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Chapter 4

Normal Forms of Nonlinear Control Systems

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4.1 Introduction

Numerous papers were published during the last decade on the normal forms of nonlinear control systems with applications in bifurcation and its control. The approach is motivated by Poincaré's theory of normal forms for classical dynamical systems using homogeneous transformations. In this paper, we summarize a variety of control system normal forms published in the literature so that the normal forms are derived in a same framework with consistent notations. Before we get into technical details, the rest of the introduction is a review of existing results on some related topics.

It is well known that there are several normal forms for a linear control system. If the system is controllable then the system can be transformed into controllable or controller normal form. If the system has a linear output map and is observable then it can be transformed into observable or observer form. The nonlinear generalization of the linear controller normal forms were extensively studied during 1980's, for instance, Krener [23], Hunt-Su [11], Jackubczyk-Respondek [10], and Brocket [3], etc. If a nonlinear control system admits a controller normal form, it can be transformed into a linear system by a change of coordinates and feedback. Therefore, the design of a locally stabilizing state feedback control law is a straightforward task. In such a case, we say the system is feedback linearizable. On the other hand, most nonlinear systems do not admit a controller normal form under change of coordinates and invertible state feedback. For systems that are not linearizable, the quadratic approximate version of controller normal form was introduced and discussed in Krener [22] and Krener-Karahan-Hubbard-Frezza [24]. It was proved that, for certain kind of nonlinear systems, there exist a quadratic change of coordinates and quadratic feedback that transform the system into the linear approximation of the plant dynamics which is accurate to at least second degree. In this case, we call the system quadratically equivalent to a linear system or quadratically feedback linearizable. But most nonlinear systems do not admit such a linear approximation. Another way of linearizing a nonlinear control system is dynamic feedback linearization. Some nonlinear systems with more than one input can be linearized by a dynamic feedback even if they are not linearizable by a state feedback. However, it was proved that a dynamic feedback cannot completely linearize a nonlinear system with single input if it is not linearizable by a state feedback (see Charlet-Lévine-Marino [4]).

Until late 80's, the problem of normal forms for nonlinear control systems that are not feedback linearizable was still largely open. On the other hand, the Poincaré normal form of nonlinear dynamic systems has been a successful theory with applications in the study of bifurcations and stability. Although the normal form of Poincaré was not applied to control systems, in Kang [12], the idea of Poincaré was applied to nonlinear control systems with a single input. A normal form was derived for the family of linearly controllable systems with a single input, including systems that are not feedback linearizable. In addition, it was proved in Kang [12] that a dynamic feedback is able to approximately linearize a controllable system to an arbitrary degree. Invariants were found in Kang [12] that uniquely determine the normal form of a control system. The homogeneous parts of degree d from two systems are equivalent under homogeneous transformations if and only if they have the same invariants. Part of the dissertation [12] were published in Kang-Krener [13], Kang [14] and [16].

Starting from early 90's, the research on normal forms moved in several related but different directions. One active research direction is to find the normal forms of systems with uncontrollable linearizations. Several authors have made contributions to this subject. Quadratic normal forms of systems with uncontrollable linearization were developed by Kang [15], [17] and [18]. The results were generalized to higher degree terms by Fitch [6], Tall-Respondek [32], and Tall-Respondek [29] for systems affine in control. In Krener-Kang-Cheng [26], the normal form and invariants of nonlinear control systems with a single input, not necessarily affine in control, is achieved through the third degree. In the following sections, the result is generalized to homogeneous terms of arbitrary degree. The proof in [26] is constructive, which is different from the existence proof adopted in most previous published work. The same constructive proof is adopted in this chapter and generalized to higher degrees. Similar to Poincaré's theory, the normal form of a control system is invariant under homogeneous transformations of the same degree. However, a normal form of degree k is not unique under transformations of degree less than k. If a normal form is unique under transformation of arbitrary degree, it is call a canonical form. Tall-Respondek [35] solved the problem of canonical form for single-input and linearly controllable systems.

For multi-input systems, their nonlinear normal forms and invariants were first studied in Kang [12]. The quadratic normal form and quadratic invariants were derived in [12] for linearly controllable systems in which the controllability indices equal each other. Without any assumption on the controllability indices, Tall-Respondek [33] found a normal form of arbitrary degree for linearly controllable systems with two inputs. The results were further generalized by Tall [34] for linearly controllable systems with any number of inputs. The normal form was derived for homogeneous parts of arbitrary degree.

Barbot, Monaco and Normand-Cyrot [2] derived a linear and quadratic normal form for linearly controllable discrete-time systems. Quadratic and cubic normal forms were derived by Krener-Li [25] for general discrete-time systems both linearly controllable and uncontrollable systems. The approaches adopted in [2] and [25] are different. As a result, the normal forms derived in the two papers are different for linearly controllable systems.

The application of normal forms and invariants of control systems is another active research topic. Based on normal forms, bifurcations and its control were studied by several authors. In Kang [17], [18], and [19], bifurcations and their classification for both open-loop and closed-loop systems were studied for systems with a single uncontrollable mode. In Krener-Kang-Cheng [26], control bifurcation for parameterized state feedback was studied. Hamzi-Kang-Barbot [8] used normal forms and invariants to characterize the orientation and stability of periodic trajectories in a Hopf Bifurcation under state feedback. Bifurcations and their control for discrete-time systems is addressed in [25] and [7].

As an application of canonical form, Respondek-Tall [29] and [30] studied the symmetry of nonlinear systems. For linearly controllable and analytic systems that are not feedback linearizable, the group of stationary symmetries contains at most two elements and the group of non stationary symmetries consist of at most two 1-parameter families. This surprising result follows from the canonical form obtained for single-input systems by Tall-Respondek [35]. Respondek [31] establishes the relationship between flatness and symmetries for two classes of systems: feedback linearizable systems and systems equivalent to the canonical contact system for curves. For these two classes of systems the minimal flat outputs determine local symmetries and vice versa.

4.2 Linearly Controllable Systems

In this section, a control system with a scalar input is defined by the following equation,

$$\dot{x} = f(x, u), \tag{4.1}$$

where $x \in I\!\!R^n$ is the state variable, and $u \in I\!\!R$ is the control input. Occasionally, it is notationally convenient to denote the control input u by x_{n+1}. We assume that the function $f(x, u)$ is C^k for sufficiently large k. An equilibrium is a pair

(x_e, u_e) that satisfies

$$f(x_e, u_e) = 0. \tag{4.2}$$

An equilibrium state x_e is one for which there exists an u_e so that (x_e, u_e) is an equilibrium. Consider the linearization of (4.1) at (x_e, u_e),

$$\begin{aligned} \dot{\delta x} &= F\delta x + G\delta u, \\ F &= \tfrac{\partial f}{\partial x}(x_e, u_e), \qquad G = \tfrac{\partial f}{\partial u}(x_e, u_e). \end{aligned} \tag{4.3}$$

A control system (4.1) is *linearly controllable* at (x_e, u_e) if its linearization (4.3) is controllable. The linear system (4.3) is controllable if

$$\text{rank}\left[\begin{array}{ccccc} G & FG & F^2G & \cdots & F^{n-1}G \end{array}\right] = n.$$

In this section, the focus is on the normal form of linearly controllable systems. The normal form of a system with an uncontrollable linearization is addressed in Section 4.3. By a translation of the (x, u) coordinate system, we can assume that the equilibrium (x_e, u_e) is the origin $(0, 0)$.

Following the method of Poincaré, we derive the normal form of (4.1) by applying homogeneous transformations to the following Taylor expansion of (4.1)

$$\dot{x} = Fx + Gu + \sum_{k=2}^{d} f^{[k]}(x, u) + O(x, u)^{d+1}. \tag{4.4}$$

In (4.4), $f_i^{[k]}(x, u)$ is a vector field in $I\!R^n$ whose components are homogeneous polynomials of degree k in (x, u). For each homogeneous part, we apply homogeneous transformations to derive the normal form. For control systems, the transformation group includes both changes of state coordinates and invertible state feedbacks. A linear transformation is defined by

$$z = Tx, \quad v = Kx + Lu, \tag{4.5}$$

where $T \in I\!R^{n\times n}$ is an invertible matrix, $K \in I\!R^n$ is a row vector, and $L \neq 0$ is a scalar. A transformation of degree $k > 1$ is defined by

$$\begin{aligned} z &= x - \phi^{[k]}(x), \\ v &= u - \alpha^{[k]}(x, u) \end{aligned} \tag{4.6}$$

A transformation of degree k may change the homogeneous term $f^{[d]}(x, u)$ in (4.4) for $d \geq k$. However, a transformation (4.6) does not change any term of degree less than k. Similar to the derivation of Poincaré normal form, we derive the linear normal form of an equilibrium of a control system using a linear transformation. Then a quadratic transformation is used to derive the quadratic normal form. Because the quadratic transformation leaves the linear part invariant, the derivation of quadratic normal form does not change the linear normal form. In general, if the normal forms of $f^{[1]}(x, u), \ldots, f^{[k-1]}(x, u)$

have been derived, a transformation of degree k is used to derive the normal form for of $f^{[k]}$ in (4.4), which leaves the normal form of $f^{[l]}(x,u)$ invariant for $1 \leq l \leq k-1$.

