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Reliability-based optimal design: problem formulations, algorithms and application (*keynote lecture*)

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ABSTRACT: Algorithms for solving three classes of reliability-based optimal design problems are presented. The algorithms address design problems for structural components, series systems, and a portfolio of series systems, where the objective and/or constraint functions involve probability terms. The proposed approach employs reformulations of the problems, in which probability terms are replaced by better-behaving functions. The reformulated problems can be solved by existing semi-infinite optimization algorithms. An important advantage of the approach is that the required reliability and optimization calculations are completely decoupled, thus allowing flexibility in the choice of the optimization algorithm and the reliability method. A comprehensive numerical example demonstrates applications of the proposed algorithms.

1 INTRODUCTION

Uncertainties and optimization are two major considerations in structural design. Uncertainties, arising from randomness in structural materials and applied loads as well as from errors in behavioral models, are inevitable and must be properly accounted for in the design of structures to assure safety and reliability. Optimization in the design of structures is desirable in order to maximize benefits and to make effective use of resources. Thus, optimal design under uncertainty is a topic of significant practical interest in structural engineering. Due to the challenges present in both probabilistic analysis and optimal design of structures, the combined problem poses significant difficulties as well as opportunities for research and innovation.

The typical single-objective optimal design problem involves an objective function that is to be minimized (or maximized), and one or more equality or inequality constraints, which define the feasible domain of the design variables. Under conditions of uncertainty, probabilistic terms may enter the objective, the constraints or both. Furthermore, the probabilistic terms may involve various measures, such as statistical moments of structural response or probabilities associated with various structural performance events.

In this paper, we present a summary of algorithms developed by the authors for solving single-objective design optimization problems involving failure probabilities (complements of reliability) as constraints, in the objective function, or both as constraints and in the objective function. Both structural component and series system problems are considered. For more detailed background on the development of these algorithms, including proofs of the various statements, the reader should consult Royset *et al.* (2001, 2002, 2003). Other relevant papers are those of Kirjner-Neto *et al.* (1998), Der Kiureghian and Polak (1998) and Polak *et al.* (2000). A comprehensive review of other works in reliability-based optimal design is presented in Royset *et al.* (2002) and will not be repeated here. However, it is important to note two distinguishing characteristics of the approach presented here relative to reliability-based optimal design algorithms developed or used by other researchers and practitioners: (a) in the proposed approach the computations for reliability and optimization are decoupled, thus allowing maximum latitude in the choice of algorithms for solving these sub-problems, (b) the developed algorithms have proven convergence properties under the stated conditions.

The paper begins with a brief review of the relevant reliability methods followed by the definition of three classes of optimal design problems. Each problem is then considered and applicable algorithms are described. The paper concludes with a comprehensive example.

2 STRUCTURAL RELIABILITY

Let $\mathbf{x}$ be an n-dimensional vector of deterministic, real-valued design variables, e.g., member sizes, maintenance times, extent of future repair. Following the well-established theory of structural reliability (Ditlevsen and Madsen 1996), we express the system reliability of a structural design by means of a set of continuously differentiable limit-state functions $G_k,(\mathbf{x}, \mathbf{v})$, $k \in \mathbf{K} = \{1, 2, \ldots, K\}$, involving $\mathbf{x}$ and an m-dimensional vector $\mathbf{v}$ of realizations of random variables $\mathbf{V}$. The event $\{G_k(\mathbf{x}, \mathbf{V}) \leq 0\}$ defines the failure of the structure in its k-th mode, a "component" event. The failure of the structure as a "series system" occurs if any of the component events $\{G_k(\mathbf{x}, \mathbf{V}) \leq 0\}$, $k \in \mathbf{K}$, occurs. As is common in reliability analysis, we use a bijective transformation $\mathbf{u} = T_{\mathbf{x}}(\mathbf{v})$ to map $\mathbf{v}$ into the realizations $\mathbf{u}$ of a standard normal random vector $\mathbf{U}$. Such transformations can be defined under weak assumptions. Replacing $\mathbf{v}$ by $T_{\mathbf{x}}^{-1}(\mathbf{u})$ gives the equivalent limit-state functions $g_k(\mathbf{x}, \mathbf{u})$, $k \in \mathbf{K}$, defined by $g_k(\mathbf{x}, \mathbf{u}) = G_k(\mathbf{x}, T_{\mathbf{x}}^{-1}(\mathbf{u}))$. Since structures usually possess high reliability, any realistic design should be safe at the mean point and hence $g_k(\mathbf{x}, \mathbf{o}) > 0$ for the problems of interest here.

The failure probability of the structural system is defined by

$$p(\mathbf{x}) = \int_{\Omega(\mathbf{x})} \varphi(\mathbf{u})\, d\mathbf{u} \tag{1}$$

where $\varphi(\mathbf{u})$ is the m-dimensional standard normal probability density function and $\Omega(\mathbf{x})$ is the failure domain. For a series structural system, $\Omega(\mathbf{x}) = \bigcup_{k \in \mathbf{K}} \{\mathbf{u} \in \mathrm{R}^m | g_k(\mathbf{x}, \mathbf{u}) \leq 0\}$. The failure probability for the k-th component, $p_k(\mathbf{x})$, is defined as in (1) with the integration domain replaced by $\Omega_k(\mathbf{x}) = \{\mathbf{u} \in \mathrm{R}^m | g_k(\mathbf{x}, \mathbf{u}) \leq 0\}$. We define the *critical component* to be the component with the largest failure probability.

In the first-order reliability method (FORM), an approximation to $p_k(\mathbf{x})$ is obtained by linearizing the limit-state function $g_k(\mathbf{x}, \mathbf{u})$ at the point in the set $\{\mathbf{u} \in \mathrm{R}^m | g_k(\mathbf{x}, \mathbf{u}) = 0\}$ closest to the origin, i.e., at

$$\mathbf{u}_k^*(\mathbf{x}) \in \underset{\mathbf{u} \in \mathrm{R}^m}{\operatorname{argmin}} \{\|\mathbf{u}\| | g_k(\mathbf{x}, \mathbf{u}) = 0\} \tag{2}$$

Such closest points are referred to as *design points*. The corresponding approximation of the component failure probability takes the form $p_k(\mathbf{x}) \approx \Phi(-\beta_{1,k}(\mathbf{x}))$, where $\beta_{1,k}(\mathbf{x}) = ||\mathbf{u}_k^*(\mathbf{x})||$ is the first-order reliability index and $\Phi(\cdot)$ is the standard normal cumulative distribution function. This probability expression is exact when $g_k(\mathbf{x}, \mathbf{u})$ is affine in $\mathbf{u}$, i.e., when $g_k(\mathbf{x}, \mathbf{u}) = b_{0,k}(\mathbf{x}) + \mathbf{b}_k(\mathbf{x})^{\mathrm{T}}\mathbf{u}$ for some positive valued function $b_{0,k}(\mathbf{x})$ and vector-valued function $\mathbf{b}_k(\mathbf{x})$. For a series system with affine component limit-state functions, the failure probability is obtained as the probability content in a polyhedral domain in the standard normal space. For non-affine component limit-state functions, the polyhedral domain defined by linearization of the individual component limit states at the respective design points provides a first-order approximation to the series system probability.

