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Cases where the Penrose limit theorem does not hold

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Abstract

Penrose's limit theorem (PLT, really a conjecture) states that the relative power measure of two voters tends asymptotically to their relative voting weight (number of votes). This property approximately holds in most of real life and in randomly generated WVGs for various measures of voting power. Lindner and Machover prove it for some special cases; amongst others they give a condition for this theorem to hold for the Banzhaf–Coleman index for a quota of 50%. We show here, by counterexamples, that the conclusion need not hold for other values of the quota. In doing this, we present an analytic proof of a counterexample recently given by Chang et al. using simulation techniques.
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JEL classification: C71; B23

1. Cases where the Penrose limit theorem does not hold

In his booklet, L.S. Penrose (1952) says that in simple weighted voting games (WVGs), if the number of voters increases indefinitely while the quota is pegged at one half of the total weight, then – under certain conditions – the ratio between the voting powers (as measured by him) of any two voters converges to the ratio between their weights. The measure he proposes later came to be known as the '(absolute) Banzhaf–Coleman index'. However, he gave no rigorous proof and there are counterexamples to his claim. Lindner and Machover (L/M, 2004) give (sufficient) conditions for this theorem to hold for the Shapley–Shubik (S–S) and the Banzhaf–Coleman (B–C) indices.

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Specifically, L/M assume an increasing chain $N_1 \subset N_2 \subset \ldots \subset N_n \subset N_{n+1} \subset \ldots$ of player sets, with players having weights (number of votes) as follows:

(a) Each player has an integer number of votes.
(b) There is an upper bound, α, such that no player has more than α votes.
(c) The set of weights that appear infinitely often (as the games get larger and larger) has the greatest common divisor 1.
(d) Decisions require a bare majority of votes (i.e. a winning coalition is any coalition with more than half the votes).

With these conditions, L/M prove that, as the size of the games grows without bound, then $\beta_1/\beta_2 \rightarrow w_1/w_2$, where β_i is the B–C power of a voter with w_i votes.

Let us analyze these conditions.

We consider (a) a reasonable simplifying assumption. (Note, also, that without this, condition (c) is meaningless.)

Next, we see that (b) is a reasonable condition. In fact, suppose this condition does not hold. Then, as the games grow larger, new, stronger players with arbitrarily large weights may join. One possibility then is that there may be a player with more than the required quota of votes. This player would assume dictatorial power. Even without dictatorial power, these arbitrarily large players can invalidate the PLT.

Example 1. Consider a sequence of games as follows: for a given integer n, the nth game has one player each with $4, 4^2, 4^3, \ldots., 4^n$ votes, and $(2 \times 4^n + 7)/3$ players with one vote each. Clearly, these games form an increasing chain, and the number of players with one vote each grows without bound, so that condition (c) above is satisfied. It is easily verified that the nth game has $2 \times 4^n + 1$ votes. Assuming that a bare majority is necessary, the strongest player, with 4^n votes, is not quite a dictator, but needs only one additional vote to win. It follows that all players other than the strongest one are equally powerful. Thus the ratio of power for any two players, no matter how many votes they have, will eventually be equal to 1.

As for condition (c), it is not a necessary condition, in the sense that the convergence may hold without it, but in its absence, something may go wrong, as shown by L/M's counterexample:

Example 2. Let there be one player with one vote, and n players with two votes each. Thus there is a total of $2n+1$ votes. Assuming a bare majority is necessary, the winning coalitions are those with $n+1$ or more votes. Now, if n is even, the player with a single vote is as powerful as any of the others. If, on the other hand, n is odd, then the player with the single vote becomes a dummy, i.e. has power 0. Thus in this case, the ratio of voting power does not even converge, but oscillates between 0 and 1.

The question is whether condition (d) is necessary. More specifically, where a super-majority, say 2/3 or 3/4 of the votes, is necessary to win, will Penrose's theorem hold?

In a recent communication, Chang et al. (2006) showed by simulation that, where super-majorities are required, the PLT does not seem to hold. We here examine a simple case to see analytically why this happens.

Example 3. We consider the following chain of games.

(a) The nth game has $n-2$ players with 1 vote each (the *weak* players), and one with 2 votes (the *strong* player). Thus there are n votes in all.

(b) A real number q is given, $0<q<1$.
(c) Winning coalitions are those with at least nq votes.