It is well known that by linear transformation (4.6), a linear control system

$$\dot{x} = Fx + Gu \tag{4.7}$$

can be brought to the Brunovsky form

$$\dot{z} = Az + Bv \tag{4.8}$$

where A and B are of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}. \tag{4.9}$$

The existence of such a linear transformation is proved in many textbooks of linear control systems.

Consider a linearly controllable system (4.4). We adopt the Brunovsky form as the linear normal form. There exists a linear transformation that brings (4.4) to the form

$$\dot{x} = Ax + Bu + f^{[2]}(x,u) + O(x,u)^3, \tag{4.10}$$

where (A,B) are defined by (4.9). In the following, we use a quadratic transformation

$$\begin{aligned} z &= x - \phi^{[2]}(x), \\ v &= u - \alpha^{[2]}(x,u) \end{aligned} \tag{4.11}$$

to simplify the quadratic nonlinear part of the system. There are two basic operations, *pull up* and *push down*, which are used to achieve this.

Consider a part of the dynamics

$$\begin{aligned} \dot{x}_{i-1} &= x_i + \dots \\ \dot{x}_i &= x_{i+1} + c x_j x_k + \cdots \end{aligned} \tag{4.12}$$

where $2 \leq i \leq n$, $1 \leq j \leq k \leq n+1$, recall $x_{n+1} = u$. The $+\cdots$ indicates other quadratic and higher degree terms. The other quadratic terms will not be changed by the operations that we will do. The higher terms may be changed but we are not interested in them at this time.

If $j < k-1$ we can *pull up* the quadratic term by defining

$$\begin{aligned} z_i &= x_i - c x_j x_{k-1} \\ z_l &= x_l \qquad \text{if } l \neq i \end{aligned} \tag{4.13}$$

Its inverse transform satisfies

$$x_i = z_i + c z_j z_{k-1} + O(z)^3 \tag{4.14}$$

Then the dynamics becomes

$$\begin{aligned} \dot{z}_{i-1} &= z_i + c z_j z_{k-1} + \cdots \\ \dot{z}_i &= z_{i+1} - c z_{j+1} z_{k-1} + \cdots \end{aligned} \tag{4.15}$$

and all the other quadratic terms remain the same. Notice that in each of the new quadratic terms, the two indices are closer together than the two indices of the original quadratic term. If $j = k - 1$ we can pull up the quadratic term by defining

$$\begin{aligned} z_i &= x_i - \tfrac{c}{2} x_j x_j \\ z_l &= x_l \qquad \text{if } l \neq i \end{aligned} \tag{4.16}$$

then the dynamics becomes

$$\begin{aligned} \dot{z}_{i-1} &= z_i + \tfrac{c}{2} z_j z_j + \cdots \\ \dot{z}_i &= z_{i+1} + \cdots \end{aligned} \tag{4.17}$$

and all the other quadratic terms remain the same. Notice that the two indices of the new quadratic term are identical.

Notice also that in either case if $i = 1$ then we can still pull up and there is no z_{i-1} dynamics to be concerned with so a term disappears.

By pulling up all the quadratic terms until the two indices are equal, we obtain

$$\dot{x}_i = x_{i+1} + \sum_{j=1}^{n+1} \epsilon_{i,j}\ x_j^2 + \cdots \tag{4.18}$$

where x denotes the new state coordinate after the pull up process. This form can be simplified further by the other operation on the dynamics, *push down*. Consider a piece of the dynamics,

$$\begin{aligned} \dot{x}_i &= x_{i+1} + c x_j x_k + \cdots \\ \dot{x}_{i+1} &= x_{i+2} + \cdots \end{aligned} \tag{4.19}$$

where $1 \leq i \leq n - 1$ and $1 \leq j \leq k \leq n$. Define

$$\begin{aligned} z_{i+1} &= x_{i+1} + c x_j x_k \\ z_l &= x_l \qquad \text{if } l \neq i+1 \end{aligned} \tag{4.20}$$

Its inverse transformation satisfies

$$x_{i+1} = z_{i+1} - c z_j z_k + O(z)^3 \tag{4.21}$$

The transformation (4.20) yields

$$\begin{aligned} \dot{z}_i &= z_{i+1} + \cdots \\ \dot{z}_{i+1} &= z_{i+2} + c z_{j+1} z_k + c z_j z_{k+1} + \cdots \end{aligned} \tag{4.22}$$

and all the other quadratic terms remain unchanged. Notice that if $i + 1 = n$ then we can absorb any quadratic terms into the control using feedback. The terms in (4.19) where $1 \leq j \leq k \leq i+1$ can be pushed down repeatedly and absorbed in the control.

If the control appears in the derivative of one of the states then we cannot push that term down any further since the control need not be differentiable. So, if the term cx_jx_k appears in the equation for $\dot{z}_i$ with j or k greater than $i+1$ and we try to repeatedly push it down cx_jx_k then the control will appear before we reach the equation for $\dot{z}_n$. For this reason, we only push down a quadratic term x_jx_k with both j and k less than or equal to $i+1$. As a result, the system (4.18) is transformed into the following quadratic normal form.

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \sum_{j=i+2}^{n+1} \epsilon_{i,j} x_j x_j + O(x,u)^3, \quad \text{for } 1 \leq i \leq n-1 \\ \dot{x}_n &= u + O(x,u)^3 \end{aligned} \tag{4.23}$$

where x represents the new state coordinates after the push down process.

Example 1 *The following is the quadratic normal form of the general two dimensional linearly controllable system.*

$$\begin{aligned} \dot{x}_1 &= x_2 + \epsilon_{1,3} u^2 + O(x,u)^3 \\ \dot{x}_2 &= u \qquad\qquad + O(x,u)^3 \end{aligned} \tag{4.24}$$

Notice there is only one coefficient that cannot be normalized to zero and this is the invariant of the system under quadratic transformations.

The following is the quadratic normal form of the general three dimensional linearly controllable system.

$$\begin{aligned} \dot{x}_1 &= x_2 + \epsilon_{1,3} x_3^2 + \epsilon_{1,4} u^2 + O(x,u)^3 \\ \dot{x}_2 &= x_3 \qquad\qquad + \epsilon_{2,4} u^2 + O(x,u)^3 \\ \dot{x}_3 &= u \qquad\qquad\qquad\qquad + O(x,u)^3 \end{aligned} \tag{4.25}$$

Now there are three coefficients that cannot be normalized to zero and these are the invariants of the system under quadratic transformations.

For the rest of the section, we use pull up and push down to prove the following theorem on general normal forms.

Theorem 1 *Suppose (4.1) is linearly controllable. Suppose the vector field $f(x,u)$ is C^{d+1}. Then by change of coordinates and feedback, (4.1) can be transformed into the following normal form*

$$\begin{aligned} \dot{z} &= Az + Bv + \sum_{k=2}^{d} \tilde{f}^{[k]}(z) + O(z,v)^{d+1} \\ \tilde{f}_i^{[k]}(z) &= \sum_{j=i+2}^{n+1} \epsilon_{i,j}^{[k-2]}(\bar{z}_j) z_j^2 \end{aligned} \tag{4.26}$$

where (A,B) is in Brunovsky form. The coefficient $\epsilon_{i,j}^{[k-2]}(\bar{z}_j)$ is a homogeneous polynomial of degree $k-2$ in the variable $\bar{z}_j = (z_1, z_2, \cdots, z_j)$. When there are no terms in the sum then it is zero as in

$$\tilde{f}_n^{[k]}(z) \quad = \quad \sum_{j=n+2}^{n+1} \epsilon_{i,j}^{[k-2]}(\bar{z}_j) z_j^2 \quad = \quad 0 \tag{4.27}$$

Proof. Consider the expansion (4.4). The proof follows by mathematical induction. We have derived the linear and quadratic normal forms. Suppose that all homogeneous parts of degree less than m in (4.4) are transformed into their normal forms, consider the homogeneous part $f^{[m]}(x)$ in (4.4). A part of the dynamics has the form

$$\begin{aligned} \dot{x}_{i-1} &= x_i + \sum_{k=2}^{m-1} \tilde{f}_{i-1}^{[k]}(x,u) + \cdots \\ \dot{x}_i &= x_{i+1} + \sum_{k=2}^{m-1} \tilde{f}_i^{[k]}(x,u) + c x_{j_1} x_{j_2} \cdots x_{j_m} + \cdots \end{aligned} \tag{4.28}$$

where $2 \leq i \leq n$, $1 \leq j_1 \leq j_2 \leq \cdots \leq j_m \leq n+1$, recall $x_{n+1} = u$. The $+\cdots$ stands for other homogeneous terms of degree m and higher. The other terms of degree m will not be affected by the operations that we do and we ignore the higher degree terms. A transformation of degree m does not change the normal form of degree less than m.