In the second-order reliability method (SORM), an approximation to $p_k(\mathbf{x})$ is obtained by replacing the limit-state function $g_k(\mathbf{x}, \mathbf{u})$ by a quadratic approximation in $\mathbf{u}$ at the design point $\mathbf{u}_k(\mathbf{x})$. The expression for the second-order approximation of $p_k(\mathbf{x})$ involves $\beta_{1,k}(\mathbf{x})$ and the principal curvatures of the surface $\{\mathbf{u} | g_k(\mathbf{x}, \mathbf{u}) = 0\}$ at the design point. A second-order approximation to the series system probability may be obtained by adjusting the distances to the faces of the first-order polyhedral approximation from $\beta_{1,k}(\mathbf{x})$ to $\beta_{2,k}(\mathbf{x})$ such that $\Phi(-\beta_{2,k}(\mathbf{x}))$ equals the SORM approximation of $p_k(\mathbf{x})$.

An important requirement in all gradient-based optimization algorithms is the existence of at least first-order derivatives of the objective and constraint functions with respect to the design variables $\mathbf{x}$. In a reliability-based optimal design problem, this translates into the requirement of differentiability of the failure probability, or the employed approximations thereof, with respect to the design variables. Unfortunately, none of the existing reliability approximation methods (FORM, SORM, response surface, importance sampling, etc.) are guaranteed to produce results that are differentiable with respect to $\mathbf{x}$. For example, one can easily show that FORM and SORM approximations of the failure probability for the limit-state function $g(x, \mathbf{u}) = 5 - 0.2(u_1 - x)^2 - u_2$ are not differentiable at $x = 0$. A similar problem exists with probability estimates based on simulation, unless special formulations are used (Royset and Polak, 2003). For the more general case of a series system, even the exact failure probability can be non-differentiable. For example, the probability of failure of the series system with the component limit-state functions $g_1(x, u_1, u_2) = 3 - u_1$, $g_2(x, u_1, u_2) = 3 - u_2$ and $g_3(x, u_1, u_2) = 3 - u_2 - x$ is not differentiable at $x = 0$.

The difficulty with differentiability means that optimization problems involving the failure probability in the objective function or the constraints may not be solvable by standard nonlinear optimization algorithms (e.g., NLPQL by Schittkowski, 1985; LANCELOT by Conn *et al.*, 1992; and NPSOL by Gill *et al.*, 1998). Ironically, most existing literature on reliability-based optimal design employs the FORM approximation in conjunction with standard nonlinear optimization algorithms. This does not mean that the solutions reported in the literature by use of these methods are necessarily wrong, but that the algorithms employed in these applications are not robust for the given problem and may fail to reach a solution for other similar problems. In short, standard nonlinear optimization algorithms appear not to be suitable for the solution of reliability-based optimal design problems. The algorithms presented in this paper circumvent this problem by a reformulation that replaces the probability terms with other better-behaving functions. The reformulation does not lead to optimization problems that can be solved by standard nonlinear optimization algorithms, but the problems can be solved by so-called semi-infinite optimization algorithms. These algorithms are well-known in the optimization literature (see, e.g., Polak, 1997).

3 DEFINITION OF OPTIMIZATION PROBLEMS

This paper addresses three classes of reliability-based optimal design problems denoted as $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$. The series system versions of these problems are denoted as $\mathbf{P}_{1,\text{sys}}$, $\mathbf{P}_{2,\text{sys}}$ and $\mathbf{P}_{3,\text{sys}}$. For $\mathbf{P}_3$, a version applicable to a "portfolio" of series systems, e.g., a group of bridges, is also formulated and denoted as $\mathbf{P}_{3,\text{por}}$. To define these problems, let $c_0(\mathbf{x})$ be the initial cost of the design, $c_k(\mathbf{x})$, $k \in \mathbf{K}$, be the cost associated with the failure of component k, and

$$\mathbf{X} = \left\{\mathbf{x} \in \mathrm{R}^n \,|\, f_j(\mathbf{x}) \leq 0,\ j = 1, \ldots, q\right\} \tag{3}$$

with $f_j(\mathbf{x})$ being continuously differentiable functions describing deterministic constraints. Problems $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are defined as follows:

$$\mathbf{P}_1: \quad \min_{x \in \mathrm{R}^n} \left\{c_0(\mathbf{x}) | p_k(\mathbf{x}) \leq \hat{p}_k, k \in \mathbf{K}, \mathbf{x} \in \mathbf{X}\right\} \tag{4}$$

$$\mathbf{P}_2: \quad \min_{x \in \mathrm{R}^n} \left\{ \max_{k \in \mathbf{K}} p_k(\mathbf{x}) \middle| \mathbf{x} \in \mathbf{X} \right\} \tag{5}$$

$$\mathbf{P}_3: \quad \min_{x \in \mathrm{R}^n} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) p_k(\mathbf{x}) \middle| p_k(\mathbf{x}) \leq \hat{p}_k, k \in \mathbf{K}, \mathbf{x} \in \mathbf{X} \right\} \tag{6}$$

As can be seen, $\mathbf{P}_1$ minimizes the cost of the design subject to upper-bound constraints on the individual component failure probabilities, $\mathbf{P}_2$ minimizes the failure probability of the critical component, and $\mathbf{P}_3$ minimizes the sum of the initial cost and the expected cost of failure of the components, assuming the component failure costs are additive, subject to constraints on the individual component failure probabilities. All three problems are also subject to the deterministic constraints $f_j(\mathbf{x}) \leq 0$, $j = 1, \ldots, q$. The series system versions of these problems are defined as

$$\mathbf{P}_{1,\text{sys}}: \quad \min_{x \in \mathrm{R}^n} \left\{c_0(\mathbf{x}) | p(\mathbf{x}) \leq \hat{p},\ p_k(\mathbf{x}) \leq \hat{p}_k, k \in \mathbf{K}, \mathbf{x} \in \mathbf{X}\right\} \tag{7}$$

$$\mathbf{P}_{s,\text{sys}}: \quad \min_{x \in \mathrm{R}^n} \{p(\mathbf{x}) | \mathbf{x} \in \mathbf{X}\} \tag{8}$$

$$\mathbf{P}_{3,\text{sys}}: \quad \min_{x \in \mathrm{R}^n} \left\{c_0(\mathbf{x}) + c(\mathbf{x}) p(\mathbf{x}) | p(\mathbf{x}) \leq \hat{p},\ p_k(\mathbf{x}) \leq \hat{p}_k, k \in \mathbf{K}, \mathbf{x} \in \mathbf{X}\right\} \tag{9}$$

