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# Minimizing the number of penetrations in a boundary defense problem (U)

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MINIMIZING THE NUMBER OF PENETRATIONS  
IN A BOUNDARY DEFENSE PROBLEM (U)

by

Gilbert T. Howard

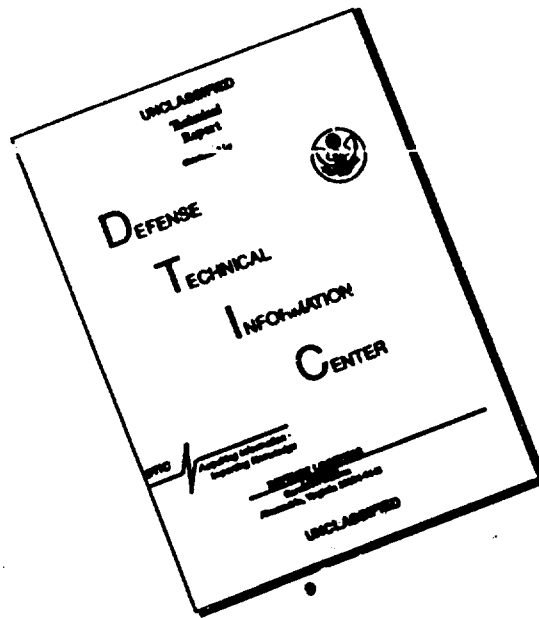
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Provost

ABSTRACT:

We consider the problem of a defender guarding a boundary from attack by a fixed number of invaders who are approaching the boundary. His objective is to intercept the maximum number of invaders before they cross the boundary. The defender is not required to remain on the boundary but he must investigate the contacts in first-come first-served order. Weights may also be assigned to each invader to reflect the value to the defender of intercepting that invader. A dynamic programming formulation is given. The multiple defender problem is also considered, and several other generalizations are discussed. Examples are included.

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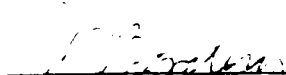
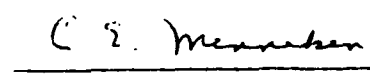
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## Executive Summary

This report deals with the problem of defending a region from penetration by a number of infiltrators who are approaching the boundary of the region. The defender is assumed to be capable of defeating each intruder which he engages. The decision problem which he faces has to do with the order in which the intruders should be engaged. The intruders are approaching from different directions at different speeds and the defender must account for the travel times between engagements. Many potential applications of the results are known, including: the defense of a coastline from infiltration as in South Vietnam, the targeting problem faced by a ship under air attack, and submarine barrier problems. The methodology employed is dynamic programming. The solution obtained is in some cases simple enough for implementation by hand calculation. In other cases implementation would require computer facilities. Examples are included, and numerous generalizations are discussed including problems with more than one defender.



## 1. Introduction

### a. Investigation Theory

This report deals with problems in the area of investigation theory. Investigation theory refers to problems in which there are a number of contacts (infiltrators, attackers, intruders, jobs) which must be investigated (classified, identified, searched, engaged in combat, or otherwise processed) by the investigator, or investigators who know the position, course, and speed of each contact. The problem involves the sequence in which the contacts should be investigated.

Several objectives are possible. The investigator may wish to travel the minimum total distance in investigating all the contacts, he may desire to investigate as many contacts as possible in some fixed time; or he may seek to engage as many as possible before they reach some prescribed region. It is this latter objective with which the majority of this report is concerned.

The term "investigation theory" was apparently originated by J. A. Neuendorffer [6] in 1961. The problem originally posed by Neuendorffer dealt with stationary points randomly located according to a two dimensional uniform density in a rectangular or circular region. Starting from some fixed point the sequence was sought which minimized the total path length traveled while investigating each contact. The sequence which ignored some

contacts while maximizing the rate of investigation was also sought. Furthermore, it was desired to compare to the two sequences just mentioned the policy of always going to the closest uninvestigated contact. Hence one problem of interest was to find a sequence which could easily be determined and implemented and which provided a nearly optimal solution.

Early efforts apparently centered around the similarity of this problem to the traveling salesman problem [7]. The similarity of these problems is evident when we require that a single investigator travel from his initial position visiting each of  $N$  stationary contacts exactly once and return to his initial position while traveling the minimum distance. The problem is only slightly modified if we do not require the investigator to return home. The results from the traveling salesman problem, particularly the branch and bound solution procedures, can be used to solve these problems for less than about 75 contacts. For a discussion of results in the traveling salesman problem see [2].

The problems that we are interested in will generally involve moving contacts, and in these problems much of the similarity to the traveling salesman problem is lost. For example, in traveling salesman problems where the distances satisfy the triangle inequality, the minimum distance tour need not cross itself. This result no longer holds for the problem where the

investigator seeks to investigate a number of moving contacts while traveling the minimum distance even if the contacts each travel with constant velocity.

