 \), since \( f(t) \) is continuous and \( |f(0)| < 1 - \epsilon \), there exists \( \tau_u \) such that

It then follows from (19) and (31a) that \( \dot{\eta}_d(\omega_r(t), \rho_r(t)) \) is bounded over \([0, \tau_u] \). Thus, \( M_d(k_1) \) in (20a) exists.

If \( |\rho_r(t) - \rho_d| < \gamma_{r_1}(t) \) is not true, since

and \( \rho_r(t) \) is continuous, there exists \( \tau \) such that

Before continuing with the proof, let us clarify the relationship between \( \tau \) and \( \tau_u \). From equation (31b), we have

Therefore, \( \tau < \tau_u \).

It follows from (33a) that

\[ \rho_d - \gamma_{r_1}(t) \leq \rho_r(t) \leq \rho_d + \gamma_{r_1}(t), \quad \forall t \in [0, \tau]. \]
Then, choosing \( \rho_d \) according to (23) leads to
\[
\rho_r(t) \geq \gamma_b. \tag{34}
\]
Thus, \( M_{\rho_r} \) in (20b) exists. Note that such selection of \( \rho_d \) is feasible, as shown in Appendix VIII.D.

Considering \( \dot{\eta}_r(t) - \dot{\eta}_d(\omega_r(t), \rho_r(t)) \), it follows from the dynamics of \( \eta_r(t) \) in (16) that
\[
\begin{aligned}
\dot{\eta}_r(t) - \dot{\eta}_d(\omega_r(t), \rho_r(t)) &= -k_2(\eta(t) - \eta_d(\omega_r(t), \rho_r(t))) \\
&+ \frac{V_r(t) \cos(\psi(t) - (\psi(t) - \eta_r(t))) - V_r(t) \cos(\psi_r(t) - (\psi(t) - \eta_r(t)))}{\rho_r(t)} - \dot{\eta}_d(\omega_r(t), \rho_r(t)). \tag{35}
\end{aligned}
\]

Summarizing the above, it follows from (20), (31a), (34) and (35) that \( \delta_{r_1}(t) \) is bounded by
\[
|\delta_{r_1}(t)| < M_{\rho_r} + M_d(k_1). \tag{36}
\]

We note that \( \eta_r(t) \) can be decomposed into two components
\[
\eta_r(t) = \eta_{r_1}(t) + \eta_{r_2}(t), \tag{37}
\]
where \( \eta_{r_1}(t) \) and \( \eta_{r_2}(t) \) are defined via:
\[
\begin{aligned}
\dot{\eta}_{r_1}(t) - \dot{\eta}_d(\omega_r(t), \rho_r(t)) &= -k_2(\eta_r(t) - \eta_d(\omega_r(t), \rho_r(t))), \quad \eta_{r_1}(0) = \eta_r(0) - \eta_d(\omega_r(0), \rho_r(0)), \tag{38a} \\
\dot{\eta}_{r_2}(t) - \dot{\eta}_d(\omega_r(t), \rho_r(t)) &= -k_2(\eta_{r_2}(t) - \eta_d(\omega_r(t), \rho_r(t))) + \delta_{r_1}(t), \quad \eta_{r_2}(0) = \eta_d(\omega_r(0), \rho_r(0)). \tag{38b}
\end{aligned}
\]
Consider (38a). We have
\[
\eta_{r_1}(t) - \eta_d(\omega_r(t), \rho_r(t)) = e^{-k_2t}(\eta_r(0) - \eta_d(\omega_r(0), \rho_r(0))). \tag{39}
\]
Consider (38b). It follows from (36) and Lemma 1 (see Appendix VIII.B) that
\[
|\eta_{r_2}(t) - \eta_d(\omega_r(t), \rho_r(t))| < \frac{M_{\rho_r} + M_d(k_1)}{k_2}. \tag{40}
\]
Equation (39), together with the inequality in (40), leads to
\[
|\eta_r(t) - \eta_d(\omega_r(t), \rho_r(t))| < e^{-k_2t}(\eta_r(0) - \eta_d(\omega_r(0), \rho_r(0))) + \frac{M_{\rho_r} + M_d(k_1)}{k_2}, \tag{41}
\]
which proves the relationship in (24b).

It follows from the dynamics of \( \rho_r(t) \) in (16) that
\[
\begin{aligned}
\dot{\rho}_r(t) &= -k_1(\rho_r(t) - \rho_d) + \beta_1(\omega_r(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t))) - \beta_1(\omega_r(t)) \sin(\eta_d(\omega_r(t), \rho_r(t)) + \beta_2(\omega_r(t))), \\
&= -k_1(\rho_r(t) - \rho_d) + \beta_1(\omega_r(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t))) - \beta_1(\omega_r(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t))) \\
&+ \beta_1(\omega_r(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t))) - \beta_1(\omega_r(t)) \sin(\eta_d(\omega_r(t), \rho_r(t)) + \beta_2(\omega_r(t))). \tag{42}
\end{aligned}
\]

Considering \( \delta_{r_2}(t) \) in (42), we have:
\[
|\delta_{r_2}(t)| = |\beta_1(\omega_r(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t))) - \beta_1(\omega_r(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t)))|,
\]
\[
\leq |\beta_1(\omega(t)) \sin(\eta_r(t) + \beta_2(\omega(t))) - \beta_1(\omega(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t)))| + |\beta_1(\omega(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t))) - \beta_1(\omega(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t)))| \tag{43}
\]
\[
+ |\beta_1(\omega(t)) \sin(\eta_r(t) + \beta_2(\omega(t))) - \beta_1(\omega(t)) \sin(\eta_r(t) + \beta_2(\omega_r(t)))| \\
\leq |\beta_1(\omega(t))| |\sin(\eta_r(t) + \beta_2(\omega(t))) - \sin(\eta_r(t) + \beta_2(\omega_r(t)))| + |\beta_1(\omega(t)) - \beta_1(\omega_r(t))|.
\]
It follows from inequalities (20c) and (43) that
\[
|\delta_{r_2}(t)| < (M_{\beta_1} \mu_{\beta_2} + \mu_{\beta_1}) \|\omega(t) - \omega_r(t)\|_{\infty}, \quad \forall \ t \geq 0. \tag{44}
\]

Considering \(\delta_{r_3}(t)\) in (42), it follows from (41) that
\[
|\delta_{r_3}(t)| < |\beta_1(\omega_r(t))| |\eta_t(t) - \eta_d(\omega_r(t), \rho_r(t))|,
\]
\[
< |\beta_1(\omega_r(t))| e^{-k_2 t}(\eta_t(0) - \eta_d(\omega_r(0), \rho_r(0)) + \frac{M_{\rho_r} + M_d(k_1)}{k_2}). \tag{45}
\]

It follows from inequalities (20c), (44) and (45) that for all \(t \geq 0\)
\[
|\delta_{r_2}(t) + \delta_{r_3}(t)| < M_{\beta_1} e^{-k_2 t}|\eta_t(0) - \eta_d(\omega_r(0), \rho_r(0))| + \frac{M_{\beta_1} (M_{\rho_r} + M_d(k_1))}{k_2} + (M_{\beta_1} \mu_{\beta_2} + \mu_{\beta_1}) \|\omega(t) - \omega_r(t)\|_{\infty}. \tag{46}
\]

Similarly, we note that \(\rho_r(t)\) can be decomposed into two components
\[
\rho_r(t) = \rho_{r_1}(t) + \rho_{r_2}(t), \tag{47}
\]
where \(\rho_{r_1}(t)\) and \(\rho_{r_2}(t)\) are defined via:
\[
\dot{\rho}_{r_1}(t) = -k_1(\rho_{r_1}(t) - \rho_d), \quad \rho_{r_1}(0) = \rho_r(0) - \rho_d, \tag{48a}
\]
\[
\dot{\rho}_{r_2}(t) = -k_1(\rho_{r_2}(t) - \rho_d) + \delta_{r_2}(t) + \delta_{r_3}(t), \quad \rho_{r_2}(0) = \rho_d. \tag{48b}
\]

Consider (48a). We have
\[
\rho_{r_1}(t) - \rho_d = e^{-k_1 t}(\rho_r(0) - \rho_d). \tag{49}
\]

To study the behavior of (48b), first consider the following scalar LTV system
\[
\dot{x}(t) = -\bar{k}_1 x(t) + \delta(t), \quad x(0) = 0, \tag{50}
\]
with
\[
|\delta(t)| < \bar{M} e^{-k_2 t}, \tag{51}
\]
where \(\bar{k}_1, \bar{k}_2\) and \(\bar{M}\) are positive constants. For the system in (50) subject to (51), we have:
\[
|x(t)| = \left| \int_0^t e^{-k_1(t-\tau)} \delta(\tau) \, d\tau \right| \leq \int_0^t e^{-k_1(t-\tau)} |\delta(\tau)| \, d\tau,
\]
\[
< \int_0^t e^{-k_1(t-\tau)} \bar{M} e^{-k_2 \tau} \, d\tau < \frac{\bar{M}}{k_1 - k_2} \left( e^{-k_2 t} - e^{-k_1 t} \right). \tag{52}
\]

