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Borg, S.F.

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ADDITIONAL INTERPRETATIONS OF THE SOLUTION OF THE STRAIGHT BEAM DIFFERENTIAL EQUATION

BY

S. F. BORG [1]

INTRODUCTION

The fundamental differential equation of the transversely loaded straight beam, the Bernoulli-Euler equation, has been subject to various interpretive solutions. Perhaps the two best known solutions of this kind are the moment-area and conjugate beam methods which solve the equation with the aid of certain properties of the $\frac{M}{EI}$ curves for the beam in question.

The present paper applies two separate mathematical methods to the solution of this equation. The first method makes use of the Green's function and obtains a solution for the beam built-in both ends. The second method utilizes the so-called "superposition theorem" which is frequently applied to problems involving transient phenonema such as those encountered in electrical network and vibration problems, and obtains a solution valid for any type of support.

GREEN'S FUNCTION

Green's functions have been used to solve the classical vibrating beam problem (1).[2] In addition, Bateman (2) obtained Green's functions for the deflections of constant moment of inertia beams subject to various end conditions and pointed out the reciprocal theorem consequences and influence line applications that follow therefrom. More recently, Shaw (3), in solving the Fredholm integral equation illustrated a Green's function-relaxation solution for the simply supported beam-column. The present paper develops a solution for the beam of variable moment of inertia built-in both ends.

Given a non-homogeneous differential equation with boundary conditions (4)

$$\begin{cases} L(y) = r(x) \\ U_i(y) = 0, (i = 1,2, \cdots n) \end{cases} \tag{1}$$

then, if the Green's function for the homogeneous system

$$\begin{cases} L(u) = 0 \\ U_i(u) = 0, (i = 1,2, \cdots n) \end{cases} \tag{2}$$

[1] Assistant Professor of Aeronautics, U. S. Naval Postgraduate School, Annapolis, Md.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

exists and is known, the solution to system (1) is completely given by the integral equation

$$y(x) = \int_a^b G(x, \xi) r(\xi) d\xi,$$

in which $G(x, \xi)$ is the appropriate Green's function.

Thus, if we consider the simply supported beam, the system corresponding to (1) is

$$\begin{cases} \dfrac{d^2y}{dx^2} = \dfrac{M(x)}{EI(x)}, \\ y(0) = 0, \\ y(l) = 0. \end{cases} \tag{3}$$

The Green's function for this system is (5)

$$\begin{aligned} G(x, \xi) &= \frac{x}{l}(\xi - l) \qquad x \leq \xi \\ G(x, \xi) &= \frac{\xi}{l}(x - l) \qquad x \geq \xi \end{aligned} \tag{4}$$

and the solution is, therefore, given as

$$y(x) = \int_0^l G(x, \xi) \frac{M(\xi)}{EI(\xi)} d\xi. \tag{5}$$

Equation 5 will be recognized as the familiar Maxwell-Mohr unit dummy load equation

$$y(x) = \int_0^l m(x, \xi) \frac{M(\xi)}{EI(\xi)} d\xi,$$

in which $m(x, \xi)$ is the moment at ξ due to a 1-lb. load at x.

An interesting solution is obtained for the system

$$\begin{cases} \dfrac{d^2y}{dx^2} = \dfrac{M(x)}{EI(x)}, \\ y(0) = 0, \\ y'(l) = 0. \end{cases} \tag{6}$$

Collatz (6) lists the Green's function for this case as

$$\begin{aligned} G(x, \xi) &= x \qquad x \leq \xi \\ G(x, \xi) &= \xi \qquad x \geq \xi \end{aligned}$$

and consequently the solution is

$$y(x) = \int_0^x \xi \frac{M(\xi)}{EI(\xi)} d\xi + \int_x^l x \frac{M(\xi)}{EI(\xi)} d\xi.$$

We may interpret this result as follows in line with the Maxwell-Mohr solution: Given any beam with boundary conditions

$$y(0) = 0,$$
$$y'(l) = 0,$$

then applying the Maxwell-Mohr relation

$$y(x) = \int_0^l m(x, \xi) \frac{M(\xi)}{EI(\xi)} d\xi,$$

where m must now be obtained for the beam loaded and supported as shown in Fig. 1. Recalling that in the Maxwell-Mohr derivation only the unit load does external work in moving through the beam deflection, it is apparent that physical requirements of statics as well as the assumed work requirements are all satisfied by the given Green's function solution which is obtained almost entirely by mathematical reasoning.