We will first describe some typical problems in investigation theory, then discuss the general features which any such problem possesses.

b. Typical Problems in Investigation Theory

The first typical problem in investigation theory comes from the Markettime Operation in South Vietnam. A section of the coastline is to be guarded from infiltration by North Vietnamese fishing boats which frequently try to unload arms and ammunition in remote spots along the South Vietnamese coast. From a distance the infiltrators are indistinguishable from legitimate fishing boats, hence all unidentified boats in the area are regarded as potential infiltrators.

The area along the coast is divided into several patrol areas, each containing a patrol boat. All traffic in the coastal area is under surveillance from a centrally located radar station which is in communication with the patrol boats. The basic problem which arises in this situation is "in what order should the patrol boats pursue the contacts in their area?"

The objective is to minimize the number of boats that reach the shore without being investigated. Some boats are traveling on a course which takes them through more than one patrol area. In

these cases there is the additional problem of determining which patrol boat should be assigned to the contact. A similar problem arises in establishing a submarine barrier.

A second problem of this type arises in the defense of a ship against an air attack. We will suppose that the appropriate defense against the attackers involves a weapon system which can engage a single contact at a time. If we assume that attackers are approaching the ship from different directions and at different speeds, the fire control system must determine the sequence in which the targets are engaged. The sequence is crucial since the weapon system takes different amounts of time to process the attackers and different amounts of time to shift from one to another.

A third problem in this area involves a number of jobs with different due dates  $d_i$ . The jobs require different amounts of processing time  $p_i$ , and the set-up time for job  $j$  given that job  $i$  was just completed is  $t_{ij}$ . The processing times can be added to the set-up times to convert the problem into one with variable set-up times  $s_{ij} = t_{ij} + p_j$  and zero processing times. In this case the machine (or job shop or investigator) which does the processing must move from job to job in that sequence which maximizes the number of jobs processed before they reach their due dates.

Scheduling problems similar to this are discussed by Moore [4]. In his problem the set-up times depend only on the job at hand and not on the preceding job. Hence the set-up times are constant, not variable. Moore solves that problem using a lemma by Jackson [3] which has the effect of ordering the jobs by their due dates.

c. Classification

It is difficult to devise a general classification scheme for investigation theory problems but the essence of such a problem can be communicated by describing the behavior of each of the contacts and the investigator and by specifying the objective. Summarized below are several considerations relating to each of these problem elements.

Contacts: (a) Are the contacts moving or stationary; and if they are moving, do they all move with the same velocity or different velocities? Are the velocities changing with time? (b) Are the set-up times sequence dependent or constant, i.e.  $s_{ij}$  or  $s_i$ ? (c) What are the processing times?

Investigator: (a) How many investigators are there? (b) What are the limitations on the motion of the investigators? Can they move freely or are they constrained?

Objective: (a) Maximize the number of contacts investigated in a fixed time. (b) Minimize the time required to investigate all contacts. (c) Minimize the distance required to investigate all contacts. (d) Maximize the number of contacts investigated before they reach some specified region.

For our discussions we assume that the contacts move independently and in complete disregard of the action of the investigator. This eliminates from our consideration any problems in the area of game theory. Certainly there is an interest in problems where this does not hold, but the subject of pursuit and evasion games is not within the scope of this study. We further assume that if there is more than one investigator, they are under the control of a single decision maker.

## 2. Analysis of Basic Problem

### a. Basic Problem Description

In the simplest problem we consider there are  $N$  points  $(x_i, y_i)$ ,  $i = 1, \dots, N$  each moving at a constant velocity  $v$  directly toward the x-axis. A portion of the x-axis, which we take to be  $[0, b]$ , is designated as the boundary. A single defender initially at  $b_0$  can move along the boundary in either direction at maximum speed of  $v_0$  and can change direction as often as he wishes. His problem is to intercept as many of the intruders (moving points) as possible before they cross the boundary. Figure 1 pictures a boundary defense problem of this type.

### b. Dynamic Programming Formulation

Our problem can be formulated as an  $N$  stage dynamic programming problem [5]. We let stage  $n$  correspond to the occurrence of the  $n + 1^{\text{st}}$  intruder crossing the boundary. The intruders are numbered such that  $n + 1$  crosses the boundary before  $n$ . Stage  $N$  corresponds to the beginning of the problem. If  $k$  intruders reach the boundary simultaneously they should all be given the same number and the problem reduces to an  $N - k + 1$  stage problem. The analysis which follows also deals with this case. We let  $X_n$ , the state variable, correspond to the position of the defender at stage  $n$ , and let  $f_n(X_n)$  be the maximum number of intruders that can be intercepted out

