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# Finding a moving fugitive. A game theoretic representation of search

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**Abstract**

We develop and analyze a "manhunting" game involving a mobile hider, who wishes to maximize his time to capture, and a mobile searcher, who wishes to minimize this same time. The game takes place within a variegated environment that offers better and worse locations to evade capture. The hider is able to move from one hide site to another at will. In choosing a hide site, he must consider the risk of discovery, the risk that he will be betrayed, and the risk that he will be captured while moving from one site to another. The searcher can select any cell to search within the fugitive's feasible hiding set. We examine the strategic behavior of both players and provide examples.
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*Keywords:* Search games; Markov process; Frobenius eigenvector

## 1. Introduction

We consider a *deductive* search game involving a fugitive, who wishes to evade capture as long as possible, and a searcher, who wishes to apprehend him as soon as possible. The model we develop builds on a base-line model presented in [1]. In this earlier work we examined the comparatively simple problem of finding an immobile hider. The fugitive is able to survey his hiding environment and select a hide site, but he is not able to move from this location once it is chosen. Given this constraint, we evaluated the optimal "hide and seek" strategies for each player.

The present article examines the much more complicated problem of finding a mobile target. The fugitive, in this case, is not only able to select a hiding location, he is able to move from one hiding site to another as frequently as he believes that it is to his advantage to do so. In making this decision we assume that he has no foreknowledge of where his opponent will search next, but that he is sensitive to the relative strengths and weaknesses of different possible hiding locations, he accounts for the fact that there is an exposure risk associated with transiting from one location to another, and he is aware of the fact that his risk of exposure and capture will increase the longer he stays in any one place. To complicate his decision process, each new hiding location he selects will be chosen, in part, with an eye toward the options it gives him for subsequent moves.

What forces a fugitive to periodically pick up and move on is the fact that the longer he remains in place the more people are likely to learn that he is there. The more people who learn of his location, the more dangerous his situation

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will become. Even if he is among allies, the risk of information leakage will increase the longer he remains in one place. Initially, this may involve nothing more than the sharing of a friendly secret. As this information continues to change hands, however, it can be expected to eventually come to the attention of someone who is willing to provide it to the authorities. The fact that a fugitive's risk of discovery increases over time if he remains in one place is what forces him to periodically move from one hideout to another. At some point, the risk of remaining in one place outweighs the marginal risk of moving to a new location.

What defines this game as a "deductive" search—in contrast to an "inductive" search—is the absence of any updates regarding where the target is likely to be located. The searcher, in this case, is not trying to follow his quarry's "signal," he is designing a search strategy that he believes will offer the best chance of uncovering such a signal in the first place. He is able to do this based on the fact that individuals who do not wish to be found hide selectively; they inevitably choose certain areas to hide and other areas to avoid. Their decision logic, in such cases, is based on the differentiated nature of their hiding environment and the search capabilities of their opponent. Based on these considerations, certain areas offer better places to hide than others, and some areas will be avoided altogether. Uncovering and applying this logic allows a searcher to design an optimal search strategy in the absence of any prior indication of his target's location.

This is one of the most important distinctions between a deductive search and traditional search games. Traditional search games, such as those formulated by Foreman [2], Gal [3], Washburn [4], Alpern and Asic [5,6] and Nakai [7], to name just a few from a well-developed literature, effectively assume that the fugitive or evader hides within a homogeneous environment. The relative characteristics of the "cell" or other hiding space within which he takes refuge are assumed to have no bearing on his ability to remain hidden from his pursuer. Such games are often resolved by giving one or both sides updates on the new location of its competitor. Examples include Halpern [8], Flynn [9], Dobbie [10], and Thomas and Washburn [11].

The approach taken in both this article and in [1] begins with the observation that, first, a fugitive's hiding environment is not homogeneous but heterogeneous and, second, that the heterogeneous nature of this environment offers better and worse places to hide. This approach is more appropriate to the "manhunting" problem we investigate here. The hiding patterns of historical and contemporary fugitives, from Jesse James to Usama bin Laden, reveal that rational fugitives use their environments to their advantage. Searchers can use this knowledge to their advantage, as well. When both players approach the game from this perspective each player's optimal strategies are mutually determined. Neither can improve his chances of success under these circumstances as long as his competitor plays the game in an optimal way.

The model we develop allows us to determine where a fugitive should hide given a set of possible hiding locations, how long he should stay in that location, when he should change locations, where he should move to next given the location he is then occupying, how long he should stay there in turn, and when and where he should move after that to maximize his time to capture. Based on this assessment, we are then able to determine how a searcher should prioritize his search efforts to minimize this time to capture. The hider takes this probable behavior of the searcher into account. An equilibrium is then reached: each player's behavior is optimal, in the game-theoretical sense.

In a dynamic game, such as this, the question of information is always of paramount importance. Given that the hider can move any time that he believes the risk of discovery has grown too great, it is important to establish what he knows. Clearly he does not know where the searcher will look after he (the fugitive) moves, but does he know where the searcher is currently looking? Our *first assumption* is that the hider knows whether the searcher is looking in the right cell (i.e., the cell where he is hiding), but no more.

In turn, the searcher's information is important. He does not have any indication where the hider is at this moment, but can we not reasonably assume that he has some information about the hider's past moves? Without this information, the searcher is at a serious disadvantage and it is difficult to model this as a tractable multi-stage game. Our *second assumption* is that the searcher knows where the hider was in the past. This information, which filters out sometime (soon) after the hider has moved on, gives the searcher an initial point of departure to begin a deductive, forward-looking search.

## 2. The model

We consider a search game in which a fugitive, whom we designate as the hider ($H$), can hide in any one of several cells. The authorities, who play the role of searcher ($S$), seek to uncover $H$'s location but do not know (in general) the cell he has chosen to occupy. If $S$ looks in the right cell, there is a relatively high probability that he will find $H$. If he looks in the wrong cell, there is still the possibility that $H$ will be betrayed and that his location will be revealed. $H$, in this respect, faces two types of risk.

As time passes, the probability of betrayal increases. This is true regardless of the particular cell that $H$ occupies. As noted above, while the likelihood of betrayal in a "safe" cell may initially prove to be low, the longer that $H$ stays in any one location, the more people in his cell and adjacent cells are likely to learn that he is there and the higher the likelihood that this information will find its way to those who are looking for him. $H$'s risk of discovery, therefore, will increase over time if he chooses to stay in one place.

Eventually, then, $H$ will choose to move. He may do so rather soon, if $S$ is indeed looking in the right cell (i.e., the one $H$ is occupying), but he will do so (albeit not so soon) even if $S$ is looking in a different cell. This motion is itself risky, and will generally leave some traces, so that $S$ can discover, after the fact, that $H$ has moved. At that moment, a new stage will begin. $S$ knows *when H* moved, and *whence* he moved, but does not know the new hiding place.

Specifically, we assume that, if $H$ is in a given cell while $S$ looks in a different cell, the probability of capture within $t$ units of time, $Q(t)$, satisfies the differential equation

$$Q'(t) = g(t)(1 - Q(t)),$$

$$Q(0) = 0, \tag{1}$$

where $g$ is a continuous, strictly increasing and unbounded function of $t$.

It is not too difficult to see that this equation has the solution

$$Q(t) = 1 - c\exp\{-G(t)\}, \tag{2}$$

where $G$ is an anti-derivative of $g$. From the initial condition, we see that, if $G(0) = 0$, then $c = 1$. Thus we have

$$Q(t) = 1 - \exp\{-G(t)\}, \quad G(t) = \int_0^t g(s)\,\mathrm{d}s \tag{3}$$

and

$$Q'(t) = g(t)\exp\{-G(t)\}. \tag{4}$$

The hider's objective, as we have said, is to maximize his survival time. This will depend, among other things, on when he decides to leave one hiding site for another.

