1987-06

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Journal of Applied Mechanics; p. 258-262; Volume 109
http://hdl.handle.net/10945/60096

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Nonlinear Beam Kinematics by Decomposition of the Rotation Tensor

A simple matrix expression is obtained for the strain components of a beam in which the displacements and rotations are large. The only restrictions are on the magnitudes of the strain and of the local rotation, a newly-identified kinematical quantity. The local rotation is defined as the change of orientation of material elements relative to the change of orientation of the beam reference triad. The vectors and tensors in the theory are resolved along orthogonal triads of base vectors centered along the undeformed and deformed beam reference axes, so Cartesian tensor notation is used. Although a curvilinear coordinate system is natural to the beam problem, the complications usually associated with its use are circumvented. Local rotations appear explicitly in the resulting strain expressions, facilitating the treatment of beams with both open and closed cross sections in applications of the theory. The theory is used to obtain the kinematical relations for coupled bending, torsion, extension, shear deformation, and warping of an initially curved and twisted beam.

1 Introduction

Beam theory has a long history (see Timoshenko, 1983). An understanding of fundamental aspects of beam theory may be obtained from the book by Wempner (1981). A summary of recent literature concerned chiefly with nonlinear beam theory is given by Hodges (1987b). In this paper our purpose is to obtain, by means of the polar decomposition theorem, an accurate but simple expression for the strain in a beam or rod undergoing large deflections. By this theorem, the change of configuration for any material element in the beam is decomposed into a pure strain and a pure rotation. A similar decomposition was obtained for thin shells by Simmonds and Danielson (1970, 1972).

The concept of decomposition was applied to beams in unpublished work by the second author in which the novel idea of separating the rotation into two parts was introduced—an arbitrarily large *global* rotation associated with the beam reference triad, and a moderately small *local* rotation associated with warp, shear, and other deformations. The physically reasonable assumptions of small strain and moderate *local* rotation led to a rather simple kinematical expression that was valid for arbitrarily large deflections and rotations of the reference triad. The original analysis was carried out entirely in terms of matrices, however, and consequently it was rather tedious, imprecise, and difficult to understand.

Interaction between the authors led to introduction of dyads into the analysis and attendant simplifications and improvements. The present paper presents a development of the theory based on Cartesian tensors, together with a comprehensive example. Hodges (1987b) embodies a rectified matrix derivation in order to accommodate engineers who are unfamiliar with tensor notation. The present derivation, however, offers far greater insight into the nature of the kinematical assumptions. It is believed that decomposition of the rotation tensor is new to beam literature and that the simplified kinematical relations obtained go beyond others in rigor and generality.

2 Beam Geometry and the Global Rotation Tensor

Let x_1 denote length along a reference line r within an undeformed beam. Let x_α denote lengths along lines orthogonal to the reference line r. (Here and throughout the paper Greek indices assume values 2 and 3 while Latin indices assume values 1, 2, and 3.) A particle of the beam is located from a fixed point in space by the position vector $\mathbf{r}(x_1, x_2, x_3)$. The covariant base vectors $\mathbf{g}_i$ are tangent to the coordinate curves:

$$\mathbf{g}_i(x_1,x_2,x_3)=\frac{\partial \mathbf{r}}{\partial x_i} \tag{1}$$

Contravariant base vectors can be obtained by standard means (see Budiansky, 1983, or Simmonds, 1982):

$$\mathbf{g}^i(x_1,x_2,x_3)=\frac{1}{2\sqrt{g}}\, e_{ijk}\mathbf{g}_j\times\mathbf{g}_k \tag{2}$$

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Contributed by the Applied Mechanics Division for presentation at the 1987 Applied Mechanics, Biomechanics, and Fluids Engineering Conference, Cincinnati, OH, June 14–17, 1987, of the American Society of Mechanical Engineers.

Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by ASME Applied Mechanics Division, May 3, 1986; final revision, September 2, 1986.

Paper No. 87-APM-17.

