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Engineering Notes

Costate Computation by a Chebyshev Pseudospectral Method

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I. Introduction

AMONG the various pseudospectral (PS) methods for optimal control [1], only the Legendre PS method has been mathematically proven to guarantee the feasibility, consistency, and convergence of the approximations [2–5]. As exemplified by its experimental and flight applications in national programs [6–10], it is not surprising that the Legendre PS method has become the method of choice [11–19] in both industry and academia for solving optimal control problems. Efforts to improve the Legendre PS methods by using either other polynomials [20–22] or point distributions [23,24] have not yet resulted in any rigorous framework for convergence of these approximations [24,25].

Compared to Legendre PS methods, Chebyshev PS methods [21,22] for optimal control are somewhat more attractive for a number of reasons. When a function is approximated, it is well known that a Chebyshev expansion is very close to the best polynomial approximation in the infinity norm [26,27]. In addition, Chebyshev polynomials have an attractive computational advantage in terms of the computation of Chebyshev–Gauss–Lobatto (CGL) nodes. Unlike the Legendre–Gauss–Lobatto (LGL) nodes, CGL nodes can be evaluated in closed form [26]. Thus, a Chebyshev PS method offers the possibility of rapid computation because it does not require the use of advanced numerical linear algebra techniques that are necessary for the calculation of LGL nodes [21]. A similar numerical advantage applies to the computation of the derivative via a fast Chebyshev differentiation scheme that is similar to a fast-Fourier-transform (FFT) computation. In the same spirit, integration is also fast because of the connection between the Clenshaw–Curtis integration and the FFT [27]. Despite these attractive properties, Chebyshev PS methods have not advanced beyond the works of [21,22]. This is, in part, due to the absence of a covector mapping theorem that is crucial for the computation of the costates and other covectors. The computation of costates and other covectors is important in solving practical optimal control problems as it provides a means for verification and validation of the computed solution [25]. Beyond verification and validation, information about covectors can also be used to facilitate the design of guidance and control algorithms [28].

In this Note, we fill the key gap of costate computation for Chebyshev PS methods by furthering the method of Fahroo and Ross [21]. We do this by combining some recent results from Clenshaw–Curtis integration [27], the unification principles proposed by Fahroo and Ross [1,29,30], and the new results of Gong et al. [23].

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II. Problem Formulation

In this Note, we consider the following nonlinear constrained optimal control problem:

$$
(B)\begin{cases}\text{Minimize} & J[x(\cdot),u(\cdot)]=E(x(-1),x(1))+\int_{-1}^{1}F(x(t),u(t))\,\mathrm{d}t\\ \text{Subject to} & \dot{x}(t)=f(x(t),u(t))\\ & e(x(-1),x(1))=0\\ & h(x(t),u(t))\le 0\end{cases}
$$

where E: $\mathbb{R}^{N_x}\times\mathbb{R}^{N_x}\to\mathbb{R}$ is the endpoint cost, F: $\mathbb{R}^{N_x}\times\mathbb{R}^{N_u}\to\mathbb{R}$ is the running cost, e: $\mathbb{R}^{N_x}\times\mathbb{R}^{N_x}\to\mathbb{R}^{N_e}$ is the endpoint constraint, and h: $\mathbb{R}^{N_x}\times\mathbb{R}^{N_u}\to\mathbb{R}^{N_h}$ is the path constraint. It is assumed that these functions are continuous with respect to their arguments and that their gradients are Lipschitz continuous over the domain. Note that the continuity of the vector fields does not exclude discontinuous optimal control. It is well known that a smooth optimal control problem may yield discontinuous solutions. Also note that, by a simple time domain transformation [21], the results hold for problems on $t\in[a,b]$ and time-free problems can be easily handled just as well.

To develop a covector mapping theorem, we apply the covector mapping principle [31] as illustrated in Fig. 1, that is, for any given optimal control problem (B), we need to generate the collection of problems illustrated in Fig. 1. The definition and generation of these problems is discussed in the remainder of this Note.

To apply the first-order necessary conditions, appropriate constraint qualifications are implicitly assumed so that the first-order necessary conditions hold. These first-order necessary conditions can be cast as the following boundary value problem.