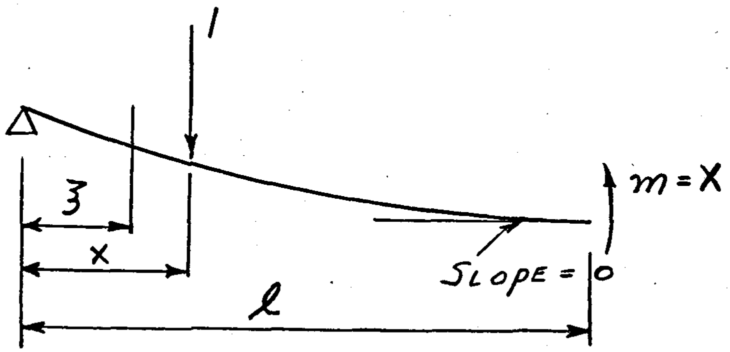

FIG. 1.

Another form of the beam differential equation is obtained by differentiating Eq. 3, and if the beam is built-in both ends we obtain

$$\begin{cases} \dfrac{d^2\theta}{dx^2} = \dfrac{d}{dx}\left(\dfrac{M(x)}{EI(x)}\right), \\ \theta(0) = 0, \\ \theta(l) = 0, \end{cases} \tag{7}$$

in which θ represents the slope of the neutral axis of the beam.

The Green's function for (7) is also given by (4), hence

$$\theta(x) = \int_0^l G(x, \xi) \frac{d}{d\xi}\left(\frac{M(\xi)}{EI(\xi)}\right) d\xi. \tag{8}$$

Equations 8 and 5 considered together give the following reciprocal relation: Given a beam A of constant section, built-in both ends and subjected to any transverse loading. Assume a second beam, B, of same length, simply supported. Then, if the $\frac{M}{EI}$ diagram of beam B is numerically equal to the $\frac{\text{Shear}}{EI}$ diagram of beam A, it follows that the deflection at any point on beam B is numerically equal to the slope at the same point on beam A.

Returning to Eq. 8, if we consider two sections of the beam a differential distance apart, we have

$$\theta(x + \Delta x) - \theta(x) = \int_0^l \{G(x + \Delta x, \xi) - G(x, \xi)\} \frac{d}{d\xi}\left(\frac{M(\xi)}{EI(\xi)}\right) d\xi. \quad (9)$$

Then, because

$$\frac{d\theta}{dx} = \frac{M}{EI}$$

$$\frac{dM}{dx} = V$$

and from the unit moment curve analogy previously discussed, we obtain in the limit

$$\frac{M(0)}{EI(0)_+} = \int_0^l \zeta(\xi) \frac{V(\xi)}{EI(\xi)} d\xi + \int_0^l \zeta(\xi) M(\xi) \frac{d}{d\xi}\left(\frac{1}{EI(\xi)}\right) d\xi, \quad (10)$$

in which $\zeta(\xi)$ is the ordinate of the "auxiliary area triangle" shown in Fig. 2. The reason for the name auxiliary area triangle will be discussed shortly.

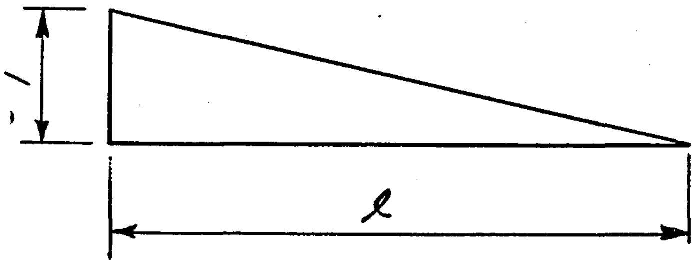

FIG. 2.