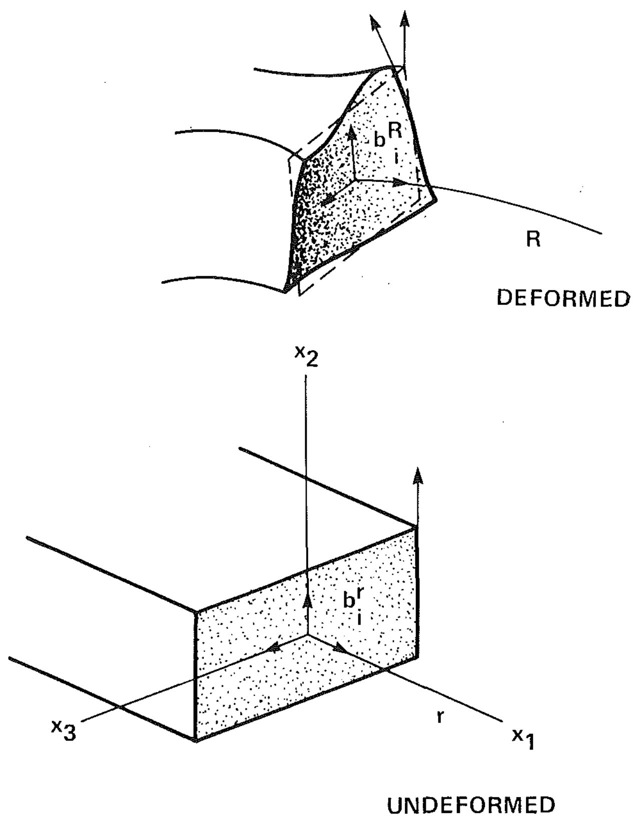

Fig. 1 Beam of thin rectangular cross section

Here

$$g = \det(\mathbf{g}_i \cdot \mathbf{g}_j) \tag{3}$$

and e_{ijk} are the components of the permutation tensor in a Cartesian coordinate system. (Repeated indices are always summed over their range.) The reference cross section at the point x_1^* is the surface whose position vector is $\mathbf{r}(x_1^*, x_2, x_3)$.

In a similar manner, consider the deformed state configuration. The locus of material points along $\mathbf{r}$ has now assumed a different curved line denoted by R. The particle which had position vector $\mathbf{r}(x_1, x_2, x_3)$ in the undeformed beam now has position vector $\mathbf{R}(x_1, x_2, x_3)$, relative to the same fixed point. The covariant base vectors $\mathbf{G}_i$ are tangent to the coordinate curves of the deformed beam:

$$\mathbf{G}_i(x_1, x_2, x_3) = \frac{\partial \mathbf{R}}{\partial x_i} \tag{4}$$

We will eventually resolve the position vectors along triads of unit base vectors on the undeformed and deformed reference lines. At each point along r define an orthogonal reference triad $\mathbf{b}_i^r(x_1)$ tangent to the coordinate curves at r with $\mathbf{b}_1^r$ tangent to r (see Fig. 1). At each point along R introduce an orthogonal reference triad $\mathbf{b}_i^R(x_1)$ such that $\mathbf{b}_\alpha^R(x_1^*) = \mathbf{G}_\alpha^*$, where $\mathbf{G}_\alpha^*$ denotes the part of $\mathbf{G}_\alpha(x_1^*, 0, 0)$ arising from the rigid body rotation of the cross section at x_1^*. Note that $\mathbf{b}_1^R = \mathbf{b}_2^R \times \mathbf{b}_3^R$ is not necessarily tangent to R unless the Euler-Bernoulli hypothesis (that the reference cross section remains normal to R when the beam is deformed) is adopted.

Rotation from $\mathbf{b}_i^r$ to $\mathbf{b}_i^R$ is accomplished by pre-dot multiplication with an orthogonal tensor which we call the global rotation tensor $\mathbf{C}^{Rr}$.

$$\mathbf{b}_i^R = \mathbf{C}^{Rr} \cdot \mathbf{b}_i^r = C_{ij}^{Rr} \mathbf{b}_j^r \tag{5}$$

The global rotation tensor can be represented as a linear combination of the dyads formed from the base vectors:

$$\mathbf{C}^{Rr} = \mathbf{b}_i^R \mathbf{b}_i^r \tag{6}$$

Rotation from $\mathbf{b}_i^R$ to $\mathbf{b}_i^r$ is accomplished by pre-dot multiplication with the rotation tensor $\mathbf{C}^{rR}$.