If $j_{m-1} < j_m - 1$ we can pull up the degree m term by defining

$$\begin{aligned} z_i &= x_i - c x_{j_1} x_{j_2} \cdots x_{j_{m-1}} x_{j_m - 1} \\ z_l &= x_l, \qquad\qquad \text{for } l \neq i \end{aligned} \tag{4.29}$$

then the dynamics becomes

$$\begin{aligned} \dot{z}_{i-1} &= z_i + \sum_{k=2}^{m-1} \tilde{f}_{i-1}^{[k]}(z,u) + c z_{j_1} z_{j_2} \cdots z_{j_{m-1}} z_{j_m - 1} + \cdots \\ \dot{z}_i &= z_{i+1} + \sum_{k=2}^{m-1} \tilde{f}_i^{[k]}(z,u) - c z_{j_1+1} z_{j_2} \cdots z_{j_{m-1}} z_{j_m - 1} - \\ &\quad c z_{j_1} z_{j_2+1} \cdots z_{j_{m-1}} z_{j_m-1} - \cdots - c z_{j_1} z_{j_2} \cdots z_{j_{m-1}+1} z_{j_m - 1} + \cdots \\ &= z_{i+1} + \sum_{k=2}^{m-1} \tilde{f}_i^{[k]}(z,u) - c \sum_{k=1}^{m-1} \frac{z_{j_1} z_{j_2} \cdots z_{j_{m-1}} z_{j_m-1}}{z_{j_k}} z_{j_k+1} + \cdots \end{aligned} \tag{4.30}$$

and all the other degree m terms remain the same. Notice that the two largest indices of the new degree m terms are closer together than those of the original degree m term.

If $j_{m-p-1} < j_{m-p} = j_{m-p+1} = \cdots j_{m-1} = j_m - 1$ we can pull up the degree m term by defining

$$\begin{aligned} z_i &= x_i - \tfrac{c}{p+1} x_{j_1} x_{j_2} \cdots x_{j_{m-p-1}} x_{j_m-1}^{p+1} \\ z_l &= x_l, \qquad\qquad \text{for } l \neq i \end{aligned} \tag{4.31}$$

then the dynamics becomes

$$\begin{aligned} \dot{z}_{i-1} &= z_i + \sum_{k=2}^{m-1} \tilde{f}_{i-1}^{[k]}(z,u) + \frac{c}{p+1} z_{j_1} z_{j_2} \cdots z_{j_{m-p-1}} z_{j_m-1}^{p+1} + \cdots \\ \dot{z}_i &= z_{i+1} + \sum_{k=2}^{m-1} \tilde{f}_i^{[k]}(z,u) - \frac{c}{p+1} \sum_{k=1}^{m-p-1} \frac{z_{j_1} z_{j_2} \cdots z_{j_{m-p-1}} z_{j_m-1}^{p+1}}{z_{j_k}} z_{j_k+1} + \cdots \end{aligned} \tag{4.32}$$

and all the other degree m terms remain the same. Notice that the two largest indices of the new degree m terms are identical.

In any case if $i = 1$ then we can still pull up and there is no z_{i-1} dynamics to be concerned with so a term disappears.

By pulling up all the degree m terms until their two largest indices are identical, we obtain

$$\dot{x}_i = x_{i+1} + \sum_{k=2}^{m-1} \tilde{f}_i^{[k]}(x,u) + \sum_{j=1}^{n+1} \epsilon_{i,j}^{[m-2]}(\bar{x}_j) x_j^2 + \cdots \tag{4.33}$$

which is almost the normal form (4.26).

By pushing down we can make $\epsilon_i^j = 0$ for $1 \leq j \leq i+1$. Consider a piece of the dynamics,

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \sum_{k=2}^{m-1} \tilde{f}_i^{[k]}(x,u) + c x_{j_1} x_{j_2} \cdots x_{j_m} + \cdots \\ \dot{x}_{i+1} &= x_{i+2} + \sum_{k=2}^{m-1} \tilde{f}_{i+1}^{[k]}(x,u) + \cdots \end{aligned} \tag{4.34}$$

If $1 \leq j_1 \leq j_2 \leq \cdots \leq j_m \leq n$, define

$$\begin{aligned} z_{i+1} &= x_{i+1} + c x_{j_1} x_{j_2} \cdots x_{j_m} \\ z_l &= x_l, \qquad \text{for } l \neq i+1 \end{aligned} \tag{4.35}$$

yielding

$$\begin{aligned} \dot{z}_i &= z_{i+1} + \sum_{k=2}^{m-1} \tilde{f}_i^{[k]}(z,u) + \cdots \\ \dot{z}_{i+1} &= z_{i+2} + \sum_{k=2}^{m-1} \tilde{f}_{i+1}^{[k]}(z,u) + c \sum_{k=1}^{m} \frac{z_{j_1} z_{j_2} \cdots z_{j_m}}{z_{j_k}} z_{j_k+1} + \cdots \end{aligned} \tag{4.36}$$

and all the other degree m terms remain unchanged. Notice that if $i+1 = n$ then we can absorb the degree m terms into the control using feedback. The terms in (4.33) where $1 \leq j_1 \leq j_2 \leq \cdots \leq j_m \leq i+1$ can be repeatedly pushed down and absorbed in the control. The result is (4.26). □

Example 2 *The following is the normal form up to the fourth degree for a general three-dimensional system.*

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 + \epsilon_{1,3}(x)x_3^2 + \epsilon_{1,4}(x,u)u^2 + O(x,u)^4 \\ \dot{x}_2 & = & x_3 \qquad\qquad\quad + \epsilon_{2,4}(x,u)u^2 + O(x,u)^4 \\ \dot{x}_3 & = & u \qquad\qquad\qquad\qquad\qquad\qquad\quad + O(x,u)^4 \end{array} \tag{4.37}$$

where $\epsilon_{i,j} = \epsilon_{i,j}^{[0]} + \epsilon_{i,j}^{[1]} + \epsilon_{i,j}^{[2]}$ *and* $\epsilon_{i,j}^{[k]}$ *is a homogeneous polynomial of degree* k.

4.3 Linearly Uncontrollable Systems

In this section, we generalize the results of § 4.2 to systems with uncontrollable linearization. Consider a control system (4.1). Suppose the controllability matrix of its linearization (4.3) has a rank $n_1 < n$. It is well known that by linear change of state coordinates and linear state feedback, the system can be brought to the form

$$\begin{array}{rcl} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \end{bmatrix} & = & \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} u \\ & & + \displaystyle\sum_{k=2}^{d} \begin{bmatrix} f_0^{[k]}(x_0,x_1,u) \\ f_1^{[k]}(x_0,x_1,u) \end{bmatrix} + O(x_0,x_1,u)^{d+1} \end{array} \tag{4.38}$$

where x_0, x_1 are n_0, n_1 dimensional, $n_0+n_1 = n$, $u \in I\!R$, A_0 is in block diagonal Jordan form, A_1, B_1 are in Brunovsky form and $f_r^{[d]}(x_0,x_1,u)$ is a vector field which is a homogeneous polynomial of degree d in its arguments. The linear change of coordinates that brings A_0 to Jordan form may be complex, in which case some of the coordinates $x_{0,i}$ are complex. The complex coordinates come in conjugate pairs. The corresponding $f_{0,i}^{[k]}$ are complex valued and come in conjugate pairs. In some formulae, the control input is treated as a state variable $u = x_{0,n_1+1}$. A nonlinear vector field $f_r^{[k]}(x_0,x_1,u)$, $r = 0,1$, has the following decomposition

$$f_r^{[k]}(x_0,x_1,u) = \sum_{|l|=k} f_r^{[l]}(x_0;x_1,u) \tag{4.39}$$

where $[l] = [l_0;l_1]$ is a multi-index and $f_r^{[l]}(x_0;x_1,u)$ denotes a polynomial vector field homogeneous of degree l_0 in x_0 and homogeneous of degree l_1 in (x_1,u), $|l| = l_0 + l_1$. A homogeneous transformation of degree k has the following form

$$\begin{array}{rcl} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} & = & \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} - \begin{bmatrix} \phi_0^{[k]}(x_0,x_1) \\ \phi_1^{[k]}(x_0,x_1) \end{bmatrix} \\ v & = & u - \alpha^{[k]}(x_0,x_1,u) \end{array} \tag{4.40}$$

