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Cristi, Roberto; Feuer, Arie

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ON THE PERSISTENCY OF EXCITATION FOR SYSTEMS HAVING PURELY DETERMINISTIC DISTURBANCES USING THE RLS ALGORITHM

ROBERTO CRISTI

Electrical and Computer Engineering Department, Naval Postgraduate School, Monterey, CA 93943-5100, U.S.A.

AND

ARIE FEUER

Department of Electrical Engineering, Technion — Israel Institute of Technology, Haifa 32000, Israel

SUMMARY

A system with purely deterministic disturbances can be represented by a non-minimal model. In this paper we establish conditions on the input to the system so that when the RLS algorithm is applied to identify its parameters, convergence is guaranteed. It should be pointed out that no boundedness conditions on this input are required.

KEY WORDS System identification Recursive least squares Deterministic disturbances

INTRODUCTION

The problem of persistency of excitation for systems having purely deterministic disturbances has been introduced in Reference 1 and addressed again in Reference 2.

Consider a single-input/single-output system described by the equation

$$\tilde{A}(q^{-1})y(t) = \tilde{B}(q^{-1})u(t) + d(t) \tag{1}$$

where q^{-1} is the backward shift operator and

$$\tilde{A}(q^{-1}) = 1 + \tilde{a}_1 q^{-1} + \cdots + \tilde{a}_{\tilde{n}} q^{-\tilde{n}}$$
$$\tilde{B}(q^{-1}) = \tilde{b}_1 q^{-1} + \cdots + \tilde{b}_{\tilde{n}} q^{-\tilde{n}}$$

are relatively prime polynomials, $d(t)$ is the disturbance for which there exists a polynomial $D(q^{-1}) = 1 + d_1 q^{-1} + \cdots + d_{n_d} q^{-n_d}$ such that

$$D(q^{-1})d(t) = 0 \tag{2}$$

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We assume that n_d is the minimal degree for which (2), is true, namely

$$\tilde{D}(q^{-1})d(t) \neq 0 \text{ and does not decay exponentially to zero}^* \tag{3}$$

for any polynomial $\tilde{D}(q^{-1})$ of degree less than n_d.

Thus multiplying (1) by $D(q^{-1})$ we get

$$A(q^{-1})y(t) = B(q^{-1})u(t) \tag{4}$$

where

$$A(q^{-1}) = \tilde{A}(q^{-1})D(q^{-1}) = 1 + a_1q^{-1} + \cdots + a_nq^{-n}$$
$$B(q^{-1}) = \tilde{B}(q^{-1})D(q^{-1}) = b_1q^{-1} + \cdots + b_nq^{-n}$$

and $n = \tilde{n} + n_d$. Clearly, (4) can be rewritten in the form

$$y(t) = \boldsymbol{\phi}(t-1)^{\mathrm{T}}\boldsymbol{\theta}_0 \tag{5}$$

where

$$\boldsymbol{\phi}(t-1) = [y(t-1), \ldots, y(t-n), u(t-1), \ldots, u(t-n)]^{\mathrm{T}}$$
$$\boldsymbol{\theta}_0 = [-a_1, \ldots, -a_n, b_1, \ldots, b_n]^{\mathrm{T}}$$

Equation (5) has the form of the 'model assumption' in Reference 3 and provides the basis for many recursive identification algorithms. To guarantee parameter convergence in any one of the algorithms, the vector $\boldsymbol{\phi}(t-1)$ must satisfy some conditions. These conditions differ from one algorithm to another.[3] A commonly used condition on $\boldsymbol{\phi}(t-1)$, termed 'persistently exciting' (PE),[1,2] can be stated as follows.

Definition 1

The vector sequence $\{\boldsymbol{\phi}(t)\}$ is said to be PE if there exist a positive integer N and a positive real ρ such that

$$\sum_{t=j}^{j+N} \boldsymbol{\phi}(t)\boldsymbol{\phi}(t)^{\mathrm{T}} \geqslant \rho\mathbf{I} \quad \text{for all (sufficiently large) } j \tag{6}$$

Considerable effort has been directed to the question of what should be required of the system input sequence $\{u(t)\}$ to guarantee the PE property of $\{\boldsymbol{\phi}(t)\}$ (see equation (4)) when $n_d = 0$. References 1 and 2 address the same problem for the case $n_d > 0$. However, before attempting to solve these problems, one should ask what is the benefit of $\boldsymbol{\phi}(t)$ being PE.

