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# A NOTE ON A COMPARISON OF CONFIDENCE INTERVAL TECHNIQUES IN TRUNCATED LIFE TESTS

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## ABSTRACT

Several approximate procedures are available in the literature for obtaining confidence intervals for the parameter $\lambda$ of an exponential distribution based on time truncated samples. This paper contains the results of an empirical study comparing three of these procedures.

## 1. INTRODUCTION

In life testing applications, it is frequently desired to obtain a confidence interval for the parameter $\lambda$ of an exponential distribution. In case a test plan is used for which all the observations are truncated at the same time point $t_0$, several approximate confidence interval procedures are available in the statistical literature. The purpose of this note is to report the results of an empirical study of the performances of three of these procedures with respect to the expected length of the interval, the variance of the interval length and the coverage probability.

The general setting of the problem is as follows: suppose the random variables $T_1, T_2, \ldots, T_n$ are independent and identically exponentially distributed with mean $\lambda^{-1}$. For $i=1, 2, \ldots, n$, let $X_i$ be equal to $T_i$ truncated at $t_0$, and let $Y_i$ be the Bernoulli random variable which is 1 if and only if $X_i < t_0$. We wish to find confidence intervals for $\lambda$, based on the $X_i$ and $Y_i$.

In what follows, three confidence interval procedures are described, and some results of an empirical study of their performances are presented.

## 2. CONFIDENCE INTERVAL PROCEDURES

PROCEDURE 1: This procedure is obtained as a special case of a solution to a more general problem that was derived by Halperin [1]. The random variable $Y=\sum_{i=1}^{n} Y_i$ has a binomial distribution with parameters $n$ and $p=1-e^{-\lambda t_0}$. Standard techniques can be used to obtain a $100(1-\alpha)$ percent confidence interval for $p$ as

$$P[a(Y)<p<b(Y)] \geqslant 1-\alpha.$$

Since $p=1-e^{-\lambda t_0}$, an inversion can be made which results in $\lambda_L=\frac{-\ln(1-a)}{t_0}$ and $\lambda_U=\frac{-\ln(1-b)}{t_0}$ as lower and upper $100(1-\alpha)$ percent confidence limits for $\lambda$.

PROCEDURE 2: Rubenstein [2] showed that

$$\hat{\lambda}=\frac{\Sigma Y_i}{\Sigma X_i}\left[1+\frac{1}{2n}\right]^{-1}$$

is an approximately unbiased estimator of $\lambda$. He noted that for $\lambda t_0 \ll 1, \Sigma Y_i$ is nearly a Poisson random variable, so a confidence interval procedure for a Poisson parameter due to Wilks [3] was used to obtain the approximate confidence limits

$$\lambda_L=[2\hat{\lambda}+z^2C-(4\hat{\lambda}z^2C+z^4C^2)^{1/2}]/2$$

$$\lambda_U=[2\hat{\lambda}+z^2C+(4\hat{\lambda}z^2C+z^4C^2)^{1/2}]/2,$$

Where $z$ is the $100(1-\alpha)$th percentage point of the standard normal distribution and $C=(\Sigma X_i)^{1/2}$.

PROCEDURE 3: We employ terminology commonly used in the literature of life testing in describing this procedure. Imagine that the random variables $X_1, X_2, \ldots, X_n$ are observed sequentially. That is, imagine that a randomly selected item is put on test and is replaced with a similar item at failure or after a period of $t_0$ has elapsed, whichever occurs first. If this process were continued, the arrival process of failures would be a Poisson process, so the time to $k^{th}$ failure (for $k$ fixed) would have a gamma distribution. (Testing to $k^{th}$ failure in this situation could be described roughly as a combination of item consoring and time truncation.) Since we are assuming that exactly $n$ items are to be tested, the experiment is stopped after a random amount of time, and the number $K$ of observed failures is a random variable. It would appear, however, that, given $K=k$, the distribution of the time $W_k$ until $k$ failures have arrived can be approximated by a gamma distribution,

$$f(w_k|K=k)\approx\frac{\lambda^k}{\Gamma(k)}w_k^{k-1}e^{-\lambda w_k};\ w_k>0.$$

Note that the distribution of $W_k$ given $K=k$ cannot be exactly gamma, since $P[W_k\leq nt_0]=1$ for any $k$. It follows that $V=2\lambda W_k$ can be approximated by a Chi-square variable with $2k$ degrees of freedom. Thus, for example, if $\chi^2_{\alpha/2}$ and $\chi^2_{(1-\alpha/2)}$ are the upper and lower $\alpha/2$ percentages points of the Chi-square distribution with $2k$ degrees of freedom, then

$$\left(\frac{\chi^2_{\alpha/2}}{2W_k},\frac{\chi^2_{1-\alpha/2}}{2W_k}\right)$$

constitutes an approximate $100(1-\alpha)$ percent confidence interval for $\lambda$.

## 3. COMPARISON OF PROCEDURES

A Monte Carlo study was made to compare the three procedures described above. One thousand samples of size $n$ ($n=30$, 40, 50) from an exponential distribution with parameter $\lambda$ ($\lambda=0.1$, 0.2, 0.8, 3, 5, 10) were generated. For each sample, 95-percent confidence intervals for $\lambda$ were obtained by the three methods (1, 2, 3) for various truncation times, $t_o$. The results are summarized in Table 1 where we give, for certain combinations of $\lambda$, $t_o$, and method, the average length of the confidence intervals, the sample variance of these lengths, and the empirical coverage probability (i.e., the proportion of intervals which actually covered $\lambda$).

