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On the Theory of Convective Currents

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Abstract

A theory of steady state, saturated, convective currents is presented which includes the transfer of heat and momentum by lateral diffusion as well as the systematic entrainment of environmental air. The system of equations governing the behavior of such convective currents is derived and numerically integrated for a variety of initial and environmental conditions. In agreement with basic physical principles, it is found that the cloud height, mass, vertical velocity, temperature excess over the environment, and liquid water content all increase with increasing (a) initial temperature, (b) environmental relative humidity, and (c) environmental lapse rate; but all decrease as the diffusion coefficient increases. Also the cloud height, vertical velocity, temperature excess, and liquid water content increase with increasing initial velocity: however, the total mass of air decreases with increasing initial velocity, reflecting a lower rate of entrainment.

Even with frictional drag in the form of momentum loss by diffusion, the updraft overshoots the level of zero buoyancy in the upper part of the cloud, giving rise to cloud temperatures at the cloud top which may be several degrees colder than the environment.

When applied to an upper-air sounding taken near the time and location of a thunderstorm, the model gives the right order of magnitude for the cloud height, vertical velocity, and temperature excess over environment for a diffusion coefficient of $k_1 = .001\ \text{sec}^{-1}$. While the life cycle of a convective cloud is certainly not a steady-state phenomenom, the theory probably affords a fair approximation of the convective cloud during the updraft stage.

1. Introduction

Convective phenomena have received considerable attention from meteorologists during the last decade resulting in drastic revisions to the simple "adiabatic parcel ascent" which forms the basis of many of the commonly-used forecasting techniques. As a result of evidence (STOMMEL, 1947) that environment air is entrained into convective updrafts, AUSTIN and FLEISHER (1948) suggested that such entrainment, called *dynamic entrainment*, is required to satisfy mass continuity. Using the first law of thermodynamics, they also obtained an expression for the temperature lapse in the rising air by assuming that it is isobarically mixed with the entrained air, and that the latter is saturated by evaporation of some of the previously condensed water.

HOUGHTON and CRAMER (1951) presented a more complete theory of both dry and saturated, steady state, frictionless, convective currents by adding the continuity and momentum equations. BUNKER (1953) further extended the theory of unsaturated thermals by including the loss of heat and momentum through lateral diffusion and body drag. In a more recent study of two tradewind cumulus clouds, MALKUS (1954) checked the convective theories by comparison of computed and observed values of temperature, moisture, and vertical velocity. Malkus hypothesized that a momentum balance is achievable by considering only (a) the momentum of the rising air, (b) the momentum of the environmental air entrained to satisfy temperature requirements (referred to as *gross entrainment*), and (c) momentum lost by systematic detrainment of air from the rising column. From continuity considerations, the dynamic entrainment then represents the difference between the gross entrainment and the detrainment. No other drag or lateral diffusion terms were included. The gross entrainment was determined by the amount of environment air needed to yield the observed convective cloud temperature instead of the temperature produced by adiabatic ascent of saturated air. Satisfactory agreement was obtained between calculated and observed vertical velocities.

The calculations of Houghton, Cramer, and Malkus were confined to the lower 5,000 ft. It therefore appears desirable to extend the study of saturated currents to elevations typical of large cumulus clouds, and to include the effects of lateral diffusion of heat and momentum, as well as systematic entrainment. Moreover, in view of favorable results achieved on the two tradewind cumulus, it is also of interest to compare the Malkus theory to the approach used here.

In the next few sections a system of equations applicable to saturated convective currents will be derived. The system will then be numerically integrated with a variety of initial and environmental conditions.

2. Continuity Equation

Consider a vertical current of air in which the velocity is horizontally uniform, or if this is not the case, the vertical velocities subsequently referred to will simply denote the mean value in the horizontal. The mass of air M flowing by any level z per unit time is

$$M = \varrho w \sigma$$

Here, ϱ is the density; w, the vertical velocity; and σ, the cross-section area of the rising column. The flow will be assumed to be steady, so that the mass rate of flow at level $z+dz$ is the sum of the inflow at level z plus the *net* air entrained from the environment dM. Thus

$$\varrho w \sigma + dM = (\varrho + d\varrho)(\sigma + d\sigma)(w + dw)$$

Expanding leads to

$$\frac{dN}{dz} = \frac{1}{w}\frac{dw}{dz} + \frac{1}{\sigma}\frac{d\sigma}{dz} + \frac{1}{\varrho}\frac{d\varrho}{dz} \qquad (1)$$

where $N = \ln M$. Differentiating the equation of state, $p = \varrho R_d T_v$ gives

$$\frac{1}{\varrho}\frac{d\varrho}{dz} = \frac{1}{p}\frac{dp}{dz} - \frac{1}{T_v}\frac{dT_v}{dz} \qquad (2)$$

T_v is the virtual temperature and R_d the gas constant for dry air. We shall now assume that the pressure in the rising air always adjusts instantaneously to that of the environment, the latter being identified by the subscript e. Hence, from the hydrostatic equation

$$\frac{1}{p}\frac{dp}{dz} = \frac{1}{p_e}\frac{\partial p_e}{\partial z} = -\frac{g}{R_d T_{ve}} \qquad (3)$$

Substituting for the density in Eq. (1) from Eqs. (2) and (3) yields the continuity equation for the described flow

$$\frac{dN}{dz} = \frac{1}{w}\frac{dw}{dz} + \frac{1}{\sigma}\frac{d\sigma}{dz} - \frac{g}{R_d T_{ve}} - \frac{1}{T_v}\frac{dT_v}{dz} \qquad (4)$$

The net rate of entrainment dN/dz, as defined here, is determined by the requirement of mass continuity. Such entrainment is referred to as *dynamic entrainment*, as mentioned earlier. In addition to this dynamic entrainment, we shall assume further *turbulent exchange* between the rising column and the adjacent environment. This lateral diffusion will normally result in a drag on the moving air through the loss of momentum. Moreover, the turbulent exchange may also be expected to effect a transfer of heat between the rising column and the environment, as well as the loss of the latent heat, required to saturate the exchanged environment air. The latter features will be considered further in Section 4.