Problem B^λ: If $(x(t),u(t))$ is the optimal solution to Problem B, then there exist $(\lambda(t),\mu(t),\nu)$ such that

$$
\begin{aligned}
\dot{x}&=f(x,u)\\
\dot{\lambda}&=-F_x(x,u)-f_x^T(x,u)\lambda-h_x^T(x,u)\mu(t)\\
0&=F_u(x,u)+f_u^T(x,u)\lambda+h_u^T(x,u)\mu(t)\\
0&=e(x(1),x(-1))\\
0&\ge h(x,u)\\
0&=\mu(t)h(x(t),u(t)),\qquad \mu(t)\ge 0
\end{aligned}
$$

$$
\lambda(-1)=-E_{x(-1)}(x(-1),x(1))-e_{x(-1)}^T(x(-1),x(1))\nu \tag{1}
$$

$$
\lambda(1)=E_{x(1)}(x(-1),x(1))+e_{x(1)}^T(x(-1),x(1))\nu \tag{2}
$$

III. Primal Chebyshev Pseudospectral Methods

In this section, we outline the primal Chebyshev PS method as proposed by Fahroo and Ross [21]. For the discretization of Problem B by a Chebyshev PS method, the CGL nodes are defined as

$$
t_k=\cos(\pi(N-k)/N),\qquad k=0,1,\ldots,N
$$

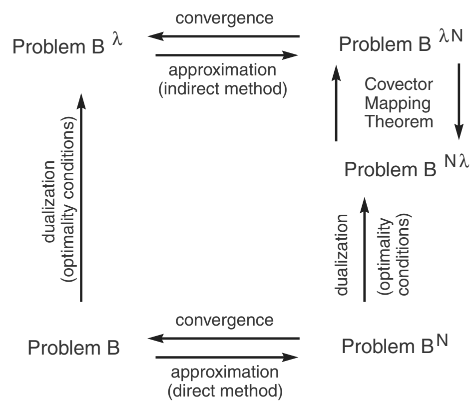

Fig. 1 Covector mapping principle and the covector mapping theorem [31].

These points lie in the interval $[-1, 1]$ and are the extrema of the Nth-order Chebyshev polynomial $T_N(t) = \cos(N\cos^{-1} t)$. The state variables at the nodes are approximated by column vectors $\bar{x}^k \in \mathbb{R}^{N_x}$, that is,

$$x(t_k) \approx \bar{x}^k = \begin{bmatrix} \bar{x}_1^k \\ \bar{x}_2^k \\ \vdots \\ \bar{x}_{N_x}^k \end{bmatrix}$$

Similarly, $\bar{u}^k$ is the approximation of $u(t_k)$. Thus, a discrete approximation of the function $x_i(t)$, the ith component of $x(t)$, across all nodes is the row vector

$$\bar{x}_i = [\bar{x}_i^1 \quad \bar{x}_i^2 \quad \cdots \quad \bar{x}_i^N]$$

Note that the discrete variables are denoted by letters with an overbar, such as $\bar{x}_i^k$ and $\bar{u}_i^k$. If k in the superscript and/or i in the subscript are missing, it represents the corresponding vector or matrix in which the indices run from minimum to maximum. For example, let

$$\bar{x} = \begin{bmatrix} \bar{x}_1^0 & \bar{x}_1^1 & \cdots & \bar{x}_1^N \\ \bar{x}_2^0 & \bar{x}_2^1 & \cdots & \bar{x}_2^N \\ \vdots & \vdots & \vdots & \vdots \\ \bar{x}_{N_x}^0 & \bar{x}_{N_x}^1 & \cdots & \bar{x}_{N_x}^N \end{bmatrix}$$

then $\bar{x}_i$ is the ith row of $\bar{x}$, which represents the discrete approximation of the ith component, $x_i(t)$, at all nodes; and $\bar{x}^i$ is the ith column of $\bar{x}$, which represents the approximation of the state, $x(t)$, at the ith node.