To begin to examine this issue, suppose that, at time $T$, the hider (if he has not yet been apprehended) moves from cell $j$ to a different cell. The random variable $X$, which is the length of time $H$ actually spends in cell $j$, then has density $Q'$ from 0 to $T$. This is followed by a probability $1 - Q(T)$ that $H$ will leave at time $T$. Thus its expected value is

$$E[X] = \int_0^T tQ'(t)\,\mathrm{d}t + (1 - Q(T))T \tag{5}$$

or, for the given $Q$,

$$E[X] = \int_0^T tg(t)\exp\{-G(t)\}\,\mathrm{d}t + T\exp\{-G(T)\}. \tag{6}$$

If $H$ makes a move, he must decide where to move, and take into consideration the fact that he might be caught in transit. Independent of the time he moves, he expects to survive an additional $V_j$ units of time after starting the move. The probability that he will actually get to move is $\exp\{-G(T)\}$, and so the total expected survival time is

$$A_j = \int_0^T tg(t)\exp\{-G(t)\}\,\mathrm{d}t + T\exp\{-G(T)\} + \exp\{-G(T)\}V_j. \tag{7}$$

As we can see, $H$'s expected time to capture while he is hiding in cell $j$ depends, in part, on the time $T$ of his departure. To maximize this value, we differentiate:

$$\mathrm{d}A_j/\mathrm{d}T = Tg(T)\exp\{-G(T)\} + \exp\{-G(T)\} - Tg(T)\exp\{-G(T)\} - g(T)\exp\{-G(T)\}V_j$$

so that

$$dA_j/dT = \exp\{-G(T)\} - g(T)\exp\{-G(T)\}V_j. \tag{8}$$

Setting this derivative equal to 0, we find

$$g(T)V_j = 1.$$

The optimal time for $H$ to move from cell $j$ to his next hiding location is, thus

$$T = g^{-1}[1/V_j]. \tag{9}$$

Since we have assumed $g$ is a strictly increasing function of $t$, this $T$ is unique.

Consider next the integral

$$\int_0^T tg(t)\exp\{-G(t)\}\,dt$$

which appears in (6). We integrate by parts, letting $u = t$, and $v = -\exp\{-G(t)\}$. This equals

$$\begin{aligned} &-t\exp\{-G(t)\}_0^T + \int_0^T \exp\{-G(t)\}\,dt \\ &\quad = -T\exp\{-G(T)\} + \int_0^T \exp\{-G(t)\}\,dt \end{aligned}$$

and so (7) takes the form

$$A_j = \int_0^T \exp\{-G(t)\}\,dt + \exp\{-G(T)\}V_j. \tag{10}$$

As mentioned above, it is reasonable to assume that, as time passes (and $H$ does not move), then the probability of discovery (over a short interval of time) increases. We will model this by assuming that it increases linearly, so that

$$g(t) = q + \alpha t, \tag{11}$$

where $q$ is the (initial) probability that $H$ will be found even if $S$ does not look in the cell in which he is hiding and $\alpha > 0$ is the rate at which this risk changes as long as $H$ remains in place.[1]

Expression (9) then gives us

$$q + \alpha T = 1/V_j$$

which reduces to

$$T = \frac{1 - qV_j}{\alpha V_j}. \tag{12}$$

As we can see, this last equation may result in a negative solution. This being the case, we will choose $T$ by (12) if this value is positive, and will set $T = 0$ otherwise.

We can also see that

$$G(t) = qt + \alpha t^2/2 \tag{13}$$

and so the integrand above takes the form

$$\exp\{-(qt + \alpha t^2/2)\} = \exp\{-\tfrac{1}{2}\alpha(t + q/\alpha)^2 + q^2/2\alpha\} = \exp\{q^2/2\alpha\}\exp\{-\tfrac{1}{2}\alpha(t + q/\alpha)^2\}.$$

[1] An advantage of this form for the function $g$ is that it allows us to calculate the integral in (10) by means of a well-known and well-tabulated function. Other forms for $g$ (and also the function $f$, introduced below) could of course be used; we only require that it be a continuous increasing function. In that case, calculation of the integral in question would be considerably more complicated without a computer.

It follows from this that the integral in (10) becomes

$$\int_0^T \exp\{-G(t)\}\,\mathrm{d}t = \exp\{q^2/2\alpha\}\int_0^T \exp\{-\tfrac{1}{2}\alpha(t+q/\alpha)^2\}\,\mathrm{d}t.$$

Making the substitution

$$u = (t+q/\alpha)\sqrt{(\alpha/2)}$$

this becomes

$$\frac{\exp\{q^2/2\alpha\}\sqrt{2}}{\sqrt{\alpha}}\int_{U_1}^{U_2} \exp\{-u^2\}\,\mathrm{d}u, \tag{14}$$

where the upper and lower limits on the integral are given by the expressions:

$$U_1 = q/\sqrt{(2\alpha)} \tag{15}$$

and

$$U_2 = q/\sqrt{(2\alpha)} + T\sqrt{(\alpha/2)}. \tag{16}$$

It may be seen that (14) actually gives the result in terms of the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\{-y^2\}\,\mathrm{d}y.$$

The function $\mathrm{erf}(x)$ is well known and extensively tabulated. (See, e.g., [12].) From it we obtain

$$\int_0^T \exp\{-G(t)\}\,\mathrm{d}t = \frac{\sqrt{\pi}\exp\{q^2/2\alpha\}[\mathrm{erf}(U_2)-\mathrm{erf}(U_1)]}{\sqrt{(2\alpha)}}$$

and find that

$$A_j = \exp\{-G(T_j)\}V_j + \frac{\sqrt{\pi}\exp\{q^2/2\alpha\}[\mathrm{erf}(U_{2j})-\mathrm{erf}(U_{1j})]}{\sqrt{(2\alpha)}}, \tag{17}$$

where $T_j$ is given by expression (9) and the limits $U_{1j}$ and $U_{2j}$ by expressions (15)–(16). Note that we have placed subscripts on the $T$, as well as on the $U_1$ and $U_2$ as these will depend on the cell where $H$ is hiding *prior to moving*.

Next, we note that the functions $g(t)$ and $G(t)$ apply only under the assumption that $S$ is looking in the *wrong* cell. The risk that $H$ is concerned about, in this case, is that he will be betrayed. It is also possible, however, that $S$ will look in the *right* cell. $H$'s risk, in this case, is that he will be captured directly. $H$, as mentioned above, faces two types of risk. To account for this, if $S$ looks in the cell where $H$ is hiding, then, $g(t)$ and $G(t)$ should be replaced by

$$f(t) = p + \alpha t \tag{18}$$

and

$$F(t) = pt + \alpha t^2/2, \tag{19}$$

where $p$ is the probability that $H$ will be found if $S$ looks in the cell in which he is hiding and $\alpha$, as before, is the rate at which this risk changes the longer he remains in place.

The probability that $H$ will be captured directly before time $t$, therefore, is given by

$$R(t) = 1 - \exp\{-F(t)\}. \tag{20}$$

Assume, then, that $S$ decides to look in cell $j$ and $H$ stays in cell $j$ until time $D_j$, when he decides to move. Our fugitive's expected survival time after leaving $j$ is still $V_j$. His total expected survival time, therefore, including the time he spends in cell $j$, is given by

$$B_j = \int_0^D tf_j(t)\exp\{-F_j(t)\}\,\mathrm{d}t + T_j'\exp\{-F_j(D_j)\} + \exp\{-F_j(D_j)\}V_j. \tag{21}$$

With $f$ and $F$ as above, this gives us

$$B_j = \exp\{-F_j(T_j')\}V_j + \frac{\sqrt{\pi}\exp\{p_j^2/2\alpha\}[\mathrm{erf}(W_{2j}) - \mathrm{erf}(W_{1j})]}{\sqrt{(2\alpha)}}, \tag{22}$$

where

$$W_{1j} = p_j/\sqrt{(2\alpha)} \tag{23}$$

and

$$W_{2j} = p_j/\sqrt{(2\alpha)} + D_j\sqrt{(\alpha/2)}. \tag{24}$$

For this purpose, we note that $D_j$ is computed in the same way as $T_j$ is, above, in (12), with $f$ and $F$ in place of $g$ and $G$. Thus, assuming these are given by (22)–(23), we have

$$D_j = \frac{1 - pV_j}{\alpha V_j} \tag{25}$$

if this quantity is positive, and $D_j = 0$ otherwise. (Note that, since $p > q$, it will follow that $D_j < T_j$, i.e., $H$ will leave cell $j$ more rapidly if $S$ is in fact looking there.)

## 3. Treatment as a stochastic game

We find, now, that this is a stochastic game, as described in [13]. A new stage starts every time that $H$ moves; at that moment, $H$ knows where he has been, and, according to our model, $S$ also knows where $H$ is moving *from*. We can, therefore, attack this problem through a process of successive approximations, as discussed in [13]. Specifically, let us use $A_j^m$ to represent our fugitive's expected survival time assuming that he starts in cell $j$ and is allowed to move $m$ times, and that $S$ is looking in the wrong cell (i.e., any cell other than $j$). Similarly, $B_j^m$ will represent $H$'s expected survival time assuming that $S$ is in fact looking in the right cell (cell $j$). After moving once, he will be allowed to move a further $m-1$ times; his expected (subsequent) survival time is then $V_j^{m-1}$.