3. The Equation of Motion

In a rising column in which there is a net entrainment (or detrainment), the mass of a rising element is changing. This situation is analagous to a rocket which is changing mass and suggests the use of the equation of motion for a variable mass. We may consider a mass M, moving at velocity w, at time t and another mass dM, moving at velocity w_e, which combine and at time $t+dt$ are moving at velocity $w+dw$. Then the momentums at time t and $t+dt$ are related by the equation

$$(M + dM)(w + dw) = Mw + EMdt + w_e\, dM$$

where E represents the external forces per unit mass. This gives the following equation of motion

$$\frac{dw}{dt} + \frac{w}{M}\frac{dM}{dt} = E + w_e \frac{1}{M}\frac{dM}{dt} \tag{5}$$

Accurate data concerning the vertical velocity of the entrained environment air is lacking; hence, for simplicity in the calculations to follow, it will be assumed that $w_e \equiv 0$. This probably produces no serious error if there is positive entrainment; however, if there is a systematic detrainment, the loss of momentum by the rising air must be considered. The external forces E include the pressure and gravity forces which may be combined in the familiar form $g\ (T_v - T_{ve})\ /\ T_{ve}$, and also the frictional force F. Thus Eq. (5) is expressible as

$$\frac{dw}{dt} + \frac{w}{M}\frac{dM}{dt} = g\frac{T_v - T_{ve}}{T_{ve}} + F \tag{6}$$

Assuming steady state conditions, this becomes

$$w\frac{dw}{dz} + w^2\frac{dN}{dz} = g\frac{T_v - T_{ve}}{T_{ve}} + F \tag{7}$$

4. Lateral Diffusion of Heat and Momentum

In a study of the diffusion of momentum and heat from a plume of hot gas, PRIESTLEY (1956) has considered the process to consist of two phases. In the first phase, the gas has a relatively high velocity and large temperature difference which promotes rapid diffusion, while during the second phase the buoyancy and velocity are considerably less. For the latter phase, Priestley has derived the following expressions for the loss of momentum (per unit time and per unit mass) and the potential temperature change per unit time, respectively

$$\begin{array}{l} -k_1\, w \\ -k_1\, (\Theta - \Theta_e) \end{array} \tag{8a}$$

Here $k_1 = Ka^{-2}$ where K is the coefficient of eddy diffusivity and a is the radius of the buoyant jet. The heat loss per unit time per unit mass is therefore $-k_1\, c_p (T/\Theta)(\Theta - \Theta_e)$, ($c_p$ is the specific heat at constant pressure); or approximately

$$c_p k_1\ (T - T_e) \tag{8b}$$

In the case of saturated air, the turbulent exchange of mass between the updraft and the environment would also require the saturation of the exchanged environment air. Thus it is necessary to add a term of the form

$$-k_1\ L\ (q - q_e) \tag{8c}$$

where q is the specific humidity and L is the latent heat of vaporization.

Using some observations, Priestley determined values of k_1 which ranged from .06 to .2 seconds^{-1} with jets of radius, 10 to 20 meters. Now the scale of this phenomenon is far different from that of a convective cloud, and the diffusion coefficients may differ. Nevertheless, values based on Priestley's determinations may be used as a starting point. For a convective updraft of radius 100 to 200 meters, the diffusion coefficient would be roughly .001 second^{-1} to .0005 second^{-1}.

With the use of the first expression in (8a) to represent the frictional drag, Eq. (7) may be written

$$w\frac{dw}{dz} + w^2\frac{dN}{dz} = g\frac{T_v - T_{ve}}{T_{ve}} - k_1 w \tag{9}$$

The dimension of second^{-1} for k_1, in Eqs. (8) and (9) implies, for example, that for a given temperature difference $(T - T_e)$, the loss of heat (or momentum) by diffusion varies linearly with time. This suggests that the diffusion is due to normal atmospheric turbulence in the vicinity of a cumulus cloud and does not depend on the speed of the updraft. Intuitively, one might expect the turbulence to increase with increasing updraft velocity. Thus, for example, it may be more reasonable to assume that the expressions (8a), (8b) and (8c) represent the loss of momentum and heat per unit vertical distance, having a diffusion coefficient, say k_2, with dimensions of cm^{-1}. In this case, the time-rate of heat and momentum loss by turbulent diffusion would increase with increasing updraft velocity. Hence the loss of momentum by the updraft per unit mass for a distance dz would be $dw = -k_2 w dz$; and the loss of momentum per unit mass per unit time (i.e., the "friction force" per unit mass) becomes