As can be seen, in $\mathbf{P}_{1,\text{sys}}$ a constraint on the system failure probability is added, whereas in $\mathbf{P}_{2,\text{sys}}$ it is the system probability that is minimized. In $\mathbf{P}_{3,\text{sys}}$, the expected failure cost is in terms of the system failure, and the system failure probability is included in the constraint set definition. To define the portfolio version of $\mathbf{P}_3$, let the superscript (l), $l \in \mathbf{L} = \{1, \ldots, L\}$, define the l-th series structural system among a portfolio of L such systems. $\mathbf{P}_{3,\text{por}}$ is then defined as

$$\mathbf{P}_{3,\text{por}} : \min_{x \in \mathrm{R}^n} \left\{ \sum_{l=1}^{L} c_0^l(\mathbf{x}) + \sum_{l=1}^{L} c^l(\mathbf{x}) p^{(l)}(\mathbf{x}) \middle| p^{(l)}(\mathbf{x}) \leq \hat{p}^{(l)}, l \in \mathbf{L}, \mathbf{x} \in \mathbf{X} \right\} \tag{10}$$

Here, $c_0^{(l)}(\mathbf{x})$ is the initial cost of the l-th series structural system, and $c^{(l)}(\mathbf{x})$ is the cost associated with failure of the l-th series structural system. For the sake of simplicity in the notation, in (10) we have dropped the constraints on the individual components of the series systems. They can be included without significantly altering the solution algorithm. This problem aims to minimize the portfolio cost of the design plus expected cost of system failures, subject to constraints on the individual system probabilities.

All the cost, limit-state and constraint functions are assumed to be continuously differentiable. Additionally, we assume that the interval (for $m = 1$), area (for $m = 2$), volume (for $m = 3$), etc., in which the limit-state function vanishes, have length, area, volume, etc., equal to zero, respectively. This is normally satisfied in realistic design problems. The precise mathematical statement of this assumption can be found as Assumption 1(iii) in Royset *et al.* (2003).

4 PROBLEMS $\mathbf{P}_1$ AND $\mathbf{P}_{1,\text{sys}}$

4.1 *Approximating problems*

Let $\hat{\beta}_k = -\Phi^{-1}(\hat{p}_k)$. If the FORM approximation is used to solve $\mathbf{P}_1$, the constraint $p_k(\mathbf{x}) \leq \hat{p}_k$ can be replaced by $\beta_{1,k}(\mathbf{x}) \geq \hat{\beta}_k$. This implies

$$\min_{\mathbf{u} \in \mathrm{R}^m} \left\{ \|\mathbf{u}\| \,\middle|\, g_k(\mathbf{x}, \mathbf{u}) \leq 0 \right\} \geq \hat{\beta}_k \tag{11}$$

or, assuming $g_k(\mathbf{x}, \mathbf{u}) > 0$, equivalently,

$$\min_{\mathbf{u} \in \mathrm{R}^m} \left\{ g_k(\mathbf{x}, \mathbf{u}) \,\middle|\, \|\mathbf{u}\| \leq \hat{\beta}_k \right\} \geq 0 \tag{12}$$

At first glance, the expression in (12) does not appear more advantageous than the one in (11). However, the left side of (12) can be interpreted as a so-called standard min-function, while the left side of (11) is a generalized min-function. Standard min-functions have been studied extensively in the literature, and there is a variety of efficient and robust algorithms available for solving optimization problems involving such functions. In contrast, generalized min-functions are significantly more difficult to deal with. Note that standard min-functions are not differentiable everywhere even if $g_k(\mathbf{x}, \mathbf{u})$ is differentiable. This fact is incorporated into the algorithms in the literature for solving optimization problems with min-functions. Motivated by this finding, we define the standard min-functions

$$\psi_{k,s_k}(\mathbf{x}) = \min_{\mathbf{u} \in \mathrm{R}^m} \left\{ g_k(\mathbf{x}, \mathbf{u}) \,\middle|\, \|\mathbf{u}\| \leq s_k \right\}, \quad k \in \mathbf{K} \tag{13}$$

where $s_k > 0$ is a parameter, and introduce the following approximation to $\mathbf{P}_1$:

$$\mathbf{P}_{1,\mathbf{s}} : \min_{x \in \mathrm{R}^n} \left\{ c_0(\mathbf{x}) \,\middle|\, \psi_{k,s_k}(\mathbf{x}) \geq 0, k \in \mathbf{K}, \mathbf{x} \in \mathbf{X} \right\} \tag{14}$$

Precise statements with proofs regarding the relation between $\mathbf{P}_1$ and $\mathbf{P}_{1,\mathbf{s}}$ are given in Kirjner-Neto *et al.* (1998), Der Kiureghian and Polak (1998), Polak *et al.* (2000), and Royset *et al.* (2002). In essence, $\mathbf{P}_1$ and $\mathbf{P}_{1,\mathbf{s}}$ have identical solutions if the limit-state functions $g_k(\mathbf{x}, \mathbf{u})$, $k \in \mathbf{K}$, are affine in $\mathbf{u}$ and $s_k = \hat{\beta}_k$, $k \in \mathbf{K}$. Furthermore, $\mathbf{P}_1$ and $\mathbf{P}_{1,\mathbf{s}}$ have identical solutions if the failure probability terms in the former are expressed in terms of their FORM approximations. If higher order probability approximations are to be used, adjustments in the parameters s_k, $k \in \mathbf{K}$, must be made for non-affine limit-state functions. Specifically, if at the solution $\hat{\mathbf{x}}$ of $\mathbf{P}_{1,\mathbf{s}}$ the FORM approximation for a component k is smaller than the corresponding higher-probability approximation, such that the latter violates the probability constraint in (4), then problem $\mathbf{P}_{1,\mathbf{s}}$ must be resolved using a larger value of s_k. Conversely, if the FORM approximation is larger than the higher-order probability approximation, then a smaller value of s_k may be used to improve the design. This process of parameter adjustment is repeated until all the component probability constraints in (4) are satisfied for the desired probability approximation level. A recursive formula for these updates of parameters s_k is given below.

The above parameter-adjustment procedure can also be employed to solve the series system problem, $\mathbf{P}_{1,\text{sys}}$. It is well known (Ditlevsen and Madsen, 1996) that the series system probability is bounded from below by the probability of the critical component, and from above by the sum of the component probabilities. In view of the upper bound, the constraint $p(\mathbf{x}) \leq \hat{p}$ can be satisfied by selecting a sufficiently large value of s_k for each component, and particularly for the critical component.