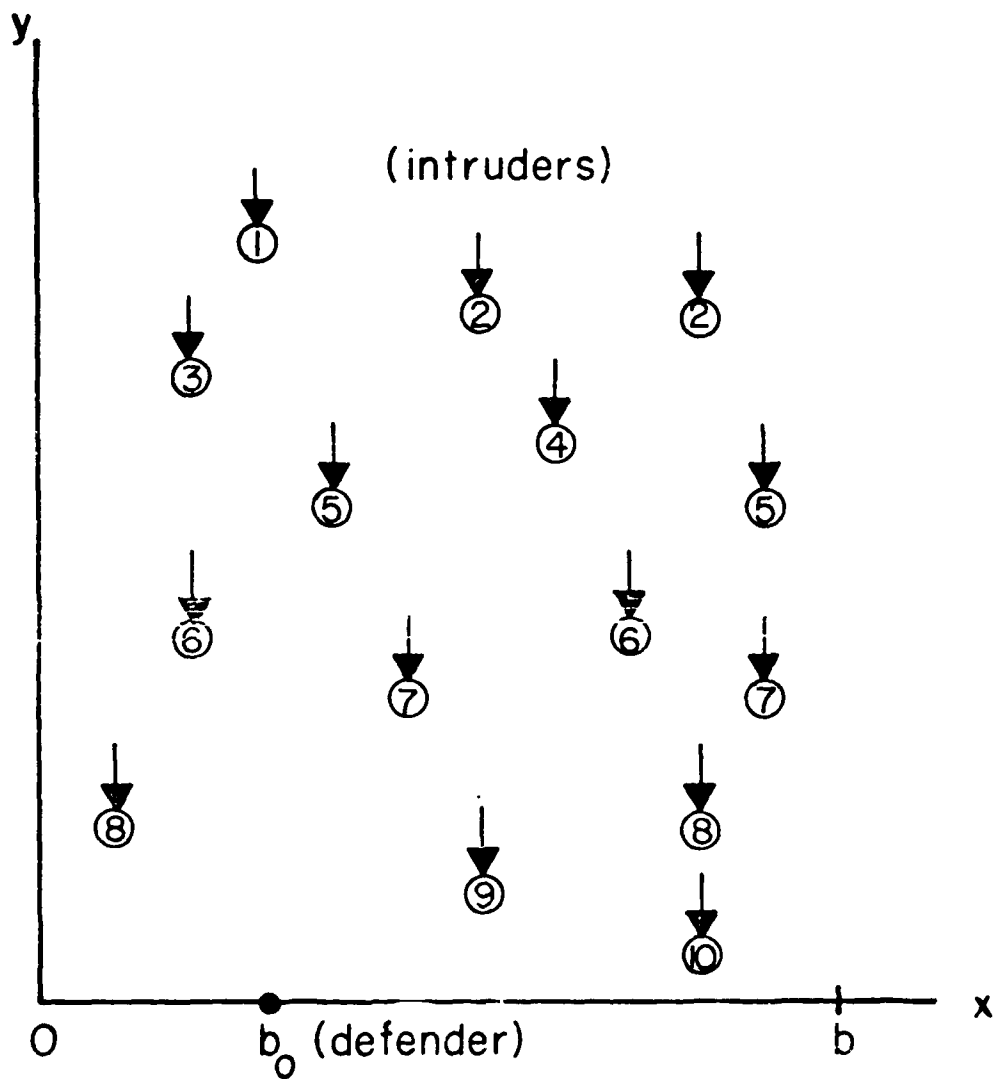


Figure 1. A boundary defense problem



of those remaining to reach the boundary. The decision variable  $d_n$  determines the position  $X_{n-1}$  to which the defender moves next. The stage return  $r_n(X_n, d_n)$  is one if you intercept contact  $n$  and zero otherwise. We let  $t'_n$  be the time at which intruder  $n$  reaches the boundary and define  $t_n = t'_n - t'_{n+1}$ .

The recursive equations are

$$f_1(X_1) = \max_{d_1} r_1(X_1, d_1)$$

and

$$f_n(X_n) = \max_{d_n} \{r_n(X_n, d_n) + f_{n-1}(X_{n-1})\}$$

where

$$X_{n-1} = X_n + d_n$$

and

$$-v_0 t_n \leq d_n \leq v_0 t_n$$

and

$$-X_n \leq d_n \leq b - X_n.$$

The first constraint on  $d_n$  simply limits the maximum distance which the defender can move during time  $t_n$ . The second constraint simply requires that the intruder remain in the interval  $[0, b]$ .

We note that the return function is

$$r_n(X_n, d_n) = \begin{cases} 1, & X_n + d_n = x_n \\ 0, & X_n + d_n \neq x_n \end{cases}$$

In the case where several intruders cross the boundary simultaneously at stage  $n-1$ ,  $r_n = 1$  if  $X_n + d_n = x_n$  for  $x_n$  corresponding to any intruder.

These recursive equations complete the dynamic programming formulation.

#### c. Graphical Solution and Example

The recursive equations just given can easily be solved to yield the optimal policy for the defender. However, they can easily be solved graphically as we will show for this case in which the defender must remain on the boundary.