We can expand it as follows

$$\begin{aligned} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} &= \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} - \sum_{|l|=k} \begin{bmatrix} \phi_0^{[l]}(x_0; x_1) \\ \phi_1^{[l]}(x_0; x_1) \end{bmatrix} \\ v &= u - \sum_{|l|=k} \alpha^{[l]}(x_0; x_1, u) \end{aligned} \tag{4.41}$$

where $\phi_r^{[l]}(x_0; x_1)$ denotes a vector field that is homogeneous of degree l_0 in x_0 and homogeneous of degree l_1 in x_1. Similarly, $\alpha^{[l]}(x_0; x_1, u)$ is a polynomial homogeneous of degree l_0 in x_0 and homogeneous of degree l_1 in (x_1, u). Under a transformation (4.41), the degree $[l]$ terms are transformed into

$$\begin{aligned} \tilde{f}_0^{[l]}(z_0; z_1, v) &= f_0^{[l]}(z_0; z_1, v) - \frac{\partial \phi_0^{[l]}}{\partial z_0}(z_0; z_1) A_0 z_0 \\ &\quad - \frac{\partial \phi_0^{[l]}}{\partial z_1}(z_0; z_1)\,(A_1 z_1 + B_1 v_1) \\ &\quad + A_0 \phi_0^{[l]}(z_0; z_1) \\ \tilde{f}_1^{[l]}(z_0; z_1, v) &= f_1^{[l]}(z_0; z_1, v) - \frac{\partial \phi_1^{[l]}}{\partial z_0}(z_0; z_1) A_0 z_0 \\ &\quad - \frac{\partial \phi_1^{[l]}}{\partial z_1}(z_0; z_1)\,(A_1 z_1 + B_1 v_1) \\ &\quad + A_1 \phi_1^{[l]}(z_0; z_1) + B_1 \alpha^{[l]}(z_0; z_1, v). \end{aligned} \tag{4.42}$$

This is still a homogeneous vector of degree $[l]$. We have proved the following lemma.

Lemma 1 *After the transformation (4.41), the new homogeneous part* $\tilde{f}_0^{[l]}$ *is completely determined by* $f_0^{[l]}$ *and* $\phi_0^{[l]}(x_0; x_1)$*. The new homogeneous part* $\tilde{f}_1^{[l]}$ *is completely determined by* $f_1^{[l]}$*,* $\phi_1^{[l]}(x_0; x_1)$*, and* $\alpha^{[l]}(x_0; x_1, v)$*.*

According to the lemma, each component of the term, $\tilde{f}_r^{[l]}$, that is homogeneous of degree $[l]$ can be considered separately in the derivation of the normal form. Following Poincaré, (4.42) is called a homological equation. In the derivation of the normal form, the quadratic transformation is first applied to (4.38) to derive the normal form of $f^{[2]}(x_0, x_1, u)$. Then, a cubic transformation is used to derive the normal form of the cubic part. In general, after the normal form of degree less than k has been found, a homogeneous transformation of degree k is used to derive the normal form of $f^{[k]}(x_0, x_1, u)$.

Theorem 2 *Consider a control system (4.38). There exist homogeneous transformations of the form (4.40) with* $k = 2, 3, \cdots, d$ *that transform the system*

(4.38) into the normal form

$$\begin{aligned}\begin{bmatrix}\dot z_0\\ \dot z_1\end{bmatrix} &= \begin{bmatrix}A_0 & 0\\ 0 & A_1\end{bmatrix}\begin{bmatrix}z_0\\ z_1\end{bmatrix}+\begin{bmatrix}0\\ B_1\end{bmatrix}v\\ &\quad+\sum_{k=2}^{d}\begin{bmatrix}\tilde f_0^{[k]}(z_0,z_1,v)\\ \tilde f_1^{[k]}(z_0,z_1,v)\end{bmatrix}+O(z_0,z_1,v)^{d+1}\end{aligned}\tag{4.43}$$

where $\tilde f_0^{[k]}$, $\tilde f_1^{[k]}$ *have the following decomposition*

$$\begin{aligned}\tilde f_0^{[k]}(z_0,z_1,v) &= \tilde f_0^{[k;0]}(z_0)+\tilde f_0^{[k-1;1]}(z_0;z_{1,1})+\sum_{l_1=2}^{k}\tilde f_0^{[k-l_1;l_1]}(z_0;z_1,v)\\ \tilde f_1^{[k]}(z_0,z_1,v) &= \sum_{l_1=2}^{k}\tilde f_1^{[k-l_1;l_1]}(z_0;z_1,v)\end{aligned}\tag{4.44}$$

The vector field $\tilde f_0^{[k;0]}(z_0)$ *is in Poincaré normal form*

$$\tilde f_{0,i}^{[k;0]}(z_0) = \sum_{\substack{|j|\,=\,k\\ j\cdot\lambda\,=\,\lambda_i}} \beta_{i,j}\ z_0^j \tag{4.45}$$

where $j=(j_1,\dots,j_{n_0})$ *is a multi-index of nonnegative integers,* $|j|=j_1+\cdots+j_{n_0}$, $j\cdot\lambda=j_1\lambda_1+\cdots+j_{n_0}\lambda_{n_0}$ *and* $z_0^j=z_{0,1}^{j_1}\cdots z_{0,n_0}^{j_{n_0}}$. *The other vector fields are as follows,*

$$\begin{aligned}\tilde f_{0,i}^{[k-1;1]}(z_0;z_{1,1}) &= \gamma_i^{[k-1]}(z_0)z_{1,1} && i=1,\dots,n_0\\ \tilde f_{0,i}^{[k-l_1;l_1]}(z_0;z_1,v) &= \sum_{j=1}^{n_1+1}\delta_{i,j}^{[k-l_1;l_1-2]}(z_0;\bar z_{1,j})z_{1,j}^2 && i=1,\dots,n_0,\ l_1=2,\dots,k\\ \tilde f_{1,i}^{[k-l_1;l_1]}(z_0;z_1,v) &= \sum_{j=i+2}^{n_1+1}\epsilon_{i,j}^{[k-l_1;l_1-2]}(z_0;\bar z_{1,j})z_{1,j}^2 && i=1,\dots,n_1\end{aligned}\tag{4.46}$$

where j *is a scalar index,* $z_{1,n_1+1}=v$, $\bar z_{1,j}=(z_{1,1},z_{1,2},\cdots,z_{1,j})$ *and* $\delta_{i,j}^{[k-l_1;l_1-2]}(z_0;\bar z_{1,j})$, $\epsilon_{i,j}^{[k-l_1;l_1-2]}(z_0;\bar z_{1,j})$ *are polynomials homogeneous of degree* $k-l_1$ *in* z_0 *and homogeneous of degree* l_1-2 *in* (z_1,v).

Proof. Suppose the homogeneous vector fields $f^{[k]}(x_0,x_1,u)$, for all $k\le d-1$, are already in normal form. Consider the homogeneous term of degree d. A transformation of degree d does not change the homogeneous parts of degree less than d. It changes the terms of degree greater or equal to d. Because of Lemma 1, we can derive the normal form for each homogeneous part $f_r^{[l]}$ separately.

Consider $f_1^{[d;0]}(x_0; x_1, u)$. Given a part of the system

$$\begin{aligned} \dot{x}_{1,i} &= x_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}_{1,i}^{[k]}(x_0, x_1, u) + c x_{0,j_1} x_{0,j_2} \cdots x_{0,j_d} + \dots \\ \dot{x}_{1,i+1} &= x_{0,i+2} + \sum_{k=2}^{d-1} f_{1,i+1}^{[k]}(x_0, x_1, u) + \dots \end{aligned} \tag{4.47}$$

The following push down

$$\begin{aligned} z_{1,i+1} &= x_{1,i+1} + c x_{0,j_1} x_{0,j_2} \cdots x_{0,j_d} \\ z_{s,t} &= x_{s,t}, \qquad \text{if } (s,t) \neq (1, i+1) \end{aligned} \tag{4.48}$$

brings (4.47) to

$$\begin{aligned} \dot{z}_{1,i} &= z_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}_{1,i}^{[k]}(z_0, z_1, u) + \dots \\ \dot{z}_{1,i+1} &= z_{1,i+2} + + \sum_{k=2}^{d-1} f_{1,i+1}^{[k]}(z_0, z_1, u) + \frac{d}{dt}(c x_{0,j_1} x_{0,j_2} \cdots x_{0,j_d}) + \dots \end{aligned} \tag{4.49}$$

Because the lowest homogeneous part of $\frac{d}{dt}(c x_{0,j_1} x_{0,j_2} \cdots x_{0,j_d})$ is still a term of degree $[d; 0]$, it can be further pushed down. When $i = n_1$, the nonlinear term is absorbed by the feedback. Therefore, all terms of degree $[d; 0]$ can be canceled by nonlinear transformations.