It follows from inequalities (46), (52) and Lemma 1 that
\[
|\rho_{r_2}(t) - \rho_d| < \frac{M_{\beta_1} |\eta_t(0) - \eta_d(\omega_r(0), \rho_r(0))|}{k_1 - k_2} \left( e^{-k_2 t} - e^{-k_1 t} \right)
\]
\[
+ \frac{(M_{\rho_r} + M_d(k_1))M_{\beta_1}/k_2 + (M_{\beta_1} \mu_{\beta_2} + \mu_{\beta_1}) \|\omega(t) - \omega_r(t)\|_{\infty}}{k_1}, \quad \forall \ t \geq 0. \tag{53}
\]

Equation (49), along with the inequalities in (14) and (53), leads to
\[
|\rho_r(t) - \rho_d| < \gamma_{r_1}(t),
\]
which contradicts (33b). Therefore, the relationship in (28) holds. Finally, it follows from (27) and (28) that the relationship in (24a) holds over \([0, \tau_{ul}]\).
Accordingly, for \( t \in [0, \tau_u] \),
\[
|f(t)| = \left| \frac{-k_1(\rho_r(t) - \rho_d)}{\beta_1(\omega_r(t))} \right| < \frac{k_1 \gamma_r(t)}{\beta_1(\omega_r(t))} < \frac{k_1}{\beta_1(\omega_r(t))} \frac{L_{\beta_r}}{k_1} (1 - \epsilon) < 1 - \epsilon, \tag{54}
\]
which contradicts (31b). Therefore, inequality (29) holds along the system trajectories in (16) subject to (22) for all \( t \geq 0 \). This completes the proof. \( \square \)

**Remark 1 (The Conditions in (22))**

- **Condition (22b)** ensures that \(|f(t)| < 1 - \epsilon\) so that \( \sin^{-1}(f(t)) \) is well defined.
- **Condition (22c)**, along with the selection of design gains \( k_2 > k_1 \) in (21a), ensures that the tracking performance bound of \( \rho_r(t) - \rho_d \) decreases monotonically per the upper bound given in (25a) for all \( t \geq 0 \).
- Since \( \rho_r(t) \) appears in the denominator of (16), \( \rho_r(t) \) needs to be regulated to be bounded away from zero. Selection of \( \rho_d \) in (23) helps to achieve this objective. **Condition (22d)** ensures that such selection of \( \rho_d \) is feasible.

**IV.B. Region of Attraction**

Let
\[
\begin{align*}
\tilde{X}_0 &= [\tilde{\omega}_0, \tilde{\rho}_0, \tilde{\eta}_0]^\top, \\
\tilde{\omega}_0 &= \omega_r(0) - \omega_0 = \dot{\omega}_0 - \omega_0, \\
\tilde{\rho}_0 &= \rho_r(0) - \rho_0 - \rho_d, \\
\tilde{\eta}_0 &= \eta_r(0) - \eta_d(\omega_r(0), \rho_r(0)) = \eta_0 - \eta_d(\dot{\omega}_0, \rho_0). 
\end{align*} \tag{55}
\]

The inequalities in (22a)–(22c) can be rewritten as
\[
\begin{align*}
\|\tilde{\omega}_0\|_\infty &< \frac{L_{\beta_r}(1 - \epsilon)}{4(M_{\beta_1}\mu_{\beta_2} + \mu_{\beta_1})}, \tag{56a} \\
|\tilde{\rho}_0| &< \frac{L_{\beta_r}(1 - \epsilon)}{k_1} - B_0 < \frac{L_{\beta_r}(1 - \epsilon)}{2k_1}, \tag{56b} \\
|\tilde{\eta}_0| &< \frac{k_1}{M_{\beta_r}} |\tilde{\rho}_0| < \frac{L_{\beta_r}(1 - \epsilon)}{2M_{\beta_r}}, \tag{56c}
\end{align*}
\]

From (25a) and (56), it is possible to conclude that for the choice of \( k_1, k_2, C(s) \) in (21) and the choice of \( \rho_d \) in (23) the domain given by
\[
D = \{ (\tilde{\omega}_0, \tilde{\rho}_0, \tilde{\eta}_0) : \|\tilde{\omega}_0\|_\infty < B_{\tilde{\omega}_0}, |\tilde{\rho}_0| < B_{\tilde{\rho}_0}, |\tilde{\eta}_0| < B_{\tilde{\eta}_0} \} \tag{57}
\]
is the region of attraction of the reference system in (16) with ultimate bound \( B_\rho \). Notice that decreasing \( k_1 \) enlarges the region of attraction at the cost of larger ultimate bound \( B_\rho \). It is worth mentioning that the conditions in (21) and (22a) ensure that
\[
\begin{align*}
B_\rho &= \frac{(M_{\rho_r} + M_d(k_1))}{k_1k_2} M_{\beta_r} + \frac{(M_{\beta_1}\mu_{\beta_2} + \mu_{\beta_1})}{k_1} \gamma_{\omega} < \frac{L_{\beta_r}(1 - \epsilon)}{4k_1} < B_{\tilde{\rho}_0}, \tag{58a} \\
B_{\eta} &= \frac{M_{\rho_r} + M_d(k_1)}{k_2} < \frac{L_{\beta_r}(1 - \epsilon)}{8M_{\beta_r}} < B_{\tilde{\eta}_0}. \tag{58b}
\end{align*}
\]
IV.C. Adaptive System

Theorem 1 characterizes both the transient and the steady-state performance of the reference system (16). In this section, we show that the trajectories of the closed-loop adaptive system can stay arbitrarily close to this reference system.

The closed-loop adaptive system, defined by application of (15) to (1) can be rewritten as:

\[
\begin{align*}
\dot{\rho}(t) &= \beta_1(\omega(t)) \sin(\eta(t) + \beta_2(\omega(t))), \\
\dot{\eta}(t) &= \frac{V_g(t) \cos(\eta(t) - V(t) \cos[\dot{\psi}(t) - (\psi(t) - \eta(t))]}{\rho(t)} + \dot{\psi}(t), \\
\dot{\psi}(t) &= \frac{V_g(t) \cos(\eta(t) - \tilde{V}(t) \cos[\dot{\psi}(t) - (\psi(t) - \eta(t))]}{\rho(t)} - k_2(\eta(t) - \eta_d(\omega_c(t), \rho(t))),
\end{align*}
\]

(59)

where \(\omega_c(t)\) is given in (9b). Notice that \(\omega_c(t)\), \(\hat{V}(t)\) and \(\hat{\psi}(t)\) are all bounded signals.

IV.D. Transient and Steady State Performance

Let

\[
\begin{align*}
\dot{\rho}(t) &= \rho(t) - \rho_r(t), \\
\dot{\eta}(t) &= \eta(t) - \eta_r(t).
\end{align*}
\]

(60)

The error dynamics between (16) and (59) can be written as:

\[
\begin{align*}
\dot{\rho}(t) &= \beta_1(\omega(t)) \sin(\eta(t) + \beta_2(\omega(t))) - \beta_1(\omega(t)) \sin(\eta_r(t) + \beta_2(\omega(t))), \\
\dot{\eta}(t) &= \frac{-k_1(\rho(t) - \rho_d) + \beta_1(\omega(t)) \sin(\eta(t) + \beta_2(\omega(t))) - \beta_1(\omega_c(t)) \sin(\eta_d(\omega_c(t), \rho(t)) + \beta_2(\omega_c(t)))}{\beta_1(\omega_c(t))} \\
&\quad + \beta_1(\omega_r(t)) \sin(\eta_d(\omega_r(t), \rho_r(t)) + \beta_2(\omega_r(t))) \\
&\quad + \beta_1(\omega_r(t)) \sin(\eta_d(\omega_r(t), \rho_r(t)) + \beta_2(\omega_r(t))) - k_2(\eta(t) - \eta_d(\omega_c(t), \rho(t)) + \beta_2(\omega_c(t))),
\end{align*}
\]

(61a)

\[
\begin{align*}
\dot{\eta}(t) &= -k_2(\eta(t) - \eta_r(t)) + k_2(\eta_d(\omega_c(t), \rho(t)) - \eta_d(\omega_r(t), \rho_r(t))) \\
&\quad + [V(t) \cos(\psi(t) - \psi(t) + \eta(t)) - \tilde{V}(t) \cos(\dot{\psi}(t) - \psi(t) + \eta(t))] / \rho(t) \\
&\quad - [V(t) \cos(\psi(t) - \psi(t) + \eta_r(t)) - \tilde{V}(t) \cos(\dot{\psi}(t) - \psi(t) + \eta_r(t))] / \rho_r(t).
\end{align*}
\]

(61b)

Next, we consider boundedness of the signals that will be used in establishing the stability of the closed-loop adaptive system in (59). Since the function \(\beta_1(t)\) is continuous, there exist finite constants \(L_{\beta_1}\) and \(\mu_c\) such that

\[
L_{\beta_1} < |\beta_1(\omega_c(t))|, \quad \left|\frac{\beta_1(\omega_c(t))}{\beta_1(\omega_r(t))}\right| \leq \mu_c \|\omega_c(t) - \omega_r(t)\|_\infty.
\]

(62a)