4.2 *Algorithms for $\boldsymbol{P}_1$ and $\boldsymbol{P}_{1,\text{sys}}$*

Problem $\mathbf{P}_{1,\mathbf{s}}$ with fixed parameters $\mathbf{s}$ belongs to a well known class of optimization problems called semi-infinite (see e.g., Polak, 1997, or Royset *et al.*, 2002) and can be solved by any of a series of well-honed algorithms with guaranteed convergence properties. To obtain approximate solutions in the case of non-affine limit-state functions and/or problems involving series systems, we repeatedly solve the approximating problem $\mathbf{P}_{1,\mathbf{s}}$ while adjusting parameters $\mathbf{s}$. This approach was originally proposed by Der Kiureghian and Polak (1998) and Polak *et al.* (2000) for problems with component failure probabilities, i.e., $\mathbf{P}_1$. In Royset *et al.* (2001) and (2002), this approach was extended to also address $\mathbf{P}_{1,\text{sys}}$. The algorithm steps are summarized as follows:

Data. Provide an initial design $\mathbf{x}_0$ and a sequence of strictly increasing integers $N_0, N_1, N_2, \ldots$.

Step 0. Set $i = 0$ and the parameters $(s_k)_0 = \hat{\beta}_k, k \in \mathbf{K}$.

Step 1. Set $\mathbf{x}_{i+1}$ to be the last iterate after N_i iterations of a semi-infinite optimization algorithm on the problem $\mathbf{P}_{1,\mathbf{s}}$, with $\mathbf{s}_i = ((s_1)_i, (s_2)_i, \ldots, (s_k)_i)$, and initialization $\mathbf{x}_i$.

Step 2. Compute appropriate estimates $\tilde{p}_k(\mathbf{x}_{i+1}), k \in \mathbf{K}$, of $p_k(\mathbf{x}_{i+1}), k \in \mathbf{K}$. If considering $\mathbf{P}_{1,\text{sys}}$, also compute an appropriate estimate $\tilde{p}_k(\mathbf{x}_{i+1})$ of the system failure probability $p(\mathbf{x}_{i+1})$.

Step 3. Update the components of $\mathbf{s}_{i+1}$ by setting

$$(s_k)_{i+1} = (s_k)_i \frac{\Phi^{-1}(\hat{p}_k)}{\Phi^{-1}(\tilde{p}_k(\mathbf{x}_{i+1}))}, k \in \mathbf{K} \tag{15}$$

If considering $\mathbf{P}_{1,\text{sys}}$, replace the updating rule for the critical component, i.e., component $k = \hat{k}$ such that $\tilde{p}_{\hat{k}}(\mathbf{x}_{i+1}) = \max_{k \in \mathbf{K}} \tilde{p}_k(\mathbf{x}_{i+1})$, by

$$(s_k)_{i+1} = (s_k)_i \max\left(\frac{\Phi^{-1}(\hat{p}_k)}{\Phi^{-1}(\tilde{p}_k(\mathbf{x}_{i+1}))}, \frac{\Phi^{-1}(\hat{p}_k)}{\Phi^{-1}(\tilde{p}_k(\mathbf{x}_{i+1}))}\right) \tag{16}$$

Step 4. Replace i by $i+1$ and go to Step 1.

With the phrase "appropriate estimate" of a failure probability in Step 2, we mean that the failure probability estimate should be computed using the same reliability method (e.g., FORM, SORM, Monte Carlo Simulation) and with the same level of accuracy as the one used to verify the final design.

5 PROBLEMS $\mathbf{P}_2$ AND $\mathbf{P}_{2,\text{sys}}$

5.1 *Approximating problems*

In $\mathbf{P}_2$, we design the structure by minimizing the probability of failure of the most critical component. This objective can be achieved approximately by maximizing the first-order reliability index of the critical component. However, considering the equivalence between (11) and (12), the latter objective can be approximately achieved by maximizing $\psi_{k,s}(\mathbf{x})$ in (13) for a given s. This is advantageous because, as opposed to $\beta_{1,k}(\mathbf{x})$, which is a generalized min-function, $\psi_{k,s}(\mathbf{x})$ is a standard min-function. Adjustments in the parameters s may be used to improve the approximation. Hence, we define the following approximation to $\mathbf{P}_2$:

$$\mathbf{P}_{2,s}: \quad \max_{\mathbf{x}\in\mathbf{X}} \left\{ \min_{k\in\mathbf{K}} \psi_{k,s}(\mathbf{x}) \right\} \tag{17}$$

Note that in $\mathbf{P}_{2,s}$ the parameter s is a scalar, whereas in $\mathbf{P}_{1,\mathbf{s}}$ there are K parameters s_k. One can show that $\mathbf{P}_2$ and $\mathbf{P}_{2,s}$ have identical solutions for any $s > 0$ when the limit-state functions are affine in $\mathbf{u}$. For non-affine limit-state functions, $\mathbf{P}_2$ and $\mathbf{P}_{2,s}$ have identical solutions if the FORM approximation is used, provided $s = \beta_{1,\hat{k}}(\mathbf{x}^*)$ for the critical component at the solution point $\mathbf{x}^*$. An improved solution relative to a higher order probability estimate may be obtained by solving $\mathbf{P}_2$ for a range of s values in the neighborhood of $\beta_{1,\hat{k}}(\mathbf{x}^*)$ and taking the best design. However, experience shows that such designs tend to be insensitive to s values in this range. Hence, a rough estimate of s is usually sufficient. Furthermore, owing to the dominance of the critical failure mode on the series system failure probability, a solution of $\mathbf{P}_{2,s}$ with s close to $\beta_{1,\hat{k}}(\mathbf{x}^*)$ is a good approximation to the solution of $\mathbf{P}_{2,\text{sys}}$ as well.

5.2 *Algorithm for* $\mathbf{P}_2$ *and* $\mathbf{P}_{2,\text{sys}}$

Data. Provide an initial design $\mathbf{x}_0$, an integer N, and a parameter s_0, with value in the neighborhood of the first-order reliability index of the critical component for the anticipated optimal design.

Step 0. Set $i = 0$.

Step 1. Set $\mathbf{x}_{i+1}$ to be the last iterate after N iterations of a semi-infinite optimization algorithm on the problem $\mathbf{P}_{2,s}$ with initialization $\mathbf{x}_i$.

Step 2. Compute appropriate estimates $\tilde{p}_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$, of $p_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$. If considering $\mathbf{P}_{2,\text{sys}}$, also compute the appropriate estimate $\tilde{p}(\mathbf{x}_{i+1})$ of $p(\mathbf{x}_{i+1})$.

Step 3. Determine $\hat{k}_{i+1}$ (the index for the critical component) such that $\tilde{p}_{\hat{k}_{i+1}}(\mathbf{x}_{i+1}) = \max_{k\in\mathbf{K}} \tilde{p}_k(\mathbf{x}_{i+1})$ and compute the corresponding FORM reliability index $\beta_{1,\hat{k}_{i+1}}(\mathbf{x}_{i+1})$.

Step 4. Set $s_{i+1} = \beta_{1,\hat{k}_{i+1}}(\mathbf{x}_{i+1})$.

