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NOTES AND DISCUSSIONS

Nonreflecting termination of a mass-and-spring lattice

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A one-dimensional mass-and-spring lattice is useful for exhibiting wave phenomena. In some numerical simulations, it would be very convenient to terminate a lattice such that there are no reflections when a wave impinges upon an end of the lattice. Such a termination may also be useful for a physical apparatus. In our case, the need has recently arisen in numerical simulations with sources that are ideally nonradiating[1,2] but which radiate when dissipation, nonuniformity, or nonlinearity are present.[3] It is possible to design an extended termination consisting of a nonuniform lattice that gradually damps a wave, thereby yielding only small reflections. It is also possible to simply employ a lattice that is sufficiently large such that the relevant motion can be observed before any reflections occur. However, to minimize computational time, and for simplicity, it is desirable to have a termination that consists of "lumped" elements. Our purpose here is to derive and numerically test such a nonreflecting termination.

If the masses of a lattice have value m, the springs have spring constant s, and the displacement of the nth mass from equilibrium is $y_n(t)$, the equation of motion for each interior mass is

$$\partial^2 y_n/\partial t^2=\omega_0^2(y_{n-1}-2y_n+y_{n+1}), \qquad (1)$$

where $\omega_0^2=s/m$. In the continuum limit, where the spatial variation of y_n is infinitesimal over the lattice spacing a between adjacent masses in equilibrium, Eq. (1) becomes the standard wave equation $\partial^2 y/\partial t^2-c^2\partial^2 y/\partial x^2=0$, where $x=na$, $y(x,t)=y_n(t)$, and the wave speed is $c=\omega_0 a$. For frequency ω and wave number k, the dispersion relation $\omega=\omega(k)$ corresponding to the wave equation (1) is determined by substituting the traveling wave expression $y_n=A\exp(i\omega t\pm ikna)$, where $k>0$, and demanding that this expression satisfy the equation. The result is

$$\omega_0=2\omega_0\sin(ka/2). \qquad (2)$$

Waves on the mass-and-spring lattice are *dispersive*; that is, the phase velocity ω/k depends upon the frequency. In the continuum limit $ka\ll 1$, the waves are *nondispersive*; the dispersion relation reduces to $\omega=ck$. In Eq. (2), the limiting minimum value of ω is 0, corresponding to waves of infinite wavelength, and the maximum value is $2\omega_0$, corresponding to the cutoff mode which has wavelength $2a$.

For the standard wave equation, which describes the continuum limit of the lattice, it is known[4] and we show below that a nonreflecting termination can be achieved for all waves by a "dashpot" that exerts a resistive force $-R\partial y/\partial t$ evaluated at the termination. No reflections occur if the mechanical resistance equals the wave impedance: $R=\rho c$, where the linear density is $\rho=m/a$ in our case. Due to the dispersion of waves in a lattice, it is natural to suspect that a nonreflecting termination can only be achieved at a definite frequency, but this is satisfactory for the common case of monofrequency waves. Because the mechanical resistance in the continuum limit is $R=\rho c$, one might guess that a nonreflecting termination for the lattice with waves of definite frequency ω has mechanical resistance $R=\rho\omega/k$. As shown below, however, this is incorrect.

To determine a nonreflecting termination of a mass-and-spring lattice, we consider a general termination at site $n=0$ consisting of the parallel combination of a mass of value M, a spring with spring constant S, and dashpot with mechanical resistance R (Fig. 1). The equation of motion for the displacement $y_0(t)$ of the termination point is then

$$M\partial^2 y_0/\partial t^2=s(y_1-y_0)-R\partial y_0/\partial t-Sy_0. \qquad (3)$$

A simple approach is to demand that no reflections occur at the termination, and to ascertain if there exist values of the termination parameters such that this is possible. Substituting into Eq. (3) the pure traveling wave $y_n=A\exp(i\omega t+ikna)$, where $k>0$ and where ω and k are related by the dispersion relation (2), yields

$$M\omega^2=s(1-e^{ika})+i\omega R+S. \qquad (4)$$

We first examine this in the continuum limit $ka\ll 1$, where $a\to 0$, $m\to 0$, and $s\to\infty$ such that the quantities $\rho=m/a$ and $c=\omega_0 a=a(s/m)^{1/2}$ remain finite and nonzero. Equation (4) then becomes $M\omega^2=-iska+i\omega R+S$. Using the expressions for ρ and c, as well as the dispersion relation $\omega=ck$, we find that the equation is satisfied if $R=\rho c$ and $M\omega^2=S$. The termination will thus be nonreflecting for *all* frequencies if we set $R=\rho c$ and $S=M=0$.