$$\mathbf{b}_i^r = \mathbf{C}^{rR} \cdot \mathbf{b}_i^R \tag{7}$$

The components C_{ij}^{Rr} are the direction cosines

$$C_{ij}^{Rr} = \mathbf{b}_i^R \cdot \mathbf{b}_j^r \tag{8}$$

The tensors $\mathbf{C}^{Rr}$ and $\mathbf{C}^{rR}$ are the transpose and inverse of each other, so that

$$\mathbf{C}^{Rr} \cdot \mathbf{C}^{rR} = \mathbf{I} \tag{9}$$

where $\mathbf{I}$ is the identity tensor. Note that $\mathbf{I}$ can be represented by a dyadic $\mathbf{b}_i \mathbf{b}_i$ in which $\mathbf{b}_i$ is any orthonormal base vector.

3 Strain and the Local Rotation Tensor

Consider the deformation gradient tensor $\mathbf{A}$ defined by

$$\mathbf{A} = \mathbf{G}_i \mathbf{g}^i \tag{10}$$

The polar decomposition theorem states that $\mathbf{A}$ can be uniquely decomposed into an orthogonal rotation tensor $\mathbf{C}$ dotted into a symmetric right stretch tensor $\mathbf{U}$ (see Ogden, 1984):

$$\mathbf{A} = \mathbf{C} \cdot \mathbf{U} \tag{11}$$

Note that

$$\mathbf{A} \cdot \mathbf{g}_i = \mathbf{C} \cdot \mathbf{U} \cdot \mathbf{g}_i = \mathbf{G}_i \tag{12}$$

This implies that $\mathbf{g}_i$ is transformed by first undergoing a pure strain to an intermediate state $\mathbf{U} \cdot \mathbf{g}_i$ and then undergoing a rotation to coincide with $\mathbf{G}_i$. The rotation may be decomposed into two parts: a large rotation equal to the *global* rotation $\mathbf{C}^{Rr}$ and an additional rotation which we call *local* rotation. Note that the local rotation may not vanish at the reference line.[2]

For the purpose of eventually allowing only small local rotation, we use the following representation for the local rotation tensor (see Sedov, 1966):

$$\exp(\tilde{\phi}) = \mathbf{I} + \tilde{\phi} + \frac{\tilde{\phi}^2}{2} + \frac{\tilde{\phi}^3}{6} + \ldots \tag{13}$$

(Here and hereafter a tilde over a tensor or matrix denotes that it is skew-symmetric) The antisymmetric tensor $\tilde{\phi}$ is related to the local rotation vector ϕ by

$$\tilde{\phi} = \phi \times \mathbf{I} \tag{14}$$

The direction of ϕ is along the axis of local rotation and the amplitude of ϕ is the angle of local rotation. Various other finite rotation vectors have been used in the literature, all differing from ϕ in amplitude only (for example, see Simmonds and Danielson, 1970, 1972; Reissner, 1973; or Kane et al., 1983).

We thus have the decomposition of the total rotation into local rotation and global rotation:

$$\mathbf{C} = \exp(\tilde{\phi}) \cdot \mathbf{C}^{Rr} \tag{15}$$

A physical feeling for this may be gained by studying Fig. 1 and by experimenting with a rectangular rubber eraser. We picture an initially straight beam of thin rectangular cross section with an arrow embedded in its side. The beam is then bent, twisted, and stretched. The final orientation of the arrow may be obtained by two rotations. First the cross section, with arrow attached, undergoes a large rigid body rotation to bring the base vectors $\mathbf{b}_i^r$ into coincidence with $\mathbf{b}_i^R$. Then the cross section, with the arrow remaining embedded in its side, undergoes a small warping to bring it into its final orientation.

