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AN INFILTRATION GAME WITH TIME DEPENDENT PAYOFF

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ABSTRACT

The problem of assigning patrol boats, subject to resource constraints, to capture or delay an infiltrator with perishable contraband attempting escape across a long, narrow strait is formulated as a two-sided time sequential game. Optimal mixed strategies are derived for the situation of one patrol boat against one smuggler. Procedures for obtaining numerical solutions for $R > 1$ patrol boats are discussed.

1. INTRODUCTION

This paper describes an application of game theory for examining strategies available to a patrol unit pursuing smugglers of perishable items who attempt escape by crossing a long, narrow strait. A number of applications of game theory to military-type problems have been reported. Recently, Moglewer and Payne [2] discussed an application of two-sided games in examining logistics allocation decisions in a combat setting. Charnes and Schroeder [1] have developed some models of tactical situations in Antisubmarine Warfare. More recently, Pugh [5] has discussed some time-sequential two person zero sum games for treating strategic and tactical decisions within a given time frame. For the application described here, we formulate a twosided time-sequential game where one side, the patrol unit, has limited resources to catch his opponent, an infiltrator, who must make his escape within a fixed time period.

2. A PATROL GAME.

We consider a long, narrow strait where smuggling activity is taking place. Let side A represent a patrol unit whose objective is to capture or reduce the value of contraband held by side B, the infiltrator or smuggler seeking escape by crossing the strait to exit from side A's territory. The contraband held by side B is perishable with a lifetime of M time units; consequently, he must make his escape within M time units in order to benefit from his infiltration. An example of the type of contraband is intelligence information. Side A is under a single command equipped with speedboats containing search radar and communication units. Side B is an individual unit with small motorboats. Although side A has search radar, due to the narrowness of the strait, side B's radar echo will be shadowed by land, thus making radar detection near the shore virtually impossible. Thus, A can detect B only if B is sufficiently far from shore. For obvious reasons, side B only attempts escape at night, and he

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departs from a point near a village or parallel to a village located on the other side of the strait. Although the patrol boats are much faster than B's, the fact that the strait is long and narrow gives side B a chance to cross successfully without being detected.

Viewed as a game, both sides would like to use their "best" strategies. The best strategy for A is that which maximizes the number of boats captured from side B. Side B views his best strategy as the one maximizing the number of trips per boat before capture. We make the following further simplifying assumptions.

Assumptions:

1. Detection information for side A is perfect in the sense that there are no errors, and once detected, B is caught.
2. A's resources are limited to $k < M$ night patrols.
3. B's success requires a single crossing of the strait during the period of M nights.
4. Both A and B know the values of k and M.

The first assumption is merely for simplification to limit the scope of the problem. As we point out later, relaxing this assumption requires only a slight extension. Implicitly, we are assuming that only light traffic exists in the channel, as one would expect for a channel that is being patrolled. Since A's boats are much faster, once B is a sufficient distance from shore to be detected, he can neither cross nor return fast enough if, in fact, he is detected. We are also excluding from our consideration those boats belonging to side B that are crossing into A's territory. Although the second assumption may seem somewhat superficial, such problems of limited resources are becoming realities for many military components.

2.1. One Patrol Boat – One Smuggler

First, we shall consider the situation where each side has a single boat. Side B decides for each night to go or not to go and attempt escape across the channel, while A similarly makes a decision on whether or not to have a patrol in the channel.

We shall denote by $\Gamma(n, k)$, $k < M$, $n = M, M-1, \ldots, 1$, the game determined by the above assumptions for the nth day before the end of the period M. The game matrix for this game is as follows:

A	B Smuggler's Actions	
	Go	No go
Assign patrol	v Game over	$\Gamma(n-1, k-1)$
No patrol	-1 Game over	$\Gamma(n-1, k)$

Whenever side A assigns a patrol and B decides to attempt escape, side A has some probability of catching side B. This probability can be determined by solving a zero sum game [3]. If side B decides to go and A does not have a patrol out, then B wins the game and we assign to A, for convenience, a payoff of -1. Likewise, we let the ultimate payoff to A for capturing B be $+1$. The remaining alternatives result in a loss of one available day, which is of benefit to side A. If side A assigns a patrol and B does not go, then both sides face the game $\Gamma(n-1, k-1)$. If A chooses not to assign a patrol while B elects not to go, then they face the game $\Gamma(n-1, k)$.