The requirement $M\omega^2=S$ can be physically understood. Given any nonreflecting termination for waves of definite frequency ω, a particle of mass M and a rigidly anchored spring of spring constant S can always be added to the termination point if $M\omega^2=S$ because free oscillations of this system have the same frequency as an incoming wave. The amplitude and phase of the mass-spring oscillations can thus be matched to the wave such that there is no effect upon the wave.

We now consider Eq. (4) for any allowed frequency $0<\omega\leqslant 2\omega_0$ of a propagating wave. By the identity $1-\cos(\theta)=2\sin^2(\theta/2)$, the real part of the equation is $M\omega^2=2s\sin^2(ka/2)+S$. Use of the dispersion relation (2) then yields $(M-m/2)\omega^2=S$. The simplest way to solve this is by choosing $M=m/2$ and $S=0$. As explained above, setting $S=0$ corresponds to removing a superfluous mass-and-spring combination from the termination, where the spring constant is S and the mass is $M-m/2$ in this case.

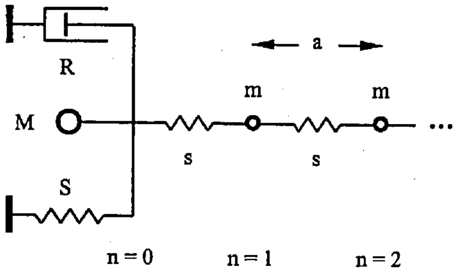

Fig. 1. Lumped-element termination of a mass-and-spring lattice. The termination consists of the parallel arrangement of a dashpot, a mass, and a spring.

The imaginary part of Eq. (4) yields $R=(s/\omega)\sin(ka)$. Use of the identities $\sin(\theta)=2\sin(\theta/2)\cos(\theta/2)$ and $\cos(\theta)=[1-\sin^2(\theta)]^{1/2}$, as well as the dispersion relation (2), yields

$$R=\rho c\sqrt{1-(\omega/2\omega_0)^2}, \tag{5}$$

where we have used the fact that $s/\omega_0=m\omega_0=(m/a)\omega_0 a=\rho c$. It should be noted that the quantities ρ and c are defined whether or not the system is behaving in the continuum limit. In this limit, note that Eq. (5) correctly reduces to $R=\rho c$. The final result, as displayed in Fig. 2, is that a nonreflecting termination for waves of frequency ω on a mass-and-spring lattice is the mass $m/2$ attached to a dashpot with mechanical resistance (5). This can also be derived with the method of impedance, although care must be taken due to the discreteness.

For the maximum frequency $\omega=2\omega_0$ of a propagating wave, Eq. (5) yields $R=0$. To physically understand this, consider first the cutoff *standing* wave mode, in which the masses m are in antiphase. Because the center of each spring is a node, two half-springs effectively act on each mass, which is thus subjected to stiffness $4s$. If we imagine splitting each mass in half with a plane that is transverse to the lattice, both the stiffness and the inertia are halved, so the frequency remains the same. Hence, the termination consisting of a mass $m/2$ will not affect the cutoff standing wave motion. Because a traveling wave can be considered as a superposition of two standing wave modes that are 90° out of phase, the termination will also have no effect upon a cutoff traveling wave. That is, the termination will be nonreflecting. This may appear to be a contradiction, because there is no resistance and yet energy appears to be absorbed. However, the group velocity $\partial\omega/\partial k$ vanishes for the cutoff mode, so there is no energy flow in this case.

The nonreflecting lattice termination holds for a pure frequency. Such a termination cannot perfectly function in all applications, however, because a source cannot operate for infinite time, and a wave cannot have infinite extent. It is thus of interest to know how well the termination performs for practical situations in which there is a small band of frequencies. We consider an initial Gaussian-modulated wave packet moving toward one end of the lattice. For a continuum with a general dispersion relation $\omega=\omega(k)$, such a packet traveling toward decreasing x can be expressed for a small time scale as $y(x,t)=A\exp\{-[x-x_0+(d\omega/dk)t]^2/\gamma^2\}\cos[k(x-x_0)+\omega t]$, where the packet has width γ and is centered at $x=x_0$ at $t=0$. The carrier moves with the phase velocity ω/k and the modulation moves with the group velocity $d\omega/dk$. The expression is valid only for a small time scale because we have neglected the fact that the width of the packet increases and the amplitude decreases at a larger time scale. The initial conditions $y(x,0)$ and