$$-k_2 w^2 \tag{10}$$

5. Thermodynamic Equation

On a typical thermodynamic diagram the saturated adiabats represent the "irreversible" expansion of saturated air. The term "irreversible" merely indicates that the condensed moisture is immediately lost during the expansion. In the convective current considered here, there is also entrainment of environment air which will be assumed to mix isobarically with the rising air and become saturated by evaporation of some of the condensed moisture. Finally, there is an additional exchange of heat through turbulent diffusion between the rising column and its environment as discussed in the previous section and represented by the expressions (8b) and (8c). This is to be distinguished from the entrainment of environment air which results in a net increase of mass in the ascending column. With these assumptions, the first law of thermodynamics may be expressed in the form

$$-MLdq - L(q-q_e)\,dM - c_p(T-T_e)\,dM - Mk_1 [c_p(T-T_e) + L(q-q_e)]dt = M[c_p dT - R_d T_v d(\ln p)] \tag{11}$$

The first term $-MLdq$ represents the heat released through condensation in the expanding column; the second term, $L(q-q_e)dM$, the latent heat required to saturate the entrained environmental air; $c_p(T-T_e)dM$, the heat required to raise the temperature of the entrained air to that of the rising column; and $Mk_1[c_p(T-T_e)+L(q-q_e)]dt$, the heat loss through diffusion to the environment during the time dt required for the current to ascend a distance dz. The right side of Eq. (11) (not term by term) represents the combined internal energy change and work done by a unit mass of the gas while expanding from pressure p to $p+dp$. In accordance with the preceding discussion, together with a substitution from Eq. (3), Eq. (11) may be expressed as

$$-L\frac{dq}{dz} - [c_p(T-T_e) + L(q-q_e)]\frac{dN}{dz} - \frac{k_1}{w}[c_p(T-T_e) + L(q-q_e)] = c_p\frac{dT}{dz} + \frac{gT_v}{T_{ve}} \tag{12}$$

If the diffusion coefficient k_2 (with dimensions of cm^{-1}) of Eq. (10) were to be used, the term in Eq. (12) representing the loss of heat by turbulent exchange with the environment would be

$$-k_2[c_p(T-T_e) + L(q-q_e)] \tag{12a}$$

6. Moisture Equation

Since the rising air is assumed to be saturated, the specific humidity of the rising column may be expressed as a function of temperature and pressure. As a good approximation, we may write $q = .622e/p$ where e is the saturation vapor pressure. Differentiation gives

$$\frac{1}{q}\frac{dq}{dz} = \frac{1}{e}\frac{de}{dz} - \frac{1}{p}\frac{dp}{dz} \tag{13}$$

The last term may be replaced by $g/R_d T_{ve}$ in accordance with Eq. (3). From Clapeyron's equation we have

$$\frac{de}{e} = \frac{LdT}{R_v T^2}$$

where R_v is the gas constant for water vapor. Making these substitutions into Eq. (13) yields the result

$$\frac{1}{q}\frac{dq}{dz} = \frac{L}{R_v T^2}\frac{dT}{dz} + \frac{g}{R_d T_{ve}} \tag{14}$$

The total liquid water content of the cloud is simply the moisture condensed minus the amount necessary to saturate the net entrained and the exchanged environmental air. Thus if l represents the "specific humidity" of the liquid water, we may write

$$d(Ml) = -Mdq - (q-q_e)dM - Mk_1(q-q_e)dt$$

Expanding the above equation leads to

$$\frac{dl}{dz} + \frac{dq}{dz} + (l+q-q_e)\frac{dN}{dz} + \frac{k_1}{w}(q-q_e) = 0 \tag{15}$$

Some additional relationships between temperature and virtual temperature, are also needed. We have the familiar (see [4]) equation

$$T_v = (1 + .61\,q)\,T \tag{16}$$

Differentiating the above equation with respect to z gives

$$\frac{dT_v}{dz} = (1+.61q)\frac{dT}{dz} + .61\,T\frac{dq}{dz}$$

Now substituting for dq/dz from Eq. (14) we obtain

$$\frac{dT_v}{dz} = \left(1 + .61q + \frac{.61qL}{R_v T}\right)\frac{dT}{dz} + \frac{.61\,Tqg}{R_d T_{ve}}$$

By using some mean values in the above equation, we may obtain as a fair approximation

$$\frac{dT_v}{dz} = (1 + 13\,q)\frac{dT}{dz} \qquad (17)$$

7. The Environment

It is of interest to determine the magnitude of the vertical velocity, rate of entrainment, liquid water content, etc., for (a) varying initial values of T, w, etc. of the updraft, (b) the diffusion coefficient, and (c) environmental conditions. With respect to the latter, some computations were made with constant values of lapse rate γ_e in the environment. In this case T_e may be expressed in the form

$$T_e = T_{e0} - \gamma_e z \qquad (18)$$

Similarly, some computations were made for some simple environmental moisture distributions consisting of a constant relative humidity. The saturation specific humidity q_{se} in the environment is obtainable from Eq. (14)

$$\frac{dq_{se}}{dz} = \left(\frac{-L\gamma_e}{R_v T_e^2} + \frac{q}{R_d T_{ve}}\right) q_{se} \qquad (19)$$

If the relative humidity r is constant, then $q_e = rq_{se}$; and from Eq. (19) it follows that

$$\frac{dq_e}{dz} + \left(\frac{-L\gamma_e}{R_v T_e^2} + \frac{q}{R_d T_{ve}}\right) q_e \qquad (19a)$$

Thus if the environmental lapse rate γ_e, the initial temperature T_{eo}, and the relative humidity r are chosen, T_e and q_e may be obtained at any level by the integration of Eqs. (18) and (19a).