At stage  $n$ , given any value of the state variable  $X_n$ , the question of whether or not we can intercept the next intruder is easily resolved by determining if  $X_n$  is contained in a certain cone whose vertex is at the position of the intruder. See Figure 2. The angle covered by the cone is determined by the quantities  $t_n$  and  $v_0$ .

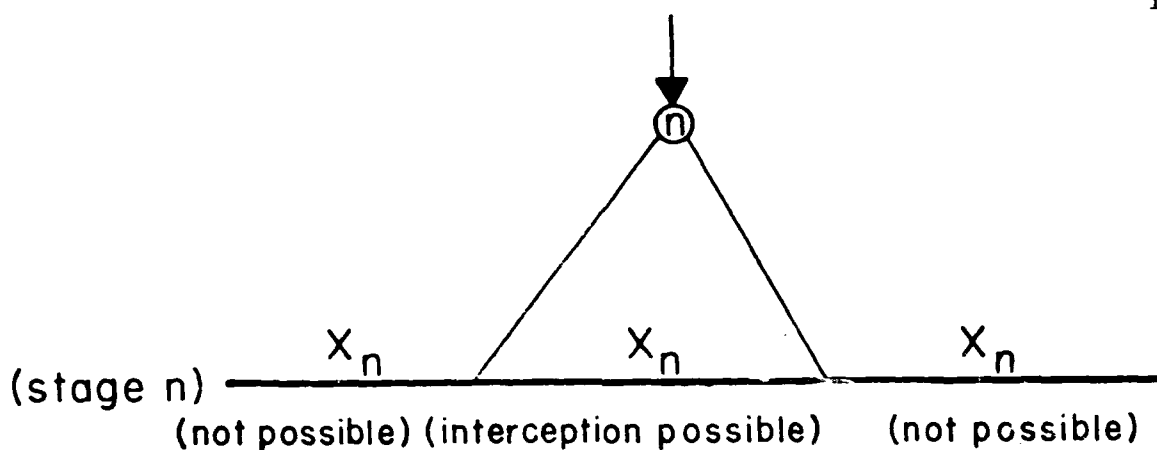


Figure 2. Cone of interception for a single intruder

Figure 2.

Of course, even if intercepting the contact is possible, it is not always optimal to do so since it may put the defender in a disadvantageous position for the remaining intruders. This is reflected by  $f_{n-1}(X_{n-1})$  and this function can easily be recorded for each point  $X_{n-1}$  on the (transposed) boundary passing through the intruder  $n$ . Thus, the graphical recursive solution simply requires that we draw the boundary through each contact and record (recursively) for each stage  $2, \dots, N$  the quantity  $f_n(X_n)$  which gives the maximum remaining number of interceptions possible. The example pictured in Figure 1 is solved below in Figure 3 using  $v = 1$  and  $v_0 = 2$ . The numbers on the horizontal lines give the values of  $f_n(X_n)$ . The optimal solution shows that the defender can at best intercept six contacts.