Step 5. The best estimate of the optimal design after $i+1$ iterations is $\hat{\mathbf{x}}_{i+1} \in \arg\min_{j=1,\ldots,i+1} \tilde{p}_{\hat{k}_j}(\mathbf{x}_j)$ (in case of $\mathbf{P}_2$) and $\hat{\mathbf{x}}_{i+1} \in \arg\min_{j=1,\ldots,i+1} \tilde{p}(\mathbf{x}_j)$ (in case of $\mathbf{P}_{2,\text{sys}}$).

Step 6. Replace i by $i+1$ and go to Step 1.

6 PROBLEMS $\mathbf{P}_3$, $\mathbf{P}_{3,\text{sys}}$ AND $\mathbf{P}_{3,\text{por}}$

6.1 *Approximating problems*

Problems $\mathbf{P}_3$, $\mathbf{P}_{3,\text{sys}}$ and $\mathbf{P}_{3,\text{por}}$, where failure probabilities appear in both the constraint definition and the objective function, are more complicated than the problems discussed earlier. The approaches for solving $\mathbf{P}_1$ and $\mathbf{P}_2$ cannot simply be "combined" to create an approach for solving $\mathbf{P}_3$. In the approximation for **P2**, we replaced the failure probability by a function that had maxima approximately for the same designs as the minima for the failure probability on the given feasible set. The actual value of the failure probability was not involved in this approximation. In $\mathbf{P}_3$, $\mathbf{P}_{3,\text{sys}}$ and $\mathbf{P}_{3,\text{por}}$ we need an approximation of the failure probability in order to estimate the objective functions. The following reformulation of these problems, which is built on the approach for solving $\mathbf{P}_1$, is developed in Royset *et al.*(2002).

We first construct approximating problems for $\mathbf{P}_3$ by replacing the failure probabilities in the objective function of $\mathbf{P}_3$ with parameters. The parameters are included in an augmented design vector and, hence, their values are automatically determined by the optimization procedure. Let $\overline{\mathbf{x}} = (\mathbf{x}, \mathbf{a})$ be an $(n+K)$-dimensional augmented design vector, where $\mathbf{x}$ is the original n-dimensional design vector and $\mathbf{a} = (a_1, a_2, \ldots, a_K)$ is a K-dimensional vector of parameters. We define the problem

$$\hat{\mathbf{P}}_3: \min_{(\mathbf{x},\mathbf{a})\in \mathbb{R}^{n+K}} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) a_k \,\middle|\, p_k(\mathbf{x}) = a_k, 0 \le a_k \le \hat{p}_k, k \in \mathbf{K}, \mathbf{x} \in \mathbf{X} \right\} \tag{18}$$

Observe that the objective function in $\hat{\mathbf{P}}_3$ is equal to the one in $\mathbf{P}_3$ when $p_k(\mathbf{x}) = a_k$. Since in $\hat{\mathbf{P}}_3$ we only consider designs $\mathbf{x}$ such that $p_k(\mathbf{x}) = a_k$ and $0 \leq a_k \leq \hat{p}_k$, $k \in \mathbf{K}$, the minima of problems $\mathbf{P}_3$ and $\hat{\mathbf{P}}_3$ must be equal. This result is stated and proven formally in Royset *et al.* (2002). It is seen from (18) that $\hat{\mathbf{P}}_3$ is a minimization problem of a smooth objective function with failure probability equality constraints. This is similar to $\mathbf{P}_1$, but $\mathbf{P}_1$ contains inequality constraints. The above reformulation removes the failure probability in the objective function. However, the failure probability is still part of the constraint set definition. Hence, one more step is needed to arrive at an optimization problem that can be solved by semi-infinite optimization algorithms.

We proceed by constructing an approximating problem with min-function constraints. Let $\mathbf{t}$ be a K-dimensional vector of positive numbers. This parameter vector is similar in nature to $\mathbf{s}$ in $\mathbf{P}_{1,\mathbf{s}}$. However, as seen below, their numerical values tend to be different. We define the approximating problem

$$\hat{\mathbf{P}}_{3,\mathbf{t}}: \min_{\bar{\mathbf{x}}=(\mathbf{x},\mathbf{a})\in\mathbb{R}^{n+K}} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x})a_k \,\middle|\, \hat{\psi}_{k,t_k}(\bar{\mathbf{x}}) \geq 0, 0 \leq a_k \leq \hat{p}_k, k \in \mathbf{K}, \mathbf{x} \in \mathbf{X} \right\} \tag{19}$$

where

$$\hat{\psi}_{k,t_k}(\bar{x}) = \min_{\mathbf{u}\in\mathbb{R}^m} \left\{ g_k(\mathbf{x}, -\Phi^{-1}(a_k)t_k\mathbf{u}) \,\middle|\, \|\mathbf{u}\| \leq 1 \right\} \tag{20}$$

Note that $\hat{\psi}_{k,t_k}(\bar{x})$ is the minimum value of the limit-state function inside a ball of radius $-\Phi^{-1}(a_k)t_k$, while $\psi_{k,s_k}(\mathbf{x})$ is the minimum value of the limit-state function inside a ball of radius s_k. Hence, the radius of the ball associated with $\hat{\psi}_{k,t_k}(\bar{\mathbf{x}})$ varies with the argument $\bar{\mathbf{x}}$. The problem $\hat{\mathbf{P}}_{3,\mathbf{t}}$ can be solved by semi-infinite optimization algorithms.

In the same way that $\mathbf{P}_1$ and $\mathbf{P}_{1,\mathbf{s}}$ were related, we find that $\hat{\mathbf{P}}_3$ and $\hat{\mathbf{P}}_{3,\mathbf{t}}$ are related: If the limit-state functions $g_k(\mathbf{x}, \mathbf{u})$, $k \in \mathbf{K}$, are affine in their second argument and $\mathbf{t} = (1, 1, \ldots, 1)$, then $\hat{\mathbf{x}}$ solves $\hat{\mathbf{P}}_3$ if and only if it solves $\hat{\mathbf{P}}_{3,\mathbf{t}}$. The mathematically precise statement and its proof can be found in Royset *et al.* (2002). In view of the above relations, the original problem $\mathbf{P}_3$ is equivalent to $\hat{\mathbf{P}}_{3,\mathbf{t}}$, when the limit-state functions are affine. For non-affine limit-state functions, $\hat{\mathbf{P}}_{3,\mathbf{t}}$ is a first-order approximation to $\mathbf{P}_3$ with parameters $\mathbf{t}$, which can be adjusted to improve the approximation.