To start with, consider $A_j^0$. $A_j^0$ defines $H$'s expectation assuming that he does not move. If that is so, then we set $V_j^{(-1)} = 0$; $H$ must remain in cell $j$ until he is captured. $T$, in this case, is effectively infinite, and, in (14) above, $U_2 = \infty$. Given the conditions on $g$, we find that $G(\infty) = \infty$. Now $\mathrm{erf}(\infty) = 1$; and $\exp\{-\infty\} = 0$, and so (16) takes the form

$$A_j^0 = \frac{\sqrt{\pi}\exp\{q^2/2\alpha\}[1 - \mathrm{erf}(U_{1j})]}{\sqrt{(2\alpha)}}. \tag{26}$$

Suppose next that $H$, starting in $j$, is allowed to move only once. As soon as this move is completed, he will then have passed to a situation in which no further moves are allowed; i.e., if he moves he then expects to survive a further $V_j^0$. We will then have

$$A_j^1 = \frac{\sqrt{\pi}\exp\{q^2/2\alpha\}[\mathrm{erf}(U_{2j}) - \mathrm{erf}(U_{1j})]}{\sqrt{(2\alpha)}} + \exp\{-G(T_j)\}V_j^0, \tag{27}$$

where $V_j^0$ is the expected survival, after leaving cell $j$, if no further moves are allowed.

We can continue in this manner, considering the situations which will hold if $H$ is allowed to move a successively larger number of $m$ times. In fact, let $A_j^m$ represent $H$'s expected survival time before word of his presence leaks out, assuming that he starts in cell $j$ and is allowed to move $m$ times. If from cell $j$ he moves to $k$ he will have only $m-1$ further moves available. His expectation in this case—if and when he gets to cell $k$—will then be $A_k^{m-1}$. It follows that the quantities $A_j^m$ will satisfy the recursion relation

$$A_j^m = \frac{\sqrt{\pi}\exp\{q^2/2\alpha\}[\mathrm{erf}(U_{2j}) - \mathrm{erf}(U_{1j})]}{\sqrt{(2\alpha)}} + \exp\{-G(T_j)\}V_j^{m-1} \tag{28}$$

with

$$T_j = \max\left\{\frac{1-qV_j^{m-1}}{\alpha V_j^{m-1}}, 0\right\}. \tag{29}$$

In a similar way, the $B_j^m$ satisfy

$$B_j^m = \frac{\sqrt{\pi}\exp\{q^2/2\alpha\}[\mathrm{erf}(W_{2j}) - \mathrm{erf}(W_{1j})]}{\sqrt{(2\alpha)}} + \exp\{-F(D_j)\}V_j^{m-1} \tag{30}$$

with

$$D_j = \max\left\{\frac{1-pV_j^{m-1}}{\alpha V_j^{m-1}}, 0\right\}. \tag{31}$$

As we proceed by computing these quantities for ever higher values of $m$, we will see that (for any given $j$) these will change very slightly after, approximately, $m = 10$. In general, the convergence will be relatively fast as $m \to \infty$. Of course, letting $m \to \infty$ in this way assumes, in effect, that $H$ can think ahead a very large number of steps. As a practical matter, he is likely to be able to think ahead a few steps, maybe 3 or 4. (Even top chess players are only able to think ahead some 3 or 4 moves.) Thus it should not be necessary to carry the process to its limit; calculation for $m = 4$ or 5 should suffice.

The quantities $B_j$ will be approximated in the same way, mutatis mutandis.

Now, for a given $m$, having computed the several $A_j^m$ and $B_j^m$, we are in a position to compute $V_j^m$. We do this by applying the results of the Algorithm developed in the Appendix.

To see how this works, let us assume that $H$ is about to leave cell $j$. If he moves to cell $i$, there is probability $s_{ji}$ that he will complete his movement (i.e., not be captured in transit). The *time of transit* is assumed negligibly small.[2] Assuming he is not captured along the way, he will survive an additional $A_i$ or $B_i$ when he arrives (depending on whether $S$ looks in cell $i$). Thus his expected survival is

$$\tau_{ji} = s_{ji}A_i \tag{32}$$

if $S$ does not look in cell $i$, and

$$\sigma_{ji} = s_{ji}B_i \tag{33}$$

if $S$ does look there.

For fixed $j$, then, the quantities $\tau_i(=\tau_{ji})$ and $\sigma_i(=\sigma_{ji})$ are the inputs to our Algorithm. This algorithm returns both a value $v$, which is the desired $V_j$, as well as an optimal hiding strategy $\mathbf{y}_j^* = (y_{1j}^*, \ldots, y_{nj}^*)$, which gives the transition probabilities $H$ should use when leaving cell $j$. The algorithm also yields an optimal search strategy $\mathbf{x}_j^*$, which tells us how $S$ should concentrate his search effort to have the best chance of finding his fugitive, assuming that $H$ has recently left cell $j$.

Incidentally, we can now see why the method of sequential approximations will converge. First of all, the $A_j$ and $B_j$ increase (or at least do not decrease) at each step, since it is better to be allowed to move $m$ times, than $m-1$ times. From the Appendix, we note that $v$ is, approximately, given by a weighted average of the terms $\tau_{ij}$ and $\sigma_{ij}$. Thus it cannot increase by more than what the greatest of these increases. But each $\tau_{ij}$ or $\sigma_{ij}$ is of the form $s_{ji}A_i$ *or* $s_{ij}B_i$, and it follows that this increase is not greater than the greatest of the $s_{ji}$, multiplied by the greatest increase in the $A_i$ or $B_i$. We thus have a non-decreasing sequence (of vector variables) which cannot increase more rapidly than a geometric sequence with ratio equal to the greatest of the $s_{ji}$. But these were all assumed smaller than 1, and there is only a finite number of them. This proves convergence.[3]

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[2] To be more precise, suppose the time of transit is $h$, and survival probability during this transit is $s$. Then a more exact result for (32) and (33) would be $\tau = sA + (h+sh)/2$ and $\sigma = sB + (h+sh)/2$.

[3] In fact, this underestimates the rate of convergence, since we have not taken the factors $\exp\{-G(t)\}$ into account. Assuming these are in the order of 0.5, the rate of convergence should be similar to that of a geometric series with ratio equal to one half of the largest $s_{ji}$.

**Example 1.** Consider a search of an area consisting of 12 cells, arranged in a 4 × 3 grid pattern:

| | | | |
|---|---|---|---|
| **1** | **2** | **3** | **4** |
| $p = 0.5$ | 0.7 | 0.4 | 0.3 |
| $q = 0.3$ | 0.4 | 0.1 | 0.2 |
| **5** | **6** | **7** | **8** |
| 0.2 | 0.5 | 0.7 | 0.6 |
| 0.1 | 0.1 | 0.4 | 0.4 |
| **9** | **10** | **11** | **12** |
| 0.8 | 0.3 | 0.4 | 0.2 |
| 0.2 | 0.1 | 0.3 | 0 |

In each cell, the bold-faced number merely names the cell; in descending order the two other numbers give $p_j$ and $q_j$, respectively. As may be seen, ranking by order of increasing $q_j$, there is one very safe cell (12), four other safe cells (3, 5, 6, 10), four risky cells (1, 4, 9, 11), and three very risky cells (2, 7, 8). We have also chosen $\alpha = 0.01$, so that the danger of remaining stationary increases rather slowly as time progresses.