Overall, the procedures appear to rank 2, 3, 1 in decreasing order of general quality of performance. This is clearly the ordering with respect to average interval length, and seems to be the best general ordering with respect to variance in interval length. All three procedures tend to be conservative in terms of coverage probability, with procedure 1 being worst in this respect. Procedure 3 is generally best in terms of more nearly attaining the "target" confidence level $1-\alpha$. Of course such a quality in a procedure is not in itself of value if it is competing with a more conservative procedure which attains comparable (or better) interval length characteristics.

## REFERENCES

[1] Halperin, M., Confidence Intervals from Censored Samples, Ann. Math. Stat., **32**, 828–37 (1961).

[2] Rubinstein, D., "Statistical Exposition of Guide Manual for Reliability Measurement Program," NavOrd 29304/Addendum (Nov. 1967).

[3] Wilks, S., Shortest Average Confidence Intervals from Large Samples, Ann. Math. Stat., **9**, 166–175 (1938).

TABLE 1. Some Characteristics of the 95-percent Confidence Intervals Obtained Using Three Procedures

| | | $n=$ | 30 | | | 40 | | | 50 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $t_0$ | PROC. | AVG. | VAR. | C.P. | AVG. | VAR. | C.P. | AVG. | VAR. | C.P. |
| 0.1 | 3 | 1 | 0.159 | 0.001 | 0.988 | 0.136 | 0.000 | 0.971 | 0.120 | 0.000 | 0.978 |
| | | 2 | 0.148 | 0.001 | 0.963 | 0.127 | 0.000 | 0.959 | 0.112 | 0.000 | 0.970 |
| | | 3 | 0.157 | 0.001 | 0.951 | 0.132 | 0.001 | 0.953 | 0.116 | 0.000 | 0.960 |
| 0.1 | 10 | 1 | 0.107 | 0.000 | 0.961 | 0.090 | 0.000 | 0.977 | 0.080 | 0.000 | 0.963 |
| | | 2 | 0.092 | 0.000 | 0.951 | 0.080 | 0.000 | 0.951 | 0.071 | 0.000 | 0.955 |
| | | 3 | 0.093 | 0.000 | 0.954 | 0.081 | 0.000 | 0.945 | 0.071 | 0.000 | 0.958 |
| 0.2 | 3.2 | 1 | 0.234 | 0.002 | 0.978 | 0.200 | 0.001 | 0.963 | 0.176 | 0.001 | 0.964 |
| | | 2 | 0.213 | 0.001 | 0.961 | 0.183 | 0.001 | 0.958 | 0.163 | 0.000 | 0.952 |
| | | 3 | 0.217 | 0.001 | 0.954 | 0.186 | 0.001 | 0.961 | 0.165 | 0.000 | 0.945 |
| 0.8 | 0.82 | 1 | 0.943 | 0.036 | 0.966 | 0.799 | 0.017 | 0.959 | 0.703 | 0.011 | 0.961 |
| | | 2 | 0.855 | 0.023 | 0.942 | 0.732 | 0.012 | 0.948 | 0.647 | 0.007 | 0.945 |
| | | 3 | 0.871 | 0.026 | 0.936 | 0.743 | 0.012 | 0.954 | 0.657 | 0.008 | 0.936 |
| 3 | 0.11 | 1 | 4.54 | 0.822 | 0.954 | 3.84 | 0.423 | 0.966 | 3.43 | 0.269 | 0.957 |
| | | 2 | 4.23 | 0.664 | 0.946 | 3.60 | 0.349 | 0.955 | 3.22 | 0.227 | 0.954 |
| | | 3 | 4.56 | 7.10 | 0.940 | 3.73 | 0.473 | 0.949 | 3.32 | 0.292 | 0.948 |
| 3 | 0.33 | 1 | 3.18 | 0.428 | 0.970 | 2.73 | 0.227 | 0.972 | 2.38 | 0.122 | 0.962 |
| | | 2 | 2.77 | 0.196 | 0.963 | 2.41 | 0.113 | 0.956 | 2.12 | 0.066 | 0.957 |
| | | 3 | 2.80 | 0.222 | 0.959 | 2.42 | 0.118 | 0.957 | 2.14 | 0.071 | 0.961 |
| 5 | 0.12 | 1 | 6.04 | 1.29 | 0.974 | 5.16 | 0.704 | 0.962 | 4.56 | 0.420 | 0.957 |
| | | 2 | 5.50 | 0.874 | 0.951 | 4.75 | 0.499 | 0.955 | 4.22 | 0.311 | 0.955 |
| | | 3 | 5.62 | 1.00 | 0.951 | 4.84 | 0.562 | 0.959 | 4.28 | 0.338 | 0.955 |
| 10 | 0.06 | 1 | 12.1 | 5.39 | 0.970 | 10.2 | 2.60 | 0.966 | 9.07 | 1.56 | 0.965 |
| | | 2 | 11.1 | 3.70 | 0.950 | 9.39 | 1.85 | 0.961 | 8.40 | 1.15 | 0.964 |
| | | 3 | 11.4 | 4.14 | 0.949 | 9.55 | 2.02 | 0.956 | 8.54 | 1.29 | 0.962 |
| 10 | 0.1 | 1 | 10.7 | 5.47 | 0.961 | 9.08 | 2.62 | 0.971 | 7.98 | 1.63 | 0.959 |
| | | 2 | 9.32 | 2.46 | 0.945 | 7.99 | 1.30 | 0.957 | 7.08 | 0.798 | 0.950 |
| | | 3 | 9.43 | 2.74 | 0.939 | 8.05 | 1.32 | 0.958 | 7.12 | 0.827 | 0.950 |

AVG. = Average interval length
VAR. = Sample variance in interval lengths
C.P. = Empirical coverage probability

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