8. The System of Equations

Assuming that environmental conditions are known, Eqs. (4), (9), (12), (14), (15), and (16) constitute a system of six equations in seven unknowns; namely, w, T, T_v, q, l, N, and σ. Hence, one more equation must be furnished in order to complete the system. The additional equation may be provided by designating the vertical variation of the cross-section area of the rising column. Observational data of this nature is scarce and somewhat confusing. Data from the THUNDERSTORM PROJECT (1949) indicated that the cross-section area of the thunderstorm, as observed by radar, first increased slightly with height between about 5 and 10 thousand feet, and then decreased markedly to the top which ranged from about 22 to 47 thousand feet. Of course, the variation of the updraft area may differ considerably from this. In the study of the two trade-wind cumuli, MALKUS (1954) found that in the first cloud, the updraft area first decreased with height and then increased, while in a second cloud there was first an increase, then a decrease followed by another increase. In the latter cases, the cloud tops extended to only 5,000—7,000 ft., and observations were confined to the lower 5,000 ft. On the basis of these studies, it would seem best, for the present, to assume a constant area, at least for the application of the theory to some hypothetical atmospheres. In this case, we have

$$\frac{d\sigma}{dz} = 0 \qquad (20)$$

On the other hand, if the system of equations is to be integrated for a specific cloud for which observations are available, then the observed variation of the cross-section area may be readily incorporated into the system.

With the use of Eq. (20), the system of equations (4), (9), (12), (14), (15), and (16) may be solved for the cloud temperature lapse as follows:

$$\frac{dT}{dz} = \left\{\frac{-Lq}{R_v T^2} - \frac{c_p}{L} + \frac{(1+13q)}{2T_v}\left[(q - q_e) + \frac{c_p}{L}(T - T_e)\right]\right\}^{-1} \left\{\frac{1}{2}\left[(q - q_e) + \frac{c_p}{L}(T - T_e)\right]\left[\frac{g}{w^2}\frac{T_v - T_{ve}}{T_{ve}} - \frac{k_1}{w} - \frac{g}{R_d T_{ve}}\right] + \frac{k_1}{w}\left[(q - q_e) + \frac{c_p}{L}(T - T_e)\right] + \frac{gT_v}{LT_{ve}} + \frac{qg}{R_d T_{ve}}\right\} \qquad (21)$$

When dT/dz has been computed by means of Eq. (21), we may proceed to a computation of dN/dz, dw/dz, etc. in the following order:

$$\frac{dN}{dz} = \frac{1}{2}\left[\frac{g}{w^2}\frac{T_v - T_{ve}}{T_{ve}} - \frac{k_1}{w} - \frac{g}{R_d T_{ve}} - \frac{1}{T}\frac{dT_v}{dz}\right] \qquad (22)$$

$$\frac{dw}{dz} = -w\frac{dN}{dz} - k_1 + \frac{g}{w}\frac{T_v - T_{ve}}{T_{ve}} \qquad (23)$$

$$\frac{dq}{dz}=\frac{Lq}{R_vT^2}\frac{dT}{dz}+\frac{gq}{R_dT_{ve}} \tag{24}$$

$$\frac{dl}{dz}=-\frac{dq}{dz}-(l+q-q_e)\frac{dN}{dz}-\frac{k_1}{w}(q-q_e) \tag{25}$$

Eqs. (16) through (19a) are used as needed. Given initial values of T, N, q, w, and l as well as those of T_e and q_e, the foregoing system of equations may be integrated numerically over sufficiently small intervals of z.

The systems of equations, (21) through (25), appears to be a reasonable physical representation as long as the velocity and mass distribution require a positive entraiment; i.e., for $dN/dz \geqq 0$. On the other hand, if $dN/dz < 0$, the preceding system implies a tendency to increase the vertical velocity, and also temperature if $T > T_e$ and $q > q_e$, as a result of the systematic detrainment. Instead, when continuity of mass no longer requires positive entrainment, it will be assumed that $dN/dz = 0$. This implies that there must be a compensating increase in the cross-section area of the updraft. If $dN/dz = 0$, the system of equations (9, 12, 14, 15), becomes

$$\frac{dT}{dz}=\left\{\frac{-Lq}{R_vT^2}-\frac{c_p}{L}\right\}^{-1}\left\{\frac{k_1}{w}\left[\frac{c_p}{L}(T-T_e)+(q-q_e)\right]+\frac{gT_v}{LT_{ve}}+\frac{gq}{R_dT_{ve}}\right\}$$

$$\frac{dw}{dz}=\frac{g}{w}\frac{T_v-T_{ve}}{T_{ve}}-k_1 \tag{26}$$

$$\frac{dq}{dz}=\frac{Lq}{R_vT^2}\frac{dT}{dz}+\frac{gq}{R_dT_{ve}}$$

$$\frac{dl}{dz}=-\frac{dq}{dz}-\frac{k_1}{w}(q-q_e)$$

The procedure consists of computing dN/dz utilizing Eqs. (21) and (22). If $dN/dz \geqq 0$, $d\omega/dz$, dq/dz, and dl/dz are computed from Eqs. (23), (24), and (25). However, if Eqs. (21) and (22) give $dN/dz<0$, then dT/dz, dw/dz, dq/dz, and dl/dz are to be computed from the system (26).