Consider $f_1^{[d-1;1]}(x_0; x_1, u)$. Given a part of the system

$$\begin{aligned} \dot{x}_{1,i-1} &= x_{1,i} + \sum_{k=2}^{d-1} \tilde{f}_{1,i-1}^{[k]}(x_0, x_1, u) + \dots \\ \dot{x}_{1,i} &= x_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}_{1,i}^{[k]}(x_0, x_1, u) + c x_{0,j_1} x_{0,j_2} \cdots x_{0,j_{d-1}} x_{1,j_d} + \dots \end{aligned} \tag{4.50}$$

If $j_d > 1$,we can pull up the degree m term by defining

$$\begin{aligned} z_{1,i} &= x_{1,i} - c x_{0,j_1} x_{0,j_2} \cdots x_{0,j_{d-1}} x_{1,j_d - 1} \\ z_{s,t} &= x_{s,t}, \qquad \text{if } (s,t) \neq (1, i) \end{aligned} \tag{4.51}$$

The new system has the form

$$\begin{aligned} \dot{z}_{1,i-1} &= z_{1,i} + \sum_{k=2}^{d-1} \tilde{f}_{1,i-1}^{[k]}(z_0, z_1, u) + c z_{1,j_1} z_{1,j_2} \cdots z_{1,j_{d-1}} z_{1,j_d - 1} + \dots \\ \dot{z}_{1,i} &= z_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}_{1,i}^{[k]}(z_0, z_1, u) - \frac{d}{dt}(c x_{0,j_1} x_{0,j_2} \cdots x_{0,j_{d-1}}) x_{1,j_d - 1} + \dots \end{aligned} \tag{4.52}$$

In all the new terms of degree $[d-1;1]$, the index of the controllable factors is $j_d - 1$, which is smaller than the original index j_d. If $i = 1$,we can cancel the degree $[d-1;1]$ term without worrying about the equation of $\dot{x}_{i-1}$. Repeat the pull up transformation until all the degree $[d-1;1]$ terms are brought to homogeneous terms in the form $x_{0,j_1}x_{0,j_2}\cdots x_{0,j_{d-1}}x_{1,1}$, in which $j_d = 1$. Now, consider a part of the system

$$\begin{array}{rcl} \dot{x}_{1,i} & = & x_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}_{2,i}^{[k]}(x_0, x_1, u) + c x_{0,j_1}x_{0,j_2}\cdots x_{0,j_{d-1}}x_{1,1} + \dots \\ \dot{x}_{1,i+1} & = & x_{1,i+2} + \sum_{k=2}^{d-1} \tilde{f}_{2,i+1}^{[k]}(x_0, x_1, u) + \dots \end{array} \tag{4.53}$$

A push down transformation

$$\begin{array}{rcll} z_{1,i+1} & = & x_{1,i+1} + c x_{0,j_1}x_{0,j_2}\cdots x_{0,j_{d-1}}x_{1,1} & \\ z_{s,t} & = & x_{s,t}, & \text{if } (s,t) \neq (1, i+1) \end{array} \tag{4.54}$$

yields

$$\begin{array}{rcl} \dot{z}_{2,i} & = & z_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}_{1,i}^{[k]}(z_0, z_1, u) + \dots \\ \dot{z}_{2,i+1} & = & z_{1,i+2} + \sum_{k=2}^{d-1} \tilde{f}_{1,i+1}^{[k]}(z_0, z_1, u) + \dfrac{d}{dt}(c x_{0,j_1}x_{0,j_2}\cdots x_{0,j_{d-1}})x_{1,1} \\ & & + c x_{0,j_1}x_{0,j_2}\cdots x_{0,j_{d-1}}x_{1,2} + \dots \end{array} \tag{4.55}$$

Repeating the push down process, all degree $[d-1;1]$ terms are finally pushed to the equation for $\dot{x}_{1,n_1}$, where they are canceled by the feedback. Therefore, $f_1^{[d-1;1]}(x_0; x_1, u)$ can be eliminated by homogeneous transformations.

Consider $f_1^{[l_0;l_1]}(x_0; x_1, u)$ with $2 \leq l_1 \leq d$. A part of the dynamics has the form

$$\begin{array}{rcl} \dot{x}_{1,i-1} & = & x_{1,i} + \sum_{k=2}^{d-1} \tilde{f}_{1,i-1}^{[d]}(x_0, x_1, u) + \dots \\ \dot{x}_{1,i} & = & x_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}_{1,i}^{[d]}(x_0, x_1, u) + c^{[l_0]}(x_0)x_{1,j_1}x_{1,j_2}\cdots x_{1,j_{l_1}} + \dots \end{array} \tag{4.56}$$

The derivation of $\tilde{f}_1^{[l_0;l_1]}$ is similar to that in section 4.2. If $j_{l_1-1} < j_{l_1} - 1$ the pull up transformation is defined by

$$\begin{array}{rcll} z_{1,i} & = & x_{1,i} - c^{[l_0]}(x_0)x_{1,j_1}x_{1,j_2}\cdots x_{1,j_{l_1-1}}x_{1,j_{l_1}-1} & \\ z_{s,t} & = & x_{s,t}, & \text{if } (s,t) \neq (1, i) \end{array} \tag{4.57}$$

then the dynamics becomes

$$\begin{aligned}\dot{z}_{1,i-1} &= z_{1,i} + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{1,i-1}(z_0, z_1, u) + c^{[l_0]}(z_0) z_{1,j_1} z_{1,j_2} \cdots z_{1,j_{l_1-1}} z_{1,j_{l_1}-1} + \dots \\ \dot{z}_{1,i} &= z_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{1,i}(z_0, z_1, u) \\ &\quad -c^{[l_0]}(z_0) \sum_{k=1}^{l_1-1} \frac{z_{1,j_1} z_{1,j_2} \cdots z_{1,j_{l_1-1}} z_{1,j_{l_1}-1}}{z_{1,j_k}} z_{1,j_k+1} \\ &\quad -\frac{d}{dt}(c^{[l_0]}(x_0)) x_{1,j_1} x_{1,j_2} \cdots x_{1,j_{l_1-1}} x_{1,j_{l_1}-1} + \dots\end{aligned} \tag{4.58}$$

The lowest terms in the time derivative of $c^{[l_0]}(x_0)$ are still polynomials of x_0 with the degree l_0. As a result of the pull up, the two largest indices of z_1 in the new terms are $j_{l_1-1}, j_{l_1} - 1$ and $j_{l_1-1} + 1, j_{l_1} - 1$, which are closer together than those of the original term. If $j_{l_1-p-1} < j_{l_1-p} = j_{l_1-p+1} = \cdots j_{l_1-1} = j_{l_1} - 1$, we define the pull up transformation by

$$\begin{aligned}z_{1,i} &= x_{0,i} - \frac{c^{[l_0]}(x_0)}{p+1} x_{1,j_1} x_{1,j_2} \cdots x_{1,j_{l_1-p-1}} x^{p+1}_{1,j_{l_1}-1} \\ z_{s,t} &= x_{s,t}, \qquad \text{for } (s,t) \neq (1,i)\end{aligned} \tag{4.59}$$

then the dynamics becomes

$$\begin{aligned}\dot{z}_{2,i-1} &= z_{1,i} + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{1,i-1}(z_0, z_1, u) \\ &\quad + \frac{c^{[l_0]}(z_0)}{p+1} z_{1,j_1} z_{1,j_2} \cdots z_{1,j_{l_1-p-1}} z^{p+1}_{1,j_{l_1}-1} + \dots \\ \dot{z}_{1,i} &= z_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{1,i}(z_0, z_1, u) \\ &\quad - \frac{c^{[l_0]}(z_0)}{p+1} \sum_{k=1}^{l_1-p-1} \frac{z_{1,j_1} z_{1,j_2} \cdots z_{1,j_{l_1-p-1}} z^{p+1}_{1,j_{l_1}-1}}{z_{1,j_k}} z_{1,j_k+1} \\ &\quad - \frac{d}{dt}(c^{[l_0]}(x_0)) x_{1,j_1} x_{1,j_2} \cdots x_{1,j_{l_1-p-1}} z^{p+1}_{1,j_{l_1}-1} + \dots\end{aligned} \tag{4.60}$$