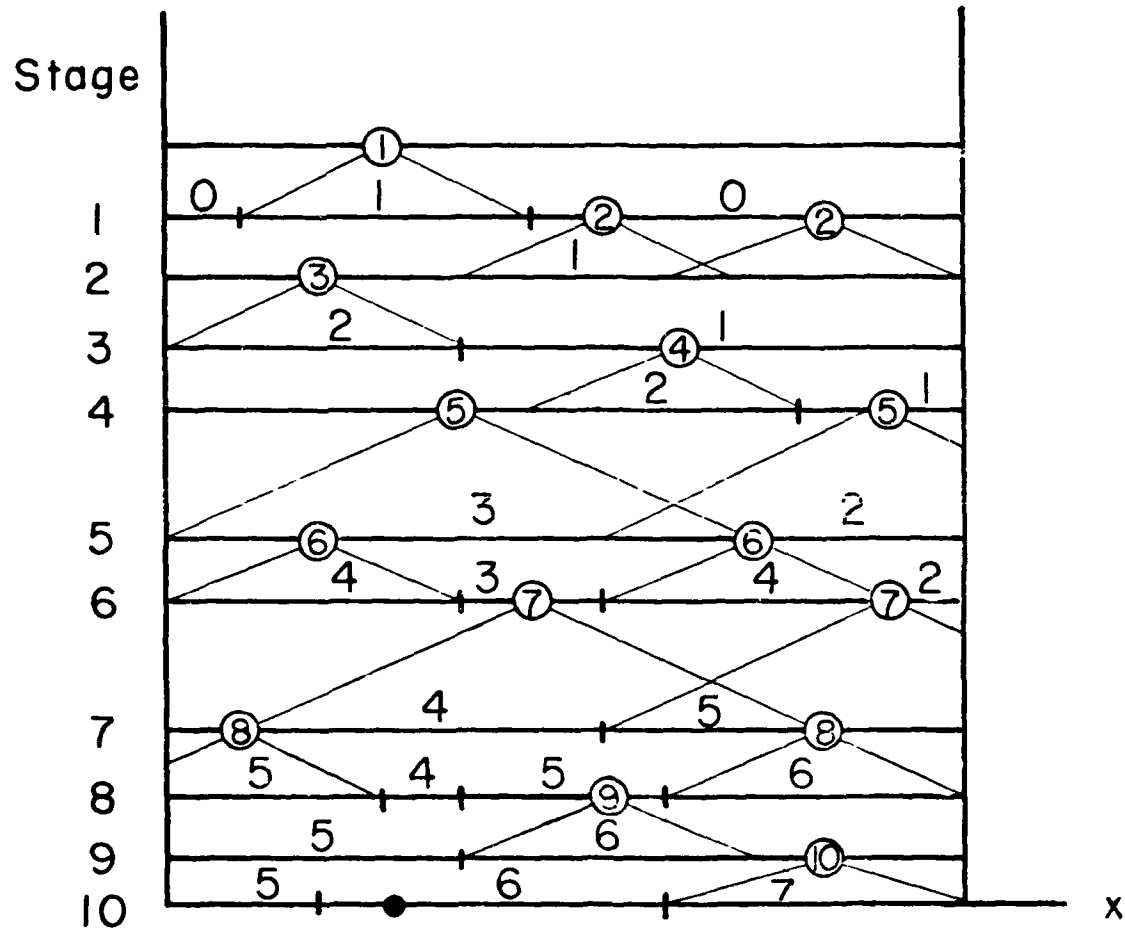


Figure 3. Graphical solution to example problem

### 3. Generalizations

The dynamic programming solution method presented for the basic problem with a single defender is computationally simple. The method can easily be extended, but with increased computational difficulty, to obtain the solution in more complex situations. In this section we describe some of these generalizations.

#### a. Multiple Defender Problems

The formulation given in this paper is also applicable to the problem of scheduling  $M$  defenders against  $N$  intruders to maximize the number of intruders intercepted. We discuss first the case in which the defenders must all stay on the boundary and can be permitted to cross one another or not as desired. Their maximum speeds are permitted to differ. To handle this generalization we simply interpret the state variable as an  $M$ -vector whose components give the positions of each defender at the time an intruder reaches the boundary. This is easily visualized, and easily computed, for the case  $M = 2$ . The state space simply becomes a portion of the plane. Regions in the space are recursively labeled to indicate the maximum number of remaining contacts which can be intercepted from that point in the space. An example is presented below for  $N = 5$ ,  $M = 2$ .

Conceptually the approach is the same when there are more defenders than two. The only difficulty is computational as  $M$  increases, but because of the relatively simple structure of the return functions the optimal solutions can easily be computed, although not by hand, for problems having three or four investigators.

When we regard the state variable  $\bar{X}$  as a vector, other slightly different interpretations of the problem are possible. Returning to the single investigator problem we can permit the boundary to be  $M$  dimensional as would be the case when the defender is guarding a portion of a plane. See Figure 4. In this case, the components of  $\bar{X}$  are simply the position of the defender on the plane.

Likewise we can interpret some components of  $\bar{X}$  to be descriptions of the physical condition of the defender. For example; we can permit the investigation of a contact to change the maximum speed at which the investigator can travel. This could be used for the case in which investigation of some contact is a dangerous operation and results in damage to the investigator. In fact, the occurrence or non-occurrence of damage could be permitted to be a random event.

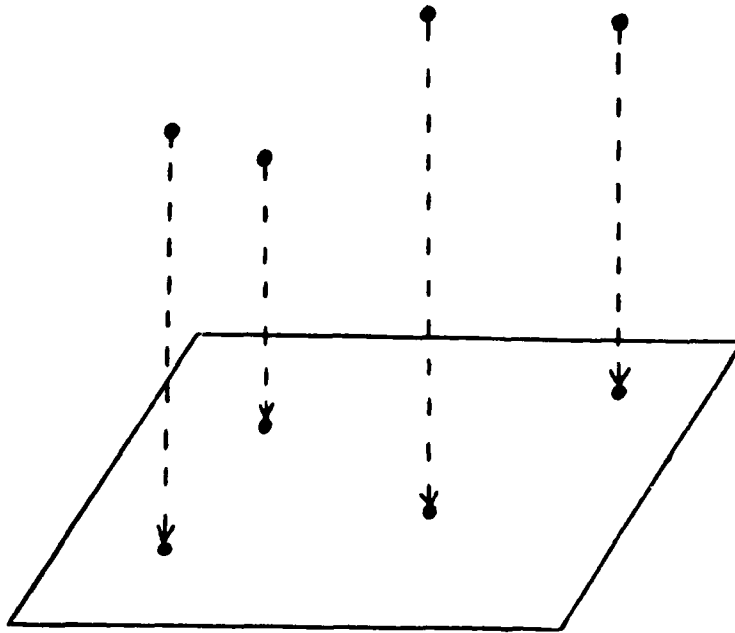


Figure 4. A two dimensional boundary defense problem

For the problem in which the defender must remain on the boundary the solution procedure can easily be modified to include several other generalizations in addition to interpreting the state variable as a vector. For example, each contact can move with a different heading and speed. The speed will affect the width of the cone and the heading will affect its projection on the boundary. In fact, it doesn't matter where the contact begins or what path it follows to reach the boundary. It can move in any manner at all as long as its point and time of crossing are known. For the cases where the contacts are approaching the boundary at different speeds, it is necessary to order the stages (number the contacts) so that they arrive in the order  $N, N-1, \dots, 1$ .