It is straightforward to show that $\boldsymbol{\phi}(k)$ being PE guarantees convergence in the recursive least squares (RLS) algorithm. However, it is not sufficient for convergence of the projection algorithm[3] and is not necessary for convergence of the RLS (without covariance resetting). The main result of this paper is the claim that when disturbances are present (i.e. $n_d > 0$), conditions are given so that an input sequence $\{u(t)\}$ not necessarily bounded yields parameter convergence of the RLS algorithm (but cannot guarantee a PE $\{\boldsymbol{\phi}(t)\}$ as was shown in Reference 2).

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*This means that $d(t)$ does not contain any exponentially decaying components. Note that the assumptions on $d(t)$ here are somewhat weaker than in References 1 and 2.

MAIN RESULT

Let us recall first the condition for parameter convergence in the RLS. This condition is[3]

$$\lim_{N \to \infty} \lambda_{\min} \sum_{t=0}^{N} \phi(t)\phi(t)^{\mathrm{T}} = \infty \tag{7}$$

where $\lambda_{\min}$ denotes the smallest eigenvalue.

Before proceeding, we need the following definition.

Definition 2

The scalar sequence $\{\nu(t)\}$ is said to be PE with richness m if the sequence of vectors $[\nu(t+m-1), \ldots, \nu(t)]^{\mathrm{T}} \in \mathbb{R}^m$ is PE.

Now we are ready to state our main result.

Theorem 1

Let $\bar{u}(t) = D(q^{-1})u(t)$ with $D(q^{-1})$ as in (2) and let $\bar{u}(t)$ be PE with richness $n + \tilde{n}$. Then the resulting sequence $\{\phi(t)\}$ satisfies (7).

The proof of this theorem is based on a partial state realization of the plant (1). In particular, since the polynomials $\tilde{A}(q^{-1})$ and $\tilde{B}(q^{-1})$ are mutually coprime of degree $\tilde{n}$, using the Bezout identity we know that there exist polynomials $F(q^{-1})$ and $G(q^{-1})$ of the same degree $\tilde{n}$ such that

$$\tilde{A}(q^{-1})F(q^{-1}) + \tilde{B}(q^{-1})G(q^{-1}) = 1 \tag{8}$$

which yields

$$d(t) = \tilde{A}(q^{-1})F(q^{-1})d(t) + \tilde{B}(q^{-1})G(q^{-1})d(t) \tag{9}$$

Combining (9) with the plant dynamics in (1) we get a partial state realization of the plant in the form

$$\begin{aligned} \tilde{A}(q^{-1})z(t) &= u(t) + G(q^{-1})d(t) \\ y(t) &= \tilde{B}(q^{-1})z(t) + F(q^{-1})d(t) \\ D(q^{-1})d(t) &= 0 \end{aligned} \tag{10}$$

On the basis of this partial state realization we can write the regression vector $\boldsymbol{\phi}(t-1)$ as

$$\boldsymbol{\phi}(t-1)=\left[\begin{array}{cccccc} 0 & \tilde{b}_1 & \cdots & \cdots & \tilde{b}_{\tilde{n}} & 0 \\ & \ddots & & & & \ddots \\ 0 & \cdots & 0 & \tilde{b}_1 & \cdots & \tilde{b}_{\tilde{n}} \\ \hline 1 & \tilde{a}_1 & \cdots & \cdots & \tilde{a}_{\tilde{n}} & 0 \\ & \ddots & & & & \ddots \\ 0 & & 1 & \tilde{a}_1 & \cdots & \tilde{a}_{\tilde{n}} \end{array}\right]\left[\begin{array}{c} z(t-1) \\ \vdots \\ z(t-n-\tilde{n}) \end{array}\right] + \left[\begin{array}{cccccc} f_0 & f_1 & \cdots & f_{\tilde{n}} & 0 \\ & \ddots & & & \ddots \\ 0 & & f_0 & f_1 & \cdots & f_{\tilde{n}} \\ \hline -g_0 & \cdots & \cdots & -g_{\tilde{n}} & 0 \\ & \ddots & & & \ddots \\ 0 & & -g_0 & \cdots & -g_{\tilde{n}} \end{array}\right]\left[\begin{array}{c} d(t-1) \\ \vdots \\ d(t-n-\tilde{n}) \end{array}\right] = \mathbf{S}\mathbf{z}(t-1)+\mathbf{V}\mathbf{d}(t-1) \tag{11}$$

with $\mathbf{S}$, $\mathbf{V}\in\mathbb{R}^{2n\times(n+\tilde{n})}$ and $\mathbf{z}(t)$, $\mathbf{d}(t)\in\mathbb{R}^{n+\tilde{n}}$.