Notice that the two largest indices of variable $x_{1,j}$ in the new degree $[l_0; l_1]$ terms are identical. In any case if $i = 1$ then we can still pull up and there is no $z_{1,i-1}$ dynamics to be concerned with so a term disappears. By pulling up all the degree $[l_0; l_1]$ terms until their two largest indices of $x_{1,j}$ are identical, we obtain

$$\dot{x}_{1,i} = x_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{1,i}(x_0, x_1, u) + \sum_{j=1}^{n_1+1} \epsilon^{[d-2]}_{i,j}(x_0, \bar{x}_{1,j}) x^2_{1,j} + \dots \tag{4.61}$$

By pushing down we can make $\epsilon_{i,j}^{[d-2]} = 0$ for $1 \leq j \leq i+1$. Consider a piece of the dynamics,

$$\begin{aligned} \dot{x}_{1,i} &= x_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}_{1,i}^{[k]}(x_0, x_1, u) + c^{[l_0]}(x_0) x_{1,j_1} x_{1,j_2} \cdots x_{1,j_{l_1}} + \dots \\ \dot{x}_{1,i+1} &= x_{1,i+2} + \sum_{k=2}^{d-1} \tilde{f}_{1,i+1}^{[k]}(x_0, x_1, u) + \dots \end{aligned} \tag{4.62}$$

If $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{l_1} \leq n_1$, define

$$\begin{aligned} z_{1,i+1} &= x_{0,i+1} + c^{[l_0]}(x_0) x_{1,j_1} x_{1,j_2} \cdots x_{1,j_{l_1}} \\ z_{s,t} &= x_{s,t}, \qquad \text{for } (s,t) \neq (1, i+1) \end{aligned} \tag{4.63}$$

yielding

$$\begin{aligned} \dot{z}_{1,i} &= z_{1,i+1} + \sum_{k=2}^{d-1} \tilde{f}_{1,i}^{[k]}(z_0, z_1, u) + \dots \\ \dot{z}_{1,i+1} &= z_{1,i+2} + \sum_{k=2}^{d-1} \tilde{f}_{1,i+1}^{[k]}(z_0, z_1, u) + c^{[l_0]}(z_0) \sum_{k=1}^{l_1} \frac{z_{1,j_1} z_{1,j_2} \cdots z_{1,j_{l_1}}}{z_{1,j_k}} z_{1,j_k+1} \\ & \quad \frac{d}{dt}(c^{[l_0]}(x_0)) x_{1,j_1} x_{1,j_2} \cdots x_{1,j_{l_1}} + \dots \end{aligned} \tag{4.64}$$

and all the other degree d terms remain unchanged. Notice that if $i+1 = n_1$ then we can absorb the degree d terms into the control using feedback. The terms in (4.61) where $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{l_1} \leq i+1$ can be repeatedly pushed down and absorbed in the control. The result is the normal form of $\tilde{f}_1^{[k]}(z_0, z_1)$ in (4.46).

Consider $f_0^{[d;0]}(x_0)$. Its homological equation (4.42) is independent of the feedback. Therefore, the normal form is the same as Poincaré normal form.

Consider $f_0^{[d-1;1]}(x_0; x_1, u)$. Given a part of the dynamics

$$\begin{aligned} \dot{x}_{0,i-1} &= \lambda_{i-1} x_{0,i-1} + \delta_{i-1} x_{0,i} + \sum_{k=2}^{d-1} \tilde{f}_{0,i-1}^{[k]}(x_0, x_1, u) + \dots \\ \dot{x}_{0,i} &= \lambda_i x_{0,i} + \delta_i x_{0,i+1} + \sum_{k=2}^{d-1} \tilde{f}_{0,i}^{[k]}(x_0, x_1, u) + c^{[d-1]}(x_0) x_{1,j} + \dots \end{aligned} \tag{4.65}$$

where $2 \leq i \leq n_0$, $1 \leq j \leq n_1 + 1$. The coefficients δ_{i-1} and δ_i equal 0 or 1. If $j > 1$ then we can pull up by defining

$$\begin{aligned} z_{0,i} &= x_{0,i} - c^{[d-1]}(x_0) x_{1,j-1} \\ z_{s,t} &= x_{s,t} \qquad \text{if } (s,t) \neq (0,i) \end{aligned} \tag{4.66}$$

so that

$$\begin{aligned}
\dot{z}_{0,i-1} &= \lambda_{i-1} z_{0,i-1} + \delta_{i-1}(z_{0,i} + c^{[d-1]}(z_0) z_{1,j-1}) + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{1,i-1}(z_0, z_1, u) + \dots \\
\dot{z}_{0,i} &= \lambda_i z_{0,i} + \delta_i z_{0,i+1} + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{0,i}(z_0, z_1, u) \\
&\quad + \lambda_i c^{[d-1]}(z_0) z_{1,j-1} - \frac{d}{dt}(c^{[d-1]}(z_0)) z_{1,j-1} + \dots
\end{aligned} \tag{4.67}$$

The new degree $[d-1;1]$ terms have last index $1, j-1$ instead of $1, j$. We can continue to pull up until $j = 1$. The result is the normal form $\tilde{f}_1^{[d-1;1]}$ in (4.46). If $i = 1$, the pull up cancels the $[d-1;1]$ term because there is no $x_{0,i-1}$.

Consider $f_0^{[l_0;l_1]}(x_0; x_1, u)$ with $2 \le l_1 \le d$. Given a part of the system

$$\begin{aligned}
\dot{x}_{0,i-1} &= \lambda_{i-1} x_{0,i-1} + \delta_{i-1} x_{0,i} + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{0,i-1}(x_0, x_1, u) + \dots \\
\dot{x}_{1,i} &= \lambda_i x_{0,i} + \delta_i x_{0,i+1} + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{0,i}(x_0, x_1, u) + c^{[l_0]}(x_0) x_{1,j_1} x_{1,j_2} \cdots x_{1,j_{l_1}} + \dots
\end{aligned} \tag{4.68}$$

where $1 \le j_1 \le j_2 \le \dots \le j_{l_1} \le n_1 + 1$. The coefficients δ_{i-1} and δ_i equal 0 or 1. If $j_{l_1-1} < j_{l_1} - 1$, then we can pull up by defining

$$\begin{aligned}
z_{0,i} &= x_{0,i} - c^{[l_0]}(x_0) x_{1,j_1} x_{1,j_2} \cdots x_{1,j_{l_1}-1} \\
z_{s,t} &= x_{s,t}, \qquad \text{if } (s,t) \neq (0,i)
\end{aligned} \tag{4.69}$$

so that

$$\begin{aligned}
\dot{z}_{0,i-1} &= \lambda_{i-1} z_{0,i-1} + \delta_{i-1}(z_{0,i} + c^{[l_0]}(z_0) z_{1,j_1} z_{1,j_2} \cdots z_{1,j_{l_1}-1}) \\
&\quad + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{1,i-1}(z_0, z_1, u) + \dots \\
\dot{z}_{0,i} &= \lambda_i z_{0,i} + \delta_i z_{0,i+1} + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{0,i}(z_0, z_1, u) \\
&\quad + \lambda_i c^{[l_0]}(z_0) z_{1,j_1} z_{1,j_2} \cdots z_{1,j_{l_1}-1} - \frac{d}{dt}(c^{[l_0]}(z_0)) z_{1,j_1} z_{1,j_2} \cdots z_{1,j_{l_1}-1} \\
&\quad - c^{[l_0]}(z_0) \sum_{k=1}^{l_1-1} \frac{z_{1,j_1} z_{1,j_2} \cdots z_{1,j_{l_1}-1}}{z_{1,j_k}} z_{1,j_k+1} + \dots
\end{aligned} \tag{4.70}$$

In the new $[l_0; l_1]$ terms, the two largest indices of $x_{0,j}$ are closer than before. If $j_{l_1-p-1} < j_{l_1-p} = j_{l_1-p+1} = \dots = j_{l_1-1} = j_{l_1} - 1$ for some $p \ge 1$, define the following pull up transformation

$$\begin{aligned}
z_{0,i} &= x_{0,i} - \frac{c^{[l_0]}(x_0)}{p+1} x_{1,j_1} \cdots x_{1,j_{l_1-p-1}} x^{p+1}_{1,j_{l_1}-1} \\
z_{s,t} &= x_{s,t}, \qquad \text{if } (s,t) \neq (0,i)
\end{aligned} \tag{4.71}$$