The quantities $s_{ij}$, in this case, depend directly on the distance between cells, although they might easily depend on other factors as well: we give these in the following table.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.452 | 0.409 | 0.452 | 0.435 | 0.402 | 0.409 | 0.402 | 0.378 | 0.371 | 0.363 | 0.349 |
| 2 | 0.452 | 0 | 0.452 | 0.435 | 0.452 | 0.435 | 0.402 | 0.409 | 0.402 | 0.363 | 0.37 | 0.363 |
| 3 | 0.409 | 0.452 | 0 | 0.402 | 0.435 | 0.452 | 0.378 | 0.402 | 0.409 | 0.349 | 0.363 | 0.37 |
| 4 | 0.452 | 0.435 | 0.402 | 0 | 0.452 | 0.409 | 0.452 | 0.435 | 0.402 | 0.409 | 0.402 | 0.378 |
| 5 | 0.435 | 0.452 | 0.435 | 0.452 | 0 | 0.452 | 0.435 | 0.452 | 0.435 | 0.402 | 0.409 | 0.402 |
| 6 | 0.402 | 0.435 | 0.452 | 0.409 | 0.452 | 0 | 0.402 | 0.435 | 0.452 | 0.378 | 0.402 | 0.409 |
| 7 | 0.409 | 0.402 | 0.378 | 0.452 | 0.435 | 0.402 | 0 | 0.452 | 0.409 | 0.452 | 0.435 | 0.402 |
| 8 | 0.402 | 0.409 | 0.402 | 0.435 | 0.452 | 0.435 | 0.452 | 0 | 0.452 | 0.435 | 0.452 | 0.435 |
| 9 | 0.378 | 0.402 | 0.409 | 0.402 | 0.435 | 0.452 | 0.409 | 0.452 | 0 | 0.402 | 0.435 | 0.452 |
| 10 | 0.37 | 0.363 | 0.349 | 0.409 | 0.402 | 0.378 | 0.452 | 0.435 | 0.402 | 0 | 0.452 | 0.409 |
| 11 | 0.363 | 0.37 | 0.363 | 0.402 | 0.409 | 0.402 | 0.435 | 0.452 | 0.435 | 0.452 | 0 | 0.452 |
| 12 | 0.349 | 0.363 | 0.37 | 0.378 | 0.402 | 0.409 | 0.402 | 0.435 | 0.452 | 0.409 | 0.452 | 0 |

In this case, the solution was generated in a single iteration of the algorithm, along with relations (28)–(31). As may be seen, the values for $A_j^2$ and $B_j^2$ are so close to their respective $A_j^1$ and $B_j^1$ that we feel safe in saying that there will be no more changes (to two decimal places) with further iterations.

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_j^0$ | 3.05 | 2.37 | 6.56 | 4.22 | 6.56 | 6.56 | 2.37 | 2.37 | 4.22 | 6.56 | 3.05 | 12.53 |
| $B_j^0$ | 1.92 | 1.40 | 2.37 | 3.05 | 4.22 | 1.92 | 1.40 | 1.63 | 1.23 | 3.05 | 2.37 | 4.22 |
| $T_j^0$ | 8.90 | 0 | 27.76 | 17.71 | 27.96 | 26.23 | 0 | 0 | 16.51 | 30.77 | 7.48 | 44.07 |
| $D_j^1$ | 0 | 0 | 0 | 7.71 | 17.96 | 0 | 0 | 0 | 0 | 10.77 | 0 | 24.07 |
| $V_j^1$ | 2.57 | 2.73 | 2.65 | 2.65 | 2.63 | 2.76 | 2.71 | 2.78 | 2.74 | 2.45 | 2.67 | 2.27 |
| $A_j^1$ | 3.06 | 2.73 | 6.56 | 4.22 | 6.56 | 6.56 | 2.71 | 2.78 | 4.22 | 6.56 | 2.85 | 12.53 |
| $B_j^1$ | 2.57 | 2.73 | 2.65 | 3.06 | 4.22 | 2.76 | 2.71 | 2.78 | 2.74 | 3.05 | 2.67 | 4.22 |

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_j^1$ | 8.82 | 0 | 27.39 | 17.69 | 27.56 | 26.18 | 0 | 0 | 16.33 | 30.74 | 7.48 | 43.71 |
| $D_j^2$ | 0 | 0 | 0 | 7.69 | 17.56 | 0 | 0 | 0 | 0 | 10.74 | 0 | 23.71 |
| $V_j^2$ | 2.58 | 2.74 | 2.67 | 2.65 | 2.66 | 2.76 | 2.71 | 2.78 | 2.75 | 2.45 | 2.67 | 2.29 |
| $A_j^2$ | 3.06 | 2.74 | 6.56 | 4.22 | 6.56 | 6.56 | 2.71 | 2.78 | 4.22 | 6.56 | 2.85 | 12.53 |
| $B_j^2$ | 2.58 | 2.74 | 2.65 | 3.06 | 4.22 | 2.76 | 2.71 | 2.78 | 2.75 | 3.05 | 2.67 | 4.22 |

We give also two tables, representing *H*'s and *S*'s strategies, in terms of places to go or look. First, suppose *H* has just left cell 1: where should he go next? Where should *S* next look?

From (32) and (33), we see that, when *S* leaves cell 1, his expected survival if he goes to the several cells (and *S* looks in the wrong cell) are $\sigma_i = s_{1i}A_i$, and, assuming *S* looks in the right cell, they will be $\tau_i = s_{1i}B_i$, which we calculate from the above results:

| $i$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma_i$ | 1.238 | 2.683 | 1.907 | 2.854 | 2.637 | 1.108 | 1.118 | 1.595 | 2.434 | 1.035 | 4.373 |
| $\tau_i$ | 1.238 | 1.084 | 1.383 | 1.836 | 1.110 | 1.108 | 1.118 | 1.040 | 1.132 | 0.969 | 1.473 |

Application of Eqs. (A.3) and (A.4) from the Appendix now give us the moving strategy **y** and search strategy **x** for the next stage, i.e., after *H* leaves cell 1:

| $i$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y_i$ | 0 | 0.240 | 0 | 0.377 | 0.251 | 0 | 0 | 0 | 0 | 0 | 0.132 |
| $x_i$ | 0 | 0.067 | 0 | 0.273 | 0.040 | 0 | 0 | 0 | 0 | 0 | 0.620 |

Continuing in this way, we complete two matrices, $Y^*$ and $X^*$: the entry $y_{ij}$, in row *j* and column *i* of matrix *Y*, tells us *H*'s probability of going to cell *i* after leaving cell *j*. Similarly, the entry $x_{ij}$, in row *j* and column *i* of matrix $X^*$, tells us the probability that *S* looks in cell *i*, after *H* leaves cell *j*.

$Y^*=$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0.240 | 0 | 0.377 | 0.251 | 0 | 0 | 0 | 0 | 0 | 0.132 |
| 2 | 0 | 0 | 0.231 | 0 | 0.386 | 0.247 | 0 | 0 | 0 | 0 | 0 | 0.135 |
| 3 | 0 | 0 | 0 | 0 | 0.520 | 0.308 | 0 | 0 | 0 | 0 | 0 | 0.171 |
| 4 | 0 | 0 | 0 | 0 | 0.363 | 0.247 | 0 | 0 | 0 | 0.267 | 0 | 0.122 |
| 5 | 0 | 0 | 0.400 | 0 | 0 | 0.396 | 0 | 0 | 0 | 0 | 0 | 0.204 |
| 6 | 0 | 0 | 0.313 | 0 | 0.524 | 0 | 0 | 0 | 0 | 0 | 0 | 0.163 |
| 7 | 0 | 0 | 0 | 0 | 0.514 | 0 | 0 | 0 | 0 | 0.330 | 0 | 0.157 |
| 8 | 0 | 0 | 0 | 0 | 0.381 | 0.244 | 0 | 0 | 0 | 0.264 | 0 | 0.111 |
| 9 | 0 | 0 | 0 | 0 | 0.537 | 0.318 | 0 | 0 | 0 | 0 | 0 | 0.145 |
| 10 | 0 | 0 | 0 | 0 | 0.518 | 0.339 | 0 | 0 | 0 | 0 | 0 | 0.143 |
| 11 | 0 | 0 | 0 | 0 | 0.538 | 0 | 0 | 0 | 0 | 0.325 | 0 | 0.137 |
| 12 | 0 | 0 | 0.224 | 0 | 0.343 | 0.208 | 0 | 0 | 0 | 0.225 | 0 | 0 |

$X^*=$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0.067 | 0 | 0.273 | 0.040 | 0 | 0 | 0 | 0 | 0 | 0.620 |
| 2 | 0 | 0 | 0.125 | 0 | 0.209 | 0.067 | 0 | 0 | 0 | 0 | 0 | 0.598 |
| 3 | 0 | 0 | 0 | 0 | 0.176 | 0.169 | 0 | 0 | 0 | 0 | 0 | 0.654 |
| 4 | 0 | 0 | 0 | 0 | 0.294 | 0.019 | 0 | 0 | 0 | 0.021 | 0 | 0.663 |
| 5 | 0 | 0 | 0.113 | 0 | 0 | 0.176 | 0 | 0 | 0 | 0 | 0 | 0.710 |
| 6 | 0 | 0 | 0.114 | 0 | 0.190 | 0 | 0 | 0 | 0 | 0 | 0 | 0.695 |
| 7 | 0 | 0 | 0 | 0 | 0.142 | 0 | 0 | 0 | 0 | 0.161 | 0 | 0.697 |
| 8 | 0 | 0 | 0 | 0 | 0.171 | 0.043 | 0 | 0 | 0 | 0.046 | 0 | 0.738 |
| 9 | 0 | 0 | 0 | 0 | 0.100 | 0.124 | 0 | 0 | 0 | 0 | 0 | 0.775 |
| 10 | 0 | 0 | 0 | 0 | 0.194 | 0.018 | 0 | 0 | 0 | 0 | 0 | 0.787 |
| 11 | 0 | 0 | 0 | 0 | 0.015 | 0 | 0 | 0 | 0 | 0.187 | 0 | 0.797 |
| 12 | 0 | 0 | 0.097 | 0 | 0.372 | 0.254 | 0 | 0 | 0 | 0.276 | 0 | 0 |