If \(\left|\frac{-k_1(\rho(t) - \rho_d)}{\beta_1(\omega_c(t))}\right| \leq 1 - \epsilon\) and \(\tilde{A}_0 \in D\), there exist finite \(L_1\) and \(L_2\) such that

\[
|\eta_d(\omega_c(t), \rho(t)) - \eta_d(\omega_r(t), \rho_r(t))| < L_1 \|\omega_c(t) - \omega_r(t)\|_\infty + L_2 \|\rho(t) - \rho_r(t)\|, \quad \forall t \geq 0.
\]

(62b)

Further, if \(\rho(t) \geq \bar{\rho}_b\), where \(\bar{\rho}_b\) is a positive constant, there exist finite constants \(\mu_p, M_{\rho_1}\) and \(M_{\rho_2}\) such that

\[
\begin{align*}
\left|\frac{2\dot{V}(t)(\dot{\psi}(t) - \psi(t)) + (\dot{V}(t) - V(t))}{\rho(t)}\right| &\leq \mu_p \|\omega_c(t) - \omega_r(t)\|_\infty, \quad \forall t \geq 0, \\
\left|\frac{2\dot{V}(t)\tilde{V}_c(t)}{\rho(t)}\right| &\leq M_{\rho_1}, \quad \left|\frac{V(t)\dot{\tilde{V}}_c(t)}{\rho(t)\rho_r(t)}\right| \leq M_{\rho_2}.
\end{align*}
\]

(62c)
We prove in Appendix VIII.E that when the inequalities in (62b) and (62c) hold with finite constants $L_1$, $L_2$, $\mu_\rho$, $M_{\rho_1}$ and $M_{\rho_2}$, the error dynamics in (61) can be written as

\[
\begin{align*}
\dot{\hat{p}}(t) &= -k_1 \hat{p}(t) + \Delta_\rho(t), \quad (63a) \\
\dot{\hat{\eta}}(t) &= -k_2 \hat{\eta}(t) + \Delta_\eta(t), \quad (63b)
\end{align*}
\]

where definitions of $\Delta_\rho(t)$ and $\Delta_\eta(t)$ follow from equation (61). In equation (63), for all $t \geq 0$,

\[
\begin{align*}
|\Delta_\rho(t)| &\leq \kappa_1 |\hat{p}(t)| + \kappa_2 |\hat{\eta}(t)| + \kappa_3 \|\omega_e(t) - \omega_r(t)\|_\infty, \quad (64a) \\
|\Delta_\eta(t)| &\leq \kappa_4 |\hat{p}(t)| + \kappa_5 |\hat{\eta}(t)| + \kappa_6 \|\omega_e(t) - \omega_r(t)\|_\infty, \quad (64b)
\end{align*}
\]

where $\kappa_i$ for $i = 1, \ldots, 6$ are positive constants chosen as:

\[
\begin{align*}
\kappa_1 &= M_{\beta_1} L_2, & \kappa_2 &= M_{\beta_1}, & \kappa_3 &= M_{\beta_1 r} + M_{\beta_1 r} \mu_{\beta_2} + \mu_{\beta_1}, \\
\kappa_4 &= M_{\rho_1}, & \kappa_5 &= M_{\rho_2} + k_2 L_2, & \kappa_6 &= \mu_\rho + k_2 L_1,
\end{align*}
\]

with $M_{\beta_1}$, $M_{\beta_1 r}$, $\mu_{\beta_1}$, $\mu_{\beta_2}$ given in (20c), $L_1$, $L_2$ in (62b), and $\mu_\rho$, $M_{\rho_1}$, $M_{\rho_2}$ given in (62c).

Let

\[
\begin{align*}
a_{\rho_i} &= k_1 - \kappa_1 - \epsilon_{k_1}, \\
a_{\eta_i} &= k_2 - \kappa_4 - \epsilon_{k_2},
\end{align*}
\]

where $\epsilon_{k_1}$ and $\epsilon_{k_2}$ are positive numbers. We have the following theorem for the transient and steady-state performance of the adaptive system in (59).

**Theorem 2** Consider the closed-loop reference system in (16), subject to (21), (22) and (23), and the closed-loop adaptive system in (59). If the controller gains $k_1$, $k_2$ and the estimation rate $\Gamma_c$ in the estimator are chosen as

\[
\begin{align*}
a_{\rho_i} a_{\eta_i} - \kappa_2 \kappa_5 &> 0, \quad (67a) \\
\Gamma_c &> \gamma_c \max \left\{ \frac{1}{B_{\beta_0}}, \frac{2(K_\rho + \epsilon_\rho)}{B_{\rho_0} - B_\rho}, \frac{2(K_\eta + \epsilon_\eta) + L_1 + L_2(K_\rho + \epsilon_\rho)}{B_{\beta_0} - B_\eta}, \frac{k_1 (K_\rho + \epsilon_\rho)}{L_{\beta_1 r} (1 - \epsilon)} + \mu_{\epsilon_\rho}, \frac{K_\rho + \epsilon_\rho}{\gamma_b - \gamma_b} \right\}, \quad (67b)
\end{align*}
\]

and the initial values $\omega_e(0)$, $\rho(0)$ and $\eta(0)$ comply with

\[
\begin{align*}
\|\omega_e(0) - \omega_0\|_\infty &= \|\dot{\omega}_0 - \omega_0\|_\infty < B_{\omega_0} - \gamma_0, \quad (68a) \\
|\rho(0) - \rho_d| &= |\rho_0 - \rho_d| < B_{\rho_0} - \gamma_\rho, \quad (68b) \\
|\eta(0) - \eta_d(\omega_e(0), \rho(0))| &= |\eta_0 - \eta_d(\dot{\omega}_0, \rho_0)| < B_{\eta_0} - \gamma_\eta, \quad (68c)
\end{align*}
\]

we have

\[
\begin{align*}
\|\omega_e - \omega_r\|_\infty &\leq \gamma_0, \quad (69a) \\
\|\rho - \rho_r\|_\infty &< \gamma_\rho, \quad (69b) \\
\|\eta - \eta_r\|_\infty &< \gamma_\eta \quad (69c)
\end{align*}
\]

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where

\[
\gamma_0 = \frac{\gamma_c}{\sqrt{T_c}},
\]
\[
\gamma_\rho = (K_\rho + \epsilon_\rho) \gamma_0,
\]
\[
\gamma_\eta = (K_\eta + \epsilon_\eta) \gamma_0,
\]
\[
K_\rho = \frac{\kappa_2 \kappa_6 + \kappa_3 \alpha_\eta}{\alpha_\rho \alpha_\eta - \kappa_2 \kappa_5},
\]
\[
K_\eta = \frac{\kappa_3 \kappa_5 + \alpha_\rho \kappa_6}{\alpha_\rho \alpha_\eta - \kappa_2 \kappa_5},
\]

and \(\epsilon_\rho, \epsilon_\eta\) are positive constants, \(\kappa_i\) (for \(i = 1, \ldots, 6\)) are given in (65), \(\alpha_\rho, \alpha_\eta\) are given in (66), \(\gamma_b > \gamma_\rho > 0, B_{\omega_0}, B_{\rho_b}, B_{\eta_b}\) are in (56), and \(\gamma_c\) is given in (13).

**Proof.** It is straightforward to check that the selection of \(\Gamma_c\) in (67b) ensures that the right-hand sides of the inequalities in (68) are greater than 0 so that (68) holds. Inequality (69a) follows from (12) immediately. Inequalities (69b) and (69c) will be proved by contradiction.

Since \(\eta(t), \eta_\rho(t), \rho(t)\) and \(\rho_\rho(t)\) are continuous and

\[
|\rho(0) - \rho_\rho(0)| = 0 \leq \gamma_\rho, \quad |\eta(0) - \eta_\rho(0)| = 0 \leq \gamma_\eta,
\]

if the relationships in (69b) and (69c) are not true, there exists \(\tau > 0\) such that

\[
|\rho(\tau) - \rho_\rho(\tau)| = \gamma_\rho, \quad \text{or} \quad |\eta(\tau) - \eta_\rho(\tau)| = \gamma_\eta,
\]

while

\[
\|(\rho - \rho_\rho)_{\tau}\|_{\mathcal{L}_\infty} \leq \gamma_\rho, \quad \|(\eta - \eta_\rho)_{\tau}\|_{\mathcal{L}_\infty} \leq \gamma_\eta.
\]

Here, \(\|\cdot\|_{\mathcal{L}_\infty}\) denotes the truncated \(\mathcal{L}_\infty\) norm, whose definition is given in (117a).