We also note that weights or priorities  $W_n$  can be assigned to each intruder to reflect the importance of intercepting him. This would be done by letting  $r_n(X_n, d_n) = W_n$  if interception is made and zero otherwise. The problem would remain one of maximizing the  $W_n$ 's summed over those contacts which are intercepted.

In another generalization we permit interception to occur if the defender is within a distance  $d$  of the intruder when he reaches boundary. It is also possible to solve the problem for the case in which each intruder remains on the boundary for some finite time before penetrating. The case in which a finite processing time is required for each intruder is also easily handled. Likewise we can easily deal with the cases where the defenders are given different maximum speeds, perhaps zero, in moving left or right.

b. Example and Solution

In this section we present and solve a problem having  $M = 2$  investigators and  $N = 5$  contacts. Table I gives the intruder data. Defender 1 has a speed of 1 unit per unit time and defender 2 has a speed of 2.

$n$	$x_n$	$t'_n$	$t_n$
1	0	8	1
2	3	7	1
3	6	6	3
4	10	3	1
5	5	2	2

Table I. Intruder data

for sample problem with  $M = 2$ .

Figures 5a through 5f show the original problem and the functions  $f_j(\bar{X}_j)$ ,  $j = 1, \dots, 5$  where  $X_{ij}$ ,  $i = 1, 2$  is the position of defender  $i$  at stage  $j$ .