It may be seen that this satisfies conditions (a), (b), and (c) above. On the other hand, condition (d) above corresponds to the special case $q=1/2$.

It is now straightforward to calculate the B–C index of power. Assume that the number of votes needed to win is k. In that case, the strong player has a swing[2] for any coalition containing either $k-2$ or $k-1$ of the $n-2$ weak voters. Thus his number of swings is

$$\beta_{\text{strong}} = \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!}. \tag{1}$$

On the other hand a weak player has a swing in two different cases: (a) any coalition containing the strong player and $k-3$ out of the remaining $n-3$ weak players, and (b) any coalition containing $k-1$ of the remaining $n-3$ weak players (but not the strong player). Thus his number of swings is

$$\beta_{\text{weak}} = \binom{n-3}{k-3} + \binom{n-3}{k-1} = \frac{(n-3)!}{(k-3)!(n-k)!} + \frac{(n-3)!}{(k-1)!(n-k-2)!}. \tag{2}$$

We wish to compare these two quantities. Since we are only interested in the ratio between them (to see whether the weak voter's power is one half the strong one's), we can multiply through by the least common denominator, which in this case is $(k-1)!(n-k)!$. Thus

$$\begin{aligned} (k-1)!(n-k)!\beta_{\text{strong}} &= (n-1)! \\ (k-1)!(n-k)!\beta_{\text{weak}} &= (n-3)![(k-2)(k-1) + (n-k)(n-k-1)]. \end{aligned}$$

Therefore we have the proportion

$$\beta_{\text{weak}}/\beta_{\text{strong}} = [(k-2)(k-1) + (n-k)(n-k-1)]/(n-1)(n-2).$$

Let us see how this behaves as the number of players goes to infinity. In the limit, k goes to nq, and only the square terms above need be considered. Thus,

$$\beta_{\text{weak}}/\beta_{\text{strong}} \rightarrow [n^2q^2 + n^2(1-q)^2]/n^2$$

and we see that the limiting ratio is

$$\beta_{\text{weak}}/\beta_{\text{strong}} \rightarrow q^2 + (1-q)^2. \tag{3}$$

Now, for $q=0.5$, this is indeed 1/2. For different values of q, however, this will be larger than 1/2. For example, if $q=2/3$, we get a limiting ratio of 5/9, and for $q=3/4$, the limiting ratio is 5/8.

[2] A *swing* for a given player is a losing coalition that becomes winning when this player joins. The B–C index effectively counts the number of swings.

Finally, to see how this behaves for relatively small n, let us look at the situation that holds with $q=2/3$. We have the following table:

n	k	$(k-2)(k-1)+(n-k)(n-k-1)$	$(n-1)(n-2)$	$\beta_{\text{weak}}/\beta_{\text{strong}}$
3	2	0	2	0
4	3	2	6	0.333
5	4	6	12	0.5
6	4	8	20	0.4
7	5	14	30	0.467
8	6	22	42	0.524
9	6	26	56	0.464
10	7	36	72	0.5
11	8	48	90	0.533
12	8	54	110	0.491
13	9	68	132	0.515
14	10	84	156	0.538
15	10	92	182	0.505
16	11	110	210	0.524
17	12	130	240	0.542
18	12	140	272	0.515
19	13	162	306	0.529
20	14	186	342	0.544
21	14	198	380	0.521
22	15	224	420	0.533
23	16	252	462	0.545
30	20	432	812	0.532
42	28	884	1640	0.539
102	68	5544	10100	0.549.

The behavior is easy to analyze. We see that, if grouped by threes, the ratio is least for $n\equiv 0$ (mod 3), increasing for $n\equiv 1$, and even more so for $n\equiv 2$. The reason here is very simple. Indeed, when $n\equiv 0$, the number of votes is a multiple of 3, and so the ratio $q=2/3$ is attainable exactly. For other n, this is not so: for $n=7$, for example, 5 votes are necessary to win, which means the actual necessary quota is 0.714 rather than 0.667. Similarly, for $n=8$, 6 votes are necessary, and now the actual quota is 0.750. Now, from (3) above, we know that the ratio increases with q over the interval [1/2, 1]. This explains the behavior *within each group of 3*. Note, however, that the differences within each group decrease (and will, in fact, approach 0) as n increases.