A continuous approximation is defined by its polynomial interpolation, denoted by $x_i^N(t)$, that is,

$$x_i(t) \approx x_i^N(t) = \sum_{k=0}^{N} \bar{x}_i^k \phi_k(t)$$

where $\phi_k(t)$ is the Lagrange interpolating polynomial:

$$\phi_k(t) = \frac{(-1)^{k+1}}{N^2 c_k} \frac{(1-t^2)\dot{T}_N(t)}{t - t_k}$$
$$c_k = \begin{cases} 2, & \text{if } k = 0, N \\ 1, & \text{if } 1 \le k \le N-1 \end{cases}$$

The derivative of $x_i^N(t)$ at the quadrature node t_k is easily computed by the following matrix multiplication:

$$[\dot{x}_i^N(t_0), \quad \dot{x}_i^N(t_1), \quad \cdots \quad \dot{x}_i^N(t_N)] = \bar{x}_i \cdot D^T$$

where the $(N+1) \times (N+1)$ differentiation matrix D is defined by

$$D_{kj} = \dot{\phi}_k(t_j) = \begin{cases} (c_k/c_j)[(-1)^{j+k}/(t_j - t_k)], & \text{if } j \ne k; \\ t_k/(2 - 2t_k^2), & \text{if } 1 \le j = k \le N-1; \\ -(2N^2+1)/6, & \text{if } j = k = 0; \\ (2N^2+1)/6, & \text{if } j = k = N \end{cases}$$

Because Chebyshev polynomials are orthogonal with respect to a nonuniform weight function [26], the discretization of the integration is done using the Clenshaw–Curtis quadrature scheme [27],

$$J[x(\cdot), u(\cdot)] \approx \bar{J}^N(\bar{x}^N, \bar{u}^N) = \sum_{k=0}^{N} F(\bar{x}^k, \bar{u}^k) w_k + E(\bar{x}^0, \bar{x}^N)$$

where w_k are the Clenshaw–Curtis quadrature weights. For N even, the weights are given by

$$w_0 = w_N = 1/(N^2 - 1)$$
$$w_s = w_{N-s} = \frac{4}{N} \sum_{j=0}^{N/2''} \frac{1}{1-4j^2} \cos\left(\frac{2\pi j s}{N}\right), \qquad s = 1, 2, \ldots, N/2$$

whereas for N odd, we have

$$w_0 = w_N = 1/N^2$$
$$w_s = w_{N-s} = \frac{4}{N} \sum_{j=0}^{(N-1)/2''} \frac{1}{1-4j^2} \cos\left(\frac{2\pi j s}{N}\right), \quad s = 1, \ldots, (N-1)/2$$

The double prime in the preceding formulas means that the first and the last elements have to be halved.

Remark 1 For $N+1$ nodes, the Legendre–Gauss integration scheme is exact for any polynomial of order $2N+1$. In contrast, the Clenshaw–Curtis integration scheme is exact for polynomials of order N. But the scheme offers computational advantages such as calculation of the weights using FFT algorithms and also the convergence of the discrete integration for any continuous function.

In fact, recently Trefethen developed a new analysis for the Clenshaw–Curtis integration and showed that its *practical* accuracy is as good as the Gauss integration for general nonpolynomial integrands [27]. Because Gauss points have well-known problems in solving optimal control problems [1,24], Trefethen's analysis implies that, for optimal control applications, we can now develop a covector mapping theorem that is similar to a Legendre PS method. To this end, we define the discretization of Problem B as follows. Let $\mathbb{X}$ and $\mathbb{U}$ be two compact sets representing the search region; then, we have the following problem.

Problem B^N: Find $\bar{x} \in \mathbb{X}$ and $\bar{u} \in \mathbb{U}$ that minimize

$$\bar{J}^N(\bar{x}, \bar{u}) = \sum_{k=0}^{N} F(\bar{x}^k, \bar{u}^k) w_k + E(\bar{x}^0, \bar{x}^N)$$

subject to the discrete dynamics

$$-\sum_{j=0}^{N} D_{kj} \bar{x}^j + f(\bar{x}^k, \bar{u}^k) = 0$$

end point constraints $e(\bar{x}^0, \bar{x}^N) = 0$, and path constraints $h(\bar{x}^k, \bar{u}^k) \le 0$ for all $k = 0, 1, \ldots, N$.

Although it is very tempting to discretize Problem B^λ in like fashion, as in the case of the Legendre PS method at LGL points, recent unification principles [1,29,30] indicate otherwise. In following these principles, we must develop an adjoint differentiation matrix, D^*, which may or may not be the same as D that appropriately discretizes the adjoint equations. This is, in fact, the heart of covector mapping theorems.

