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Hamiltonian Form of the Kemmer Equation*

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The Hamiltonian form of the relativistic wave equation for bosons of spin 0 or 1 was first given by Kemmer. The problems associated with the redundant components of the wave function were later resolved by Heitler, who eliminated the redundant components by means of projection operators. We present an alternative treatment which yields essentially the same results as obtained by Heitler, but which retains all components of the wave function.

I. INTRODUCTION

A FIRST-ORDER, relativistic wave equation for bosons of spin 0 or 1 was originally derived by Kemmer.[1] The Kemmer equation is similar in appearance to the Dirac equation, except that the matrices involved, the so-called β matrices, obey different commutation relations from Dirac's γ matrices. Also, the existence of redundant components in the Kemmer wave function introduces complications in the specification of observables and their expectation values. For example, Kemmer noted an ambiguity in the definition of the expectation value. In addition, Kemmer's Hamiltonian for a boson in an electromagnetic field is not Hermitian, and, in fact, as we shall see, it is not even unique.

In order to resolve these, and related difficulties, Heitler proposed a theory in which the redundant components of the wave function are eliminated.[2] In Heitler's theory, a new Hamiltonian, involving only the dynamical components, is obtained. The purpose of the present paper is to present an alternative approach which is in many respects similar to that of Heitler, except that we retain all components of the wave function. In our approach, we also obtain a new Hamiltonian, whose dynamical components are, in fact, the same as Heitler's Hamiltonian.

II. THE KEMMER EQUATION

The Kemmer equation for a free boson of rest mass m is[3]

$$(i\partial_\mu\beta^\mu+m)\psi(x)=0\,, \tag{1}$$

where the matrices β^μ satisfy

$$\beta^\mu\beta^\nu\beta^\sigma+\beta^\sigma\beta^\nu\beta^\mu=\beta^\mu g^{\nu\sigma}+\beta^\sigma g^{\nu\mu}\,. \tag{2}$$

If a 5×5 representation of the β matrices is used, ψ describes a spinless boson, and if a 10×10 representation is used, ψ describes a boson of spin 1. Defining

$$\eta^0=2(\beta^0)^2-1$$

and

$$\bar\psi=-\psi^\dagger\eta^0\,,$$

the four-vector current density can be taken to be

$$j^\mu=\bar\psi\beta^\mu\psi\,,$$

where

$$\partial_\mu j^\mu=0\,.$$

In particular, since, from Eq. (2),

$$\eta^0\beta^0=\beta^0\eta^0=\beta^0\,,$$

the charge density is

$$j^0=-\psi^\dagger\beta^0\psi\,. \tag{3}$$

The requirement

$$\int_V j^0d^3x=\delta_E\,, \tag{4}$$

where V is the volume of a normalization box, and

$$\delta_E=+1 \text{ for positive-energy solutions}$$
$$=-1 \text{ for negative-energy solutions,}$$

determines the normalization of ψ. (The fact that $\int j^0d^3x$ has the sign of the energy will be shown below.)

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[1] N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939).

[2] W. Heitler, Proc. Roy. Irish Acad. **A49**, 1 (1943).

[3] We use the conventions $c=\hbar=1$ and $\partial_\mu=\partial/\partial x^\mu$, where $x^0=t$, $x^1=x$, $x^2=y$, $x^3=z$. Also, $x_\mu=g_{\mu\nu}x^\nu$, where $g_{00}=1$, $g_{11}=g_{22}=g_{33}=-1$, and $p^\mu=g^{\mu\nu}p_\nu=(E,\mathbf{p})$. Repeated Greek indices are summed from 0–3; Latin indices from 1–3.