Thus we find that:

(1) $H$ will never stay in one of the very risky cells (2, 7, 8). If by any chance he should find himself in one of these cells, he will immediately leave for one of the safe or very safe cells (3, 5, 6, 10, 12).
(2) If $H$ is in one of the risky cells (1, 4, 9, 11), he will stay there a while (between 8 and 18 time units), and then leave for one of the safe or very safe cells. $H$ will never return to these risky cells in a future move.
(3) If $H$ is in one of the safe cells (3, 5, 6, 10), he will stay there between 26 and 31 time units, and then move to another safe or very safe cell.
(4) If $H$ is in the very safe cell (12), he will stay there 44 time units and then move to one of the other safe cells.
(5) All the above hold only in case $S$ is looking in the wrong cell. If $S$ is, in fact, looking in the right cell, then $H$ will leave immediately, except for cells 4, 5, 10, or 12. In those cells, he will stay between 8 and 24 time units, and then leave.

In general, we find in this case that moving between cells is so risky (note all $s_{ij} < 0.5$) that our fugitive will stay put for a while, unless he finds himself in a very risky cell (2, 7, 8), or in case $S$ is looking in the right cell. $H$, in short, will move only when he is forced to do so, i.e., when the risk of remaining in place has become so great that it outweighs the significant risk of changing locations. The probability that he will survive long enough to actually make a move, however, is low. Under the best possible circumstances, $H$'s expected time to capture is fairly short.

It is interesting to note that $H$ does not go to cell 12 too frequently, even though that is the safest. Instead, he seems to go to cell 5 much more frequently. (It is true of course that, once he gets to 12, $H$ stays there much longer.)

For his part, $S$ will never look in the risky or very risky cells (1, 2, 4, 7, 8, 9, 11). In fact, he will concentrate his search, most of the time, in cell 12—unless, of course, he is told that $H$ has just left this cell.

**Example 2.** In Example 2 we use the same 12-cell grid as in Example 1, with the same values for $p_j$ and $q_j$. However, movement is much safer, as evidenced by the following values for $s_{ij}$:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.904 | 0.818 | 0.904 | 0.869 | 0.803 | 0.818 | 0.803 | 0.756 | 0.741 | 0.726 | 0.698 |
| 2 | 0.904 | 0 | 0.904 | 0.869 | 0.904 | 0.869 | 0.803 | 0.818 | 0.803 | 0.726 | 0.741 | 0.726 |
| 3 | 0.818 | 0.904 | 0 | 0.803 | 0.869 | 0.904 | 0.756 | 0.803 | 0.818 | 0.698 | 0.726 | 0.741 |
| 4 | 0.904 | 0.869 | 0.803 | 0 | 0.904 | 0.818 | 0.904 | 0.869 | 0.803 | 0.818 | 0.803 | 0.756 |
| 5 | 0.869 | 0.904 | 0.869 | 0.904 | 0 | 0.904 | 0.869 | 0.904 | 0.869 | 0.803 | 0.818 | 0.803 |
| 6 | 0.803 | 0.869 | 0.904 | 0.818 | 0.904 | 0 | 0.803 | 0.869 | 0.904 | 0.756 | 0.803 | 0.818 |
| 7 | 0.818 | 0.803 | 0.756 | 0.904 | 0.869 | 0.803 | 0 | 0.904 | 0.818 | 0.904 | 0.869 | 0.803 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 0.803 | 0.818 | 0.803 | 0.869 | 0.904 | 0.869 | 0.904 | 0 | 0.904 | 0.869 | 0.904 | 0.869 |
| 9 | 0.756 | 0.803 | 0.818 | 0.803 | 0.869 | 0.904 | 0.818 | 0.904 | 0 | 0.803 | 0.869 | 0.904 |
| 10 | 0.741 | 0.726 | 0.698 | 0.818 | 0.803 | 0.756 | 0.904 | 0.869 | 0.803 | 0 | 0.904 | 0.818 |
| 11 | 0.726 | 0.741 | 0.726 | 0.803 | 0.818 | 0.803 | 0.869 | 0.904 | 0.869 | 0.904 | 0 | 0.904 |
| 12 | 0.698 | 0.726 | 0.741 | 0.756 | 0.803 | 0.818 | 0.803 | 0.869 | 0.904 | 0.818 | 0.904 | 0 |

For this exercise, seven iterations of the Algorithm were necessary to get results to two decimal places. The main reason is that there is a reasonably high probability of survival while moving locations, so that changing cells really does improve *H*'s expectations. More specifically, since the largest $s_{ji}$ is of the order of 0.9, we expect convergence similar to a geometric series with ratio 0.45 which is not very fast.

We give final results below, dispensing with the intermediate iterations. Note that the quantities $V_j$ are much larger here than in Example 1. One consequence of this is that, by (25), all $D_j = 0$. Thus, *H* will immediately leave his cell if *S* looks there, and, because of this, all $B_j = V_j$.

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_j$ | 0 | 0 | 6.11 | 0 | 6.99 | 5.79 | 0 | 0 | 0 | 7.84 | 0 | 18.16 |
| $V_j$ | 5.90 | 6.28 | 6.21 | 6.17 | 6.25 | 6.33 | 6.05 | 6.28 | 6.34 | 5.60 | 6.14 | 5.51 |
| $A_j$ | 5.90 | 6.28 | 7.14 | 6.17 | 7.16 | 7.19 | 6.05 | 6.28 | 6.34 | 6.90 | 6.14 | 12.72 |
| $B_j$ | 5.90 | 6.28 | 6.21 | 6.17 | 6.25 | 6.33 | 6.05 | 6.28 | 6.34 | 5.60 | 6.14 | 5.51 |

$Y^*=$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0.864 | 0 | 0 | 0 | 0 | 0 | 0 | 0.136 |
| 2 | 0 | 0 | 0.458 | 0 | 0.468 | 0 | 0 | 0 | 0 | 0 | 0 | 0.074 |
| 3 | 0 | 0 | 0 | 0 | 0.462 | 0.470 | 0 | 0 | 0 | 0 | 0 | 0.068 |
| 4 | 0 | 0 | 0 | 0 | 0.869 | 0 | 0 | 0 | 0 | 0 | 0 | 0.131 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0.882 | 0 | 0 | 0 | 0 | 0 | 0.118 |
| 6 | 0 | 0 | 0.462 | 0 | 0.472 | 0 | 0 | 0 | 0 | 0 | 0 | 0.066 |
| 7 | 0 | 0 | 0 | 0 | 0.548 | 0 | 0 | 0 | 0 | 0.369 | 0 | 0.083 |
| 8 | 0 | 0 | 0 | 0 | 0.884 | 0 | 0 | 0 | 0 | 0 | 0 | 0.116 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0.893 | 0 | 0 | 0 | 0 | 0 | 0.107 |
| 10 | 0 | 0 | 0 | 0 | 0.890 | 0 | 0 | 0 | 0 | 0 | 0 | 0.110 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.860 | 0 | 0.140 |
| 12 | 0 | 0 | 0 | 0 | 0.367 | 0.381 | 0 | 0 | 0 | 0.252 | 0 | 0 |