Let $g(n, k)$ represent the value of the game $\Gamma(n, k)$. It follows (from [6, p. 173]) that $g(n, k)$ is given by the recursive relationship

$$g(n, k)=\frac{v\cdot g(n-1, k)+g(n-1, k-1)}{v+g(n-1, k+1)-g(n-1, k-1)} \tag{1}$$

with the boundary conditions $g(n, 0)=-1$ and $g(n, n)=v, \forall n>0$. We note that for the last period $(n=1)$, the game matrix is

$$\Gamma(1, 0)=\begin{bmatrix} v & 1 \\ -1 & 1 \end{bmatrix}$$

where $|v|<1$. By dominance, side B must choose the go strategy, which implies that he always elects to go in the last period if he has not attempted escape before. Now if side A has $k=n$ available nights to assign patrols, then clearly he will use them all. Hence, the value of the game is v, since we know that B must go on one of these nights. If side A has $k>n$ available nights for patrol, then with a single patrol boat, he can assign at most n of them.

THEOREM 2.1: The solution to the difference equation (1) for the game $\Gamma(n, k)$, $k<M$, $n=M, M-1, \ldots, 1$ is

$$g^*(n, k)=\frac{k(v+1)-n}{n}. \tag{2}$$

PROOF: The proof follows directly by substituting (1) into (2) and simplifying.

We can now apply this result to our game matrix to obtain

$$\Gamma(n, k)=\begin{bmatrix} v & \dfrac{(k-1)(v+1)-(n-1)}{n-1} \\ & \\ -1 & \dfrac{k(v+1)-(n-1)}{n-1} \end{bmatrix} \tag{3}$$

The optimal mixed strategies for A and B can be determined from (3). Let x_k^n be the probability that A will "assign a patrol" and y_k^n the probability that B will "go" when n nights remain and A's resources allow k patrols until the end of the period. It follows that the optimal choices for these probabilities are given by

$$x_k^n=k/n, \quad (k<M;\ n=M, M-1, \ldots, 1) \tag{4a}$$

$$y_k^n=1/n, \quad (k<M;\ n=M, M-1, \ldots, 1). \tag{4b}$$

We conclude that in order to obtain the value of the game, side A must allocate his available nights that he can assign patrols such that his probability of assigning a patrol is equal to the ratio of the number of search periods available to him to the total number of remaining periods. For side B, a uniform distribution over the remainder of the time period will provide him the value of the game. Note, in particular, that this probability, y_k^n, does not depend on the number of available days that A has for assigning patrols.

EXAMPLE: In order to demonstrate these results, consider the game $\Gamma(n, k)$, whereby in any period if side A allocates a patrol when side B has elected to "go," then A receives a payoff of 0.5. Suppose further that 10 nights remain until the end of the period, but A has only six nights available to him for assigning patrols.

Thus, we have $v=0.5$, $n=10$, and $k=6$ for which we get from (4a) and (4b) that $x_6^{10}=0.6$ and $y_6^{10}=0.1$, and from (2) that the value of the game is $g^*(10, 6)=-0.1$. Now if A did not "assign a patrol" and B did not "go," then A and B face the new game $\Gamma(9,6)$ for which: $x_6^9=0.667$, $y_6^9=0.112$, and $g^*(9,6)=0$.

2.2. Two Patrol Boats—One Smuggler

Unfortunately, we do not have such closed form results for situations where A and B have more than one boat. We shall, however, discuss formulations for deriving numerical solutions when A has two patrol boats. Let $k_1 < M$ and $k_2 < M$ represent the number of patrols that can be assigned to these boats due to limited resources. There are two cases to be considered.