Suppose, then, that we consider only the cases where $n\equiv 0$ (mod 3), since these are the ones that admit a quota of exactly 2/3. For these, we notice that, for smaller games, the strong player is disproportionately strong (i.e. the ratio is less than 0.5). This effect decreases until, above $n=15$, this same strong player becomes disproportionately weak. The ratio will approach $5/9\approx 0.5556$ as $n\to\infty$.

We do this once again with $q=0.75$. In this case we will only consider $n\equiv 0$ (mod 4), so that the exact ratio 3/4 is attainable.

n	k	$(k-2)(k-1)+(n-k)(n-k-1)$	$(n-1)(n-2)$	$\beta_{\text{weak}}/\beta_{\text{strong}}$
4	3	2	6	0.333
8	6	22	42	0.524
12	9	62	110	0.564
16	12	122	210	0.581

20	15	202	342	0.591
32	24	562	930	0.604
100	75	6002	9702	0.619

We see the same sort of behavior. The limiting ratio is 5/8=0.625.

Example 4. A more general case

More generally, let us consider the following situation:

There is a relative quota, q, $0<q<1$. This is kept fixed as the number of voters increases.

There is one strong voter with s votes, where $s \geq 2$ is an integer. This s is kept fixed.

There are $n-s$ weak voters with one vote each. This n increases without bound.

For each n, we let k be the smallest integer $\geq qn$. Thus the nth game has a total of n votes, with k votes required for a winning coalition.

As before, we let β_{weak} be the number of swings for a weak voter, and β_{strong} be the number of swings for the strong voter.

Now, for a small voter, a swing will be either (1) a set with $k-1$ of the remaining $n-s-1$ small voters, or (2) a set with the large voter, and $k-s-1$ of the remaining small voters. Thus,

$$\beta_{\text{weak}} = \gamma_1 + \gamma_2$$

where

$$\gamma_1 = \binom{n-s-1}{k-1}$$

and

$$\gamma_2 = \binom{n-s-1}{k-s-1}.$$

For the large voter, a swing will be a set with at least $k-s$ and at most $k-1$ of the $n-s$ small voters. Thus

$$\beta_{\text{strong}} = \alpha_1 + \alpha_2 + \ldots + \alpha_s$$

where for each $i=1, 2, \ldots, s$,

$$\alpha_i = \binom{n-s}{k-i}.$$

We now consider the behavior of these quantities as $n \to \infty$.

Note first that

$$\frac{\alpha_{s-1}}{\alpha_s} = \frac{n-k}{k-s+1}.$$

As $n \to \infty$, k/n will approach q, while s becomes negligible, and so

$$\frac{\alpha_{s-1}}{\alpha_s} \to \frac{1-q}{q}.$$

In a similar way, $\alpha_{s-i-1}/\alpha_{s-i} \to (1-q)/q$ for all i, and we see that $\beta_{\text{strong}}/\alpha_s$ will converge to the geometric series

$$\frac{\beta_{\text{strong}}}{\alpha_s} \to \sum\nolimits_{0 \le i \le s-1} [(1-q)/q]^i = \frac{1-[(1-q)/q]^s}{(2q-1)/q}.$$

Next, note that $\gamma_2/\alpha_s = (k-s)/(n-s) \to q$, and also $\gamma_1/\alpha_1 = (n-s-k+1)/(n-s) \to 1-q$. Thus $\gamma_1/\alpha_s \to (1-q)\ [(1-q)/q]^{s-1}$. We conclude that

$$\frac{\beta_{\text{weak}}}{\alpha_s} \to q + (1-q)[(1-q)/q]^{s-1}$$

and thus,

$$\frac{\beta_{\text{weak}}}{\beta_{\text{strong}}} \to \frac{[q^s + (1-q)^s](2q-1)}{q^s-(1-q)^s}. \tag{4}$$

For $q=1/2$, both numerator and denominator in this expression vanish. In this case, an application of L'Hôpital's rule will give us the value $1/s$. Again we see that Penrose's theorem holds for $q=1/2$. Once again however, the ratio is greater[3] than $1/s$ if $q \neq 1/2$. For instance, if we set $s=3$, $q=0.6$, we have $\beta_{\text{weak}}/\beta_{\text{strong}} \to 7/19 \approx 0.3684$, and for $s=3$, $q=0.8$, the ratio converges to $13/21 \approx 0.619$. In all cases (i.e. for each fixed s no matter how large), the ratio approaches 1 as $q \to 1$.