The Jaumann-Biot-Cauchy strain tensor Γ is defined by

[2] It is possible to formulate the theory so that the local rotation always vanishes at the reference line. We could choose $\mathbf{b}_i^R = \mathbf{C}(x_1, 0, 0) \cdot \mathbf{b}_i^r$. Hodges (1987b) defines the global rotation at the reference axis to be equal to the total rotation there. The present analysis results in the simplest algebra.

$$\mathbf{\Gamma}=\mathbf{U}-\mathbf{I} \tag{16}$$

Other strain tensors based on $\mathbf{U}$ could be chosen. With this definition when the strains are small, as in subsequent sections, the components of $\mathbf{\Gamma}$ are simply the relative elongations and shears of material elements lying along the coordinate curves. Use of equations (9), (11), (15), and the above leads us to

$$\mathbf{C}^{Rr}\cdot\mathbf{\Gamma}\cdot\mathbf{C}^{rR}=\exp(-\tilde{\boldsymbol{\phi}})\cdot\mathbf{A}\cdot\mathbf{C}^{rR}-\mathbf{I} \tag{17}$$

The ensuing formulas will look simpler if we now write the tensors in component form. Since $\mathbf{\Gamma}$ is a Lagrangean strain tensor, it is appropriate to resolve it along the *undeformed* beam reference triad $\mathbf{b}_i^r$ yielding

$$\mathbf{\Gamma}=\mathbf{b}_i^r\gamma_{ij}\mathbf{b}_j^r \tag{18}$$

Since the left side of equation (17) then becomes $\mathbf{b}_i^R\gamma_{ij}\mathbf{b}_j^R$, it is appropriate to resolve the tensor $\tilde{\boldsymbol{\phi}}$ along the *deformed* beam triad $\mathbf{b}_i^R$ yielding

$$\tilde{\boldsymbol{\phi}}=\mathbf{b}_i^R\tilde{\phi}_{ij}\mathbf{b}_j^R \tag{19}$$

Comparison of equations (14) and the above leads to resolution of the local rotation vector $\boldsymbol{\phi}$ along the triad $\mathbf{b}_i^R$ yielding

$$\boldsymbol{\phi}=\phi_i\mathbf{b}_i^R=-\frac{1}{2}\,e_{ijk}\tilde{\phi}_{jk}\mathbf{b}_i^R \tag{20}$$

It follows from equation (17) that the deformation gradient tensor is resolved along the *mixed* bases

$$\mathbf{A}=\mathbf{b}_i^R A_{ij}\mathbf{b}_j^r \tag{21}$$

Our formula (17) thus becomes in matrix form

$$\gamma=\exp(-\tilde{\phi})A-I \tag{22}$$

4 Simplifications for Small Strain and Local Rotation

Our expression (22) for the strain is exact but very complex. Let ϵ and ϕ denote the maximum absolute values of the components of the matrices γ and $\tilde{\phi}$, respectively:

$$\begin{aligned}&\max|\gamma_{ij}(x_1,x_2,x_3)|=\epsilon<<1\\&\max|\tilde{\phi}_{ij}(x_1,x_2,x_3)|=\phi<1\end{aligned} \tag{23}$$

We will retain only terms of the lowest order in ϵ and ϕ. The Taylor expansion of $\exp(-\tilde{\phi})$ is easily obtained from equation (13):

$$\exp(-\tilde{\phi})=I-\tilde{\phi}+\frac{\tilde{\phi}^2}{2}-\frac{\tilde{\phi}^3}{6}+O(\phi^4) \tag{24}$$

To expand A we first break it up into symmetric and antisymmetric components:

$$A=I+E+\tilde{A} \tag{25}$$

Here we have defined

$$\begin{aligned}E&=\frac{A+A^T}{2}-I\\\tilde{A}&=\frac{A-A^T}{2}\end{aligned} \tag{26}$$

where A^T denotes the transpose of A. Substituting the above into equation (22) and solving for E, we obtain

$$E=\exp(\tilde{\phi})(I+\gamma)-I-\tilde{A} \tag{27}$$

Noting that the left side of equation (27) is symmetric, we can obtain an equation for $\tilde{A}$ by equating the right side of equation (27) with its transpose:

$$\tilde{A}=\frac{1}{2}[\exp(\tilde{\phi})(I+\gamma)-(I+\gamma)\exp(-\tilde{\phi})] \tag{28}$$

Using equation (24), we now expand the above in powers of $\tilde{\phi}$:

$$\tilde{A}=\tilde{\phi}+\frac{\tilde{\phi}^3}{6}+\frac{1}{2}(\gamma\tilde{\phi}+\tilde{\phi}\gamma)+O(\phi^4,\phi^2\epsilon) \tag{29}$$