9. Comparison to the Malkus Model

As indicated in the introduction, MALKUS (1954) has made some comparisons between observation and theory with reasonably good results. It is, therefore, of interest to compare the equations derived here to those of Malkus. If dM_i represents the gross entrained air between levels z and $z+dz$, and dM_0 the detrained air from the updraft, then the net entrainment dM (dynamic entrainment) is

$$dM=dM_i-dM_0 \tag{27}$$

The continuity equation is then identical to Eq. (4). The momentum equation is given by

$$\frac{d(Mw)}{dt}+\frac{dM_i}{dt}w_e-\frac{dM_0}{dt}w+Mg\frac{T_v-T_{ve}}{T_{ve}}$$

If we take $w_e \approx 0$, the above equation becomes

$$\frac{dw}{dt}+\frac{w}{M}\frac{dM}{dt}=g\frac{T_v-T_{ve}}{T_{ve}}-\frac{w}{M}\frac{dM_0}{dt} \tag{28}$$

The thermal equation may be obtained by assuming that the total entropy of the system comprising the buoyant jet and the entrained environment air is conserved from z to $z+dz$. Thus

$$\begin{aligned}M[(l-q)s_d+qs_v]+dM_i[(1-q_e)s_{de}+q_es_{ve}]&=\\=dM_0[(1-q)s_d+q\,s_v]+(M+dM)\times\\\times[(1-q-dq)(s_d+ds_d)+(q+dq)(s_v+ds_v)]-\\-Ms_ldq-(q-q_e)s_ldM_i\end{aligned} \tag{29}$$

Here s refers to the entropy per unit mass and the subscripts d, v, and l denote dry air, water vapor, and liquid water, respectively. With some relatively minor approximations, Eq. (29) may be written in the form

$$\begin{aligned}-Ldq-[L(q-q_e)+c_p(T-T_e)]\frac{dM}{M}-\\-[L(q-q_e)+c_p(T-T_e)]\frac{dM_0}{M}=\\=c_pdT-R_d\,T_v\,d\ln p\end{aligned} \tag{30}$$

Comparison of Eqs. (28) and (30) to Eqs. (9) and (12) shows that the pairs are identical if

$$k_1=\frac{1}{M}\frac{dM_0}{dt}$$

Thus the detrainment is equivalent to assuming a loss of heat and momentum by lateral diffusion. The rate of detrainment may vary in the vertical, and so also may the diffusion coefficient. However, in this study the latter has been assumed constant for any particular convective current.

10. Computations and Results

After appropriate scaling, the system of equations was numerically integrated by the Runge-Kutta-Gill method (1951) on a National Cash Register 102A electronic computer. Values were computed for every 10 meters and each of the latter steps, in turn, was obtained through four iterations.

As a preliminary check on the integration procedure, the system of equations was integrated with $k_1 = 0$ and $dN/dz \equiv 0$. These conditions imply zero loss of heat and momentum by lateral diffusion and zero entrainment of environment air, corresponding to saturated adiabatic expansion. The temperatures so obtained were then compared to the saturated adiabats on a conventional thermodynamic diagram. Also, the vertical velocities were independently calculated for the particular values at hand by means of the following equation, (see [4]), which is equivalent to Eq. (10) for the special case considered,

$$\frac{w^2}{2} - \frac{w_0^2}{2} = R_d \int_{p}^{p_0} (T_v - T_{ve})\, d \ln p$$

The temperatures and vertical velocities obtained by the two methods were in excellent agreement.

A further check on the numerical integration was possible by comparison of the computed specific humidities to the saturated values corresponding to the computed temperatures, and again, very good agreement was obtained.

As indicated earlier, computations were made for a variety of initial and environmental conditions, some of which are indicated below. The values of q_0 and q_{e0} were computed for a pressure of 1,000 mb. and the various initial temperatures indicated. An appreciably lower initial pressure would increase the values of q_0 and q_{e0}, and thus increase the values of w, M, l and the cloud height H. The results of the numerical integrations are expressed graphically in terms of curves of w, M/M_0, l, T_e and $\Delta T = T - T_e$. The plot of M/M_0 indicates the rate of increase of mass due to