In the following, we show that

\[
\left|\frac{k_1(\rho(t) - \rho_\rho)}{\beta_1(\omega_\rho(t))}\right| < 1 - \epsilon \quad \text{and} \quad \rho(t) \geq \gamma_b \quad \text{for} \quad t \in [0, \tau].
\]

Considering

\[
\left|\frac{k_1(\rho(t) - \rho_\rho)}{\beta_1(\omega_\rho(t))}\right| \leq \left|\frac{k_1(\rho(t) - \rho_\rho)}{\beta_1(\omega_\rho(t))}\right| + \left|\frac{k_1(\rho_\rho(t) - \rho_\rho)}{\beta_1(\omega_\rho(t))}\right|
\]

\[
\leq \frac{k_1}{\|\beta_1(\omega_\rho(t))\|} \gamma_\rho + \frac{k_1}{\|\beta_1(\omega_\rho(t))\|} \frac{L_{\beta_\rho}}{k_1} (1 - \epsilon),
\]

\[
= \frac{k_1}{\|\beta_1(\omega_\rho(t))\|} (K_\rho + \epsilon_\rho) \gamma_0 + \frac{L_{\beta_\rho}}{\|\beta_1(\omega_\rho(t))\|} \left|\frac{\beta_1(\omega_\rho(t))}{\beta_1(\omega_\rho(t))}\right| (1 - \epsilon),
\]

\[
\leq \frac{k_1}{\|\beta_1(\omega_\rho(t))\|} (K_\rho + \epsilon_\rho) \gamma_0 + \frac{\beta_1(\omega_\rho(t))}{\beta_1(\omega_\rho(t))} (1 - \epsilon),
\]

\[
\leq \left(\frac{k_1}{L_{\beta_\rho}} (K_\rho + \epsilon_\rho) + \mu_\epsilon (1 - \epsilon)\right) \gamma_0 < 1 - \epsilon,
\]

where the relationships in (20c), (24a), (62a), (67b), (69) and (73) have been used. Considering \(\rho(t)\), it follows from (73) that

\[
\rho_\rho(t) - \gamma_\rho \leq \rho(t) \leq \rho_\rho(t) + \gamma_\rho, \quad \forall t \in [0, \tau].
\]

The initial conditions in (68), together with (73), ensure that \(\tilde{X}_0 \in \mathcal{D}\). Thus, for the reference system we have \(\rho_\rho(t) \geq \gamma_b\) subject to (21), (22) and (23). It then follows from (75) and the
selection of $\Gamma_c$ in (67b) that $\rho(t) \geq \tilde{\gamma}_b$ for $\gamma_b > 0$. Therefore, the finite constants $L_1$, $L_2$, $\mu_\rho$, $M_{\rho_1}$ and $M_{\rho_2}$ in (62) exist and (63)–(66) hold.

Next, consider (63a). It follows from (63a), (64a) and Lemma 1 that

$$\|\tilde{\rho}_r\|_{L_\infty} \leq \frac{\kappa_2 \|\tilde{\eta}_r\|_{L_\infty} + \kappa_3 \gamma_0}{a_{\rho_t}}.$$  \hfill (76)

Similarly, it follows from (63b), (64b) and Lemma 1 that

$$\|\tilde{\eta}_r\|_{L_\infty} \leq \frac{\kappa_3 \|\tilde{\rho}_r\|_{L_\infty} + \kappa_6 \gamma_0}{a_{\eta_t}}.$$  \hfill (77)

Summarizing the above two inequalities, we have

$$\|\tilde{\rho}_r\|_{L_\infty} \leq \frac{\kappa_2 \kappa_5 \|\tilde{\rho}_r\|_{L_\infty} + \kappa_6 \gamma_0}{a_{\eta_t}} \leq \frac{\kappa_3 \gamma_0}{a_{\rho_t} a_{\eta_t}} \leq \left( \frac{\kappa_2 \kappa_5 + \kappa_6}{a_{\rho_t} a_{\eta_t}} \right) \gamma_0.$$ \hfill (78)

The condition in (67a) ensures that

$$\|\tilde{\rho}_r\|_{L_\infty} \leq \frac{\kappa_2 \kappa_5 + \kappa_6}{a_{\rho_t} a_{\eta_t}} \gamma_0.$$ \hfill (79)

Similarly, we have

$$\|\tilde{\eta}_r\|_{L_\infty} \leq \frac{\kappa_3 \kappa_5 + \kappa_6}{a_{\rho_t} a_{\eta_t}} \gamma_0.$$ \hfill (80)

Thus,

$$\|\tilde{\rho}_r\|_{L_\infty} \leq K_\rho \gamma_0,$$

$$\|\tilde{\eta}_r\|_{L_\infty} \leq K_\eta \gamma_0,$$ \hfill (81)

where $K_\rho$ and $K_\eta$ are given in (70). The inequality in (81), together with the definitions of $\gamma_\rho$ and $\gamma_\eta$ in (70), implies that

$$\|\tilde{\rho}_r\|_{L_\infty} < \gamma_\rho,$$

$$\|\tilde{\eta}_r\|_{L_\infty} < \gamma_\eta,$$ \hfill (82)

which contradicts (72). Hence, the relationships in (69b) and (69c) must hold. This completes the proof.\hfill \square

**Remark 2** From Theorems 1 and 2, it is possible to conclude that for the choice of $k_1$, $k_2$, $C(s)$, $\Gamma_c$ in (21) and (67), and the choice of $\rho_d$ in (23),

$$\Omega = \{ (\tilde{\omega}_0, \tilde{\rho}_0, \tilde{\eta}_0) : \|\tilde{\omega}_0\|_{L_\infty} < B_\omega \rho_0 - \gamma_0, \ |\tilde{\rho}_0| < B_{\rho_0} - \gamma_\rho, \ |\tilde{\eta}_0| < B_{\eta_0} - \gamma_\eta \}$$ \hfill (83)

is the region of attraction of the adaptive system in (59) with the ultimate bound $B_\rho + \gamma_\rho$.

**Remark 3** Let us have a look at the five terms in (67b). Condition $\Gamma_c > \frac{\kappa_3}{\kappa_5}$ ensures that the right-hand side of inequality (68a) is greater than 0 so that (68a) is valid. Conditions $\Gamma_c > \gamma_c \frac{2(K_\rho + \epsilon_\rho)}{B_{\rho_0} - B_{\rho}}$ and $\Gamma_c > \gamma_c \frac{2(\epsilon_\rho + \epsilon_\eta + L_1 + L_2(K_\rho + \epsilon_\rho))}{B_{\eta_0} - B_{\rho}}$ ensure that

$$B_\rho + \gamma_\rho < B_{\rho_0} - \gamma_\rho,$$

$$B_\eta + \gamma_0 \left[ (K_\rho + \epsilon_\rho) + L_1 + L_2(K_\rho + \epsilon_\rho) \right] < B_{\eta_0} - \gamma_\eta,$$ \hfill (84)

which will be used in Sec. V when analyzing the system’s stability in the presence of out-of-frame events. Condition $\Gamma_c > \gamma_c \left( \frac{k_1(K_\rho + \epsilon_\rho)}{\mu_\rho} + \mu_\rho \right)$ is imposed so that $\left| \frac{k_1(\rho(t) - \rho_0)}{\beta_\rho(\omega_\rho(t))} \right| < 1 - \epsilon$ for all $t \geq 0$ over $\Omega$. Condition $\Gamma_c > \gamma_c \frac{K_\rho + \epsilon_\rho}{B\rho - B_\rho}$ helps to ensure that $\rho(t) \geq \tilde{\gamma}_b$ for all $t \geq 0$. 

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V. Stability in the Presence of Out-of-Frame Events

In vision-based applications, continuous extraction of the target’s information is often unavailable due to environmental factors, limited field of view of the camera, or failure in the image processing module. These phenomena are commonly referred to as out-of-frame events. The out-of-frame event cannot be avoided due to the complexity of a real outdoor application scenario. The problems due to temporary target loss need to be addressed explicitly.

For this purpose, we study the performance degradation of the closed-loop system, casting it into the framework of switching systems. The switching system includes two subsystems, as shown in Fig. 2. One subsystem corresponds to the case when the visual measurements are available, while the other subsystem corresponds to the situation when the visual measurements are not available. The switching signal is the signal defining the out-of-frame event.

For the problem at hand, it is intuitive that stability of the subsystem in the presence of out-of-frame event cannot be guaranteed. To characterize the stability of the closed-loop switched system, following Ref. 4, the concept of “brief instabilities” is exploited to model the out-of-frame events.

Define the tracking loss as a binary signal

\[
s(t) := \begin{cases} 
0 & \text{out-of-frame event at time } t, \\
1 & \text{camera tracks the target at time } t.
\end{cases}
\]  

(85)

Following Ref. 4, let \( T_s(\tau, t) \) denote the amount of time in the interval \([\tau, t]\) when \( s(t) = 0 \). Then, formally,

\[
T_s(\tau, t) := \int_\tau^t (1 - s(t')) dt'.
\]

We say that the image processing experiences brief target loss event if

\[
T_s(\tau, t) \leq T_0 + \alpha (t - \tau), \quad \forall t \geq \tau \geq 0,
\]

for some \( T_0 \geq 0 \) and \( \alpha \in [0, 1] \). The scalar \( T_0 \) is called the instability bound and \( \alpha \) is called the asymptotic instability ratio (Ref. 4, page 891). Note that \( \alpha \) provides an asymptotic upper bound on the ratio \( T_s(\tau, t)/(t - \tau) \) as \( (t - \tau) \to \infty \).

V.A. Estimator in the Presence of Out-of-Frame Events

When the target is out-of-frame, we do not have the measurement for \( x(t) = [p_x(t), p_y(t)]^\top \). In that case, the guidance law uses the latest available estimate \( \hat{\omega}(t) \) for the unknown parameters, treating it as constant. That is, referring to (8), we let \( \hat{\omega}(t) = 0 \) when \( s(t) = 0 \), which is equivalent...
to assuming $\tilde{x}(t) = 0$ during the out-of-frame event. Suppose that the measurements become available at time instant $t_i$. The initial state of $\tilde{x}(t_i)$ is then set as $\tilde{x}(t_i) = x(t_i)$.