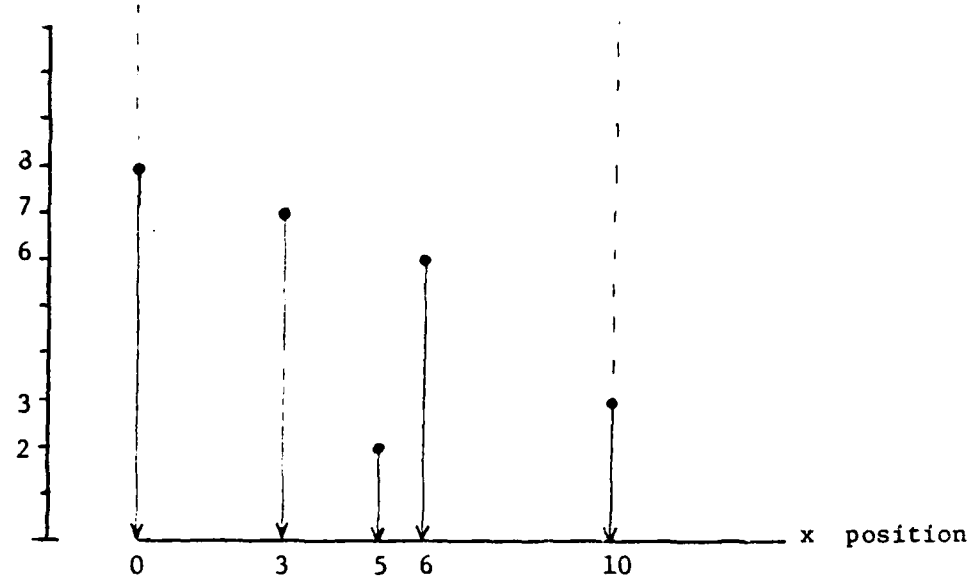


Figure (5a). Problem data for example with  $M = 2$

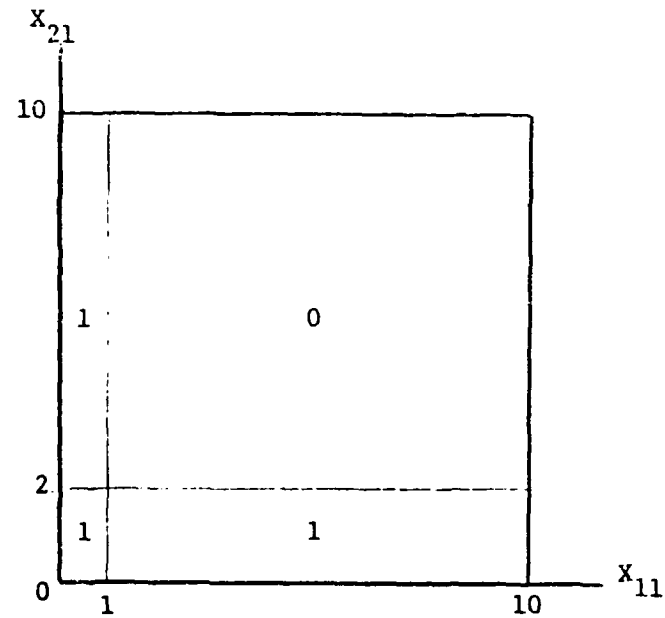


Figure (5a).  $f_1(\bar{X}_1)$  for example problem

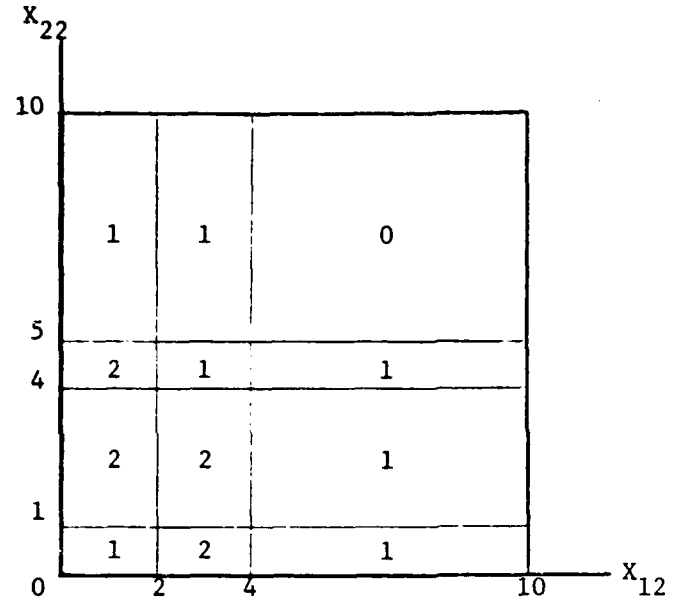


Figure (5).  $f_2(\bar{x}_2)$  for example problem

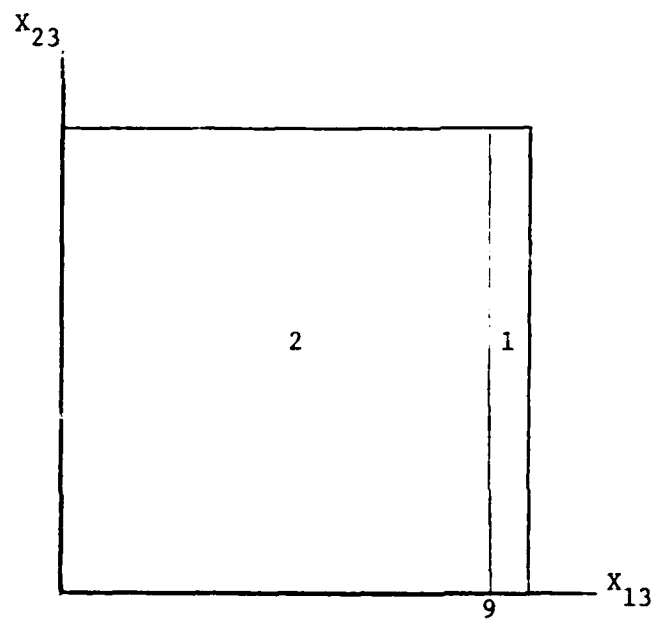


Figure (5).  $f_3(\bar{x}_3)$  for example problem

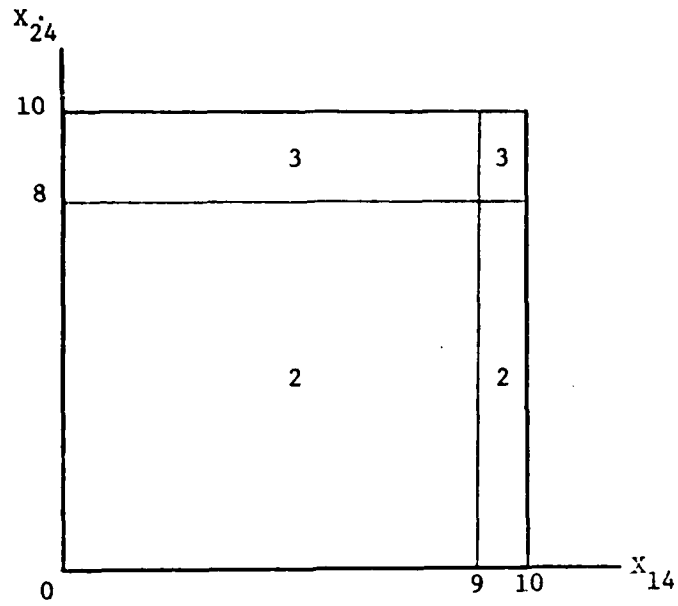


Figure (5).  $f_4(\bar{X}_4)$  for example problem

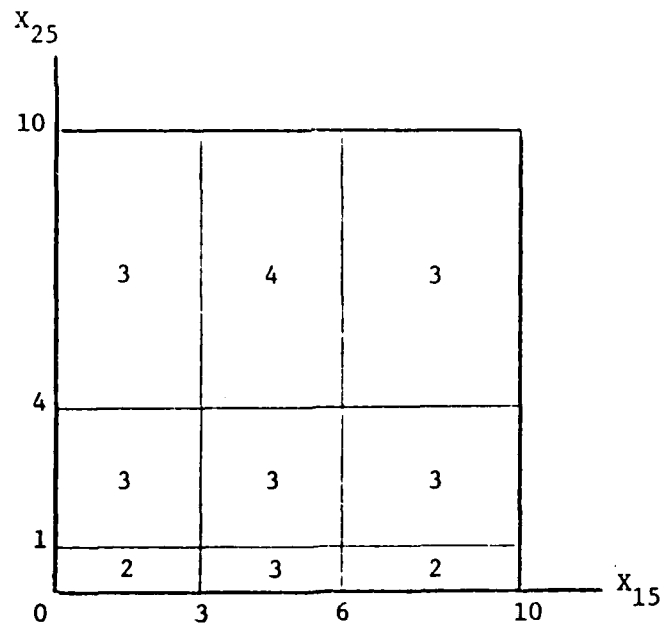


Figure (5).  $f_5(\bar{X}_5)$  for example problem

We note from function  $f_5(\bar{X}_5)$  that the two defenders can at best intercept four intruders and this can occur only if defender one has a starting position between three and six and defender two has a starting position to the right of four. In the worst case, corresponding to the regions labeled with a two, the defenders will be able to intercept only two intruders.

c. Single Investigator Off the Boundary

We now discuss the case in which the defender is permitted to move off the boundary. For ease of discussion we will consider only the single investigator problem. In this case we will assume that the defender is permitted to move in some region  $R$  in any direction he chooses at a maximum speed on  $v_0$ . There are  $n$  contacts in the region, contact  $i$  starting at position  $p_i$  and moving with a constant velocity of  $v_i \leq v_0$  directly toward his objective. The investigator must process the contacts in some specified order deleting some contacts as necessary to minimize the number who reach their objective without being investigated. All  $n$  contacts may have the same objective but need not. It is most convenient to think of the region  $R$  as a rectangular region in  $E^2$  with all the contacts moving directly toward one of the sides of the region, but more generality is permitted.

To solve this problem by dynamic programming the state variable  $\bar{X}$  must have  $m + 1$  components where  $m$  is the dimension of the region  $R$ . Each additional investigator increases the dimension of  $\bar{X}$  by  $m$ , consequently dynamic programming becomes impractical very rapidly. The first  $m$  components give the position of the defender

and the other gives the time. Stage  $n$  is again identified by the  $n + 1$  <sup>st</sup> contacts reaching his objective. The decision is which contact to investigate next. The desired point of contact would be the feasible point occurring first in time. This is not necessarily true however if  $v_i > v_o$  for some  $i$ . The solution by dynamic programming would require imposing an  $m$  dimensional grid over the region  $R$  and then performing the standard dynamic programming calculations.