If the moment of inertia of the beam is constant, the last term in Eq. 10 disappears. If the moment of inertia varies in steps, Eq. 10 takes the form

$$\frac{M(0)}{EI(0)_+} = \int_0^l \zeta(\xi) \frac{V(\xi)}{EI(\xi)} d\xi + \Sigma\, \zeta(a) M(a) \left[\frac{1}{EI(a)_+} - \frac{1}{EI(a)_-}\right], \quad (11)$$

in which the summation is extended over all points, a, of step variation in moment of inertia, to the right of the left support.

Equation 11 is equivalent to

$$0 = \int_0^l \zeta(\xi) \frac{V(\xi)}{EI(\xi)} d\xi + \Sigma\, \zeta(a) M(a) \left[\frac{1}{EI(a)_+} - \frac{1}{EI(a)_-}\right], \quad (11a)$$

in which the summation term now includes all points of step variation in moment of inertia including the two supports, the walls being assumed of infinite moment of inertia. The signs of M and V follow the usual beam theory convention.

It may be verified that Eqs. 11 are equivalent to the first moment-area relation concerning the area of the $\frac{M}{EI}$ diagram. For this reason the auxiliary triangle is called the "auxiliary area triangle." Equations 11 are not sufficient to determine the two redundant quantities, M and V, at the support. To permit the calculation of both redundants, we note that the auxiliary area triangle is a form of operator such that its ordinate at any point, when multiplied by the appropriate shear, moment and stiffness values and summed over the entire beam, will give the area of the $\frac{M}{EI}$ curve. If these ordinates are integrated we obtain a parabola, and the ordinates of this parabola when multiplied by

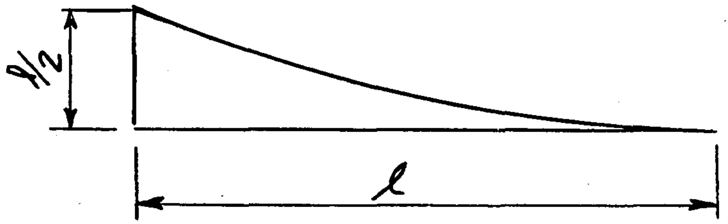

FIG. 3.

the same quantities will give an expression corresponding to the second moment area relation, that is, the moment about the right support of the $\frac{M}{EI}$ diagram.

Therefore, for the second independent relation, we have

$$0 = \int_0^l \eta(\xi) \frac{V(\xi)}{EI(\xi)} d\xi + \Sigma \eta(a) M(a) \left[\frac{1}{EI(a)_+} - \frac{1}{EI(a)_-} \right], \quad (12)$$

in which η is the ordinate of the "auxiliary moment parabola," Fig. 3, the other terms being as in Eq. 11.

THE SUPERPOSITION THEOREM

Given a constant coefficient differential equation

$$p(D)y = f \quad (13)$$

with

$$p(D) = C_0 D^n + C_1 D^{n-1} + \cdots C_n$$

then, following Murnaghan's (7) development we find

$$y(x) = f(0)_+ r_1(x) + \int_0^x r_1(x - \tau) Df(\tau) d\tau, \tag{14}$$

in which r_1 is the unit impulse "rest solution," that is, the solution of Eq. 13 corresponding to $f = 1$, and we must satisfy the initial conditions

$$y(0) = Dy(0) = \cdots = D^{n-1}y(0) = 0.$$

Equation 14 is the well known superposition theorem of electrical network and vibration theory, useful in the transient state solution.