$X^*=$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0.409 | 0 | 0 | 0 | 0 | 0 | 0 | 0.591 |
| 2 | 0 | 0 | 0.205 | 0 | 0.231 | 0 | 0 | 0 | 0 | 0 | 0 | 0.563 |
| 3 | 0 | 0 | 0 | 0 | 0.021 | 0.378 | 0 | 0 | 0 | 0 | 0 | 0.601 |
| 4 | 0 | 0 | 0 | 0 | 0.368 | 0 | 0 | 0 | 0 | 0 | 0 | 0.632 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0 | 0 | 0 | 0 | 0 | 0.318 | 0 | 0 | 0 | 0 | 0 | 0.682 |
| 6 | 0 | 0 | 0.136 | 0 | 0.172 | 0 | 0 | 0 | 0 | 0 | 0 | 0.692 |
| 7 | 0 | 0 | 0 | 0 | 0.223 | 0 | 0 | 0 | 0 | 0.164 | 0 | 0.613 |
| 8 | 0 | 0 | 0 | 0 | 0.238 | 0 | 0 | 0 | 0 | 0 | 0 | 0.762 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0.209 | 0 | 0 | 0 | 0 | 0 | 0.791 |
| 10 | 0 | 0 | 0 | 0 | 0.198 | 0 | 0 | 0 | 0 | 0 | 0 | 0.802 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.084 | 0 | 0.916 |
| 12 | 0 | 0 | 0 | 0 | 0.334 | 0.535 | 0 | 0 | 0 | 0.131 | 0 | 0 |

In this case, the situation is somewhat simpler than in Example 1. We find that:

(1) $H$ will leave immediately from any of the risky or very risky cells (1, 2, 4, 7, 8, 9, 11) for one of thesafe or very safe cells (3, 5, 6, 10, 12).
(2) If $H$ is in one of the safe cells (3, 5, 6, 10), he will remain there about 6 to 8 time units, and move to one of the other safe or very safe cells.
(3) If $H$ is in cell 12, he will stay there for 18 time units, and then move to one of the safe cells.
(4) $H$ will always leave immediately from any cell if $S$ looks in that cell.

In contrast to the situation we saw in Example 1, there is now a non-negligible probability that $H$ will survive long enough to move—about 34–47% from the safe cells, and 19% from the very safe cell.

## 4. Long-term considerations

Given what we now know about $H$'s best practices for remaining hidden, where should $S$ concentrate his search efforts? The answer to this question will be relatively straightforward if $S$ has information on $H$'s last hiding location. In this case, $H$'s optimal hiding strategy $\mathbf{y}_j^*$ will tell $S$ where $H$ is likely to go next, and based on this, $S$ will be able to compute his own optimal search strategy, $\mathbf{x}_j^*$. (See Appendix A for the full derivation of these equilibrium strategies). What if $S$ has no indication of where $H$ may have been hiding last, only that he was and is somewhere within the area encompassed by our 12 possible cells? How should he concentrate his search efforts in this case? We can begin to answer this question by approaching this problem as a Markov process.

We begin to consider this by noting that, if $S$ has not been looking at all, then (by our Assumption 1 in the Introduction) $H$ will know only that $S$ is not looking in the right cell. Thus $H$ will move according to the times $T_j$. Now, the vector $\mathbf{y}_j^* = (y_{1j}^*, \ldots, y_{nj}^*)$ gives $H$'s next-cell probabilities: given that $H$ is in cell $j$, there is probability $y_{ij}^*$ that he will next move to cell $i$. Now, let $\Delta$ be a small interval of time (small relative to $T_j$).[4] If all that is known about $H$ is that he is in cell $j$ (but not how long he has been there) then (assuming he is not caught during this small interval) the probability that he will still be in $j$ at the end of the interval is $\delta_j = 1 - \Delta/T_j$. There is probability $1 - \delta_j$ that $H$ will leave in that interval, and, conditional on this (and also assuming $H$ is not caught), there is probability $y_{ij}^*$ that he will then move to cell $i$. Thus the vector $\mathbf{z}_j^* = (z_{1j}^*, \ldots, z_{nj}^*)$, given by

$$z_{ij} = \begin{cases} (1-\delta_j)y_{ij}^* & \text{if } i \neq j, \\ \delta_j & \text{if } i = j \end{cases} \tag{34}$$

represents $H$'s transition probabilities expressed in terms of a Markov chain.

It is of course possible, however, that $T_j = 0$. For such $j$, set

$$z_{ij} = \begin{cases} y_{ij}^* & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases} \tag{35}$$

[4] See Appendix B for more on the choice of this $\Delta$.

Now, the $n \times n$ matrix

$$\mathbf{Z} = (z_{ij})$$

whose columns are the given vectors, is then the transition matrix for this Markov process. (Note that the transition values of this matrix will depend, in part, on the particular $\Delta$ chosen. The $\Delta$ should be chosen small by comparison to all the non-zero $T_j$.)

Since $\mathbf{Z}$ is the transition matrix for a Markov chain, one of its eigenvalues is equal to 1. In the non-degenerate case, this eigenvalue will have multiplicity 1, and its eigenvector $\mathbf{u}$ will have all its components non-negative. (See, e.g., [14] for this.) This is the *Frobenius* eigenvector for $\mathbf{Z}$, and is a non-zero solution of the equation

$$\mathbf{u} = \mathbf{Z}\mathbf{u}.$$

Normalized so that its components sum to 1, this $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ represents the long-run probabilities that $H$ will be in each of our 12 possible cells. Thus, in the absence of any further information regarding $H$'s location (some recent sightings of $H$, say), $\mathbf{u}$ gives us the likely(probabilistic) position of $H$.

A last question arises: can we, from the data, find also a long-term distribution for $S$? In the long run, how frequently will $S$ be in each of the cells?

Note that the answer to this question is not as simple as the previous one. In the previous case, we had a standard Markov process, since $H$'s moves depend on his current state. On the other hand, $S$'s moves do not depend on his state; whereas, in our model, $S$ looks in a new cell which depends on the cell which $H$ had previously occupied. Thus $S$'s long-term frequency depends on the probability that $H$ moves from the given cells.

To analyze this, we note that, if (for example) $H$ is in cell $j$ twice as frequently as in cell $k$, this does not mean that he moves out of (or into) cell $j$ twice as frequently as out of cell $k$. (It might mean, instead, that, each time he visits, he stays in cell $j$ twice as long as in cell $k$.) Thus, the ratio of number moves into $j$ to moves into $k$ should be equal to

$$(u_j/T_j)/(u_k/T_k).$$

In the long run, the number of times that $H$ enters cell $j$ (as a fraction of all his moves) is given by

$$\varphi_j = \frac{u_j/T_j}{\sum_k u_k/T_k}. \tag{36}$$

Since the vector $\boldsymbol{\varphi}$ tells us the frequency with which $H$ visits (and leaves) the given cells, and the matrix $X^*$ tells us where $S$ will search next, it follows that the product

$$\mathbf{w} = \boldsymbol{\varphi} X^* \tag{37}$$

tells us how frequently $S$ looks in the given cells.

**Example 3.** Consider the data of Example 1. Letting $\Delta = 1$, we obtain the transition matrix $\mathbf{Z}$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 887 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 027 | 231 | 963 | 0 | 015 | 012 | 0 | 0 | 0 | 0 | 0 | 005 |
| 4 | 0 | 0 | 0 | 943 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 043 | 386 | 019 | 021 | 964 | 020 | 514 | 381 | 033 | 017 | 072 | 008 |
| 6 | 028 | 247 | 011 | 014 | 014 | 962 | 0 | 244 | 019 | 011 | 0 | 005 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 939 | 0 | 0 | 0 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0 | 0 | 0 | 015 | 0 | 0 | 330 | 264 | 0 | 967 | 043 | 005 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 866 | 0 |
| 12 | 015 | 135 | 007 | 007 | 007 | 006 | 157 | 111 | 009 | 005 | 019 | 977 |

(in which all entries should be divided by 1000). It is easily checked that this matrix has eigenvalue 1. The corresponding Frobenius eigenvector is then

$$\mathbf{u} = (0, 0, 0.223, 0, 0.303, 0.216, 0, 0, 0, 0.034, 0, 0.224)$$

which represents the long-term probabilities that $H$ will be in any one of our 12 possible cells. We note that $H$ spends more time in the safe cell 5 than in the very safe cell 12. The reason for this is that $S$ has more to gain from looking in cell 12 than in 5, and so is likely to spend more time looking in 12. The result is that cell 5 is a better hiding place.

As far as $S$'s long-term search strategies, we can now compute the vector $\boldsymbol{\varphi}$:

$$\boldsymbol{\varphi} = (0, 0, 0.242, 0, 0.325, 0.247, 0, 0, 0, 0.034, 0, 0.153).$$

And from this, the vector $\mathbf{w}$:

$$\mathbf{w} = (0, 0, 0.080, 0, 0.153, 0.138, 0, 0, 0, 0.042, 0, 0.587).$$

Thus, $S$ spends more than half his time looking in cell 12. This is so, even though $H$ spends more time in other cells, for example in cell 5. The reason is that, since $q_{12}$ is so much smaller than the other $q_j$, it is important to look here. But it is precisely for that reason that $H$ does not spend much time there.