IV. Primal–Dual Chebyshev Discretization

It is fairly straightforward to develop the Karush–Kuhn–Tucker conditions for Problem B^N; these conditions are summarized as follows.

Problem $B^{N\lambda}$: Find $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$, such that

$$
\begin{aligned}
&-\sum_{j=0}^{N} D_{kj}\bar{x}^j + f(\bar{x}^k, \bar{u}^k) = 0\\
&e(\bar{x}^0, \bar{x}^N) = 0\\
&h(\bar{x}^k, \bar{u}^k) \le 0\\
&-\sum_{j=0}^{N} D_{jk}\bar{\lambda}^j + f_x^T(\bar{x}^k, \bar{u}^k)\bar{\lambda}^k + F_x(\bar{x}^k, \bar{u}^k)w_k\\
&\quad + h_x^T(\bar{x}^k, \bar{u}^k)\bar{\mu}^k + c_k = 0\\
&F_u(\bar{x}^k, \bar{u}^k)w_k + f_u^T(\bar{x}^k, \bar{u}^k)\bar{\lambda}^k + h_u^T(\bar{x}^k, \bar{u}^k)\bar{\mu}^k = 0\\
&\bar{\mu}^k \cdot h(\bar{x}^k, \bar{u}^k) = 0, \qquad \bar{\mu}^k \ge 0
\end{aligned} \tag{3}
$$

where $c_i = 0$ for $1 \le i \le N - 1$ and

$$
c_0 = \frac{\partial E}{\partial x^0}(\bar{x}^0, \bar{x}^N) + \left(\frac{\partial e}{\partial x^0}(\bar{x}^0, \bar{x}^N)\right)^T \bar{\nu}
$$
$$
c_N = \frac{\partial E}{\partial x^N}(\bar{x}^0, \bar{x}^N) + \left(\frac{\partial e}{\partial x^N}(\bar{x}^0, \bar{x}^N)\right)^T \bar{\nu}
$$

Note that the index of the differentiation matrix in Eq. (3) is D_{jk}, not D_{kj}. *This is a key point because*

$$\sum_{j=0}^{N} D_{kj}\bar{x}^j \approx \dot{x}(t_k)$$

but

$$\sum_{j=0}^{N} D_{jk}\bar{x}^j$$

is not an approximation of $\dot{x}(t_k)$. *Indeed, the difference is very large.* Therefore, Eq. (3) is not a discretization of Problem B^λ. As a point of comparison, in the Legendre differentiation matrix, D^L, with LGL weights, w_j^L, the following relations [32]

$$
\begin{aligned}
w_j^L D_{jk}^L &= -w_k^L D_{kj}^L, \qquad \text{if } k \ne j\\
D_{jj}^L &= 0, \qquad \text{if } j \ne 1, N
\end{aligned} \tag{4}
$$

allow us to switch the index of D^L. For a Chebyshev differentiation matrix with Clenshaw–Curtis weights, relation (4) does not hold. This is the main technical point that prevented the completion of the Chebyshev PS method for optimal control.

In following the new principles laid out by Fahroo and Ross [1,29,30], we claim that

$$D^* = -W^{-1}D^T W + W^{-1}\Delta \tag{5}$$

where $W = \text{diag}(w_0, w_1, \ldots, w_N)$ (w_i are Clenshaw–Curtis weights), and

$$
\Delta = \begin{bmatrix} -1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ & & \vdots & \\ 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 1 \end{bmatrix}
$$

is an approximation to a differentiation matrix. In lieu of a complete rederivation of this result along lines similar to those developed in [1,23,29,30], we illustrate this point in Fig. 2.

Remark 2: The analysis Gong et al. [23] shows that the accuracy of D^* as a differentiation matrix depends highly on the accuracy of the integration scheme. Because the Clenshaw–Curtis integration used in a Chebyshev PS method is practically as accurate as the Gauss quadrature integration [27], D^* provides a reasonably good estimation of the derivatives as demonstrated in Fig. 2.