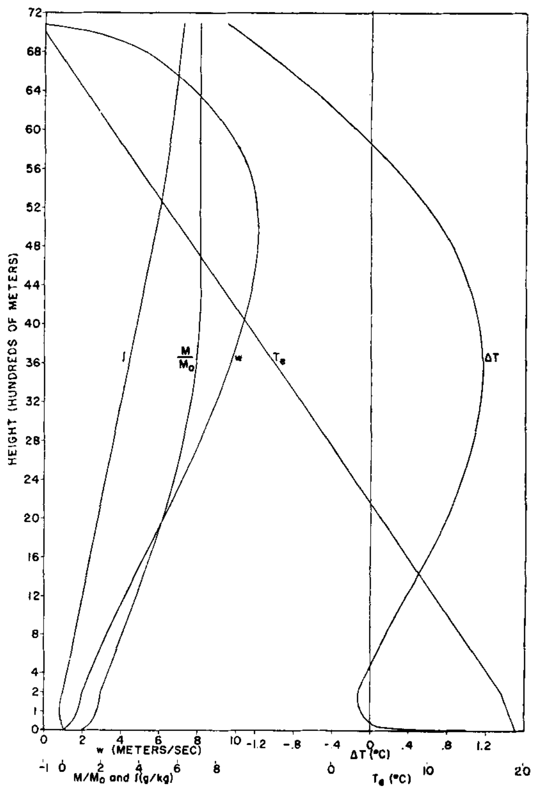

Figure 1. Vertical velocity w, fractional mass increase M/M_0, liquid water content l, temperature excess over environment ΔT and environment temperature T_e as functions of height for the convective current of Case (a), corresponding to $\gamma_e = .7°$ C/100 m., $r = 80$ %, $T_e = 20°$ C, $T_{e0} = 19°$ C, $w_0 = 1$ mps., $k_1 = .001$ sec.$^{-1}$. The dashed curve is ΔT for Case (c), $r = 70$ %.

entrainment; e.g., a value of 2 indicates that the mass has doubled etc. Included here is the following representative sample of cases.

Case (a): (Figure 1):

$\gamma_e = .7°$ C/100 meters, $r = 80$ %
$w_0 = 1$ meter/sec., $T_0 = 20°$ C
$T_{e0} = 19°$ C, $k_1 = .001$ sec.$^{-1}$

Case (b):

Same as Case (a) except that $r = 90$ %

Case (c):

Same as Case (a) except that $r = 70$ %

Case (d): (Figure 3):

Same as Case (a) except that $\gamma_e = .6°$ C/100 meters

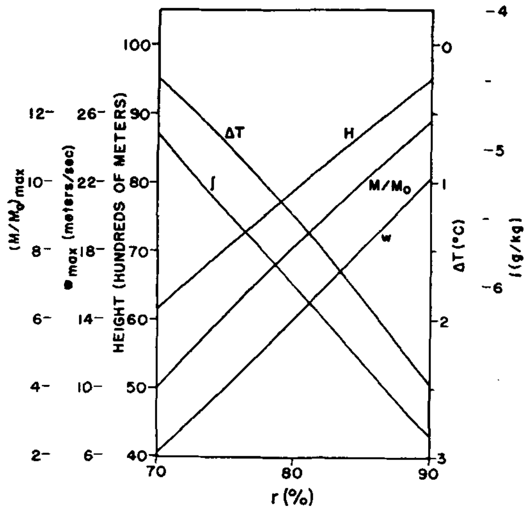

Figure 2: Cloud height H and maximum$_0$ w, M/M_0 l and ΔT as functions of environmental relative humidity r.

Case (e):

Same as Case (a) except that $T_0 = 30°$ C, $T_{e0} = 29°$ C

Case (f):

Same as Case (a) except that $T_0 = 10°$ C, $T_{e0} = 9°$ C

Case (g): (Figure 6):

Same as Case (a) except that $w_0 = 2$ meters/sec.

Case (h):

Same as Case (a) except that $k_1 = .0005$ sec^{-1}

Case (i):

Same as Case (a) except that $T_{e0} = 19.5°$ C

Case (j): (Figure 7):

Same as Case (a) except that $k_2 = .0000025$ cm^{-1}

Cases (a) through (c) show that cloud height H, vertical velocity w, the entrainment indicated by M/M_0 and liquid water content l all increase with increasing relative humidity of the environment, other things being equal. Figure 2 gives H and the maximum M/M_0, w, ΔT, and l as functions of environmental relative humidity, as obtained from these three cases. The curves here, as elsewhere, were faired between computed points.

In determining the maximum ΔT, the first few hundred meters including the cloud-base value $\Delta T = 1$ were excluded. It may be noted from Figure 2 that the parameters plotted vary almost linearly with relative humidity. Figure 1 shows that the most rapid changes in the variables w, M, and ΔT take place in the first 100 meters. Here the rapid acceleration results in a large rate of entrainment while the low velocity gives a greater length of time (per unit height) for diffusion, both processes causing ΔT to decrease rapidly. As the relative humidity of the environment increases, the entrained air is less effective in cooling the updraft. Figure 1 also includes the curve for Case (c) while Figure 5 shows the ΔT curves for Cases (b), (e), (f), and (h). For Case (b), ΔT is positive in the lower $^1/_3$ of the cloud and then the cloud becomes colder than the environment. With environmental humidities of 80 % and 70 %, ΔT is negative in a layer in the lower part of the cloud; but ΔT is positive in the middle portion. In this cases, acceleration nevertheless occurs in the lower part of the cloud, where $\Delta T < 0$, because the virtual temperature of