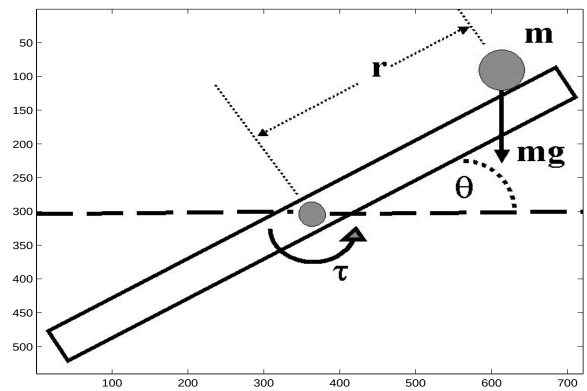

Figure 4.1: The configuration of ball and beam system

Then

$$\begin{aligned}
\dot{z}_{0,i-1} &= \lambda_{i-1} z_{0,i-1} + \delta_{i-1}(z_{0,i} + \frac{c^{[l_0]}(z_0)}{p+1} z_{1,j_1} \cdots z_{1,j_{l_1-p-1}} z^{p+1}_{1,j_{l_1}-1}) \\
&\quad + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{0,i-1}(z_0, z_1, u) + \dots \\
\dot{z}_{0,i} &= \lambda_i z_{0,i} + \delta_i z_{0,i+1} + \sum_{k=2}^{d-1} \tilde{f}^{[k]}_{0,i}(z_0, z_1, u) \\
&\quad + \lambda_i \frac{c^{[l_0]}(z_0)}{p+1} z_{1,j_1} \cdots z_{1,j_{l_1-p-1}} z^{p+1}_{1,j_{l_1}-1} \\
&\quad - \frac{d}{dt}(\frac{c^{[l_0]}(x_0)}{p+1}) x_{1,j_1} \cdots x_{1,j_{l_1-p-1}} x^{p+1}_{1,j_{l_1}-1} \\
&\quad - \frac{c^{[l_0]}(z_0)}{p+1} \sum_{k=1}^{l_1-p-1} \frac{z_{1,j_1} \cdots z_{1,j_{l_1-p-1}} z^{p+1}_{1,j_{l_1}-1}}{z_{1,j_k}} z_{1,j_k+1} + \dots
\end{aligned} \tag{4.72}$$

In the new $[l_0; l_1]$ terms, the last two indices of $x_{1,j}$ are equal. We repeat the pull up process until all $[l_0; l_1]$ terms have the form $c(z_0, \bar{z}_{1,j}) z^2_{1,j}$. □

4.4 Examples of Normal Form

The derivation of normal forms for specific engineering systems is not necessarily a complicated process. In the following, we introduce three examples. In each example, the normal form can be easily derived through simple transformations of push up and pull down.

4.4.1 The Normal Form of Ball and Beam

Consider the ball and beam experiment shown in Figure 4.1. The system model adopted in this section is from [9]. We assume that the beam rotates around the axis at its center. The ball rolls along the beam. The control input of the system is τ, the angular acceleration of the beam. The state variables are r, the distance from the center of the ball to the axis, and θ, the angle of the beam. Let J be the moment of inertia of the beam. Let m be the mass of the ball. Let g be the acceleration of gravity. The equations of motion are

$$\begin{array}{rcl} 0 & = & \ddot{r} + g\sin\theta - r\dot{\theta}^2 \\ \tau & = & (mr^2+J)\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos\theta \end{array} \qquad (4.73)$$

Let

$$\tau = 2mr\dot{r}\dot{\theta} + mgr\cos\theta + (mr^2+J)u \qquad (4.74)$$

This is an invertible feedback under which the system (4.73) is equivalent to

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -g\sin x_3 + x_1x_4^2 \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & u \end{array} \qquad (4.75)$$

where $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$, and $x_4 = \dot{\theta}$. The origin $(x_1, x_2, x_3, x_4) = (0,0,0,0)$ is an equilibrium point of the system. The linearization of the system at the origin is

$$\begin{array}{rcl} \delta\dot{x}_1 & = & \delta x_2 \\ \delta\dot{x}_2 & = & -g\delta x_3 \\ \delta\dot{x}_3 & = & \delta x_4 \\ \delta\dot{x}_4 & = & \delta u \end{array} \qquad (4.76)$$

Obviously, the linearization is controllable. So, the model (4.75) of ball and beam system is linearly controllable at the origin. In the following, we derive the normal form for the system (4.75). At first, we focus on the nonlinear term $g\sin x_3$. We will handle the term $x_1x_4^2$ later. Instead of dealing with the homogeneous terms separately, system (4.75) allows us to push down all the homogeneous terms in $g\sin x_2$ simultaneously. The push down transformation is

$$z_3 = -g\sin x_3 \qquad (4.77)$$

after which the system becomes

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & z_3 + x_1x_4^2 \\ \dot{z}_3 & = & -gx_4\cos x_3 \\ \dot{x}_4 & = & u. \end{array} \qquad (4.78)$$

One more step of pushing down by

$$z_4 = -gx_4\cos x_3 \qquad (4.79)$$

yields

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= z_3 + x_1 x_4^2 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -gu\cos x_3 + g x_4^2 \sin x_3. \end{aligned} \tag{4.80}$$

If $-\frac{\pi}{2} < x_3 < \frac{\pi}{2}$ we can define an invertible feedback

$$v = -gu\cos x_3 + g x_4^2 \sin x_3$$

and then the system becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= z_3 + x_1 x_4^2 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= v. \end{aligned} \tag{4.81}$$

Now, we have to deal with the term $x_1 x_4^2$ in (4.81). From (4.77) and (4.79), the inverse transformation satisfies

$$\begin{aligned} x_3 &= \arcsin(-\frac{z_3}{g}) \\ x_4 &= -\frac{z_4}{g\cos(\arcsin(-\frac{z_3}{g}))} \end{aligned} \tag{4.82}$$

Define $z_1 = x_1$, $z_2 = x_2$, (4.81) is equivalent to

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 + \frac{z_1 z_4^2}{g^2\cos^2(\arcsin(-\frac{z_3}{g}))} \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= v. \end{aligned} \tag{4.83}$$

However,

$$\begin{aligned} \cos^2(\arcsin(-\frac{z_3}{g})) &= 1 - \sin^2(\arcsin(-\frac{z_3}{g})) \\ &= 1 - \frac{z_3^2}{g^2}. \end{aligned}$$

So, the system (4.83) is equivalent to

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 + \frac{z_1}{g^2 - z_3^2} z_4^2 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= v. \end{aligned} \tag{4.84}$$

This system is in normal form. Its homogeneous parts of any degree can be found in the following Taylor expansion

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 + \sum_{k=0}^{\infty} \frac{1}{g^{2k+2}} z_1 z_3^{2k} z_4^2 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= v. \end{aligned} \tag{4.85}$$

4.4.2 Engine Compressor

The second example is the Moore-Greitzer three state model of an axial flow compressor. The model is a typical example of a control system with both classical and control bifurcations. When the engine compressor is operated around the equilibrium with the maximum pressure rise, a classical bifurcation occurs in its uncontrolled dynamics. There is also a control bifurcation in the control system. On a branch of the bifurcated equilibria, the system exhibits rotating stall which can cause severe vibrations with rapid and catastrophic consequences. In the following, a model of engine compressor is introduced. Then the normal form of the model is derived at the point where rotating stall occurs.

The Moore-Greitzer model of an engine compressor described in [5] is

$$\begin{aligned} \frac{dA}{d\xi} &= \frac{3\alpha H}{2W} A \left(1 - (\frac{\Phi}{W} - 1)^2 - \frac{A^2}{4W^2} \right) \\ \frac{d\Phi}{d\xi} &= \frac{1}{l_c} \left(-\Psi + \Phi_c(\frac{\Phi}{W} - 1) - \frac{3HA^2}{4W^2}(\frac{\Phi}{W} - 1) \right) \\ \frac{d\Psi}{d\xi} &= \frac{1}{4 l_c B^2} \left(\Phi - F_T^{-1}(\Psi) \right) \end{aligned} \tag{4.86}$$

where ξ is the scaled time. The compressor and throttle characteristics are

$$\begin{aligned} \Phi_c(y) &= \psi_0 + H(1 + \frac{3}{2}y - \frac{1}{2}y^3) \\ F_T^{-1}(\Psi) &= K_T \sqrt{\Psi}. \end{aligned} \tag{4.87}$$

The three states in the system are A, the scaled amplitude of the rotating stall cell; Φ, the scaled annulus averaged mass flow; Ψ, the scaled annulus averaged pressure rise. The throttle parameter is K_T. When viewed as a dynamical system, K_T is a parameter and a classical bifurcation occurs at a critical value. When viewed as a control system, K_T is the control input and a control bifurcation occurs at the same critical value. The other parameters ψ_0, H, B, α, l_c and W are constants. More details on the meaning of the variables and the parameters are discussed in [5] and [28]. We focus on the following equilibrium point for our discussion. It is actually the stall inception point of the compressor model.

$$A_0 = 0, \quad \Phi_0 = 2W, \quad \Psi_0 = \psi_0 + 2H, \quad K_{T0} = \frac{2W}{\sqrt{\psi_0 + 2H}}, \tag{4.88}$$