Lemma 1

Let $\boldsymbol{\Phi}_k=[\boldsymbol{\phi}(kN),\ldots,\boldsymbol{\phi}(kN+N-1)]$ with $k=0,1,\ldots$ and N an integer (the 'block length'). Then there exists a $\lambda>0$, independent of k, such that

$$\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Phi}_k\boldsymbol{\Phi}_k^{\mathrm{T}}\boldsymbol{\theta}>\frac{1}{\lambda}\,\boldsymbol{\theta}^{\mathrm{T}}\mathbf{S}\overline{\mathbf{U}}_k\overline{\mathbf{U}}_k\mathbf{S}^{\mathrm{T}}\boldsymbol{\theta} \tag{12}$$

where $\bar{\mathbf{u}}(t)=[\bar{u}(t),\ldots,\bar{u}(t-n-\tilde{n}+1)]^{\mathrm{T}}\in\mathbb{R}^{n+\tilde{n}}$ and $\overline{\mathbf{U}}_k=[\bar{\mathbf{u}}(kN+n),\ldots,\bar{\mathbf{u}}(kN+N-1)]$.

Proof of Lemma 1. From (11) we can write

$$\boldsymbol{\Phi}_k=\mathbf{S}\mathbf{Z}_k+\mathbf{V}\mathbf{D}_k \tag{13}$$

where $\mathbf{Z}_k=[\mathbf{z}(kN),\ldots,\mathbf{z}(kN+N-1)]$ and $\mathbf{D}_k=[\mathbf{d}(kN),\ldots,\mathbf{d}(kN+N-1)]$. Now recall $A(q^{-1})=\tilde{A}(q^{-1})D(q^{-1})=1+a_1q^{-1}+\cdots+a_nq^{-n}$ and define

$$\mathbf{W}=\left[\begin{array}{ccc} a_n & 0 & \\ a_{n-1} & & \ddots \\ \vdots & a_n & \ddots \\ 1 & a_{n-1} & \\ & \vdots & \ddots \\ 0 & 1 & \end{array}\right]\in\mathbb{R}^{N\times(N-n)} \tag{14}$$

Then from (10) we have $A(q^{-1})z(t)=D(q^{-1})u(t)=\bar{u}(t)$ so that we can write

$$\mathbf{Z}_k\mathbf{W}=\overline{\mathbf{U}}_k \tag{15}$$

Also, since $A(q^{-1})$ has $D(q^{-1})$ as a factor,

$$\mathbf{D}_k\mathbf{W}=0 \tag{16}$$

for all k. Combining (15) and (16) with (13) we conclude that

$$\mathbf{\Phi}_k\mathbf{W} = \mathbf{S}\mathbf{U}_k. \tag{17}$$

Using (17) we can now write that for all $\boldsymbol{\theta} \in \mathbb{R}^{2n}$

$$\boldsymbol{\theta}^{\mathrm{T}}\mathbf{\Phi}_k\mathbf{\Phi}_k^{\mathrm{T}}\boldsymbol{\theta} \geqslant \frac{1}{\lambda_{\mathrm{M}}}\boldsymbol{\theta}^{\mathrm{T}}\mathbf{\Phi}_k\mathbf{W}\mathbf{W}^{\mathrm{T}}\mathbf{\Phi}_k\boldsymbol{\theta} = \frac{1}{\lambda_{\mathrm{M}}}\boldsymbol{\theta}^{\mathrm{T}}\mathbf{S}\bar{\mathbf{U}}_k\bar{\mathbf{U}}_k^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\boldsymbol{\theta}$$

where λ_{M} is the largest eigenvalue of $\mathbf{W}\mathbf{W}^{\mathrm{T}}$, and this completes the proof of the lemma. □

Proof of Theorem 1. To show that (7) holds we have to show that

$$\lim_{N\to\infty} \boldsymbol{\theta}^{\mathrm{T}} \sum_{t=0}^{N} \boldsymbol{\phi}(t)\boldsymbol{\phi}(t)^{\mathrm{T}}\boldsymbol{\theta} < \infty \quad \text{iff } \boldsymbol{\theta} = 0 \tag{18}$$