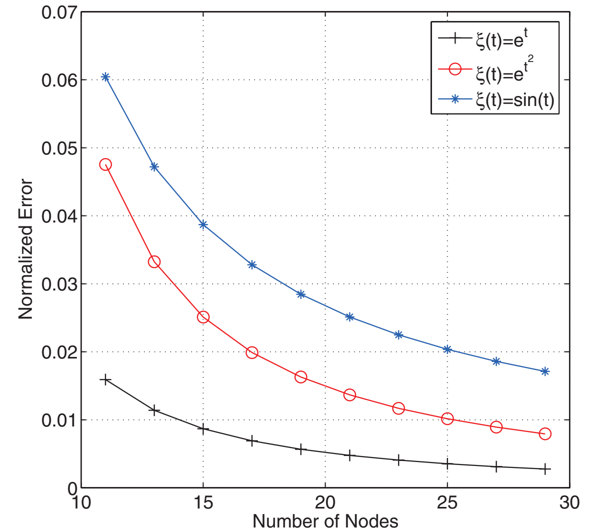

Fig. 2 ***D*** **as a differentiation matrix; the normalized error is defined as** $e = \|D^* \cdot [\xi(t)] - [\dot{\xi}(t)]\|_2 / \|[\dot{\xi}(t)]\|_\infty$**, where** $[\xi(t)]$ **and** $[\dot{\xi}(t)]$ **are shorthand notations for** $[\xi(t_0), \ldots, \xi(t_N)]^T$ **and** $[\dot{\xi}(t_0), \ldots, \dot{\xi}(t_N)]^T$**.**

Using D^* as an adjoint Chebyshev differentiation matrix, we define the primal–dual PS discretization of Problem B^λ as follows.

Problem $B^{\lambda N}$: Find $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$, such that

$$
\begin{aligned}
&-\sum_{j=0}^{N} D_{kj}\bar{x}^j + f(\bar{x}^k, \bar{u}^k) = 0\\
&\sum_{j=0}^{N} D_{kj}^*\bar{\lambda}^j = -f_x^T(\bar{x}^k, \bar{u}^k)\bar{\lambda}^k - F_x(\bar{x}^k, \bar{u}^k)w_k - h_x^T(\bar{x}^k, \bar{u}^k)\bar{\mu}^k\\
&F_u(\bar{x}^k, \bar{u}^k)w_k + f_u^T(\bar{x}^k, \bar{u}^k)\bar{\lambda}^k + h_u^T(\bar{x}^k, \bar{u}^k)\bar{\mu}^k = 0\\
&e(\bar{x}^0, \bar{x}^N) = 0\\
&h(\bar{x}^k, \bar{u}^k) \le 0\\
&\bar{\mu}^k \cdot h(\bar{x}^k, \bar{u}^k) = 0, \qquad \bar{\mu}^k \ge 0\\
&\bar{\lambda}^0 = -\frac{\partial E}{\partial x^0}(\bar{x}^0, \bar{x}^N) - \left(\frac{\partial e}{\partial x^0}(\bar{x}^0, \bar{x}^N)\right)^T \bar{\nu}\\
&\bar{\lambda}^N = \frac{\partial E}{\partial x^N}(\bar{x}^0, \bar{x}^N) + \left(\frac{\partial e}{\partial x^N}(\bar{x}^0, \bar{x}^N)\right)^T \bar{\nu}
\end{aligned}
$$

V. Covector Mapping Theorem

From (5), it follows that

$$
\begin{cases}
D_{jk} = -\frac{w_k}{w_j} D_{kj}^*, & \text{unless } j = k = 0, \text{ or } j = k = N,\\
D_{00} = -D_{00}^* - 1/w_0 & \\
D_{NN} = -D_{NN}^* + 1/w_N &
\end{cases} \tag{6}
$$

Substituting these equations in Eq. (3) and denoting

$$\hat{\lambda}^j = \frac{\bar{\lambda}^j}{w_j}, \qquad \hat{\mu}^j = \frac{\bar{\mu}^j}{w_j}, \qquad \hat{\nu} = \bar{\nu} \tag{7}$$

we transform Problem $B^{N\lambda}$ as follows.