As shown by Kemmer, the equations

$$i\partial_0\psi=(i\partial_k[\beta^k,\beta^0]-m\beta^0)\psi \tag{5}$$

and

$$\{i\partial_k\beta^k(\beta^0)^2+m[1-(\beta^0)^2]\}\psi=0 \tag{6}$$

are together equivalent to Eq. (1). Setting

$$H'=i\partial_k[\beta^k,\beta^0]-m\beta^0 \tag{7}$$

and making use of

$$\beta^0\beta^k\beta^0=0\,, \tag{8}$$

which follows from Eq. (2), Eqs. (5) and (6) become

$$i\partial_0\psi=H'\psi\,, \tag{9}$$

$$(H'\beta^0+m)\psi=0\,. \tag{10}$$

Equation (10) can be taken as an initial condition and Eq. (9) a dynamical equation in Hamiltonian form. Kemmer takes H' as the free-particle Hamiltonian. If H' is to be Hermitian, we must have

$$\beta^{0\dagger}=\beta^0,\quad \beta^{k\dagger}=-\beta^k. \tag{11}$$

To obtain the wave equation for a charged boson in an electromagnetic field, we replace $i\partial_\mu$ by $(i\partial_\mu-eA_\mu)$ in Eq. (1), obtaining

$$[(i\partial_\mu-eA_\mu)\beta^\mu+m]\psi(x)=0\,. \tag{12}$$

This equation can also be put in Hamiltonian form,[1] the two equations

$$i\partial_0\psi=\{eA_0+(i\partial_k-eA_k)[\beta^k,\beta^0]-m\beta^0 -ie(2m)^{-1}F_{\mu\nu}(\beta^\mu\beta^0\beta^\nu-g^{\mu0}\beta^\nu)\}\psi \tag{13}$$

and

$$\{(i\partial_k-eA_k)\beta^k(\beta^0)^2+m[1-(\beta^0)^2]\}\psi=0\,, \tag{14}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,,$$

together being equivalent to Eq. (12). The operator on the right-hand side of Eq. (13) could be taken to be the Hamiltonian for a charged boson in an electromagnetic field. However, if Eqs. (11) are satisfied, then this Hamiltonian is not Hermitian.

III. OBSERVABLES

Such problems as the determination of observables (as, for example, the above Hamiltonians) and the existence of the subsidiary conditions (10) and (14) arise in the Kemmer theory because of the excessive number of dimensions in the spinor space, there being more components in the wave function than necessary to specify the internal degrees of freedom. Heitler[2] develops a consistent theory by separating the spinor space into two subspaces, which he calls subspace I and subspace II. Subspace I involves those components of the wave function which determine the dynamical state of the boson, and subspace II involves the nondynamical, or redundant components. The separation is achieved by means of the projection operators

$$\Gamma_{\mathrm{I}}=(\beta^0)^2$$

and

$$\Gamma_{\mathrm{II}}=1-(\beta^0)^2\,,$$

which project out the components of subspace I and subspace II, respectively. In so doing, Heitler obtains a Hamiltonian which acts only in subspace I, but which involves second derivatives with respect to the space coordinates, so that the theory loses its relativistic appearance (though not its Lorentz invariance).

We present here an alternative approach, in which all components of the wave function are retained. We obtain a new Hamiltonian, which, like Heitler's, involves second derivatives, and, in fact, reduces to Heitler's if we perform his reduction to subspace I.

We begin by noting that because of the form of the charge density, Eq. (3), we can define the inner product of any two wave functions ψ_1 and ψ_2 by[4]

$$(\psi_1,\psi_2)=-\int\psi_1{}^\dagger\beta^0\psi_2d^3x. \tag{15}$$

We then say that an operator Q is "neo-Hermitian" provided

$$(\psi_1,Q\psi_2)=(Q\psi_1,\psi_2)$$

for all ψ_1 and ψ_2. From Eq. (15), we see that this is equivalent to the requirement

$$\beta^0Q=Q^\dagger\beta^0. \tag{16}$$

We then require that all observables, and in particular the Hamiltonian, be neo-Hermitian.[5]

The eigenvalues of a neo-Hermitian operator are not necessarily real, but an eigenstate corresponding to a nonreal eigenvalue must satisfy $j^0=-\psi^\dagger\beta^0\psi\equiv0$, and hence cannot represent a real boson. To see this, suppose Q is neo-Hermitian, and $Q\psi=q\psi$. Then

$$\psi^\dagger\beta^0Q\psi=q\psi^\dagger\beta^0\psi.$$

But $\psi^\dagger\beta^0Q\psi$ and $\psi^\dagger\beta^0\psi$ are both real, so if $\psi^\dagger\beta^0\psi\neq0$, q must be real.