It is convenient to transfer the equilibrium point to the origin by the following change of coordinates

$$\begin{aligned} \Phi &= \phi + 2W, \\ \Psi &= \psi + \psi_0 + 2H, \\ K_T &= \frac{2W}{\sqrt{\psi_0 + 2H}} + u \end{aligned} \tag{4.89}$$

where, u is the new control input. The resulting system under the new coordinates (A, ϕ, ψ) has the following form

$$\begin{aligned} \frac{dA}{d\xi} &= \frac{3\alpha H}{2W} A \left(1 - (\frac{\phi}{W} + 1)^2 - (\frac{A}{2W})^2 \right), \\ \frac{d\phi}{d\xi} &= \frac{1}{l_c} \left(-\psi - \psi_0 - 2H + \Phi_c(\frac{\phi}{W} + 1) - \frac{3HA^2}{4W^2}(\frac{\phi}{W} + 1) \right), \\ \frac{d\psi}{d\xi} &= \frac{1}{4l_c B^2} \left(\phi + 2W - (\frac{2W}{\sqrt{\psi_0 + 2H}} + u)\sqrt{\psi + \psi_0 + 2H} \right) \end{aligned} \tag{4.90}$$

It is equivalent to

$$\begin{aligned} \frac{dA}{d\xi} &= \frac{3\alpha H}{2W} A \left(-\frac{\phi^2}{W^2} - \frac{2\phi}{W} - \frac{A^2}{4W^2} \right), \\ \frac{d\phi}{d\xi} &= \frac{1}{l_c}(-\psi - \frac{3H}{2W^2}\phi^2 - \frac{3H}{4W^2}A^2 - \frac{H}{2W^3}\phi^3 - \frac{3H}{4W^3}A^2\phi), \\ \frac{d\psi}{d\xi} &= \frac{1}{4l_c B^2} \left(\phi + 2W - \frac{2W}{\sqrt{\psi_0 + 2H}}\sqrt{\psi + \psi_0 + 2H} + \sqrt{\psi + \psi_0 + 2H}u \right) \end{aligned} \tag{4.91}$$

The variables ψ and ϕ constitute the linearly controllable part. The normal form of the controllable part can be obtained by pushing down. Let

$$x_{0,1} = A, \quad x_{1,1} = \phi, \quad x_{1,2} = \frac{1}{l_c}\left(-\psi - \frac{3H}{2W^2}\phi^2 - \frac{3H}{4W^2}A^2 - \frac{H}{2W^3}\phi^3 - \frac{3H}{4W^3}A^2\phi \right). \tag{4.92}$$

The resulting system is

$$\begin{aligned}\frac{dx_{0,1}}{d\xi} &= -\frac{3\alpha H}{W^2}\left(x_{0,1}x_{1,1} + \frac{1}{8W}x_{0,1}^3 + \frac{1}{2W}x_{0,1}x_{1,1}^2\right)\\ \frac{dx_{1,1}}{d\xi} &= x_{1,2}\\ \frac{dx_{1,2}}{d\xi} &= a(x_{0,1},x_{1,1},x_{1,2}) + b(x_{0,1},x_{1,1},x_{1,2})u\end{aligned} \tag{4.93}$$

where $a(x_{0,1},x_{1,1},x_{1,2}) + b(x_{0,1},x_{1,1},x_{1,2})u$ is defined by

$$\begin{aligned}\frac{dx_{1,2}}{d\xi} &= \frac{1}{l_c}(-\frac{d\psi}{d\xi} - \frac{3H}{W^2}\phi\frac{d\phi}{d\xi} - \frac{3H}{2W^2}A\frac{dA}{d\xi}\\ &\quad -3\frac{H}{2W^3}\phi^2\frac{d\phi}{d\xi} - \frac{3H}{4W^3}(2A\phi\frac{dA}{d\xi} + A^2\frac{d\phi}{d\xi}))\end{aligned} \tag{4.94}$$

If we define the new control input by

$$v = a(x_{0,1},x_{1,1},x_{1,2}) + b(x_{0,1},x_{1,1},x_{1,2})u \tag{4.95}$$

then we have

$$\begin{aligned}\frac{dx_{0,1}}{d\xi} &= -\frac{3\alpha H}{W^2}\left(x_{0,1}x_{1,1} + \frac{1}{8W}x_{0,1}^3 + \frac{1}{2W}x_{0,1}x_{1,1}^2\right)\\ \frac{dx_{1,1}}{d\xi} &= x_{1,2}\\ \frac{dx_{1,2}}{d\xi} &= v\end{aligned} \tag{4.96}$$

In this system, the controllable part is in normal form. The dynamics of $x_{0,1}$ is not linearly controllable. However, this equation is already in its normal form. So, (4.96) is the normal form of the engine compressor model (4.86). Although the feedback (4.95) is complicated, only the linear and quadratic parts of a and b are critical to the bifurcations of the system ([19]). Their linear and quadratic Taylor expansions are

$$\begin{aligned}a(x_{0,1},x_{1,1},x_{1,2}) &= -\frac{1}{4l_c^2B^2}x_{1,1} - \frac{W}{4l_c^2B^2(\psi_0+2H)}x_{1,2}\\ &\quad -\frac{3H}{16l_c^2B^2W(\psi_0+2H)}x_{0,1}^2 - \frac{3H}{8l_c^2B^2W(\psi_0+2H)}x_{1,1}^2\\ &\quad -\frac{3H}{l_cW^2}x_{1,1}x_{1,2} - \frac{W}{16B^2(\psi_0+2H)^2}x_{1,2}^2\\ &\quad +O(x)^3,\\ b(x_{0,1},x_{1,1},x_{1,2}) &= \frac{\sqrt{\psi_0+2H}}{4l_c^2B^2} - \frac{1}{8l_cB^2\sqrt{\psi_0+2H}}x_{1,2} + O(x)^2.\end{aligned} \tag{4.97}$$

4.4.3 Controlled Lorenz Equation

It is known that circuit systems can be designed to approximate chaotic behavior such as the one exhibited by the Lorenz system. In [36] and [27], the following controlled Lorenz equation is studied,

$$\begin{array}{rcl} \dot{x} & = & a(y-x) \\ \dot{y} & = & cx - xz - y + u \\ \dot{z} & = & xy - bz \end{array} \tag{4.98}$$

where a, b, and c are constant numbers. It is shown in [36] and [27] that several state feedbacks exist under which the closed-loop system exhibit at least three fundamentally different chaos. In the following, we use a globally invertible transformation to derive the normal form of (4.98). As a result, the entire family of control systems with the same normal form has chaotic trajectories equivalent to those found in [36] and [27].

The transformation is simple

$$\begin{array}{rcl} x_1 & = & x \\ x_2 & = & a(y-x) \\ x_0 & = & z - \dfrac{1}{2a}x^2 \\ v & = & a(cx - xz - y - ay + ax + u) \end{array} \tag{4.99}$$

Its inverse transformation is defined as follows

$$\begin{array}{rcl} x & = & x_1 \\ y & = & x_1 + \dfrac{1}{a}x_2 \\ z & = & x_0 + \dfrac{1}{2a}x_1^2 \\ u & = & (1-c)x_1 + (1+\dfrac{1}{a})x_2 + x_1x_0 + \dfrac{1}{2a}x_1^3 + \dfrac{1}{a}v \end{array} \tag{4.100}$$

In (4.99), x is the same as x_1. The second equation in (4.99) is a push down. The transformation of x_0 is a pull up to cancel the term $\frac{1}{a}x_1x_2$ in the equation of $\dot{x}_0$. Under this transformation, it is easy to check

$$\begin{array}{rcl} \dot{x}_0 & = & -bx_0 + (1 - \dfrac{b}{2a})x_1^2 \\ \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & v \end{array} \tag{4.101}$$

It is in normal form, with only one nonzero invariant, the coefficient of x_1^2. If $b \neq 0$ and $2a$, the equilibrium set of the system is a parabola. The system is linearly controllable at all its equilibrium points except for the origin. So, local control of such a system is relatively simple. However, its global behavior needs further study due to the chaotic behavior under certain state feedbacks.

4.5 Conclusions

In this paper, normal forms of single input control systems are summarized. The system is nonlinear and the input is non-affine. The family of systems addressed in this paper is the most general one relative to existing published normal forms of single input systems based on a similar approach. In addition, examples of normal forms are shown to illustrate the elementary transformation of push up and pull down in the derivation of normal forms. Due to page limitation, applications of the normal forms are not addressed in the paper. However, interested readers are referred to the related publications in the references for results on bifurcation control, invariants, symmetries, and practical stabilization of nonlinear systems based on normal form approach.

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