The state estimator and the adaptive law in equations (7) and (8) in the presence of target loss events lead to the following:

- **State Estimator:**
  \[
  \dot{\hat{x}}(t) = s(t) A_m \tilde{x}(t) - V_g(t) \begin{bmatrix} \sin \psi(t) \\ \cos \psi(t) \end{bmatrix} + \dot{\omega}(t), \quad \tilde{x}(t) = \hat{x}(t) - x(t),
  \]
  where $s(t)$ is defined in (85).

- **Adaptive Law:**
  \[
  \dot{\hat{\omega}}(t) = \Gamma_c \text{Proj} (\hat{\omega}(t), -s(t) P^T \tilde{x}(t)).
  \]
  The low-pass filter in (10) and the extraction of unknown parameters in (11) remain the same.

V.B. Controller in the Presence of Out-of-Frame Events

In the controller design, we notice that the signals related to visual measurements become unavailable in the presence of target loss. Considering (15), these signals include $\rho(t)$, $\eta(t)$ and $\eta_d(\omega_r(t), \rho(t))$, which are therefore kept frozen during the out-of-frame time interval. In the presence of out-of-frame event, the controller in (15) operates with frozen estimates:

**Controller using frozen estimates:**

\[
\begin{aligned}
\dot{\psi}(t) &= V_g(t) \cos \bar{\eta} - \tilde{V}_t \cos [\tilde{\psi}_t - (\psi(t) - \bar{\eta})] - k_2(\bar{\eta} - \bar{\eta}_d), \\
\bar{\eta}_d &= \sin^{-1} \left( \frac{-k_1(\bar{\rho} - \rho_d)}{\beta_1(\tilde{V}_t, \psi_t)} \right) - \beta_2(\tilde{V}_t, \tilde{\psi}_t),
\end{aligned}
\]

where

- $\tilde{V}_t$ is a constant, frozen from pervious estimate $\hat{V}_t(t)$.
- $\tilde{\psi}_t$ is a constant, frozen from pervious estimate $\hat{\psi}_t(t)$.
- $\bar{\rho}$ is a constant, frozen from pervious quantity $\rho(t)$.
- $\bar{\eta}$ is a constant, frozen from pervious quantity $\eta(t)$.
- $\bar{\eta}_d$ is a constant, frozen from pervious quantity $\eta_d(\omega_r(t), \rho(t))$.

V.C. Stability of the Two Subsystems

We have two subsystems. One subsystem corresponds to the case when the visual measurements are available, referred to as $G_1$ hereafter. The other subsystem $G_2$ corresponds to the out-of-frame event. The subsystem $G_1$ is achieved by applying the controller in (15) to the plant in (1), using estimates in (7)–(11) that are obtained from continuous visual measurements. The closed-loop form of $G_1$ is given in equation (59). The subsystem $G_2$ is achieved via application of the controller in (89)
to the plant, where signals related to visual measurements are kept frozen. These two subsystems are listed below:

**Subsystem \( \mathcal{G}_1 \):**

\[
\begin{align*}
\dot{\rho}(t) &= \beta_1(V_t(t), \psi(t)) \sin(\eta(t) + \beta_2(V_t(t), \psi(t))), \\
\dot{\eta}(t) &= -\frac{V_g(t) \cos \eta(t) - V_t(t) \cos[\hat{\psi}_t(t) - (\psi(t) - \eta(t))]}{\rho(t)} \\
&\quad + \frac{V_g(t) \cos \eta(t) - \hat{V}_t(t) \cos[\hat{\psi}_t(t) - (\psi(t) - \eta(t))]}{\rho(t)} \\
&\quad - k_2(\eta(t) - \eta_d(\omega_e(t), \rho(t))),
\end{align*}
\]

where \( \hat{V}_t(t) \) and \( \hat{\psi}_t(t) \) are obtained through (7)–(11), and

**Subsystem \( \mathcal{G}_2 \):**

\[
\begin{align*}
\dot{\rho}(t) &= \beta_1(V_t(t), \psi(t)) \sin(\eta(t) + \beta_2(V_t(t), \psi(t))), \\
\dot{\eta}(t) &= -\frac{V_g(t) \cos \eta(t) - V_t(t) \cos[\hat{\psi}_t(t) - (\psi(t) - \eta(t))]}{\rho(t)} \\
&\quad + \frac{V_g(t) \cos \eta(t) - \hat{V}_t(t) \cos[\hat{\psi}_t(t) - (\psi(t) - \eta)]}{\rho} - k_2(\eta - \bar{\eta}_d),
\end{align*}
\]

where \( \hat{V}_t, \hat{\psi}_t, \bar{\rho}, \bar{\eta} \) and \( \bar{\eta}_d \) are given in (89).

Considering \( \mathcal{G}_1 \), it has been shown in Theorems 1 and 2 that, upon proper selection of the controller gains \( k_1, k_2 \) per (67a) and the adaptation rate \( \Gamma_i \) per (67b), for \( \hat{X}_0 \in \Omega \), the closed-loop system \( \mathcal{G}_1 \) can be designed to be arbitrarily close to the bounded reference system in (16). Let us first consider \( \lim_{t \to \infty} |\rho(t) - \rho_d| \). It follows from (25a) and (69b) that

\[
\lim_{t \to \infty} |\rho(t) - \rho_d| \leq \lim_{t \to \infty} |\rho(t) - \rho_r(t)| + \lim_{t \to \infty} |\rho_r(t) - \rho_d| < B_\rho + \gamma_\rho.
\]

Similarly, we have

\[
\lim_{t \to \infty} |\eta(t) - \eta_d(\omega_e(t), \rho(t))| = \lim_{t \to \infty} |\eta(t) - \eta_r(t) + \eta_r(t) - \eta_d(\omega_e(t), \rho_r(t))| \\
+ \lim_{t \to \infty} \eta_d(\omega_e(t), \rho_r(t)) - \eta_d(\omega_e(t), \rho(t))|,
\]

\[
< B_\eta + \gamma_\eta + L_1 \gamma_0 + L_2 \gamma_\rho, \quad \text{for } \hat{X}_0 \in \Omega \text{ and } \rho(t) \geq \tilde{\gamma}_b,
\]

where the relationships in (25b), (62b) and (69c) have been used. Consider the following candidate Lyapunov function

\[
V(t) = \frac{(\rho(t) - \rho_d)^2}{2} + \frac{(\eta(t) - \eta_d(\omega_e(t), \rho(t)))^2}{2}.
\]

It then follows from (92) and (93) that for \( \hat{X}_0 \in \Omega \) and \( \rho(t) \geq \tilde{\gamma}_b \), there exist \( \alpha_1 > 0 \) and \( \xi_1 > 0 \) such that

\[
\dot{V}(t) \leq -\alpha_1 V(t) + \xi_1, \quad \text{for } \hat{X}_0 \in \Omega \text{ and } \rho(t) \geq \tilde{\gamma}_b,
\]

with

\[
\begin{align*}
\alpha_1 &= 1, \quad \xi_1 = \frac{(B_\rho + \gamma_\rho)^2}{2} + \frac{(B_\eta + \gamma_0([K_\eta + \epsilon_\eta] + L_1 + L_2[K_\rho + \epsilon_\rho])^2}{2},
\end{align*}
\]
Considering $G_2$, it follows from equations (2) and (91) that $\dot{r}(t)$ is bounded. The frozen estimates $\bar{V}_t(t)$ and $\dot{\bar{V}}_t(t)$ are also bounded, leading to boundedness of $\bar{\dot{r}}(t)$ in (91) for $r(t) \geq \bar{r}_b$. Hence, for all $X_0 \in \Omega$ and $r(t) \geq \bar{r}_b$, there exist positive constants $M_{s1}$ and $M_{s2}$ such that

$$|\bar{\dot{r}}(t) - \bar{\dot{r}}_d(\omega_e(t), r(t))| \leq M_{s1}, \quad |\dot{r}(t)| \leq M_{s2}. \tag{96}$$

Using the same candidate Lyapunov function in (94), we have

$$\dot{V}(t) = (r(t) - r_d) \dot{r}(t) + (\eta(t) - \eta_d(\omega_e(t), r(t))) (\bar{\dot{r}}(t) - \bar{\dot{r}}_d(\omega_e(t), r(t))),$$

\[
\leq M_{s1} |r(t) - r_d| + M_{s2} |\eta(t) - \eta_d(\omega_e(t), r(t))|,
\]

\[
\leq \frac{1}{2} (r(t) - r_d)^2 + \frac{1}{2} M_{s1}^2 + \frac{1}{2} (\eta(t) - \eta_d(\omega_e(t), r(t)))^2 + \frac{1}{2} M_{s2}^2,
\]

\[
= \frac{1}{2} V(t) + \frac{M_{s1}^2 + M_{s2}^2}{2}.
\]

Therefore,

$$\dot{V}(t) \leq \alpha_2 V(t) + \xi_2, \tag{98a}$$

with

$$\alpha_2 = \frac{1}{2}, \quad \xi_2 = \frac{M_{s1}^2 + M_{s2}^2}{2}. \tag{98b}$$

V.D. Stability of the Closed-Loop Switched System

Combining inequalities (95a) and (98a), we have:

$$\dot{V}(t) \leq \begin{cases} -\alpha_1 V(t) + \xi_1 = - \left( \alpha_1 - \frac{\xi_1}{V(t)} \right) V(t), & \text{Subsystem } G_1, \\ \alpha_2 V(t) + \xi_2 = \left( \alpha_2 + \frac{\xi_2}{V(t)} \right) V(t), & \text{Subsystem } G_2. \end{cases} \tag{99}$$

Let

$$V_b = \frac{\xi_1}{\alpha_1} + \epsilon_b,$$

where $\epsilon_b > 0$ is a small positive constant. If $V(t) > V_b$, we have:

$$\dot{V}(t) \leq \begin{cases} -\lambda_0 V(t), & \lambda_0 = \alpha_1 - \xi_1/V_b, & \text{Subsystem } G_1, \\ \mu V(t), & \mu = \alpha_2 + \xi_2/V_b, & \text{Subsystem } G_2. \end{cases} \tag{100}$$

The next theorem establishes the stability of the closed-loop switched system consisting of the two subsystems $G_1$ and $G_2$.