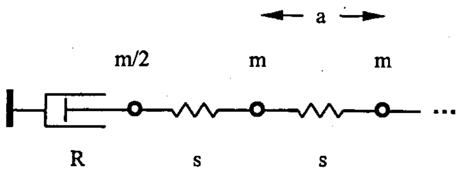

Fig. 2. Nonreflecting termination of a mass-and-spring lattice for waves of definite frequency ω. The mechanical resistance of the dashpot is $R=\rho c[1-(\omega/\omega_0)^2]^{1/2}$, where $\rho=m/a$ and $c=\omega_0 a=(s/m)^{1/2}a$, so $\rho c=(sm)^{1/2}$.

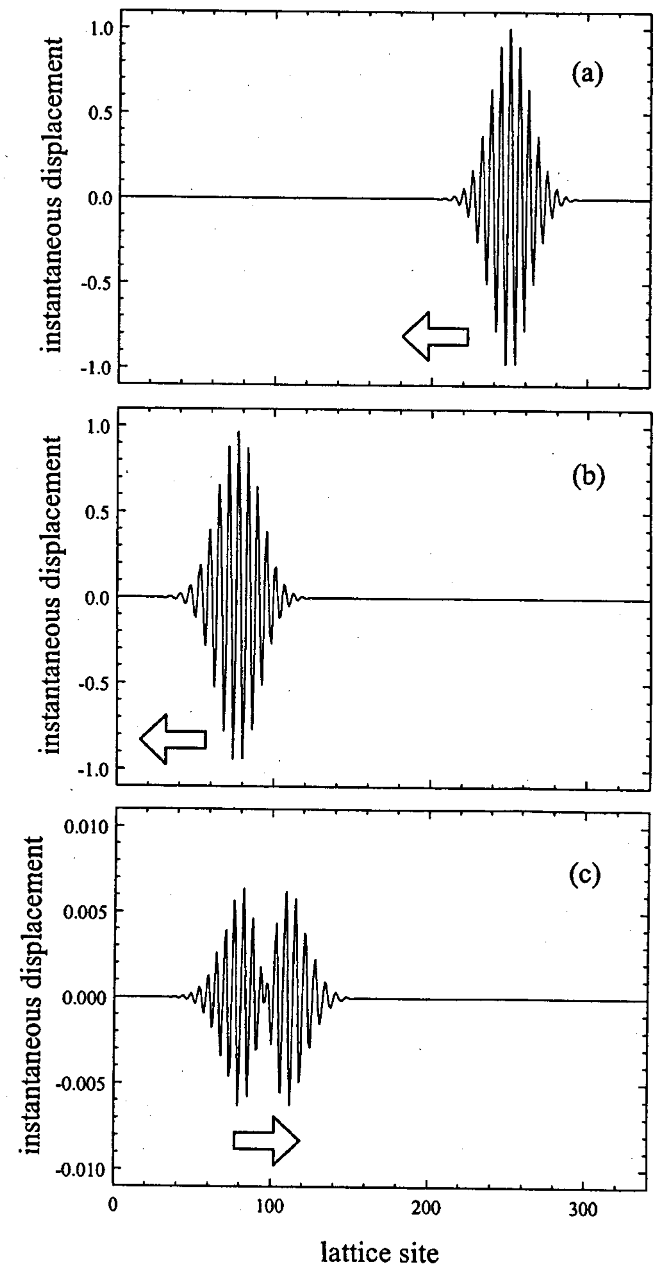

Fig. 3. Gaussian-modulated wave packet incident on an ideally nonreflecting termination (a) initially ($t=0$), (b) prior to incidence ($t=200$), and (c) after incidence ($t=400$). Note that the ordinate in (c) has been expanded by a factor of 100.

$dy(x,0)/dt$ give rise to the above wave. To yield a Gaussian-modulated wave packet in a lattice, we thus employ the initial conditions

$$y_n(0)=Ae^{-(n-n_0)^2a^2/\gamma^2}\cos[ka(n-n_0)], \quad (6)$$

$$\frac{dy_n(0)}{dt}=-Ae^{-(n-n_0)^2a^2/\gamma^2}\left\{\omega\sin[ka(n-n_0)] + \frac{2a(n-n_0)}{\gamma^2}\frac{d\omega}{dk}\cos[ka(n-n_0)]\right\}, \quad (7)$$