**Example 4.** Consider the data of Example 2. Letting $\Delta = 1$, we obtain the transition matrix $\mathbf{Z}$:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 458 | 836 | 0 | 0 | 080 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 864 | 468 | 075 | 869 | 857 | 082 | 548 | 884 | 0 | 114 | 0 | 020 |
| 6 | 0 | 0 | 077 | 0 | 126 | 826 | 0 | 0 | 893 | 0 | 0 | 021 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 369 | 0 | 0 | 872 | 860 | 014 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 136 | 074 | 011 | 131 | 017 | 012 | 083 | 116 | 107 | 014 | 140 | 945 |

In this case, the Frobenius eigenvector is

$$\mathbf{u} = (0, 0, 0.154, 0, 0.308, 0.316, 0, 0, 0, 0.022, 0, 0.200).$$

In this case, $H$ is most likely to be found in cells 6 or 5, followed by cells 12, 3, and 10. He will never spend time in cells 1, 2, 4, 7, 8, 9, and 11.

Finally, the vector of long-term search patterns is

$$\mathbf{w} = (0, 0, 0.054, 0, 0.102, 0.213, 0, 0, 0, 0.010, 0, 0.620).$$

## Appendix A. A simple search game

We consider a game in which a hider $(H)$ can hide in any of $n$ cells. A searcher ($S$) looks for him in any one of the cells. $H$ wishes to maximize the time to capture. Assuming $H$ hides in cell $j$, let $\sigma_j$ be the expected time to capture if $S$ looks in cell $j$, and let $\tau_j$ be the expected time to capture if $S$ looks in a different cell. We will assume that, for each $j$, $0 < \sigma_j < \tau_j$.

We represent this game by an $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \sigma_j \quad \text{if } i = j, \\ \qquad \tau_j \quad \text{if } i \neq j.$$

Each row or column of the matrix is a *pure strategy* of the game. It is understood that $S$ chooses the row, $i$, while $H$ chooses the column, $j$. The payoff $a_{ij}$ is the expected time to capture of $H$, which $S$ wishes to minimize (and $H$, to maximize). [The reader will note that we are here going against game-theoretic convention, which traditionally has the row-player seeking to maximize, while the column-player seeks to minimize, the payoff.]

We look here for optimal strategies of the two-person game. A *mixed strategy* for $S$ is defined as a vector $\mathbf{x} = (x_1, \ldots, x_n)$, with non-negative components whose sum is equal to 1. Similarly, a *mixed strategy* for $H$ is a vector $\mathbf{y} = (y_1, \ldots, y_n)$, also with non-negative components adding to 1. The components of $\mathbf{x}$ are the probabilities that $S$ will look in each of the cells; those of $\mathbf{y}$, the probabilities that $H$ will hide in each of the cells.

Suppose $S$ chooses the mixed strategy $\mathbf{x}$ while $H$ chooses column $j$. Then the expected payoff is

$$E(\mathbf{x}, j) = \sum_i x_i a_{ij}.$$

Similarly, if $H$ chooses the mixed strategy $\mathbf{y}$ while $S$ chooses row $i$, then the expected payoff is

$$E(i, \mathbf{y}) = \sum_j a_{ij} y_j.$$

By the *minimax theorem*, there exist mixed strategies $\mathbf{x}^*$ and $\mathbf{y}^*$, and a number, $v$, such that, for each $j = 1, \ldots, n$,

$$E(\mathbf{x}^*, j) \leqslant v$$

and, for each $i = 1, \ldots, n$,

$$E(i, \mathbf{y}^*) \geqslant v.$$

Such an $\mathbf{x}^*$ and $\mathbf{y}^*$ are the *optimal strategies* for the game, while $v$ is the *value* of the game. It is not difficult to prove that, if $x_i^* > 0$, then $E(i, \mathbf{y}^*) = v$; similarly, if $y_j^* > 0$, then $E(\mathbf{x}^*, j) = v$.

To look for the solution of this game (optimal strategies and a value), we will list the cells in order of decreasing $\tau_j$, so that $\tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_n$. For purposes of our analysis, set also $\tau_{n+1} = 0$.

Next, for each $k$, $1 \leqslant k \leqslant n$, define

$$L_k = \sum_{i=1}^{k} 1/(\tau_i - \sigma_i), \tag{A.1}$$

$$v_k = \frac{\sum_{i=1}^{k} \tau_i/(\tau_i - \sigma_i) - 1}{L_k}. \tag{A.2}$$

**Lemma 1.** *Suppose that, for some $k$, $1 \leqslant k \leqslant n$, we have $\tau_k \geqslant v_k \geqslant \tau_{k+1}$. In this case, the game has value $v_k$, and optimal strategies $\mathbf{x}^*$, $\mathbf{y}^*$ for $S$ and $H$, respectively, given by*

$$x_i^* = \frac{\tau_i - v_k}{\tau_i - \sigma_i} \quad \textit{for } 1 \leqslant i \leqslant k, \tag{A.3}$$

*and*

$$\begin{aligned} &0 \quad \textit{for } k+1 \leqslant i \leqslant n \\ &y_j^* = 1/(\tau_j - \sigma_j)L_k \quad \textit{for } 1 \leqslant j \leqslant k \\ &\qquad 0 \quad \textit{for } k+1 \leqslant j \leqslant n. \end{aligned} \tag{A.4}$$

**Proof.** We note first of all that all components of $\mathbf{x}^*$ and $\mathbf{y}^*$ thus defined are non-negative.

Next, note that

$$\sum y_j^* = \sum{}' [1/(\tau_j - \sigma_j)L_k] = \left[\sum{}' 1/(\tau_j - \sigma_j)\right] L_k,$$

where the prime symbol on the summation means that we only sum over those $j$ from 1 to $k$. But the sum inside the bracket in this last expression is precisely $L_k$. Thus the sum of the components is equal to 1, and we see that $\mathbf{y}^*$ is a strategy.

Now, for $1 \leqslant i \leqslant k$,

$$\begin{aligned} E(i, \mathbf{y}^*) &= \sum_j a_{ij} y_j^* \\ &= \sum{}' [a_{ij}/(\tau_j - \sigma_j)L_k] \\ &= \left[\sigma_i/(\tau_i - \sigma_i) + \sum_{j \neq i}{}' \tau_j/(\tau_j - \sigma_j)\right] \Big/ L_k \\ &= \left[(\sigma_i - \tau_i)/(\tau_i - \sigma_i) + \sum{}' \tau_j/(\tau_j - \sigma_j)\right] \Big/ L_k \\ &= v_k. \end{aligned}$$

On the other hand, for $i > k$,

$$E(i, \mathbf{y}^*) = \sum{}' \tau_j y_j^* \geqslant \tau_k \sum{}' y_j^* = \tau_k \geqslant v_k.$$

Thus $E(i, \mathbf{y}^*) \geqslant v_k$ for all $i$.

Next, note that, for $1 \leqslant i \leqslant k$,

$$\begin{aligned} L_k x_i^* &= \frac{L_k \tau_i - L_k v_k}{\tau_i - \sigma_i} \\ &= \frac{1 - \sum' \tau_j/(\tau_j - \sigma_j) + \tau_i \sum' 1/(\tau_j - \sigma_j)}{\tau_i - \sigma_i} \\ &= 1/(\tau_i - \sigma_i) - \sum{}' (\tau_j - \tau_i)/(\tau_j - \sigma_j)(\tau_i - \sigma_i), \end{aligned}$$

where, once again, the prime symbol on the summation means that the sum is taken over all $j$ from 1 to $k$.

If we now sum these terms with respect to $i$, from 1 to $k$, we obtain

$$\sum L_k x_i^* = \sum{}' \left\{1/(\tau_i - \sigma_i) - \sum{}' (\tau_j - \tau_i)/(\tau_j - \sigma_j)(\tau_i - \sigma_i)\right\}.$$

Note that each of the terms

$$(\tau_j - \tau_i)/(\tau_j - \sigma_j)(\tau_i - \sigma_i)$$

appears once, with positive sign, in the expression for $x_i^*$, and once again, with negative sign, in the expression for $x_j^*$. Thus, in adding, all of these terms cancel out, and all we are left with is the simple sum

$$\sum L_k x_i^* = \sum{}' 1/(\tau_i - \sigma_i).$$

The expression on the right-hand side here is precisely $L_k$. Thus the components of $\mathbf{x}^*$ add up to 1, and we see $\mathbf{x}^*$ is a strategy.