Transformed Problem $B^{N\lambda}$: Find $(\bar{x}, \bar{u}, \hat{\lambda}, \hat{\mu}, \hat{\nu})$, such that

$$
\begin{aligned}
&-\sum_{j=0}^{N} D_{kj}\bar{x}^j + f(\bar{x}^k, \bar{u}^k) = 0\\
&e(\bar{x}^0, \bar{x}^N) = 0\\
&h(\bar{x}^k, \bar{u}^k) \le 0\\
&\sum_{j=0}^{N} D^*_{kj}\hat{\lambda}^j + f_x^T(\bar{x}^k, \bar{u}^k)\hat{\lambda}^k + F_x(\bar{x}^k, \bar{u}^k) + h_x^T(\bar{x}^k, \bar{u}^k)\hat{\mu}^k + \hat{c}_k = 0\\
&F_u(\bar{x}^k, \bar{u}^k) + f_u^T(\bar{x}^k, \bar{u}^k)\hat{\lambda}^k + h_u^T(\bar{x}^k, \bar{u}^k)\hat{\mu}^k = 0\\
&\hat{\mu}^k \cdot h(\bar{x}^k, \bar{u}^k) = 0, \qquad \bar{\mu}^k \ge 0
\end{aligned}
\tag{8}
$$

where $\hat{c}_i = 0$ for $1 \le i \le N-1$ and

$$
\begin{aligned}
\hat{c}_0 &= \frac{1}{w_0}\left[\hat{\lambda}^0 + \frac{\partial E}{\partial x^0}(\bar{x}^0, \bar{x}^N) + \left(\frac{\partial e}{\partial x^0}(\bar{x}^0, \bar{x}^N)\right)^T \hat{\nu}\right]\\
\hat{c}_N &= \frac{1}{w_N}\left[-\hat{\lambda}^N + \frac{\partial E}{\partial x^N}(\bar{x}^0, \bar{x}^N) + \left(\frac{\partial e}{\partial x^N}(\bar{x}^0, \bar{x}^N)\right)^T \hat{\nu}\right]
\end{aligned}
$$

If a solution to Problem $B^{\lambda N}$ exists, it is clear that a solution to the transformed Problem $B^{N\lambda}$ exists with the added condition that

$$
\hat{c}_0 = 0 \quad \text{and} \quad \hat{c}_N = 0 \tag{9}
$$

Thus, we have the following result, also illustrated in Fig. 1.

Covector Mapping Theorem: Let $\chi := \{\bar{x}, \bar{u}\}$, $\Lambda := \{\bar{\nu}, \bar{\mu}, \bar{\lambda}\}$. and $\tilde{\Lambda} := \{\tilde{\nu}, \tilde{\mu}, \tilde{\lambda}\}$.

Define the following multiplier sets corresponding to χ:

$$
\mathbb{M}^{\lambda N}(\chi) := \{\Lambda\colon \Lambda \text{ satisfies conditions of Problem } B^{\lambda N}\} \tag{10}
$$

$$
\mathbb{M}^{N\lambda}(\chi) := \{\tilde{\Lambda}\colon \tilde{\Lambda} \text{ satisfies conditions of Problem } B^{N\lambda}\} \tag{11}
$$

$$
\hat{\mathbb{M}}^{N\lambda}(\chi) := \{\tilde{\Lambda} \in \mathbb{M}^{N\lambda}(\chi)\colon \tilde{\Lambda} \text{ satisfies Eq. (9)}\} \tag{12}
$$

Then, $\hat{\mathbb{M}}^{N\lambda}(\chi) \sim \mathbb{M}^{\lambda N}(\chi)$. That is, under the closure conditions given by Eq. (9), every solution to Problem $B^{N\lambda}$ is also a solution to Problem $B^{\lambda N}$.

Remark 3: There are many different ways to incorporate the closure conditions [30,32,33]. In [23], a primal-only collection of conditions is proposed that is equivalent to the primal–dual conditions of Eq. (9).

VI. Illustrative Examples

We present two examples. The first example is a simple nonlinear problem from [5] with a known analytic solution:

$$
\begin{cases}
\text{Minimize} & J[x(\cdot), u(\cdot)] = 4x_1(2) + x_2(2) + 4\int_0^2 u^2(t)\,\mathrm{d}t\\
\text{Subject to} & \dot{x}_1(t) = x_2^3(t)\\
& \dot{x}_2(t) = u(t)\\
& (x_1(0), x_2(0)) = (0, 1)
\end{cases}
$$

The necessary conditions uniquely determine the optimal solution

$$
x_1(t) = -\frac{64}{5(2+t)^5} + \frac{2}{5}; \qquad x_2(t) = \frac{4}{(2+t)^2}
$$

$$
u(t) = -\frac{8}{(2+t)^3}; \qquad \lambda_1(t) = 4 \qquad \lambda_2(t) = \frac{64}{(2+t)^3}
$$

The problem is solved using the Chebyshev PS method. The accuracy of the computed solution is listed Table 1, in which the column labeled N denotes the number of nodes used in the discretization. The errors are the maximum relative errors between the discrete and exact solutions. From the results listed in the table, it is clear that the Chebyshev PS method provides accurate solutions for both the primal and dual variables.