where ω and k are related by the dispersion relation (2). For numerical simulations, we choose all parameters of the lattice to be unity (which incurs no essential loss of generality): $a=m=s=1$, so $\omega_0=1$ and propagating frequencies have the range $0<\omega\leq 2$. We choose the carrier frequency of the packet to be $\omega=1$, and employ a mechanical resistance (5) based on this value. By the dispersion relation (2), the value of the wave number is $k=\pi/3$ and the wavelength is $\lambda=2\pi/k=6$ lattice spacings. We choose the width to be $\gamma=3\lambda=18$ and the amplitude to be $A=1$. By the uncertainty relation $\Delta k\Delta x=1/2$, where $\Delta x=\gamma/2^{1/2}$, the bandwidth of the wave packet centered on $\omega=1$ is calculated to be $\Delta\omega=(3/2)^{1/2}/2\gamma=0.034$, which is small.

We employed the Euler–Cromer method[5] with time step 0.001 to simulate motion of a 501-site lattice ($n=0$ to $n=500$). Figure 3 shows the evolution for the initial conditions (6) and (7) with the center of the wave packet initially at the middle site $n=250$. For clarity, only line segments between successive data points are shown. (The labels for the points are not shown.) Figure 3(a) shows the initial displacements, and Fig. 3(b) shows the displacements at a later time before reflection. Note that there is a slight spreading of the wave packet, which is due to the dispersion of lattice waves. To the right of Fig. 3(b) is a very small-amplitude (roughly 0.0004) packet propagating to the right, which occurs because the initial conditions are exact for a unidirectional packet only for a continuum. Figure 3(c) shows the displacements after the packet has interacted with the termination. Note that the ordinate has been expanded by a factor of 100. The reflected wave amplitude is only 0.6% of the incident amplitude. By contrast, a reflected amplitude of 25% occurs for a termination consisting of a dashpot with mechanical resistance $R=\rho c$ connected to a mass m.

The shape of the reflected packet can be understood as follows. The Fourier transform of the incident packet is a Gaussian centered at the carrier frequency ω. The termination acts as a notch filter, removing energy at and near the carrier frequency. The mechanical resistance of the dashpot is smaller than that required to completely absorb the energy at greater frequencies, and larger at lesser frequencies. The Fourier transform of the reflected packet thus vanishes at the center frequency, and has a positive peak at a slightly greater frequency and a negative peak at a slightly smaller frequency. These spectral peaks yield a Gaussian-modulated beat pattern in space, where the center is a beat due to the opposite polarities of the peaks. The packet in Fig. 3(c) indeed has this shape.

The relative amplitude of the reflected wave is expected to decrease as the incident wave approaches a monofrequency wave. To check this, we repeated the above simulation with the same parameters except for the width, which was increased by a factor of 10 to $\gamma=30\lambda$, and the number of lattice sites, which was increased from 501 to 1001 to accommodate the longer wave packet. In addition, the time step was reduced by a factor of 5. The reflected amplitude was observed to decrease to 0.06% of the incident amplitude. That is, an order of magnitude increase in the width of the wave packet causes an order of magnitude decrease in the amplitude of the reflected wave, which is reasonable. In practice, tests should be performed to ensure that the frequency spread is sufficiently small that the reflections are negligible for the intended application.

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[1]Michael Berry, John T. Foley, Greg Gbur, and Emil Wolf, "Nonpropagating string excitations," Am. J. Phys. **66**, 121–123 (1998).
[2]Bruce Denardo, "A simple explanation of simple nonradiating sources in one dimension—Comment on 'Nonpropagating string excitations,' by M. Berry, J. T. Foley, G. Gbur, and E. Wolf [Am. J. Phys. **66** (2), 121–123 (1998)]," Am. J. Phys. **66**, 1020–1021 (1998).
[3]Bruce Denardo, "Quasi-nonradiating wave sources in one dimension" (unpublished).
[4]A. P. French, *Vibrations and Waves* (Norton, New York, 1971), pp. 259–264.
[5]Nicholas J. Giordano, *Computational Physics* (Prentice–Hall, Upper Saddle River, NJ, 1997), pp. 46–47, 84, 351.

Comment on "Field pattern of a magnetic dipole" [Am. J. Phys. 68 (6), 577–578 (2000)] by J. P. Mc Tavish

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The present comment is meant as an alternative to J. P. Mc Tavish's recent derivation of the shape of the field lines of a magnetic point dipole. Mc Tavish tackles the question by representing the field configuration in Cartesian coordinates. I often discuss this same classroom/assignment problem with my students, but I represent the field in spherical coordinates instead, so the students can better appreciate the types of situations where these coordinates may be useful.