Substituting equations (29) and (24) into (27), we obtain

$$E=\gamma+\frac{\tilde{\phi}^2}{2}+\frac{1}{2}(\tilde{\phi}\gamma-\gamma\tilde{\phi})+O(\phi^4,\phi^2\epsilon) \tag{30}$$

Finally, we solve equation (30) for the strain

$$\gamma=E-\frac{\tilde{\phi}^2}{2}+\frac{1}{2}(E\tilde{\phi}-\tilde{\phi}E)+O(\phi^4,\phi^2\epsilon) \tag{31}$$

The question now is when can the higher order terms be neglected? We assume that ϵ can be neglected in comparison with unity. We also assume that $\phi = O(\epsilon^r)$. Two cases are of interest:

(1) *Small local rotation*: $r \geq 1$. A theory based on this assumption would be suitable for solid beams without thin cross sections or thick-walled beams with open or closed cross sections, where effects of local rotations would be expected to be negligible. The strain for this case reduces to

$$\gamma=E \tag{32}$$

(2) *Moderate local rotation*: $1/2 \leq r < 1$. A theory based on this assumption would be suitable for thin cross sections such as thin-walled open cross sections, thin strips, rotor blades, etc., where local rotations could be appreciable. This case yields the rotation from equation (29) and strain from equation (31) as

$$\begin{aligned}&\tilde{\phi}=\tilde{A}\\&\gamma=E-\frac{\tilde{\phi}^2}{2}+\frac{1}{2}(E\tilde{\phi}-\tilde{\phi}E)\end{aligned} \tag{33}$$

The matrices E and $\tilde{A}$ are related to A by the definitions (26). The components of A follow from equations (10) and (21):

$$A_{ij}=(\mathbf{b}_i^R\cdot\mathbf{G}_k)(\mathbf{g}^k\cdot\mathbf{b}_j^r) \tag{34}$$

These components are rather easily obtained because in a beam theory it is convenient to resolve the base vectors of the undeformed state in the directions of the $\mathbf{b}_i^r$, and to resolve the base vectors of the deformed state in the directions of the $\mathbf{b}_i^R$. This will be illustrated by a comprehensive example in the next section.

5 Application to an Initially Curved and Twisted Beam

The position vector to points in any undeformed beam may be written as

$$\mathbf{r}(x_1,x_2,x_3)=\bar{\mathbf{r}}(x_1)+x_\alpha\mathbf{b}_\alpha^r \tag{35}$$

where $\bar{\mathbf{r}}(x_1) = \mathbf{r}(x_1, 0, 0)$ is the position vector to points on the reference line r. With this choice of coordinates the reference cross section is planar. The covariant base vectors are obtained from equation (1) by differentiation of equation (35). This may be accomplished using the formulas

$$\begin{aligned}&\bar{\mathbf{r}}'=\mathbf{b}_1^r\\&(\mathbf{b}_i^r)'=\mathbf{k}\times\mathbf{b}_i^r=\tilde{\mathbf{k}}\cdot\mathbf{b}_i^r\end{aligned} \tag{36}$$

where primes denote differentiation with respect to x_1. Here $\mathbf{k} = k_i\mathbf{b}_i^r$ is the curvature vector of the undeformed beam (k_1 is the pretwist of the beam while k_α are components of the curvature of the reference line (see Love, 1944)) and $\tilde{\mathbf{k}}$ is the curvature tensor of the undeformed beam defined by

$$\tilde{\mathbf{k}}=\mathbf{k}\times\mathbf{I}=\mathbf{b}_i^r\tilde{k}_{ij}\mathbf{b}_j^r=-\mathbf{b}_i^r e_{ijl}k_l\mathbf{b}_j^r \tag{37}$$