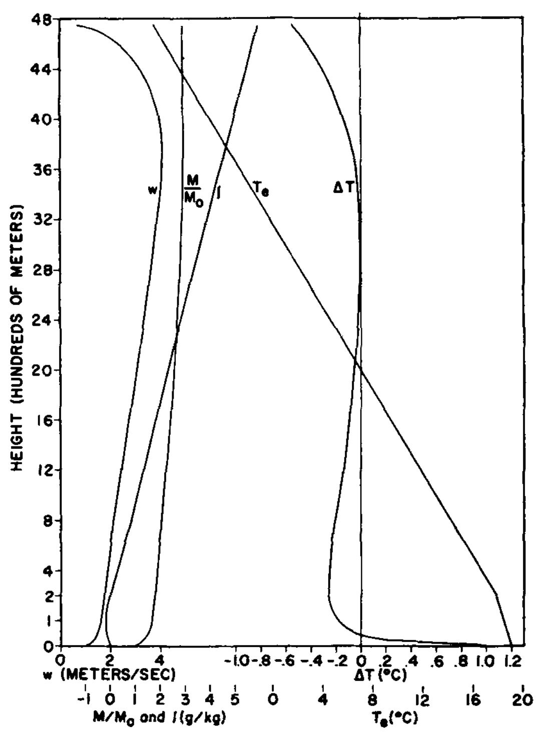

Figure 3. Similar to Fig. 1 except that $\gamma_e = .6°$ C/100 m.

the updraft exceeds that of the environment. It may be noted in Figure 1 that the liquid water content is zero or slightly negative in the first few hundred meters. This indicates that the condensed water is entirely used to saturate the entrained environmental air. A negative l implies that the convective current briefly becomes unsaturated during which no condensation would tend to occur. This minor discrepancy was neglected, however, since the lapse rate is already strongly super-adiabatic as a result of entrainment and diffusion, and saturation would occur again almost immediately.

Case (d) (Figure 3) differs from Case (a) (Figure 1) only in the environmental lapse rates, which are .6° C/100 m. and .7° C/100 m., respectively. This difference in lapse rate is sufficient to produce substantial changes in the cloud height, vertical velocity, etc., the higher values occurring with the larger lapse rate, as would be expected. Note in Figure 3 that ΔT is negative throughout the cloud; nevertheless, the cloud reaches a height of 4,750 meters. The rate of entrainment here is quite low resulting in total mass increase of less than three-fold.

Figure 4 illustrates the variation of the significant parameters with initial environmental temperature. For the most part, the

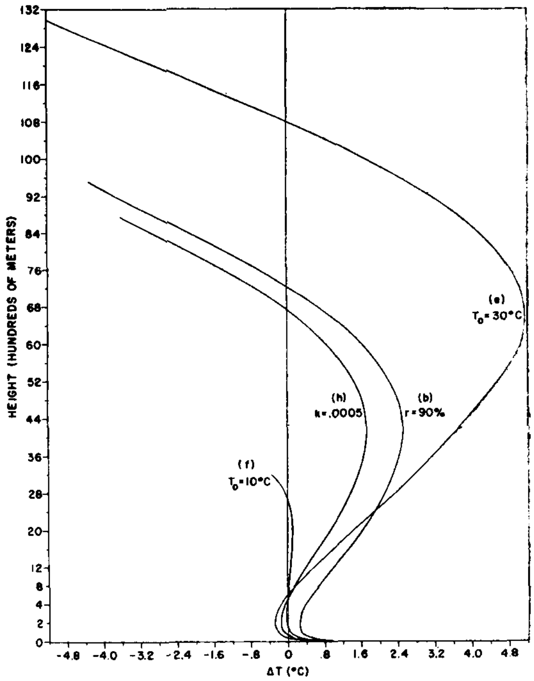

Figure 5. Temperature excess $\Delta T = T - T_e$ of updraft over environment as a function of height for Cases (b), (e), (f) and (h).

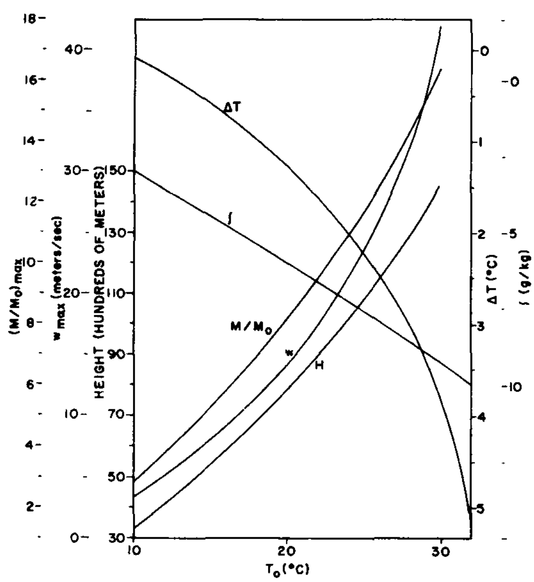

Figure 4. Cloud height H and maximum w, M/M_0, l, and ΔT as functions of initial temperature.

curves are definitely non-linear; and the rate of increase of H, w_{max}, $(M/M_0)_{max}$, $(\Delta T)_{max}$, and l_{max} increases with initial temperature. The ΔT curves for Cases (e) and (f) are included in Figure 5. A maximum ΔT of 5.14° C occurs with Case (e), which may be in excess of observed values; however, lapse rates of .7° C/100 meters are probably rare at these high initial temperatures.

Figure 6 (Case g) shows the result of taking a larger initial velocity w_0 at the base of the cloud. A comparison of Figure 6 to Figure 1 shows an increase in H, w, ΔT, and l; however, there is a *decrease* in M. Note also that with the higher initial velocity, ΔT remains positive in the lower portion of the cloud since there is less time for loss of heat by diffusion and also smaller entrainment.