Suppose, next, that $1 \leqslant j \leqslant k$. Then

$$\begin{aligned} E(\mathbf{x}^*, j) &= \sum a_{ij} x_i^* = \sigma_j x_j^* + \tau_j (1 - x_j^*) \\ &= \tau_j - (\tau_j - \sigma_j) x_j^* \\ &= \tau_j + \frac{(\tau_j - \sigma_j)(v_k - \tau_j)}{\tau_j - \sigma_j} \\ &= v_k. \end{aligned}$$

On the other hand, for $j > k$,

$$\sum a_{ij} x_i^* = \tau_j \leqslant v_k.$$

Thus $E(\mathbf{x}^*, j) \leqslant v_k$ for all $j$. It follows that $\mathbf{x}^*$ and $\mathbf{y}^*$ are the optimal strategies, and $v_k$ the value, of the game. $\square$

Note: if $k = n$, we need not worry about the cases where either $i$ or $j$ is greater than $k$.

We next prove that we can obtain the desired $k$ in a finite number of steps. The idea is to discard successively the most dangerous cells (i.e. those for which $\tau$ is smallest). These are cells in which $H$ will never seek sanctuary. This gives us the following algorithm:

**Algorithm.**

1. Let $k = n$.
2. Let $v = v_k$, computed by (A.1)–(A.2).
3. If $v \leqslant \tau_k$, proceed to step 6.
4. If $v > \tau_k$, let (new)$k = \max\{j | \tau_j > v\}$.
5. Return to step 2.
6. Compute $\mathbf{x}^*$ by (A.3) and $\mathbf{y}^*$ by (A.4).

**Lemma 2.**

*The algorithm terminates after at most n iterations.*

**Proof.** We know the $\tau_j$'s are in decreasing order. If, then, $\tau_k > v$, it must follow, from step 4 above, that (new)$k \leqslant$ (old)$k-1$. Thus $k$ decreases by at least one unit at each iteration.

On the other hand, we see from (A.2) that $v_k$ is smaller than a weighted mean of the terms $\tau_1, \ldots, \tau_k$, and so we always have $v < q_1$. Thus the procedure terminates, at the latest, when $k = 1$, and this will require at most $n$ iterations.

Now, suppose that the procedure terminates at iteration $s$, with $v_k \leqslant \tau_k$. We need to prove that, at that point, $v_k \geqslant \tau_{k+1}$. If $s = 1$, then the current value of $k$ is $n$, and nothing more need be proved.

If $s \geqslant 2$, then for purposes of notation, we let $h$ be the value of $k$ at iteration $s - 1$, and $K$ be the value of $k$ at iteration $s$. By step 4 in the algorithm, we know $v_h \geqslant \tau_{K+1}$. Thus

$$\frac{\sum_{i=1}^{h} \tau_i / (\tau_i - \sigma_i)}{L_h} \geqslant \tau_{K+1},$$

$$\sum_{i=1}^{h} \tau_i / (\tau_i - \sigma_i) \geqslant \tau_{K+1} L_h,$$

$$\sum_{i=1}^{h} \tau_i / (\tau_i - \sigma_i) \geqslant \sum_{i=1}^{h} \tau_{K+1} / (\tau_i - \sigma_i),$$

$$\sum_{i=1}^{K} \tau_i / (\tau_i - \sigma_i) + \sum_{i=K+1}^{h} \tau_i / (\tau_i - \sigma_i) \geqslant \sum_{i=1}^{K} \tau_{K+1} / (\tau_i - \sigma_i) + \sum_{i=K+1}^{h} \tau_{K+1} / (\tau_i - \sigma_i).$$

Now (because of the way the $\tau_j$ were ordered) the second term on the left-hand side of this inequality is at most equal to the second term on the right-hand side. It follows that subtracting those (second) terms will decrease the left-hand side less than it decreases the right-hand side, and consequently

$$\sum_{i=i}^{K} \tau_i/(\tau_i - \sigma_i) \geqslant \tau_{K+1} L_K.$$

But this means $v_K \geqslant \tau_{K+1}$. Thus $K$ is the desired value of $k$, and we have the solution of the game. □

**Example.** Consider a game with six cells, and

$$\sigma_1 = 3, \quad \tau_1 = 10,$$

$$\sigma_2 = 2, \quad \tau_2 = 8,$$

$$\sigma_3 = 3, \quad \tau_3 = 7,$$

$$\sigma_4 = 4, \quad \tau_4 = 5,$$

$$\sigma_5 = 3, \quad \tau_5 = 4,$$

$$\sigma_6 = 1, \quad \tau_6 = 2.$$

Starting with $k = 6$, we have

$$L_6 = 0.143 + 0.167 + 0.25 + 1 + 1 + 1 = 3.560,$$

$$v_6 = (1.429 + 1.333 + 1.75 + 5 + 4 + 2 - 1)/3.560 = 4.077.$$

Since $v_6 > \tau_6$, we proceed to step 4 in our algorithm. Noting that $\tau_4 > v > \tau_5$, we must now choose a new $k = 4$. Now we have

$$L_4 = 0.143 + 0.167 + 0.25 + 1 = 1.560,$$

$$v_4 = (1.429 + 1.333 + 1.75 + 5 - 1)/1.560 = 5.456.$$

Now we see $v_4 > \tau_4$, and so we once again proceed to step 4. Noting that $\tau_3 > v > \tau_4$, we now choose a new $k = 3$. Now we have

$$L_3 = 0.1429 + 0.1667 + 0.25 = 0.5596,$$

$$v_4 = (1.429 + 1.333 + 1.75 - 1)/0.5596 = 6.276.$$

Now we see $v_3 < \tau_3$, so we proceed to step 6. The value of the game is then, 6.276, and the optimal strategies are

$$x_1^* = (10 - 6.276)/7 = 0.532,$$

$$x_2^* = (8 - 6.276)/6 = 0.287,$$

$$x_3^* = (7 - 6.276)/4 = 0.181,$$

$$x_4^* = x_5^* = x_6^* = 0$$

for $S$, and

$$y_1^* = 1/7(0.5596) = 0.255,$$

$$y_2^* = 1/6(0.5596) = 0.298,$$

$$y_3^* = 1/4(0.5596) = 0.447,$$

$$y_4^* = y_5^* = y_6^* = 0$$

for $H$.

## Appendix B. The role of the time interval Δ

In Section 4, above, we introduce a quantity, $\Delta$, "a small interval of time," saying only that it should be positive, and small relative to all the $T_j$. It may seem that we have, in effect, discretized time. This is not so.

In fact, the value of $\Delta$ is not important, so long as it is positive and smaller than the smallest of the $T_j$'s. To understand why this is so, note first of all that the Frobenius eigenvector (representing as it does the long-term probabilities) will depend only on those cells to which $H$ moves—essentially, what we have called the safe cells, and what in Markov theory are called the persistent states. For each of these, of course, $T_j > 0$. (In our example, these would be cells 3, 5, 6, 10, and 12.) Thus, we can disregard all other rows and columns of the transition matrix $Z$.

Now, restricted to the safe cells' rows and columns, we see from Eq. (34) that $Z = I + \Delta H$, where $H$ is given by $h_{ii} = -1/T_i$, and $h_{ij} = y_{ij}/T_i$ for $i \neq j$. Remembering that the Frobenius eigenvector corresponds to the eigenvalue 1, we have the equation $Z\mathbf{u} = \mathbf{u}$, which becomes

$$(I + \Delta H)\mathbf{u} = \mathbf{u},$$

$$\mathbf{u} + \Delta H\mathbf{u} = \mathbf{u},$$

$$\Delta H\mathbf{u} = \mathbf{0}.$$

Since $\Delta > 0$, it follows that $H\mathbf{u} = \mathbf{0}$, and hence the same $\mathbf{u}$ (namely, one in the null-space of $H$) will satisfy the eigenvalue equation, regardless of the value of $\Delta$. Assuming that this is a regular Markov chain, i.e., that it is possible (using the optimal strategies) to go from any one of the persistent states to any other (possibly employing several moves), this eigenvector will be unique (up to multiplication by a constant). Thus the Frobenius eigenvector is independent of the choice of $\Delta$. (Note, however, that if $\Delta$ is too large, or if it is negative, then the matrix $W$ will have some negative entries, and can no longer be considered a transition matrix.)

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