Another thing to notice is that, for fixed $q>0.5$, the limiting ratio (4) approaches $2q-1$ as $s \to \infty$. Thus, for example, with $q=2/3$, we would find that a very strong player, say one with 100 votes, surrounded by 5000 weak players, would only have 3 times as much power as one of the weak players. We find this a very counterintuitive result.[4]

2. Note: contrast with the Shapley–Shubik index

An interesting point is that, for the Shapley value, the ratios will converge to $1/s$, as desired, for this particular chain of games. This result is in accordance with L/M (2004) who prove a general result for the Shapley–Shubik index for $0<q<1$. Specifically, they show that the ratio of voting power converges to the ratio of weights; this convergence holds under conditions (a) and (b), provided the two voters whose S–S indices and weights are compared are "replicative" (i.e., roughly speaking, belong to types that occur sufficiently often).

The obvious question is as to the difference in behavior of the two indices (B–C and S–S). The answer is, more or less, as follows.

Both indices try to measure the probability that a given voter, i, will be decisive, i.e., that she will somehow, be faced with a coalition that represents a *swing* for her: the coalition loses without her, and wins with her votes. The question is then as to how these probabilities are calculated.

[3]The proof of this statement, which is quite technical, is available upon request from the authors. Essentially, it uses the inequality that relates the geometric and arithmetic means. We do not include it here because it is a bit too long and is really not that important to the development of this article.

[4]Essentially, what happens for this case is that the strong player's power can be represented by 100 terms of a rapidly decreasing series, and thus is only about three times the first term.

The B–C measure assumes *a priori* that all voters other than i vote independently of one another, each voting "yes" or "no" with probability 1/2. In terms of the multilinear extension, (MLE, see Owen, 1972, 1975) we obtain the expression

$$\beta_i = F_i(1/2, 1/2, \dots, 1/2) \tag{5}$$

where F_i is the ith partial derivative of the MLE.

On the other hand, as pointed out in Straffin (1982, 297–299), the S–S index can be derived by a model in which a probability t is first obtained from a uniform distribution in [0, 1]; once t is given, the voters vote independently, each voting "yes" or "no" with probabilities t and $1-t$ respectively. In terms of the MLE, this gives us the integral

$$\varphi_i = \int_0^1 F_i(t, t, \dots, t)\mathrm{d}t. \tag{6}$$

This thoroughly probabilistic approach to both indices allows us to apply powerful results of stochastics, primarily the central limit theorem. The answer as to the difference in behavior of the two indices seems to lie in the convergence properties of the sum, X, of several independent identically distributed variables. As is well known, the cumulative distribution of the normalized sum (i.e., subtract the mean, divide by the standard deviation) will converge to the standard normal distribution. However, this deals with the function itself; it does not deal with the ratios of probabilities of small intervals, i.e., with something of the form $\mathrm{Prob}\{a \le X \le b\}/\mathrm{Prob}\{c \le X \le d\}$. In fact, this ratio should still converge properly, so long as the intervals in question remain fixed, or at least do not shift in position too much.

Thus, from (5), the B–C index essentially looks at a distribution with mean at 0.5, and this works very well so long as $q=0.5$. When $q \ne 0.5$, however, the difference between q and the mean, when measured in standard deviations, is of the order of $(q-0.5)\sqrt{n}$. Since this increases without bound, it follows that the behavior is quite different from what one would expect. In analytical terms, the ratios converge pointwise, but not uniformly, and the intervals in question are shifting towards a region of slow convergence.

As opposed to that, we see from (6) the S–S index seems to be looking at many distributions, with means at t, where t takes on all values from 0 to 1. However, it turns out that most of the growth in the MLE will take place near $t=q$, i.e., the partial derivatives will be very close to 0 if t is not close to q. Thus, the integral in question can be replaced, with very small error, if we integrate, instead, over the interval $[t-\varepsilon, t+\varepsilon]$ rather than [0, 1]; this ε goes to 0 as $n \to \infty$. Thus the intervals in question are all very near the mean of the distribution, where convergence of the ratios is fast.

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