The next example is the following orbit transfer problem formulated in normalized units:

$$
\begin{cases}
\text{Minimize} & \int_0^{20}(u_r(t)^2 + u_t(t)^2)\,\mathrm{d}t\\
\text{Subject to} & \dot{r} = v_r\\
& \dot{\theta} = \frac{v_t}{r}\\
& \dot{v}_r = \frac{v_t^2}{r} - \frac{1}{r^2} + u_r\\
& \dot{v}_t = -\frac{v_r v_t}{r} + u_t\\
& |u_r| \le 0.05; |u_t| \le 0.05\\
& (r(0), v_r(0), v_t(0)) = (1, 0, 1)\\
& (r(t_f), v_r(t_f), v_t(t_f)) = (4, 0, 0.5)
\end{cases}
$$

where r is the radial distance, θ is the true anomaly, v_r and v_t are the velocities in the radial and transverse directions, respectively, and u_r and u_t are the radial and transverse thrust components, respectively.

Figure 3 shows the candidate optimal trajectory and the candidate optimal control computed by the Chebyshev PS method with 64 nodes. Also shown in Fig. 3 are the states obtained from a numerical (RK4/5) propagation of the discrete-time optimal controller. The solid lines are the propagated trajectories generated by a linear interpolation of the controls. Clearly, the discrete optimal states match the propagated trajectory very accurately, which numerically demonstrates the feasibility and accuracy of the discrete optimal solution.

From the minimum principle, it is straightforward to derive the following adjoint equations:

$$
\begin{cases}
\dot{\lambda}_1 &= \lambda_2\frac{v_t}{r^2} + \lambda_3\left(\frac{v_t^2}{r^2} - \frac{2}{r^3}\right) - \lambda_4\frac{v_r v_t}{r^2}\\
\dot{\lambda}_2 &= 0\\
\dot{\lambda}_3 &= -\lambda_1 + \lambda_4\frac{v_t}{r}\\
\dot{\lambda}_4 &= -\frac{\lambda_2}{r} - \frac{2\lambda_3 v_t}{r} + \frac{\lambda_4 v_r}{r}
\end{cases}
$$

and the corresponding initial transversality conditions:

$$
\begin{cases}
\lambda_1(0) &= \nu_1^0\\
\lambda_2(0) &= 0\\
\lambda_3(0) &= \nu_3^0\\
\lambda_4(0) &= \nu_4^0
\end{cases}
$$

Except for λ_2, an analytic solution for the costates is not available. From the covector mapping theorem, we find that

Table 1 Illustrating the accuracy of the Chebyshev PS method

N	e_{x1}	e_{x_2}	e_u	$e_{\lambda 1}$	$e_{\lambda 2}$
10	0.0043	0.0087	0.0014	6.7290e − 007	0.0014
20	3.3720e − 004	0.0015	4.1194e − 004	6.8339e − 007	9.3467e − 005
30	9.9324e − 005	6.2443e − 004	2.3521e − 005	1.3826e − 008	2.3492e − 005

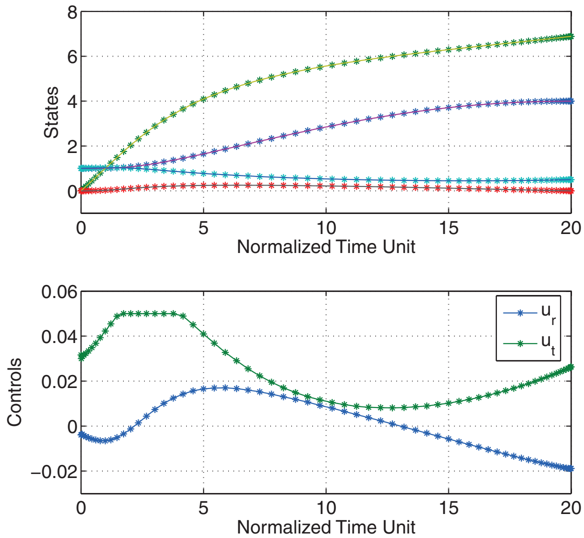

Fig. 3 Primal-feasible solutions as candidate optimal solutions.