The contravariant base vectors are obtained from equation (2). The final result is

$$\mathbf{g}^1=\frac{\mathbf{b}_1^r}{\sqrt{g}}$$

$$\mathbf{g}^2=\frac{x_3k_1\mathbf{b}_1^r}{\sqrt{g}}+\mathbf{b}_2^r \tag{38}$$

$$\mathbf{g}^3=\frac{-x_2k_1\mathbf{b}_1^r}{\sqrt{g}}+\mathbf{b}_3^r$$

where we have obtained from equation (3) the relation

$$\sqrt{g}=1-x_2k_3+x_3k_2 \tag{39}$$

We assume that the reference cross section does not distort in its plane.[3] Thus, the position vector to points in the deformed beam can be represented by

$$\mathbf{R}(x_1,x_2,x_3)=\bar{\mathbf{r}}+\mathbf{u}+x_\alpha\mathbf{b}_\alpha^R+\lambda\psi\mathbf{b}_1^R \tag{40}$$

Here $\mathbf{u}=u_i\mathbf{b}_i^r$ is the displacement vector of points on the reference line $\mathbf{r}$, $\lambda(x_2, x_3)$ is the Saint-Venant warping function for the local cross section, and $\psi(x_1)$ is the warp amplitude. The $\mathbf{b}_\alpha^R$ are chosen tangent to the x_α coordinate curves at R. The reference axis is chosen so that $\lambda(0, 0) = \lambda_\alpha(0, 0) = 0$ where $\lambda_\alpha = \partial\lambda/\partial x_\alpha$. The covariant base vectors of the deformed beam are obtained from equation (4) by differentiation of equation (40). This is accomplished using formulas analogous to equations (36):

$$\bar{\mathbf{R}}'=(1+\bar{\gamma}_{11})\mathbf{b}_1^R+2\bar{\gamma}_{1\alpha}\mathbf{b}_\alpha^R$$
$$(\mathbf{b}_i^R)'=\mathbf{K}\times\mathbf{b}_i^R=\tilde{\mathbf{K}}\cdot\mathbf{b}_i^R \tag{41}$$

Here $\mathbf{K}=K_i\mathbf{b}_i^R$ is the curvature vector of the deformed beam (the components of $\mathbf{K}$ are $(1+\bar{\gamma}_{11})$ times the twist and curvatures of the deformed beam), and the components $\tilde{K}_{ij} = -e_{ijl}K_l$ of the curvature tensor $\tilde{\mathbf{K}}$ of the deformed beam are in matrix form

$$\tilde{K}=-(C^{Rr})'C^{rR}+C^{Rr}\tilde{k}C^{rR} \tag{42}$$

The strains $\bar{\gamma}_{1i}(x_1)=\gamma_{1i}(x_1, 0, 0)$ at the reference line are obtained by evaluation of equations (33)–(34) at the reference line

$$\bar{\gamma}_{11}=C_{11}^{Rr}-1+C_{1i}^{Rr}(u_i'+\tilde{k}_{ij}u_j)$$
$$2\bar{\gamma}_{1\alpha}=C_{\alpha 1}^{Rr}+C_{\alpha i}^{Rr}(u_i'+\tilde{k}_{ij}u_j) \tag{43}$$

We thus have

$$\begin{aligned}\mathbf{G}_1&=(1+\bar{\gamma}_{11}-x_2K_3+x_3K_2+\lambda\psi')\mathbf{b}_1^R\\&\quad+(2\bar{\gamma}_{12}-x_3K_1+\lambda\psi K_3)\mathbf{b}_2^R\\&\quad+(2\bar{\gamma}_{13}+x_2K_1-\lambda\psi K_2)\mathbf{b}_3^R\\\mathbf{G}_\alpha&=\mathbf{b}_\alpha^R+\lambda_\alpha\psi\mathbf{b}_1^R\end{aligned} \tag{44}$$

The components A_{ij} of the deformation gradient matrix may now be calculated from equations (34), (38), and (44). The result is

$$A_{11}=\frac{1+\bar{\gamma}_{11}+x_3K_2-x_2K_3+\lambda\psi'+\psi k_1(x_3\lambda_2-x_2\lambda_3)}{\sqrt{g}}$$

$$A_{21}=\frac{2\bar{\gamma}_{12}-x_3(K_1-k_1)+\lambda\psi K_3}{\sqrt{g}}$$

$$A_{31}=\frac{2\bar{\gamma}_{13}+x_2(K_1-k_1)-\lambda\psi K_2}{\sqrt{g}} \tag{45}$$