Case 1—Identical Patrol Boats

Suppose the two patrol boats are identical and A can make assignments of patrols according to some optimal plan. Let V_1^* and V_2^* be payoffs to A for assigning one and two patrols, respectively, on a given night when B chooses to "go". Side A now has three alternatives, and the game matrix is of the form

$$\Gamma(n, k)=\begin{bmatrix} V_2^* & \Gamma(n-1, k-2) \\ V_1^* & \Gamma(n-1, k-1) \\ -1 & \Gamma(n-1, k) \end{bmatrix}, \tag{5}$$

with the boundary conditions leading to

$$\Gamma(n, 1)=\begin{bmatrix} V_1^* & -1 \\ -1 & \Gamma(n-1, 1) \end{bmatrix} \quad \text{and} \quad \Gamma(1, k)=\begin{bmatrix} V_2^* & 1 \\ V_1^* & 1 \\ -1 & 1 \end{bmatrix},$$

noting that on the last day of the period

$$g(1, k)=\begin{cases} V_2^* & \text{if} \quad k \geq 2 \\ V_1^* & \quad\quad k=1 \\ -1 & \quad\quad k=0. \end{cases}$$

The fact that the two patrol boats are identical allows us to lump together the remaining available patrols for the period. The solution to this game can be derived through a recursive equation of the form

$$g(n, k) = f(V_1^*, V_2^*, \Gamma(n-1, k), \Gamma(n-1, k-1), \Gamma(n-1, k-2)),$$

from (5). This calls for the solution of a 3×2 game, which typically is solved by linear programing. We note that for this particular structure, the dual to the standard LP problem is

$$\text{(6)} \qquad \begin{aligned} &\min W \\ &\text{s.t. } V_2^* \cdot y + g(n-1, k-2) \cdot (1-y) \leq W \\ &\quad V_1^* y + g(n-1, k-1) \cdot (1-y) \leq W \\ &\quad -y + g(n-1, k) \cdot (1-y) \leq W. \end{aligned}$$

In order to have dominance, it is necessary that $V_2^* \geq V_1^* \geq -1$ and $g(n-1, k) \geq g(n-1, k-1) \geq g(n-1, k-2)$. The LP given by (6) can conveniently be solved graphically.

Case 2 – Nonidentical Patrol Boats

Consider now the case where the two patrol boats are not identical. Let V_1^*, V_2^*, and V_3^* represent the expected payoffs when boat number 1, boat number 2, and both boats, respectively, are assigned patrols according to some optimal allocation procedure. Denote by $\Gamma(n, k_1, k_2)$ our game played when n periods remain and side A has k_1 available patrols for patrol boat number 1 and k_2 for boat number 2. The game matrix then is

$$\text{(7)} \qquad \Gamma(n, k_1, k_2) = \begin{bmatrix} V_3^* & \Gamma(n-1, k_1-1, k_2-1) \\ V_2^* & \Gamma(n-1, k_1-1, k_2) \\ V_1^* & \Gamma(n-1, k_1, k_2-1) \\ -1 & \Gamma(n-1, k_1, k_2) \end{bmatrix},$$

with the boundary conditions

$$\Gamma(1, k_1, k_2) = \begin{cases} V_3^*, & \text{if } k_1 \geq 1, k_2 \geq 1 \\ V_2^*, & k_1 \geq 1, k_2 = 0 \\ V_1^*, & k_1 = 0, k_2 \geq 1 \\ -1, & k_1 = 0, k_2 = 0 \end{cases}$$

$$\Gamma(n, 0, 0) = -1.$$

The solution to this game can be obtained numerically using the procedures described for Case 1.

3. CONCLUDING REMARKS

As with any model, the models presented here are mere abstractions of reality. Thus, the major gains provided are through insights from identifying and examining various relationships among operational parameters. It is of interest in maintaining a patrol capability (side A) to know what will happen if certain conditions are changed. In particular, one is concerned with how side A's effectiveness and best strategy vary if he changes the number of patrol boats available for assignment to the strait.

There are a host of extensions that could be made to the present study. In principle, the situation where side A has $R > 2$ patrol boats when B has one boat can be treated in a similar fashion as Case 2, i.e., with two nonidentical patrol boats, only the game matrix is $2^R \times 2$. One must define all combinations of payoffs and numerically solve a recursive relationship for the value of the game.

A much more difficult problem, but indeed one of interest, is the case where side B has more than one boat. Depending upon the payoffs involved, it might be more reasonable from B's point of view to send out a number of boats, some of which are missioned to deceive or confuse A. From A's viewpoint, this problem begins to take the form of a type of search and detection problem (see Pollock [4]). In the present study we assumed that side A had a constant detection capability and that perfect information was gained through detection. Although admittedly it is a more difficult problem computationally, conceptually one can extend these games to allow for false alarms.

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