$$\begin{cases} \lambda_1(0) &= -0.074339205563738 \\ \lambda_2(0) &= 0.000000067035108 \\ \lambda_3(0) &= 0.004932786034396 \\ \lambda_4(0) &= -0.062702841556571 \end{cases}$$

Using these numbers, we propagate the costates through the adjoint equations. This result is shown in Fig. 4 (solid lines). Also shown in this figure are the costates obtained at the CGL points by a direct application of the covector mapping theorem (stars). It is clear that the propagated costates as well as the mapped costates are in close agreement. Note also from this figure that the Hamiltonian is a constant equal to zero as required by the minimum principle. Furthermore, the adjoint equation for λ_2 indicates that this costate must also be a constant and equal to zero; this point is also verifiable from Fig. 4.

From the Hamiltonian minimization condition, and the fact that the constraint on $u_r(t)$ is never active (see Fig. 3), we have

$$2u_r + \lambda_3 = 0 \qquad 2u_t + \lambda_4 = -\mu_2$$

Figure 5 demonstrates that the computed covectors satisfy the Hamiltonian minimization condition within numerical tolerances. Thus, the optimality of the computed solution is verified by a joint application of the minimum principle and the covector mapping principle.

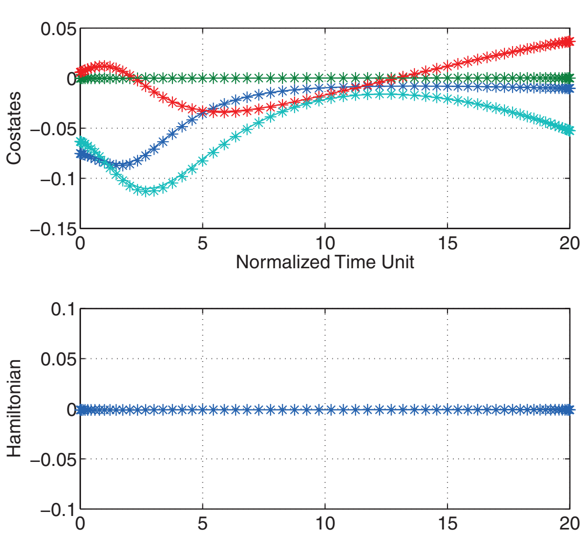

Fig. 4 Dual feasible solution by the Chebyshev PS method.

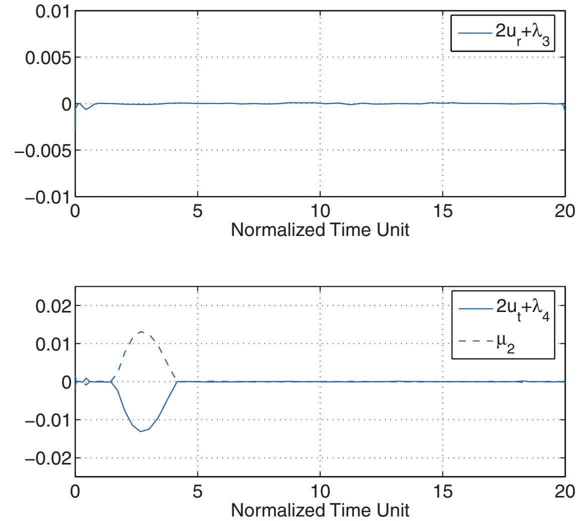

Fig. 5 Verification of Hamiltonian minimization condition.

VII. Conclusions

In this Note, we have presented results on the long-standing problem of costate computation by a Chebyshev PS method. This solution was facilitated by an application of recent unifying principles on PS methods. A key result that was used in this note is the use of an adjoint differentiation matrix that is different from the one used to discretize the state dynamics. This concept overcomes the difficulty of the nonunity weight function associated with the orthogonal property of Chebyshev polynomials and provides a way to compute costates through the covector mapping principle.

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