The situation for $\mathbf{P}_{3,\text{sys}}$ and $\mathbf{P}_{3,\text{por}}$ is similar to the one for $\mathbf{P}_3$. We first define

$$\hat{\mathbf{P}}_{3,\text{por}}: \min_{(\mathbf{x},\mathbf{a})\in\mathbb{R}^{n+L}} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}) + \sum_{l=1}^{L} c^{(l)}(\mathbf{x})a_l \,\middle|\, p^{(l)}(\mathbf{x}) = a_l, 0 \leq a_l \leq \hat{p}^{(l)}, l \in \mathbf{L}, \mathbf{x} \in \mathbf{X} \right\} \tag{21}$$

Royset *et al.* (2002) have shown that the minimum value of $\mathbf{P}_{3,\text{por}}$ is equal to the minimum value of $\hat{\mathbf{P}}_{3,\text{por}}$. Next, we define the approximating problem

$$\hat{\mathbf{P}}_{3,\text{por},\mathbf{t}}: \min_{\bar{\mathbf{x}}=(\mathbf{x},\mathbf{a})\in\mathbb{R}^{n+L}} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}) = \sum_{l=1}^{L} c^{(l)}(\mathbf{x})a_l \,\middle|\, \hat{\psi}_{t_l}^{(l)}(\bar{\mathbf{x}}) \geq 0, 0 \leq a_l \leq \hat{p}^{(l)}, l \in \mathbf{L}, \mathbf{x} \in \mathbf{X} \right\} \tag{22}$$

where

$$\hat{\psi}_{t_l}^{(l)}(\bar{\mathbf{x}}) = \min_{k\in\mathbf{K}_l} \min_{\mathbf{u}\in\mathbb{R}^m} \left\{ g_k^{(l)}(\mathbf{x}, -\Phi^{-1}(a_l)t_l\mathbf{u}) \,\middle|\, \|\mathbf{u}\| \leq 1 \right\} \tag{23}$$

We are not able to prove equivalence between $\hat{\mathbf{P}}_{3,\text{por}}$ and $\hat{\mathbf{P}}_{3,\text{por},\mathbf{t}}$ similar to that between $\hat{\mathbf{P}}_3$ and $\hat{\mathbf{P}}_{3,\mathbf{t}}$ for affine limit-state functions. However, if all the limit-state functions $g_k^{(l)}(\mathbf{x}, \mathbf{u})$ are affine in their respective second arguments, then $\hat{\psi}_{t_l}^{(l)}(\bar{\mathbf{x}}) \geq 0$ implies that the critical failure component, say $\hat{k}_l$, of the l-th structure has failure probability $p_{\hat{k}_l}^{(l)}(\mathbf{x}) \leq \Phi(-\Phi^{-1}(a_l)t_l)$. Hence, when $t_l = 1, p_{\hat{k}_l}^{(l)}(\mathbf{x}) \leq a_l$. Due to the close relation between the failure probability of the critical component and the failure probability of the series system we can adjust t_l such that $p^{(l)}(\mathbf{x}) \approx a_l$ whenever $\hat{\psi}_{t_l}^{(l)}(\bar{\mathbf{x}}) = 0$. Hence, $\hat{\mathbf{P}}_{3,\text{por},\mathbf{t}}$ is a good approximation to $\hat{\mathbf{P}}_{3,\text{por}}$ for a suitable selection of $\mathbf{t}$.

In view of the above discussion, we can approximately solve $\mathbf{P}_{3,\text{por}}$ by solving the semi-infinite optimization problem $\hat{\mathbf{P}}_{3,\text{por},\mathbf{t}}$. We present algorithms for $\mathbf{P}_3$ and $\mathbf{P}_{3,\text{por}}$ in the next section. Since $\mathbf{P}_{3,\text{sys}}$ is similar to $\mathbf{P}_{3,\text{por}}$ (set $L=1$ in $\mathbf{P}_{3,\text{por}}$ and add component failure probability constraints), it is straightforward to develop an algorithm for $\mathbf{P}_{3,\text{sys}}$ based on the ones for $\mathbf{P}_{3,\text{por}}$ and $\mathbf{P}_1$.

6.2 *Algorithms for* $\mathbf{P}_3$ *and* $\mathbf{P}_{3,\text{por}}$

To solve $\mathbf{P}_3$ and $\mathbf{P}_{3,\text{sys}}$, we repeatedly solve the approximating problem $\hat{\mathbf{P}}_{3,\mathbf{t}}$ as described below.

Data. Provide an initial design $\mathbf{x}_0$ and a sequence of strictly increasing integers $N_0, N_1, N_2, \ldots$.
Step 0. Set $i=0$, $\mathbf{a}_0 = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_K)$, $\mathbf{t}_0 = (1, 1, \ldots, 1)$ and $\overline{\mathbf{x}}_0 = (\mathbf{x}_0, \mathbf{a}_0)$.
Step 1. Set $\overline{\mathbf{x}}_{i+1}$ to be the last iterate after N_i iterations of a semi-infinite optimization algorithm on the problem $\hat{\mathbf{P}}_{3,\mathbf{t}_i}$ with initialization $\overline{\mathbf{x}}_i$.
Step 2. Compute appropriate estimates $\tilde{p}_k(\mathbf{x}_{i+1})$ of $p_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$. If considering $\mathbf{P}_{3,\text{por}}$, compute appropriate estimates $\tilde{p}^{(l)}(\mathbf{x}_{i+1})$, $l \in \mathbf{L}$, of $p^{(l)}(\mathbf{x}_{i+1})$, $l \in \mathbf{L}$.
Step 3. Update the components of $\mathbf{t}_{i+1}$ by setting

$$(t_k)_{i+1} = (t_k)_i \frac{\Phi^{-1}((a_k)_{i+1})}{\Phi^{-1}(\tilde{p}_k(\mathbf{x}_{i+1}))}, \quad k \in \mathbf{K} \tag{24}$$

If considering $\mathbf{P}_{3,\text{por}}$, use the updating rule

$$(t_l)_{i+1} = (t_l)_i \frac{\Phi^{-1}((a_l)_{i+1})}{\Phi^{-1}(\tilde{p}^{(l)}(\mathbf{x}_{i+1}))}, \quad l \in \mathbf{L} \tag{25}$$

Step 4. Replace i by $i+1$ and go to Step 1.

7 EXAMPLE APPLICATION

Consider a highway bridge with reinforced concrete girders of the type shown in Figure 1. The objective is to find the optimal design for one such girder using the material and load data from Lin and Frangopol (1996) and Frangopol *et al.* (1997). The nine design variables are collected in the vector