The orthogonality of eigenstates of a Hermitian operator corresponding to different eigenvalues can also be generalized to neo-Hermitian operators. Suppose Q is neo-Hermitian, and

$$Q\psi_a=a\psi_a \quad\text{and}\quad Q\psi_b=b\psi_b.$$

Then

$$\int\psi_a{}^\dagger\beta^0Q\psi_bd^3x=b\int\psi_a{}^\dagger\beta^0\psi_bd^3x=a^*\int\psi_a{}^\dagger\beta^0\psi_bd^3x.$$

Hence, if $a^*\neq b$, then $\int\psi_a{}^\dagger\beta^0\psi_bd^3x=0$.

[4] For simplification, we drop the reference to the normalization volume V.

[5] A similar requirement is imposed by Heitler in his theory.

Now the Hamiltonian H' is not neo-Hermitian. Note, however, that in place of Eqs. (9) and (10), we could write

$$i\partial_0\psi=[H'+A(H'\beta^0+m)]\psi\,, \qquad (17)$$

$$(H'\beta^0+m)\psi=0\,, \qquad (10)$$

where the operator A is completely arbitrary. Every simultaneous solution of Eqs. (9) and (10) is also a simultaneous solution of Eqs. (17) and (10), and vice versa. Hence, Eqs. (17) and (10) are also together equivalent to Eq. (1). Since the operator on the right-hand side of Eq. (17) represents the free-particle Hamiltonian, we see that this Hamiltonian is not unique. In particular, by appropriately choosing the operator A, we can make the Hamiltonian neo-Hermitian. It is easily verified that a suitable choice is

$$A=m^{-1}i\partial_k\beta^0\beta^k\,,$$

yielding the neo-Hermitian Hamiltonian

$$H=H'+m^{-1}i\partial_k\beta^0\beta^k(H'\beta^0+m)\,. \qquad (18)$$

Equation (17) becomes

$$i\partial_0\psi=H\psi\,.$$

If we restrict ourselves to wave functions satisfying Eq. (10), then an eigenstate of H' corresponding to eigenvalue E is also an eigenstate of H corresponding to the same eigenvalue E. Similarly, an eigenstate of H corresponding to eigenvalue E is also an eigenstate of H' corresponding to the same eigenvalue E. For such a state, we have

$$0=\int\psi^\dagger(H'\beta^0+m)\psi d^3x=E\int\psi^\dagger\beta^0\psi d^3x+m\int\psi^\dagger\psi d^3x\,.$$

Hence

$$E\int\psi^\dagger\beta^0\psi d^3x=-m\int\psi^\dagger\psi d^3x<0\,,$$

so that $E\neq 0$, and $\int j^0d^3x=-\int\psi^\dagger\beta^0\psi d^3x\neq 0$ and has the sign of E. As shown by Kemmer,[1] the nonzero eigenvalues of H' are $E=\pm(p^2+m^2)^{1/2}$.