**Theorem 3** Assume that the switched system consisting of (90) and (91) has brief instability with instability bound $T_0$ and asymptotic instability ratio $\alpha$ that satisfy

$$\alpha < \alpha^* = \frac{\lambda_0}{\lambda_0 + \mu}, \tag{101a}$$

$$T_0 < \log \left( \frac{V_0}{V_b} \right) / (\lambda_0 + \mu), \tag{101b}$$

where

$$V_b = \frac{\xi_1}{\alpha_1} + \epsilon_b. \tag{102}$$
subject to 

\[ 0 < \epsilon_b < V_\Omega - \xi_1/\alpha_1, \]  

(103)

with \( V_\Omega = \frac{(B_{b0} - \gamma_o)^2 + (B_{by} - \gamma_o)^2}{2} \) defined via \( B_{b0}, B_{by} \) given in (56) and \( \gamma_o, \gamma_v \) given in (70). Then, the switched system consisting of subsystems \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) is uniformly ultimately bounded with the ultimate bound given by \( e^{(\lambda_0 + \mu) T_0 V_b} \) for all initial conditions verifying \( X_0 \in \Omega \), where \( \Omega \) is given in (83).

**Proof.** First we notice that the condition (84) ensures that \( \xi_1/\alpha_1 < V_\Omega \) so that the choice of \( \epsilon_b \) in (103) is valid. Next, we show by contradiction that the trajectory of the switched system remains inside the region of attraction \( \Omega \) if it starts inside \( \Omega \). Then, we quantify the corresponding ultimate bound.

Since \( V(0) \in \Omega \) and \( V(t) \) is piecewise continuous along the trajectory of the switched system consisting of \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \), if \( V(t) \) does not remain inside \( \Omega \) for all \( t \geq 0 \), there exists \( \tau_u > 0 \) such that

\[
V(t) \leq V_\Omega, \quad t \in [0, \tau_u],
\]

(104a)

\[
V(\tau_u) = V_\Omega.
\]

(104b)

Since \((\rho(t) - \rho_d)^2 \leq 2V(t)\), it follows from (104a) that \(|\rho(t) - \rho_d| \leq \sqrt{2V_\Omega} \) for \( t \in [0, \tau_u] \), which leads to

\[
\rho_d - \sqrt{2V_\Omega} \leq \rho(t).
\]

Hence, the choice of \( \rho_d \geq \sqrt{2V_\Omega} + \bar{\gamma}_b \)

(105)

ensures that \( \rho(t) \geq \bar{\gamma}_b \)

(106)

where \( \bar{\gamma}_b \) is a positive constant. It then follows from inequalities (104a) and (106) that equations (95) and (98) hold. Therefore, for the \( V_b \) in (102) subject to (103), we have

\[
V(t) \leq e^{-\lambda_0(t-\tau-T_0)+\mu T_0}V(\tau) \leq e^{-\lambda_0(t-\tau) + (\lambda_0 + \mu) T_0 + \alpha(t-\tau)}V(\tau),
\]

(107)

\[
eq e^{(\lambda_0 + \mu) T_0 - [\lambda_0 - \alpha(\lambda_0 + \mu)](t-\tau)}V(\tau) = e^{(\lambda_0 + \mu) T_0 - \lambda(t-\tau)}V(\tau),
\]

where

\[
\lambda = \lambda_0 - \alpha(\lambda_0 + \mu).
\]

(108)

It is clear that the assumption in (101a) ensures that \( \lambda > 0 \). Next, we show that \( V(t) \) is upper bounded by \( e^{(\lambda_0 + \mu) T_0 V_b} \) over \([0, \tau_u]\).

From (107), for any \( V(\tau) \geq V_b \), we have \( V(t) \leq e^{(\lambda_0 + \mu) T_0 - \lambda(t-\tau)}V(\tau) \), for \( \tau \leq t \leq \tau_u \). Suppose that at time instant \( t_1 \in [\tau, \tau_u] \) we have \( V(t_1) \leq e^{(\lambda_0 + \mu) T_0 - \lambda(t_1-\tau)}V(\tau) \leq e^{(\lambda_0 + \mu) T_0} V_b \). Then, for any \( \tau \leq t_1 \leq t_2 \leq \tau_u \), we have

\[
V(t_2) \leq e^{(\lambda_0 + \mu) T_0 - \lambda(t_2-\tau)}V(\tau) = e^{(\lambda_0 + \mu) T_0 - \lambda(t_1-\tau)}V(\tau) e^{-\lambda(t_2-t_1)},
\]

(109)

\[
\leq e^{(\lambda_0 + \mu) T_0} V_b e^{-\lambda(t_2-t_1)} \leq e^{(\lambda_0 + \mu) T_0 V_b}.
\]

Hence, if for any \( \tau \in [0, \tau_u] \) we have \( V(\tau) \geq V_b \), then \( V(t) \) is ultimately upper bounded by \( e^{(\lambda_0 + \mu) T_0 V_b} \) for \( t \in [\tau, \tau_u] \). In summary, we have \( V(t) \leq e^{(\lambda_0 + \mu) T_0 V_b} \) for \( t \in [0, \tau_u] \). Since the relationships in (101b) and (103) ensure that \( e^{(\lambda_0 + \mu) T_0 V_b} < V_\Omega \), we have

\[
V(t) \leq e^{(\lambda_0 + \mu) T_0 V_b} < V_\Omega, \quad \forall t \in [0, \tau_u],
\]

(110)

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which contradicts \((104b)\). Therefore, the trajectory of the switched system remains inside \(\Omega\) for all \(V(0)\in \Omega\). This is illustrated in Fig. 3.

Therefore, for any \(V(0)\in \Omega\) we have \(V(t)\in \Omega, \forall t \geq 0\). Hence, inequalities \((105)\)–\((110)\) hold for all \(t \geq 0\) and we can further conclude that the switched system is uniformly ultimately bounded by \(e^{(\lambda_0 + \mu)T_0}V_b\) over \(\Omega\). This completes the proof.

\[\text{Region of Attraction } \Omega\]

\[\text{Figure 3. Proving Lyapunov stability.}\]

**Remark 4** Recall that the adaptive system in \((59)\) has only local stability for all \(\tilde{X}_0 \in \Omega\). The two inequalities in \((101)\), by restricting \(T_0\) and \(\alpha\) with respect to \(\Omega\), ensure that when out-of-frame events happen, the system trajectory does not leave \(\Omega\). Otherwise, it is not guaranteed that the controller for \(G_1\) will bring the system trajectory back to \(\Omega\) even when the visual measurements become available.

**Remark 5** From \((101a)\) and \((108)\), it can be seen that smaller \(\alpha\) leads to larger \(\lambda\), and as a result, smaller \(e^{(\lambda_0 + \mu)T_0 - \lambda(t-\tau)} V(\tau)\). Therefore, in the presence of out-of-frame events, the tracking performance bound is upper bounded by an exponentially growing term with the exponent being proportional to \(\alpha\), so that the system has no finite escape time. We further notice that smaller \(\alpha\) implies more robust measurement from the camera.

**V.E. Tracking Performance vs. Instability Ratio**

For \(T_0 = 0\) we illustrate the trade-off between the tracking performance, represented by \(V_b\), and the ability to handle out-of-frame events, represented by \(\alpha^*\). Recall that \(\alpha^*\) is the upper bound on the instability ratio defined in \((101a)\), where \(\lambda_0\) and \(\mu\) are given in \((100)\). Thus we have

\[
\alpha^* = \frac{\lambda_0}{\lambda_0 + \mu} = \frac{1}{\frac{1}{\lambda_0} + \frac{\mu}{\lambda_0}} = \frac{1}{1 + \frac{\alpha_2 + \xi_2/V_b}{\alpha_1 - \xi_1/V_b} f(V_b)}.
\]

Consider

\[
f(V_b) = \frac{\alpha_2 + \xi_2/V_b}{\alpha_1 - \xi_1/V_b}.
\]

Since

\[
f'(V_b) = -\frac{\alpha_1 \xi_2 + \alpha_2 \xi_1}{V_b^2 (\alpha_1 - \xi_1/V_b)^2} < 0,
\]

\[
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\]
we have

\[ V_b \text{ increases } \Rightarrow f(V_b) \text{ decreases } \Rightarrow \alpha^* \text{ increases. } \tag{114} \]

From (111)–(114), it can be concluded that less restrictive requirement on the tracking performance allows for more “brief instabilities”.