A special case of this problem for which a particularly simple solution procedure exists is the case in which  $v_i = v$  for all  $i$ . In this case a simple computational procedure has been developed by Balut [1].

#### 4. Summary and Recommendations for Future Work

The results presented in this paper provide a means of solving the basic investigation theory problem in which the intruders all move with velocity  $v$  and the defender is required to remain on the boundary. Several generalizations to this basic problem were considered including the generalization to  $M$  defenders.

The problem in which the investigator can move off the boundary and the contacts all move toward different objectives with different velocities can also be solved by dynamic programming to minimize the number who reach their objective without being investigated, provided that the investigator must process the contact in some specified order.

A valuable addition to this work from a practical point of view would be the development of heuristic solution procedures which could easily be implemented and which would provide nearly optimal solutions. This need is not great for the basic problem with a single investigator who is required to remain on the boundary, since the solution developed here is easily computed, but it is pronounced for the multiple defender problems where the computational burden is increased and for the problem in which the defender is permitted to move off the boundary.

Solutions are also needed for the problem in which the investigator is free to move off the boundary and pursue the contacts in any order desired. This is expected to be a very difficult

problem, and since no proven means is available for computing the optimal investigation sequence, there is no standard for comparison of heuristic procedures which might be developed. A branch and bound algorithm has been developed for computing the optimal sequence in this problem but its efficiency has not been proven.

Another area of possible future work involves consideration of other objectives. This work has dealt exclusively with the objective of minimizing the number of contacts who reach their goal without being investigated. The "jailbreak" problem has a different objective. It involves  $N$  contacts with initial positions  $p_i$ ,  $i = 1, \dots, n$  and constant velocities  $v_i$ ,  $i = 1, \dots, n$ . The investigator desires to investigate all the contacts either in minimum time or while traveling the minimum distance.

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