The form (18) for the free-particle Hamiltonian is still not unique. We could add to H the operator

$$B(H'\beta^0+m)\,, \qquad (19)$$

where B is any dimensionless operator such that (19) is neo-Hermitian. For example, we could take $B=1$. But this additional term would not change the eigenvalues of H corresponding to those eigenstates satisfying Eq. (10).[6] Indeed, using Heitler's viewpoint, the quantity (19) acts entirely in subspace II. To see this, we must show that

$$(\beta^0)^2B(H'\beta^0+m)(\beta^0)^2=0$$

for all B such that (19) is neo-Hermitian. Note first that from Eq. (2) we have

$$(\beta^0)^3=\beta^0\,, \qquad (20)$$

and from Eqs. (7), (8), and (20), we have

$$\beta^0H'\beta^0=-m\beta^0\,. \qquad (21)$$

Since $B(H'\beta^0+m)$ must be neo-Hermitian, we have

$$\beta^0B=B^\dagger\beta^0-m^{-1}\beta^0BH'\beta^0+m^{-1}\beta^0H'B^\dagger\beta^0\,. \qquad (22)$$

Using Eq. (22), we have

$$(\beta^0)^2B(H'\beta^0+m)(\beta^0)^2 =\beta^0(B^\dagger\beta^0-m^{-1}\beta^0BH'\beta^0+m^{-1}\beta^0H'B^\dagger\beta^0) \times(H'\beta^0+m)(\beta^0)^2\,. \qquad (23)$$

Multiplying out the right-hand side of Eq. (23) and using Eqs. (20) and (21), we find that all the terms cancel. Thus, the quantity (19) acts entirely in Heitler's subspace II. It must therefore represent the ambiguity in the Hamiltonian associated with the redundant, or nondynamical, components, and has no physical manifestation.

A neo-Hermitian Hamiltonian for a boson in an electromagnetic field can be found in the same manner as in the free-particle case. We add to the operator on the right-hand side of Eq. (13) an operator of the form

$$A\{(i\partial_k-eA_k)\beta^k(\beta^0)^2+m[1-(\beta^0)^2]\}\,,$$

where, in this case,

$$A=m^{-1}(i\partial_i-eA_i)\beta^0\beta^i\,.$$

This yields the neo-Hermitian Hamiltonian

$$H_{\rm em}=eA_0+(i\partial_k-eA_k)\beta^k\beta^0 -m\beta^0-ie(2m)^{-1}F_{\mu\nu}(\beta^\mu\beta^0\beta^\nu-g^{\mu0}\beta^\nu) +m^{-1}(i\partial_i-eA_i)(i\partial_k-eA_k)\beta^0\beta^i\beta^k\,. \qquad (24)$$

A term could be added to $H_{\rm em}$ analogous to the term (19) for the free-particle Hamiltonian, only involving the operator on the left-hand side of Eq. (14). As in the case of the free-particle Hamiltonian, such a term would act only in Heitler's subspace II, and would have no physical consequence. One also readily finds that the operator on the left-hand side of Eq. (14) itself acts entirely in subspace II, so that Eq. (14) is a restriction on the redundant components only.

If we project out of $H_{\rm em}$ the part which acts only in Heitler's subspace I, we obtain

$$(\beta^0)^2H_{\rm em}(\beta^0)^2=eA_0(\beta^0)^2-m\beta^0 +m^{-1}(i\partial_i-eA_i)(i\partial_k-eA_k)\beta^0\beta^i\beta^k\,,$$

which is the same as Heitler's Hamiltonian.[2]

IV. EXPECTATION VALUES

Because of the form (3) for j^0, and the normalization (4), the expectation value of any observable Q in the

[6] Neither would it change the expectation value of the time derivative of any observable, as we shall see below.

state ψ is given by

$$\langle Q\rangle = -\delta_E \int \psi^\dagger \beta^0 Q \psi d^3x, \tag{25}$$

which is real because Q is neo-Hermitian. There can be no ambiguity here in the ordering of the factors β^0 and Q, since the order $Q\beta^0$ would not in general give a real expectation value. Taking the time derivative of Eq. (25), we have

$$\begin{aligned} d\langle Q\rangle/dt &= -i\delta_E \int \psi^\dagger (H^\dagger \beta^0 Q - \beta^0 Q H)\psi d^3x \\ &= -i\delta_E \int \psi^\dagger \beta^0 [H,Q]\psi d^3x \\ &= \langle i[H,Q]\rangle, \end{aligned}$$