VI. Simulations and Experiments

This section shows some experimental data and numerical simulations. In the numerical simulations, the initial conditions, estimation parameters, and control parameters are selected as follows:

- Initial Conditions: \( h(t) = 100, \rho_d = 30, \rho(0) = 10, \eta(0) = \pi/6, V_g(t) = 30, \psi(0) = \pi/6. \)
- Time-Varying Target Velocity:
  \[
  V_t(t) = 12 + 10 \sin \left( \frac{\pi}{12} t \right), \quad \psi_t(t) = \sin \left( \frac{\pi}{12} t \right). \tag{115}
  \]
- Estimation Parameters: \( A_m = -I_{2 \times 2}, P = 1/2I_{2 \times 2}, \Gamma_c = 2 \times 10^8, c = 10, \hat{\omega}(0) = (0.1, 0.1). \)
- Control Parameters: \( k_1 = 3, k_2 = 30. \)

VI.A. Experimental Test of Target Motion Estimation

The motion estimation algorithm has been tested via experimental data collected during flight tests. Comparison with a Linear Parametrically-Varying (LPV) filter that is reported in Ref. 4 for estimation of the relative 2D range is shown in Fig. 4. The improvement of the proposed fast estimator over the LPV filter is obvious.

![Figure 4](image-url)  
**Figure 4.** Comparison between fast estimation (\( L_1 \)) and an early-developed filter (LPV) in estimating the 2D horizontal range using experimental data collected during flight tests.
VI.B. Analysis of the Guidance Law

Figure 5 shows the relationship between the Lyapunov function \( V(t) \) as given in (94) and the controller gain \( k_2 \in \{5, 10, 15, 20, 25, 30\} \) for the closed-loop reference system (16). It can be observed that as \( k_2 \) increases, \( V(t) \) decreases. Accordingly, \( (\rho(t) - \rho_d)^2 \) decreases.

![Figure 5. Relationship between Lyapunov function \( V(t) \) and controller gain \( k_2 \), for the reference system (16).](image)

VI.C. Performance Study in the Presence of Out-of-Frame Event

Figures 6–8 show the tracking performance of the switched system when the out-of-frame signal is of 10%, 20% and 30%, of every 2-second time interval. In these three figures, the target’s velocity is given in (115). It can be observed that the tracking performance degrades as the out-of-frame duration increases.

Figures 9–11 show simulation results of the following target motion dynamics

\[
V_t(t) = 10, \quad \psi_t(t) = \frac{\pi}{5} + 0.8 \sin \left( \frac{\pi}{2} t \right),
\]

using the same design. It can be observed that the tracking objective is achieved. The system performance degrades as the out-of-frame duration increases.

VII. Conclusion

This paper extends the early work of authors on vision-based target tracking of a ground vehicle moving with unknown time-varying velocity. The control objective is to regulate the 2D horizontal range between the UAV and the target to a constant. The paper has complete performance and robustness analysis, including the out-of-frame events.
Figure 6. Tracking performance of (115) in the presence of out-of-frame event: 10% of every 2 seconds.
Figure 7. Tracking performance of (115) in the presence of out-of-frame event: 20% of every 2 seconds.
Figure 8. Tracking performance of (115) in the presence of out-of-frame event: 30% of every 2 seconds.

Figure 9. Tracking performance of (116) in the presence of out-of-frame event: 10% of every 2 seconds.
Figure 10. Tracking performance of (116) in the presence of out-of-frame event: 20% of every 2 seconds.

Figure 11. Tracking performance of (116) in the presence of out-of-frame event: 30% of every 2 seconds.
VIII. Appendix

VIII.A. Basic Definitions

We recall some basic definitions from linear systems theory.

**Definition 1** For a signal $\xi(t)$, $t \geq 0$, $\xi(t) \in \mathbb{R}^n$, its truncated $L_{\infty}$ norm and $L_{\infty}$ norms are defined as

\[
\|\xi\|_{L_{\infty}} = \max_{i=1,\ldots,n} \left( \sup_{0 \leq \tau \leq t} |\xi_i(\tau)| \right),
\]

\[
\|\xi\|_{L_{\infty}} = \max_{i=1,\ldots,n} \left( \sup_{\tau \geq 0} |\xi_i(\tau)| \right),
\]

where $\xi_i(t)$ is the $i^{th}$ component of $\xi(t)$.

**Definition 2** The $L_1$ gain of a stable proper single-input single-output system $H(s)$ is defined as:

\[
\|H(s)\|_{L_1} = \int_0^\infty |h(t)| dt,
\]

where $h(t)$ is the impulse response of $H(s)$.

VIII.B. Lemma 1

**Lemma 1** Consider a linear time-varying (LTV) system:

\[
\dot{x}(t) = -a(t)x(t) + g(t) + b(t), \quad x(0) = 0,
\]

where

\[
|g(t)| \leq a_0|x(t)|, \quad 0 < a_l \leq a(t) - a_0 \leq a_u, \quad |b(t)| \leq b,
\]

and $a_l$, $a_u$, $a_0$ and $b$ are positive constants. We have

\[
|x(t)| \leq \frac{b}{a_l}, \quad \forall t \geq 0.
\]

**Proof**: Since $x(0) = 0$ and $x(t)$ is continuous, if (121) is not true, there exists $\tau \geq 0$ such that

\[
x(\tau) = \frac{b}{a_l}, \quad (122a)
\]

\[
\dot{x}(\tau) > 0, \quad (122b)
\]

or

\[
x(\tau) = -\frac{b}{a_l}, \quad (123a)
\]

\[
\dot{x}(\tau) < 0. \quad (123b)
\]

It follows from (120) and (122a) that

\[
\dot{x}(\tau) = -a(\tau)x(\tau) + g(\tau) + b(\tau) \leq \frac{b}{a_l}(-a(\tau) + a_0) + b \leq \frac{b}{a_l}(-a_l) + b \leq 0, \quad (124)
\]

which implies $\dot{x}(\tau) \leq 0$ and further contradicts (122b). Similarly, it follows from (120) and (123a) that

\[
\dot{x}(\tau) = -a(\tau)x(\tau) + g(\tau) + b(\tau) \geq \frac{b}{a_l}(a(\tau) - a_0) - b \geq a_l \frac{b}{a_l} - b \geq 0, \quad (125)
\]

which implies $\dot{x}(\tau) \geq 0$ and further contradicts (123b). Therefore, inequality (121) must be true. $\square$
VIII.C. Derivation of (26)

We show that inequality (26) holds under the conditions in (21a) and (22c). Let

\[ g(t) = |\tilde{\rho}_0| e^{-k_1 t} + \frac{M_{\beta_1} |\tilde{\eta}_0|}{k_1} \left( e^{-k_2 t} - e^{-k_1 t} \right), \]  \hspace{1cm} (126)

\[ \tilde{\rho}_0 = \rho_r(0) - \rho_d, \quad \tilde{\eta}_0 = \eta_r(0) - \eta_d(\omega_r(t), \rho_r(t)). \]

Compute \( \dot{g}(t) \) from (126)

\[ \dot{g}(t) = k_2 M_{\beta_1} |\tilde{\eta}_0| e^{-k_2 t} - \left( \frac{k_1 M_{\beta_1} |\tilde{\eta}_0|}{k_2 - k_1} + k_1 |\tilde{\rho}_0| \right) e^{-k_1 t} \]
\[ = \frac{k_2 M_{\beta_1} |\tilde{\eta}_0|}{k_2 - k_1} \left( e^{-k_2 t} - e^{-k_1 t} \right) + (M_{\beta_1} |\tilde{\eta}_0| - k_1 |\tilde{\rho}_0|) e^{-k_1 t}. \]  \hspace{1cm} (127)

It follows from (21a), (22c) and (127) that \( \dot{g}(t) < 0 \) for all \( t \geq 0 \). Thus, \( g(t) \) achieves its maximum value at \( t = 0 \). Moreover, the signal \( \frac{M_{\beta_1} \mu_{\beta_2} + \mu_{\beta_1}}{k_1} \| \omega_0 - \hat{\omega}_0 \|_\infty e^{-ct} \) decreases monotonically for all \( t \geq 0 \). Therefore, it follows from (25a) that

\[ \gamma_{r_1}(t) = g(t) + \frac{M_{\beta_1} \mu_{\beta_2} + \mu_{\beta_1}}{k_1} \| \omega_0 - \hat{\omega}_0 \|_\infty e^{-ct} + B_\rho \]
\[ \leq g(0) + \frac{M_{\beta_1} \mu_{\beta_2} + \mu_{\beta_1}}{k_1} \| \omega_0 - \hat{\omega}_0 \|_\infty + B_\rho = |\rho_r(0) - \rho_d| + B_0 = \bar{\gamma}_{r_1}. \]  \hspace{1cm} (128)

Thus, the relationship in (26) holds under (21a) and (22c).