$$A_{12}=\lambda_2\psi \quad A_{13}=\lambda_3\psi$$

$$A_{22}=A_{33}=1 \quad A_{23}=A_{32}=0$$

Now let

$$\kappa_i=K_i-k_i \tag{46}$$

Components of the symmetric matrix E are obtained from equations (26) and (45):

$$\begin{aligned}E_{11}&=\frac{\bar{\gamma}_{11}+x_3\kappa_2-x_2\kappa_3+\lambda\psi'+\psi k_1(x_3\lambda_2-x_2\lambda_3)}{\sqrt{g}}\\E_{12}=E_{21}&=\frac{2\bar{\gamma}_{12}-x_3\kappa_1+\lambda_2\psi\sqrt{g}+\lambda\psi(\kappa_3+k_3)}{2\sqrt{g}}\\E_{13}=E_{31}&=\frac{2\bar{\gamma}_{13}+x_2\kappa_1+\lambda_3\psi\sqrt{g}-\lambda\psi(\kappa_2+k_2)}{2\sqrt{g}}\\E_{22}&=E_{33}=E_{23}=E_{32}=0\end{aligned} \tag{47}$$

Components of the antisymmetric matrix $\tilde{\phi}$ are obtained from equations (26), (33), and (45):

$$\begin{aligned}\tilde{\phi}_{12}=-\tilde{\phi}_{21}=-\phi_3&=\frac{-2\bar{\gamma}_{12}+x_3\kappa_1+\lambda_2\psi\sqrt{g}-\lambda\psi(\kappa_3+k_3)}{2\sqrt{g}}\\\tilde{\phi}_{13}=-\tilde{\phi}_{31}=\phi_2&=\frac{-2\bar{\gamma}_{13}-x_2\kappa_1+\lambda_3\psi\sqrt{g}+\lambda\psi(\kappa_2+k_2)}{2\sqrt{g}}\\\tilde{\phi}_{11}=\tilde{\phi}_{22}=\tilde{\phi}_{33}=0;&\quad\tilde{\phi}_{23}=\tilde{\phi}_{32}=\phi_1=0\end{aligned} \tag{48}$$

From equation (32) the strain components for the small local rotation theory are the elements of the matrix E. The strain components for the moderate local rotation theory can now be obtained directly by substitution of E and $\tilde{\phi}$ into the second of equations (33). Note that the matrix $\tilde{\phi}$ contains some terms of the order of strains (i.e., $\bar{\gamma}_{1\alpha}$). Care should be taken to discard these terms when squared and when multiplied by any of the terms of matrix E. Note that it is not necessary, however, to introduce any *ad hoc* arguments to remove terms of the order of squares and products of the strain components. It may be desirable to make some simplifying assumptions about the magnitude of the initial curvatures, which affect the order of magnitude of the quantity $\sqrt{g}-1$.

The elements of E have clear physical significance. We see that E_{11} is the extensional strain for the beam; $\bar{\gamma}_{11}$ is the strain of the reference line, terms involving κ_α are the bending strains, and the remaining terms are extensional strains related to warping. The off-diagonal terms $E_{1\alpha}$ are shear strains; $\bar{\gamma}_{1\alpha}$ is the transverse shear of the reference line and the remaining terms concern shear strain due to torsion and warp. The direction cosine matrix C^{Rr} (and thus κ_i) are taken as given in the present analysis. They may be expressed in a variety of ways as discussed by Hodges (1987a).

The present strain expressions for the small local rotation theory are very similar to those obtained by Wempner (1981). The last term in E_{11} is missing from Wempner's equations (8-29). This term was shown by Hodges (1980) to be important in correctly predicting the untwist of pretwisted beams under an axial tension force. The terms in $E_{1\alpha}$ involving $\lambda\psi\kappa_\beta$ are missing from Wempner's final expressions but are probably not very important anyway. It is unknown if there are any further differences in predictive capability between Wempner's theory and our small local rotation theory. Moreover, most of the nonlinear terms in equation (33) of the moderate local rotation theory are not included in Wempner's book.