so that we may take as the operator representing the time derivative of Q,

$$\dot{Q} = i[H,Q]. \tag{26}$$

We note that $\dot{Q}$ is neo-Hermitian, for

$$\begin{aligned} (\beta^0\dot{Q})^\dagger &= -i(\beta^0 HQ - \beta^0 QH)^\dagger = -i(Q^\dagger\beta^0 H - H^\dagger \beta^0 Q) \\ &= -i(\beta^0 QH - \beta^0 HQ) = \beta^0\dot{Q}. \end{aligned}$$

We also note that a term of the form (19) added to the free-particle Hamiltonian would have no effect on $\langle\dot{Q}\rangle$ for states satisfying Eq. (10), because of the fact that (19) is neo-Hermitian. An analogous statement holds for H_{em}.

As an example, a straightforward calculation of the time derivative of the position operator x^i for a free boson yields

$$\dot{x}^i = [\beta^0,\beta^i] - im^{-1}\partial_k \beta^0\beta^k\beta^i - m^{-1}\beta^0\beta^i(H'\beta^0 + m).$$

As in the Dirac theory, the quantity $\dot{x}^i$ is not directly measurable, but its expectation value is. Suppose we calculate $\langle\dot{x}^i\rangle$ in a simultaneous eigenstate of the free-particle Hamiltonian and the linear momentum, corresponding to eigenvalues E and $\mathbf{p}$, respectively. We have, making use of Eqs. (8) and (10),

$$\langle\dot{x}^i\rangle = -\delta_E \int \psi^\dagger (\beta^0)^2 \beta^i \psi d^3x + \delta_E m^{-1} p_k \int \psi^\dagger (\beta^0)^2 \beta^k \beta^i \psi d^3x. \tag{27}$$

Using Eqs. (7) and (2), the first term on the right-hand side of Eq. (27) can be written

$$-\delta_E E^{-1} \int \psi^\dagger (\beta^0)^2 \beta^i H' \psi d^3x = -\delta_E E^{-1} p_k \int \psi^\dagger \beta^i \beta^k \beta^0 \psi d^3x,$$

and the second term can be written

$$\delta_E (mE)^{-1} p_k \int \psi^\dagger H' (\beta^0)^2 \beta^k \beta^i \psi d^3x = -\delta_E E^{-1} p_k \int \psi^\dagger \beta^0 \beta^k \beta^i \psi d^3x.$$

Adding these together gives

$$\langle\dot{x}^i\rangle = p^i/E,$$

which is the classical result.

V. CONCLUSION

The problems associated with the redundant components of the Kemmer wave function can be resolved without resorting to Heitler's reduction to a subspace containing only the dynamical components. This result is achieved by defining a new Hamiltonian, and restricting the redundant components of the wave function via Eqs. (10) or (14). Just as in Heitler's theory, one obtains the customary expression (26) for the time derivative of an observable, and the unambiguous expression (25) for the expectation value of an observable.[7]

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[7] Actually, Heitler omits the sign factor δ_E from the definition of the expectation value, but multiplies the eigenvalues of an observable by δ_E to get the "measurable values." Thus, for a negative-energy eigenstate of the momentum operator corresponding to eigenvalue $\mathbf{p}$, the "measurable value" of the momentum would be $-\mathbf{p}$, and the expectation value of the momentum in this state would also be $-\mathbf{p}$. This system has the advantage that the "measurable values," and the expectation values, of the Hamiltonian are positive, even for negative-energy solutions. The same system could be adopted here, but it seems to us somewhat more artificial than the customary method of including the "norm" of the state in the definition of the expectation value. The problem of the negative-energy solutions is no different here than in the Dirac equation, and is similarly resolved by second quantization. [See W. B. Zeleny and A. O. Barut, Phys. Rev. **121**, 908 (1961).]