Let us assume that $\boldsymbol{\theta}$ is such that

$$\lim_{N\to\infty} \boldsymbol{\theta}^{\mathrm{T}} \sum_{t=0}^{N} \boldsymbol{\phi}(t)\boldsymbol{\phi}(t)^{\mathrm{T}}\boldsymbol{\theta} < \infty$$

or, equivalently,

$$\boldsymbol{\theta}^{\mathrm{T}} \sum_{k} \mathbf{\Phi}_k\mathbf{\Phi}_k^{\mathrm{T}}\boldsymbol{\theta} < \infty \tag{19}$$

Then, if $\bar{\mathbf{u}}(t)$ is PE with richness $n + \tilde{n}$, we have by Lemma 1 for the appropriate N

$$\boldsymbol{\theta}^{\mathrm{T}}\mathbf{\Phi}_k\mathbf{\Phi}_k^{\mathrm{T}}\boldsymbol{\theta} \geqslant \varepsilon\boldsymbol{\theta}^{\mathrm{T}}\mathbf{S}\mathbf{S}^{\mathrm{T}}\boldsymbol{\theta} \tag{20}$$

where $\varepsilon = \rho/\lambda$ a positive constant independent of k and $\boldsymbol{\theta}$. Therefore if (19) holds we must have

$$\boldsymbol{\theta}^{\mathrm{T}}\mathbf{S} = 0 \tag{21}$$

with S as in (11). Denoting $\boldsymbol{\theta} = [\alpha_1, \ldots, \alpha_n, -\beta_1, \ldots, -\beta_n]^{\mathrm{T}}$, it is straightforward to show that the $n + \tilde{n}$ entries of the vector $\boldsymbol{\theta}^{\mathrm{T}}\mathbf{S}$ are the coefficients of the polynomial

$$C(q^{-1}) = \alpha(q^{-1})\tilde{B}(q^{-1}) - \beta(q^{-1})\tilde{A}(q^{-1}) \tag{22}$$

of degree not exceeding $n + \tilde{n} - 1$, where $\alpha(q^{-1}) = \alpha_1 + \alpha_2 q^{-1} + \cdots + \alpha_n q^{-n+1}$ and $\beta(q^{-1}) = \beta_1 + \beta_2 q^{-1} + \cdots + \beta_n q^{-n+1}$. Therefore $\boldsymbol{\theta}^{\mathrm{T}}\mathbf{S} = 0$ in (21) and the coprimeness of $\tilde{A}(q^{-1})$ and $\tilde{B}(q^{-1})$ imply that

$$\begin{aligned} \alpha(q^{-1}) &= \tilde{D}(q^{-1})\tilde{A}(q^{-1}) \\ \beta(q^{-1}) &= \tilde{D}(q^{-1})\tilde{B}(q^{-1}) \end{aligned} \tag{23}$$

for some polynomial $\tilde{D}(q^{-1})$ of degree not exceeding $n_{\mathrm{d}} - 1$. Therefore each term $\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\phi}(t)$ in (9) can be written as

$$\boldsymbol{\theta}^{T}\boldsymbol{\phi}(t) = \tilde{D}(q^{-1})[\tilde{A}(q^{-1})y(t) - \tilde{B}(q^{-1})u(t)] = \tilde{D}(q^{-1})d(t)$$

Hence using (3) we see that

$$\lim_{N\to\infty} \boldsymbol{\theta}^{\mathrm{T}} \sum_{t=0}^{N} \boldsymbol{\phi}(t)\boldsymbol{\phi}(t)^{\mathrm{T}}\boldsymbol{\theta} = \lim_{N\to\infty} \sum_{t=0}^{N} |\tilde{D}(q^{-1})d(t)|^2 < \infty \quad \text{iff } \tilde{D}(q^{-1}) = 0$$

which completes the proof. □

CONCLUDING REMARKS

The counterexample in Reference 2 needs a small modification to serve its purpose and convince the reader that boundedness of input is necessary to guarantee PE in the sense of Definition 1. In our paper, however, we point out that this PE is more than is required for parameter convergence in an RLS algorithm. We proceed then to establish the conditions on the input to guarantee parameter convergence for the RLS algorithm. These conditions are weaker than in Reference 1 and do not require boundedness of the input.

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