Equation 14 is applied to the stationary beam problem as follows: Given

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI(x)}.$$

In this case $r_1(x)$ corresponds to the deflection caused by a distributed couple of value $EI(x)$ acting on a cantilever, hence

$$r_1(x) = \frac{x^2}{2}.$$

The initial conditions are satisfied if we choose as the reference line from which deflections are measured, a tangent to the deflected beam at $x = 0$, the deflection at this point being taken as zero. Then Eq. 14 takes the form

$$y(x) = \theta(0)x + \frac{M(0)x^2}{2EI(0)_+} + \int_0^x \frac{(x - \tau)^2}{2} \frac{V(\tau)}{EI(\tau)} d\tau + \int_0^x \frac{(x - \tau)^2}{2} M(\tau) D\left(\frac{1}{EI(\tau)}\right) d\tau, \tag{15}$$

$\theta(0)$ being the slope of the beam at $x = 0$.

Consider also

$$\frac{d\theta}{dx} = \frac{M(x)}{EI(x)}.$$

In this case $r_1(x)$ corresponds to the slope caused by a distributed couple of value $EI(x)$ acting on a cantilever, hence

$$r_1(x) = x.$$

The initial conditions are satisfied if we choose as the reference line, from which slopes are measured, a tangent to the deflected beam at $x = 0$. Then Eq. 15 takes the form

$$\theta(x) = \theta(0) + \frac{M(0)x}{EI(0)_+} + \int_0^x (x - \tau) \frac{V(\tau)}{EI(\tau)} d\tau + \int_0^x (x - \tau) M(\tau) D\left(\frac{1}{EI(\tau)}\right) d\tau. \tag{16}$$

If EI is constant the last terms of Eqs. 15 and 16 fall out. If EI varies in steps, Eqs. 15 and 16 take the form

$$y(x) = \theta(0)x + \int_0^x \frac{(x-\tau)^2}{2}\frac{V(\tau)}{EI(\tau)}\,d\tau + \sum \frac{(x-\alpha)^2}{2} M(\alpha)\left[\frac{1}{EI(\alpha)_+} - \frac{1}{EI(\alpha)_-}\right]. \quad (15a)$$

$$\theta(x) = \theta(0) + \int_0^x (x-\tau)\frac{V(\tau)}{EI(\tau)}\,d\tau + \sum (x-\alpha)M(\alpha)\left[\frac{1}{EI(\alpha)_+} - \frac{1}{EI(\alpha)_-}\right], \quad (16a)$$

in which α represents the distance from the left support to the points of step variation in EI, including the changes at the built-in ends, the walls in this case being assumed of infinite moment of inertia.

Equations 15 and 16 are somewhat more general than similar equations derived in a different manner by Compton and Dohrenwend (8) and may be interpreted physically as follows:

(*a*) The deflection at any point, measured from the reference line, is given by one-half the moment of inertia of the $\left\{\frac{V}{EI} + MD\left(\frac{1}{EI}\right)\right\}$ curves between the left end and the point, about the point of desired deflection. When EI changes in steps, the moment at the step, multiplied by the change in $\frac{1}{EI}$ at the step is considered as a concentrated $\frac{V}{EI}$ area.

(*b*) The slope at any point, measured from the reference tangent, is given by the moment of the $\left\{\frac{V}{EI} + MD\left(\frac{1}{EI}\right)\right\}$ curves between the left end and the point, about the point of desired slope. When EI varies in steps, the moment at the step is treated as in (*a*) above.

CONCLUSION

Two methods for solving the Bernoulli-Euler beam differential equation have been discussed. The first method used a Green's function and developed a solution for the beam built-in both ends in which the $\frac{V}{EI}$ and $MD\left(\frac{1}{EI}\right)$ diagrams, when operated on by an auxiliary area triangle and an auxiliary moment parabola gave expressions corresponding to the two moment area relations. The second method applied the superposition theorem to the problem and in this case the solution was obtained by taking appropriate moments of inertia and moments of the $\frac{V}{EI}$ and $MD\left(\frac{1}{EI}\right)$ curves.

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