$$\mathbf{x} = (A_s, b, h_f, b_w, h_w, A_v, S_1, S_2, S_3) \tag{26}$$

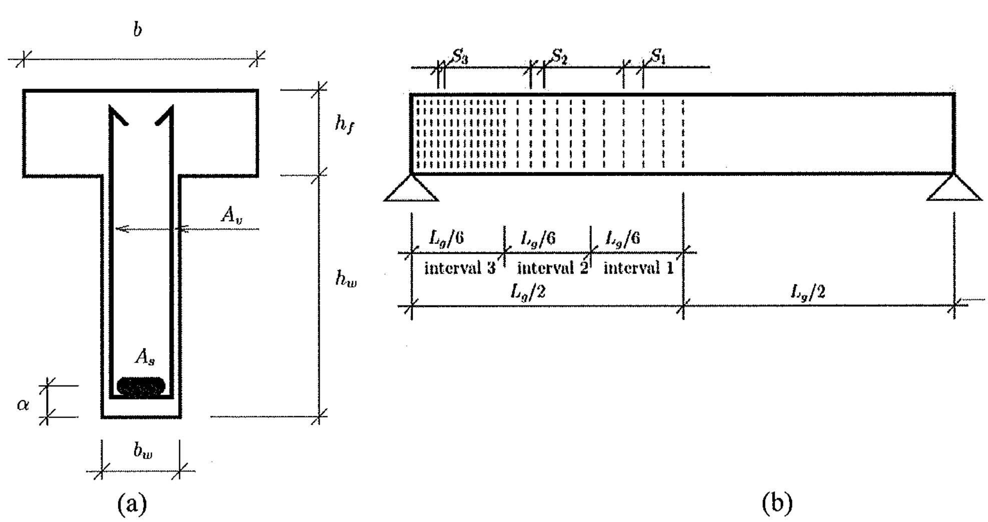

Figure 1. Example reinforced concrete girder: (a) cross section, (b) side view with shear reinforcement.

where A_s is the area of the tension steel reinforcement, b is the width of the flange, h_f is the thickness of the flange, b_w is the width of the web, h_v is the height of the web, A_v is the area of the shear reinforcement (twice the cross-section area of a stirrup), and S_1, S_2 and S_3 are the spacings of shear reinforcements in intervals 1, 2 and 3, respectively (Figure 1b). The random variables describing the loading and material properties are

$$\mathbf{V} = (f_y, f_c', P_D, M_L, P_{S1}, P_{S2}, P_{S3}, W) \tag{27}$$

where f_y is the yield strength of the reinforcement, f_c' is the compressive strength of concrete, P_D is the dead load excluding the weight of the girder, M_L is the live load moment, P_{S1}, P_{S2} and P_{S3} are the live load shear forces in intervals 1, 2 and 3, respectively, see Figure 1b, and W is the unit weight of concrete. Following Lin and Frangopol (1996), all the random variables are considered to be independent and normally distributed with the means and coefficients of variation listed in Table 1. The girder length is $L_g = 18.30$ m, and the distance from the bottom fiber to the centroid of the tension reinforcement is $\alpha = 0.1$ m, see Figure 1.

As in Lin and Frangopol (1996), we assume that the reinforced concrete girder fails if it exceeds its flexure capacity or its shear capacity in one of three sections of the girder (see Figure 1b). Hence, the reliability of the girder is defined by a series structural system with four components. The limit-state functions associated with the four failure modes are given in Royset *et al.* (2002).

Suppose that the objective is to minimize the material cost of the reinforced concrete girder subject to a constraint on the system failure probability, i.e., a design problem of the type $\mathbf{P}_{1,\mathrm{sys}}$. Let $C_s = 50$ and $C_c = 1$ be the unit costs of steel reinforcement and concrete per cubic meter, respectively. As in Lin and Frangopol (1996), we define the objective function to be

$$c_0(\mathbf{x}) = 0.75 C_s L_g A_s + C_s n_S A_v (h_f + h_w - \alpha + 0.5 b_w) + C_c L_g (b h_f + b_w h_w) \tag{28}$$

where $n_S = L_g(1/S_1 + 1/S_2 + 1/S_3)/3$ is the total number of stirrups. The first term in the above expression represents the cost of the bending reinforcement. The factor 0.75 appears due to the assumption that the total amount of bending reinforcement is placed only within a length $L_g/2$ centered at the middle point of the girder, and the remaining part is reinforced with $0.5A_s$. The second and third terms in (28) represent the costs of shear reinforcement and concrete, respectively. Let the constraint on the system failure probability be $p(\mathbf{x}) \leq 0.001350$.

This problem is solved by using the algorithm for $\mathbf{P}_{1,\mathrm{sys}}$. The results are summarized in Table 2 as Case 1, where the design vector $\mathbf{x}_i$, the objective $c_0(\mathbf{x}_i)$, and the system failure probability $P(\mathbf{x}_i)$ are listed. The system failure probability is evaluated using Monte Carlo simulation with a c.o.v. of 0.01.

Now suppose we wish is to minimize the initial cost plus the expected cost of failure of the reinforced concrete girder described above. Additionally, we assume a constraint on the system failure probability, i.e., a design problem of the type $\mathbf{P}_{3,\mathrm{sys}}$. Let the initial cost of the design be as in (28) and the cost of failure be $c(\mathbf{x}) = 500c_0(\mathbf{x})$. Also let the constraint on the system failure probability be $p(\mathbf{x}) \leq 0.001350$, with no constraints on the component failure probabilities. We solve this problem by using the algorithm described in Section 6.2 and the results are listed in Table 2 as Case 2 are listed. The system failure probability is evaluated using Monte Carlo simulation with a c.o.v. of 0.01. Relative to Case 1, a significant increase in the initial cost of the design is observed due to the consideration of the failure cost. On the other hand, the design failure probability is almost one order of magnitude smaller.

Table 1. Statistics of normal random variables in girder example.

Variable	Mean	c.o.v.
f_y	413.4 MPa	0.150
f_c'	27.56 MPa	0.150
P_D	13.57 kN/m	0.200
M_L	929 kNm	0.243
P_{S1}	138.31 kN	0.243
P_{S2}	183.39 kN	0.243
P_{S3}	228.51 kN	0.243
W	22.74 kN/m^3	0.100

Table 2. Results for optimal design of reinforced concrete girder.

Design variable or function	Case 1	Case 2	Case 3	Case 4
A_s	0.00983 m^2	0.0116 m^2	0.0161 m^2	0.0144 m^2
b	0.418 m	0.492 m	0.686 m	0.612 m
h_f	0.415 m	0.415 m	0.415 m	0.415 m
b_w	0.196 m	0.196 m	0.197 m	0.196 m
h_w	0.785 m	0.785 m	0.785 m	0.785 m
A_v	0.000186 m^2	0.000227 m^2	0.000255 m^2	0.000255 m^2
S_1	0.508 m	0.502 m	0.549 m	0.550 m
S_2	0.224 m	0.226 m	0.246 m	0.247 m
S_3	0.140 m	0.142 m	0.154 m	0.155 m
c_a	N/A	N/A	0.050 m	0.050 m
m_1	N/A	N/A	N/A	0.105
m_2	N/A	N/A	N/A	0.243
$c_0(\mathbf{x})$	13.664	15.558	20.434	18.678
$c(\mathbf{x})p(\mathbf{x})$	N/A	1.459	2.514	1.824
$c_m(\mathbf{x})$	N/A	N/A	N/A	1.699
$p(\mathbf{x})$	0.00131	0.000188	0.000246	0.000195
Total expected cost	13.664	17.017	22.948	22.201

Table 3. Statistics of lognormal random variables describing corrosion.