VIII.D. Choice of \( \rho_d \) in (23)

The choice of \( \rho_d \) in (23) says that \( \rho_d \) needs to be selected to satisfy

\[ \rho_d \geq \bar{\gamma}_{r_1} + \gamma_b = |\rho_r(0) - \rho_d| + B_0 + \gamma_b. \]  \hspace{1cm} (129)

- If \( \rho_r(0) \geq \rho_d \), choosing \( \rho_d \geq \frac{\rho_r(0) + B_0 + \gamma_b}{2} \) leads to (129) directly.
- If \( \rho_r(0) < \rho_d \), it follows from (22d) that

\[ B_0 + \gamma_b < \rho_r(0) \Rightarrow \rho_d - \rho_r(0) + B_0 + \gamma_b < \rho_d \Rightarrow |\rho_d - \rho_r(0)| + B_0 + \gamma_b < \rho_d. \]  \hspace{1cm} (130)

Thus, (129) holds.

VIII.E. Derivation of (64)

Suppose there exist finite constants \( L_1, L_2, \mu_\rho, M_{\rho_1} \) and \( M_{\rho_2} \) such that (62b) and (62c) hold. Consider (61a) and rewrite it as:

\[ \dot{\hat{\rho}}(t) = -k_1 \bar{\rho}(t) + \frac{\Delta_{\rho_1}(t) + \Delta_{\rho_2}(t)}{\Delta_{\rho}(t)}. \]  \hspace{1cm} (131)

where

\[ \Delta_{\rho_1}(t) = \beta_1(\omega(t)) \sin(\eta(t) + \beta_2(\omega(t))) - \sin(\eta_r(t) + \beta_2(\omega(t)))], \]
\[ \Delta_{\rho_2}(t) = \beta_1(\omega_r(t)) \sin(\eta_d(\omega_r(t), \rho_r(t)) + \beta_2(\omega_r(t))) - \beta_1(\omega_r(t)) \sin(\eta_d(\omega_r(t), \rho(t)) + \beta_2(\omega_r(t))). \]  \hspace{1cm} (132)

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We have:

\[ |\Delta_{\rho_1}(t)| = M_{\beta_1} |\eta(t) - \eta_r(t)|, \]
\[ |\Delta_{\rho_2}(t)| = |\beta_1(\omega_r(t)) \sin(\eta_d(\omega_r(t), \rho(t)) + \beta_2(\omega_r(t))) - \beta_1(\omega_r(t)) \sin(\eta_d(\omega_r(t), \rho(t)) + \beta_2(\omega_r(t)))| \]
\[ \leq M_{\beta_1, L_1} |\eta_d(\omega_r(t), \rho(t)) - \eta_1(\omega_r(t), \rho_r(t))| + (M_{\beta_1, \mu_{\beta_2}} + M_{\beta_1}) \|\omega_r(t) - \omega_r(t)\|_\infty, \]
\[ \leq M_{\beta_1, L_2} |\rho(t) - \rho_r(t)| + (M_{\beta_1, L_1} + M_{\beta_1, \mu_{\beta_2}} + M_{\beta_1}) \|\omega_r(t) - \omega_r(t)\|_\infty, \]

where \( M_{\beta_1}, M_{\beta_1, \mu_{\beta_1}}, \mu_{\beta_2}, L_1, \) and \( L_2 \) are given in (20c) and (62), respectively. From the above, the inequality in (64a) holds by choosing \( \kappa_i \) (for \( i = 1, 2, 3 \)) as

\[ \kappa_1 = M_{\beta_1, L_2}, \quad \kappa_2 = M_{\beta_1}, \quad \kappa_3 = M_{\beta_1, \mu_{\beta_2}} + M_{\beta_1}. \tag{134} \]

Next, consider (61b) and rewrite it to be:

\[ \tilde{\dot{y}}(t) = -k_2\bar{\eta}(t) + \Delta\eta_1(t) + \Delta\eta_2(t) + k_2(\eta_d(\omega_r(t)) - \eta_d(\omega_r(t)) \big), \tag{135} \]

where

\[ \Delta\eta_1(t) = \frac{\tilde{V}_l(t) \cos(\tilde{\psi}_l(t) - \psi(t) + \eta(t)) - \tilde{V}_r(t) \cos(\psi_r(t) - \psi(t) + \eta_r(t))}{\rho(t)} \]
\[ \Delta\eta_2(t) = \frac{V_l(t) \cos(\psi_l(t) - \psi(t) + \eta_l(t)) - V_r(t) \cos(\psi_r(t) - \psi(t) + \eta_r(t))}{\rho(t)}. \tag{136} \]

Since

\[ \Delta\eta_1(t) = \frac{\tilde{V}_l(t) \cos(\tilde{\psi}_l(t) - \psi(t) + \eta(t)) - \tilde{V}_r(t) \cos(\psi_r(t) - \psi(t) + \eta_r(t))}{\rho(t)} + \frac{\tilde{V}_l(t) \cos(\psi_l(t) - \psi(t) + \eta_l(t)) - \tilde{V}_r(t) \cos(\psi_r(t) - \psi(t) + \eta_r(t))}{\rho(t)}, \]
\[ = \frac{\tilde{V}_l(t)}{\rho(t)} \left[ \cos(\tilde{\psi}_l(t) - \psi(t) + \eta(t)) - \cos(\psi_l(t) - \psi(t) + \eta_l(t)) \right] + \cos(\psi_l(t) - \psi(t) + \eta_l(t)) \left( \frac{\tilde{V}_l(t)}{\rho(t)} - \frac{V_l(t)}{\rho_r(t)} \right), \]
\[ = -2\frac{\tilde{V}_l(t)}{\rho(t)} \sin \left( \frac{\tilde{\psi}_l(t) + \eta_l(t) + \psi_l(t) + \eta_l(t)}{2} - \psi(t) \right) \sin(\tilde{\psi}_l(t) - \psi(t)) + (\eta_l(t) - \eta_l(t)) \]
\[ + \frac{\tilde{V}_l(t)}{\rho(t)} (\rho_r(t) - \rho(t)) + \frac{1}{\rho_r(t)} (\tilde{V}_l(t) - V_l(t)), \tag{137} \]

and

\[ \Delta\eta_2(t) = V_l(t) \left[ \frac{1}{\rho(t)} \cos(\psi_l(t) - \psi(t) + \eta_l(t)) - \frac{1}{\rho(t)} \cos(\psi_l(t) - \psi(t) + \eta_l(t)) \right] \]
\[ + V_r(t) \left[ \frac{1}{\rho(t)} \cos(\psi_r(t) - \psi(t) + \eta_r(t)) - \frac{1}{\rho_r(t)} \cos(\psi_r(t) - \psi(t) + \eta_r(t)) \right], \tag{138} \]
\[ = -2\frac{V_l(t)}{\rho(t)} \sin \left( \frac{\psi_l(t) + \eta_l(t) + \psi_l(t) + \eta_l(t)}{2} \right) \sin(\eta_l(t) - \eta_l(t)) \]
\[ + \frac{V_l(t) \cos(\psi_l(t) - \psi(t) + \eta_l(t))}{\rho(t)} (\rho_r(t) - \rho(t)), \]
it follows from \((20c)\) that
\[
|\Delta_{\theta}(t)| = |\Delta_{\eta_1}(t) + \Delta_{\eta_2}(t) + \Delta_{\eta_3}(t)|,
\]
\[
\leq 2\frac{\hat{V}_1(t)}{\rho(t)}|\hat{\psi}(t) - \psi_r(t)| + |\eta(t) - \eta_r(t)| + \frac{\hat{V}_1(t)}{\rho(t)\rho_r(t)}|\rho(t) - \rho_r(t)| + \frac{1}{\rho(t)}|\hat{V}_1(t) - V_r(t)| + 
\]
\[
+ 2\frac{\hat{V}_1(t)}{\rho(t)}|\eta(t) - \eta_r(t)| + \frac{\hat{V}_1(t)}{\rho(t)\rho_r(t)}|\rho_r(t) - \rho(t)| + k_2L_1\|\omega_e(t) - \omega_r(t)\|_{\infty} + k_2L_2|\rho(t) - \rho_r(t)|,
\]
\[
\leq 2\frac{(\hat{V}_1(t) + V_r(t))}{\rho(t)}|\eta(t) - \eta_r(t)| + \left(\frac{\hat{V}_1(t) + V_r(t)}{\rho(t)\rho_r(t)} + k_2L_2\right)|\rho(t) - \rho_r(t)| + 
\]
\[
+ \frac{1}{\rho(t)}(2\hat{V}_1(t)|\hat{\psi}(t) - \psi_r(t)| + \hat{V}_1(t) - V_r(t)| + k_2L_1\|\omega_e(t) - \omega_r(t)\|_{\infty}.
\]
\[(139)\]

It follows from \((62)\) and \((139)\) that there exist finite constants \(\kappa_i\) (for \(i = 3, 4, 5\)), chosen as
\[
\kappa_4 = M_{\rho_1}, \quad \kappa_5 = M_{\rho_2} + k_2L_2, \quad \kappa_6 = \mu_{\rho} + k_2L_1,
\]
\[(140)\]
such that the inequality in \((64b)\) holds.

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