Perhaps as significant as any differences in the results is the number of approximations invoked in Wempner's arguments in order to obtain his final result. The quadratic terms not shown in his equations (8-25) are neglected. The gradual change of the cross section strains is neglected (the underlined terms of equations (8-25)). Further approximations are made based on the thinness of the beam. Although the present approach involves fewer approximations, the results are simpler.

The simplicity of the present analysis, despite the generality of the example problem, is noteworthy. Indeed, the present

[3] This assumption is made strictly for illustrative purposes. The general theory does not require such an assumption. Nonclassical effects such as transverse normal strains and distortion shear can be treated by assumption of a more general displacement field.

expressions for strain were obtained with effort comparable to that expended by Hodges (1980) in an analysis involving only initial twist, extension, and torsion. Like Wempner, Hodges had to invoke several *ad hoc* approximations in order to simplify his result.

6 Conclusion

The analysis was based solely on the two assumptions that the strain components can be neglected compared to unity and that the local rotation components are no larger than the square root of the strain. If local rotation components are allowed to be as small as the strains, which might be the case for a beam with a closed or thick cross section, a very simple theory results. Unlike previous analyses, the removal of higher-order terms based on subjective criteria is unnecessary.

The theory was applied to a slender, precurved, pretwisted beam undergoing bending, torsion, and extension, as well as shear deformation and warping of arbitrary amplitude, which could be important for composite beams. The strain components are explicitly functions of x_α and depend implicitly upon seven functions of x_1 alone: $\bar{\gamma}_{1i}$, κ_i, and ψ. These could be used as the generalized strains of a complete engineering beam theory. Such a theory will be developed in a later paper.

Acknowledgments

The first author was supported by the Naval Postgraduate School Foundation Research Program while on leave at the Aeroflightdynamics Directorate, US Army Aviation Research and Technology Activity (AVSCOM), Ames Research Center, Moffett Field, California (April–June, 1986). The authors thank Howard E. Hinnant of the Rotorcarft Dynamics Division, Aeroflightdynamics Directorate, who helped design the figure.

References

Budiansky, B., 1983, "Tensor Analysis," *Handbook of Applied Mathematics*, Pearson, C. E., ed., Van Nostrand Reinhold, Chapter 4.

Hodges, D. H., 1980, "Torsion of Pretwisted Beams Due to Axial Loading," ASME Journal of Applied Mechanics, Vol. 47, pp. 393–397.

Hodges, D. H., 1987a, "Finite Rotations and Nonlinear Beam Kinematics," *Vertica*, Vol. 11, No. 1–2, p. 297–307.

Hodges, D. H., 1987b, "Nonlinear Beam Kinematics for Small Strains and Finite Rotations," *Vertica*, Vol. 11, No. 3, to be published.

Kane, T. R., Likins, P. W., and Levinson, D. A., 1983, *Spacecraft Dynamics*, McGraw-Hill, Chapter 1.

Love, A. E. H., 1944, *A Treatise on the Mathematical Theory of Elasticity*, Dover, Chapter 18.

Ogden, R. W., 1984, *Non-Linear Elastic Deformations*, Ellis Horwood, Chichester, Section 2.2.

Reissner, E., 1973, "On One-Dimensional Large-Displacement Finite-Strain Beam Theory," *Studies in Applied Mathematics*, Vol. 52, No. 2, pp. 87–95.

Sedov, L. I., 1966, *Foundations of the Non-linear Mechanics of Continua*, Pergamon Press, Chapter 1.

Simmonds, J. G., and Danielson, D. A., 1970, "Nonlinear Shell Theory with A Finite Rotation Vector," *Proceedings, Koninkl. Nederl. Akademie Van Wettenschappen*, Series B, Vol. 73, No. 5, pp. 460–478.

Simmonds, J. G., and Danielson, D. A., 1972, "Nonlinear Shell Theory with Finite Rotation and Stress-Function Vectors," ASME Journal of Applied Mechanics, Vol. 39, pp. 1085–1090.

Simmonds, J. G., 1982, *A Brief on Tensor Analysis*, Springer-Verlag, Chapter 2.

Timoshenko, S. P., 1983, *History of Strength of Materials*, Dover.

Wempner, G. A., 1981, *Mechanics of Solids with Application to Thin Bodies*, Sijthoff and Noordhoff, The Netherlands, Chapters 6, 8, and 10.