Variable	Mean	c.o.v.
A	5 years	0.20
B	300 years/m	0.20
V	0.000040 m/years	0.30

Now suppose that the girder is subject to corrosion of its longitudinal reinforcement. We adopt a corrosion model similar to that used in Frangopol *et al.* (1997), where the diameter $D_b(T)$ of a longitudinal reinforcement bar at time T is given by

$$D_b(T) = \begin{cases} D_{b0} - 2v(T - T_I), T > T_I \\ D_{b0} \qquad\qquad\qquad , T \leq T_I \end{cases} \tag{29}$$

with D_{b0} being the initial diameter, v being the corrosion rate, and T_I being the corrosion initiation time. The factor 2 in (29) takes into account that the reinforcement bar is subject to corrosion from all sides. We assume $T_I = A + Bc_a$, where A is a lognormal random variable with mean 5 years and c.o.v. equal to 0.20, representing the time it takes to initiate corrosion with a 10 mm concrete cover, B is a lognormal random variable with mean 300 years/m and c.o.v. equal to 0.20, representing the additional time it takes to initiate corrosion per meter additional concrete cover, and c_a is the concrete cover in meters in addition to the 10 mm minimum cover. The additional concrete cover c_a is considered a design variable and is included in the design vector $\mathbf{x}$. We assume that the corrosion rate v is lognormally distributed with mean 0.000040 m/years and c.o.v. 0.30. All the random variables are assumed to be statistically independent and lognormally distributed with the parameters as in Table 3.

As seen in (29), the area of bending reinforcement is reduced over time. Hence, the reinforced concrete girder is now a time-varying structure. Since the area of the bending reinforcement is monotonically decreasing over time, the failure probability in a given time period is equal to the failure probability at the end of the time period. Based on this assumption and a projected girder lifetime of $T_L = 60$ years, limit-state functions can be defined corresponding to the four failure modes of the girder. Details about this can be found in Royset *et al.* (2002). We obtain a design problem of the form $\mathbf{P}_{3,\mathrm{sys}}$ where the initial cost now is

$$c_0(\mathbf{x}) = 0.75C_sL_gA_s + C_sn_SA_v(h_f + h_w - \alpha + 0.5b_w) + C_cL_g(bh_f + b_wh_w) + C_cL_gb_wc_a \tag{30}$$

and the cost of failure is $c(\mathbf{x}) = 500c_0(\mathbf{x})$. Let the constraint on the system failure probability be $p(\mathbf{x}) \leq 0.001350$, with no constraints on the component failure probabilities. The deterministic constraints defining $\mathbf{X}$ are as above except that we also include the two constraints $c_a \leq 0.05$ and $c_a \geq 0$.

We solve this instance of $\mathbf{P}_{3,\text{sys}}$ by means of the algorithm in Section 6.2 and the result is given in Table 2 as Case 3. The system failure probability is evaluated using Monte Carlo simulation with c.o.v. 0.01. Note that the constraint associated with maximum concrete cover is active, i.e., the use of maximum concrete cover is most cost efficient. Relative to Cases 1 and 2, the total expected cost of the design is much higher due to the effect of deterioration in the strength with time.

Now suppose it is decided to maintain the structure in intervals of 20 years, i.e., at 20 and 40 years after its construction. The time of maintenance can be incorporated as a design variable, but in this example we have fixed those times for simplicity. Let m_1 and m_2 be two design variables characterizing the maintenance effort at 20 years and 40 years, respectively. Let $m_i = 0$ denote no maintenance, and $m_i = 1$ denote full maintenance, i.e., restoration to the initial state of the structure. Furthermore, we consider m_1 as the fraction of the aging of the structure from initial construction ($T = 0$) to the first maintenance action ($T = 20$ years), which is restored to its initial condition. Thus, $40 - 20m_1$ years is the effective age of the structure before the second maintenance action at $T = 40$ years. Similarly, m_2 is the fraction of the aging of the structure from initial construction ($T = 0$) to the second maintenance action ($T = 40$ years), which is mitigated by the second maintenance effort, i.e., $20 + (40 - 20m_1)(1 - m_2)$ years is the effective age of the structure at $T = 60$ years. We add the two variables m_1 and m_2 to the vector of design variables, i.e.,

$$\mathbf{x} = (A_s, b, h_f, b_w, h_w, A_v, S_1, S_2, S_3, c_a, m_1, m_2) \tag{31}$$

We ensure the safety of the girder by imposing the constraint that the system failure probability over the 60 years lifetime be less than 0.00135. This probability is obtained as the probability of the union of the failure events during the intervals 0–20 years, 20–40 years and 40–60 years. For the reasons mentioned earlier, the event of failure within each interval is identical to the failure event at the end of the interval. The design is subject to the deterministic constraints as above with the additional constraints $m_i \leq 1$ and $0 \leq m_i$, $i = 1, 2$. Let the initial cost of the structure be as in (30), the cost of failure be $c(\mathbf{x}) = 500c_0(\mathbf{x})$, and the cost of maintenance be

$$c_m(\mathbf{x}) = c_y[20m_1 + (40 - 20m_1)m_2] \tag{32}$$

where $c_y = 0.15$ represents the cost of complete restoration of the girder after a year's worth of corrosion. Note that the factor in front of m_2 represents the effective age of the structure at 40 years. We solve this particular instance of $\mathbf{P}_{3,\text{sys}}$ by using the algorithm in Section 6.2 and the result is listed in Table 2 as Case 4. The system failure probabilities are evaluated using Monte Carlo simulation with c.o.v. 0.01.

We observe in Table 2 that the expected total cost of the design is smaller for the case with the option of maintenance (Case 4) than for the case without this option (Case 3). Also in the case with maintenance, there is a significant decrease in the initial cost, at the expense of a subsequent maintenance cost. The optimal solution suggests a larger maintenance effort at 40 year than at 20 years. It is noted that the solutions in Table 2 were obtained by repeated decoupled optimization and reliability analyses, as described in the applicable algorithms. The number of such analyses varied between 4 and 25 for the four problems considered.

8 CONCLUSIONS

Algorithms are described for solving three classes of optimal structural design problems with functions representing the failure probability in the objective function and the constraint set definition. The failure probabilities can describe component or series structural system failures. Based on a first-order approximation to the failure probability, we have constructed approximating problems that can be solved repeatedly to obtain an approximation to a solution of the original design problem. By the use of higher-order reliability methods in the iterative scheme, e.g., second-order or Monte Carlo simulation, the approximating solution can be made to satisfy failure probability constraints for corresponding reliability measures.

The algorithms have stronger convergence properties than other algorithms found in the literature. Hence, the proposed algorithms are expected to be numerically more efficient and robust than algorithms based on heuristics. A significant advantage of the new algorithms is the flexibility in the selection of the reliability method. The approximating problems are semi-infinite optimization problems that can be solved using algorithms from the literature.

An extensive numerical example demonstrates that the new algorithms can be used in design and maintenance planning and with models involving both time-invariant and time-variant failure probabilities.

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