Case (h) is similar to Case (a) except that the diffusion coefficient is decreased by a factor of 2. This results in a 13 % increase in cloud height, a 30 % increase in w_{max}, a 24 % increase in $(M/M_0)_{max}$, a 40 % increase in $(\Delta T)_{max}$, and a 5 % increase in l_{max}. The ΔT curve for Case (h) is included in Figure 5.

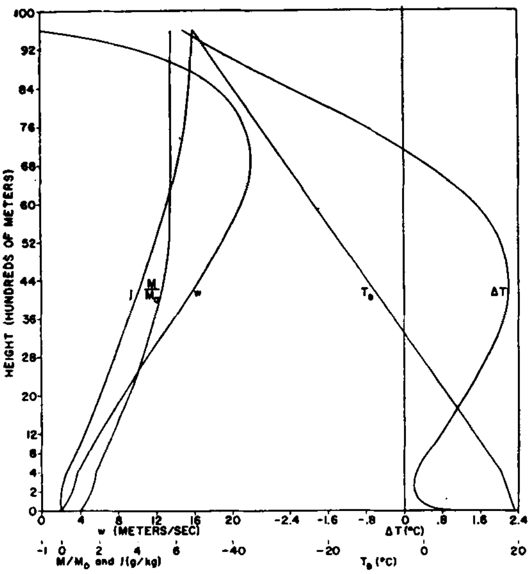

Figure 6. Similar to Fig. 1 except that $w_0 = 2$ meter/sec.

The purpose of Case (i) was to ascertain the effect of reducing the initial temperature difference ΔT. By taking $T_{e0} = 292.5°$ C with the other parameters the same as Case (a), the initial $\Delta T = .5°$ C is just half of that of Case (a). As a result, the initial acceleration and rate of entrainment of Case (i) is somewhat smaller than Case (a); however, differences between like parameters of the two cases diminish fairly rapidly. At a height of about 800 meters, the parameters are virtually identical and so remain to the top of the cloud. These results are not surprising when one takes note of the rapid decrease in the updraft temperature in the initial stages in all of the cases studied. Any initial temperature excess usually disappears or at least is sharply decreased while the other variables change more slowly. It should be noted here that the set of all variables (including updraft and environment) at any height may be considered as a set of initial values, and subsequent values depend only on these values and the system of equations. Thus, in a sense, the system of curves for each case include an infinite array of different initial conditions. In connection with the liquid water content of the cloud, note that not only l, but also M, increases so that the total liquid water content of the cloud Ml shows a rapid increase.

Table I.

Cloud height H, and maximum values of w, M/M_0, l, and ΔT, for Cases (a) through (j).

Case	H (meters)	w_{max} (meter/sec.)	$\left(\frac{M}{M_0}\right)_{max}$	ΔT_{max} (°C)	l_{max} (g/kg)
(a)	7900	14.12	8.13	1.26	6.0
(b)	9500	22.28	11.8	2.47	7.1
(c)	6150	6.25	4.0	.25	4.9
(d)	4750	4.07	2.88	—.01	5.9
(e)	14,600	41.88	16.4	5.14	10.0
(f)	3350	3.55	2.84	.10	3.0
(g)	9600	22.08	5.82	2.21	7.0
(h)	8900	18.4	10.0	1.70	6.3
(i)	Essentially the same as Case (a)				
(j)	7100	11.10	7.09	1.15	6.15

Table I summarizes some of the pertinent data from the computations cited thus far and gives the cloud height H, and the maximum values of w, M/M_0, l, and ΔT.

11. An Estimate of the Diffusion Coefficient

The results described in the preceding section give a fairly complete picture of the influence of initial and environmental conditions on the properties of the convective current. It would also be desirable to obtain an estimate of the diffusion coefficient by comparisons of computed and observed data. Unfortunately, detailed measurements of vertical velocity, temperature, etc., in and near convective clouds are lacking. Some data on thunderstorms have been published (1949); however, the initial vertical velocity and temperature difference are unknown, and only sparse data on vertical velocity and temperature are available. Consequently, only a very rough estimate of the diffusion coefficient may be made. The storm of August 17, 1947 was selected at random from the Thunderstorm Project data. The upper air sounding for 0300 G.C.T. from Huntington, W. Va., was used to represent the environmental conditions, and for lack of observational data, initial values of ΔT and w were taken as 1° C and 1 meter/sec. respectively. It may be recalled, however, that the choice of the initial value of ΔT was of less significance, at least in the range .5 to 1° C, than the selection of w_0. With the above choices, a value of $k_1 = .001$ sec.$^{-1}$ gave a cloud height 44,600 feet and a maximum vertical velocity of 20 meters/sec. Also, the computed values of temperature gave a cloud which was warmer than the environment up to 36,500 feet, with ΔT mainly in the range from .64° to 1.18° C, and then ΔT became negative, reaching a value of − 3° C at 44,600 feet. Radar echoes from Cell I of the actual thunderstorm of Aug. 17 indicated a maximum height of 50,000 feet, which lowered to 20,000 ft. within an hour or so. Such a lowering could be brought about or at least aided by a settling of the denser air at the cloud top which resulted from overshooting from below. Cell II of this storm gave a radar height of only 29,000 ft., presumed to be due to drier air which had moved in aloft. The maximum vertical velocity indicated below 25,000 ft. for this storm was about 12 meters/sec. On the other hand, the computations gave a value of 15 meters/sec. at 25,000 ft. with lesser values below. These comparisons offer no detailed check on the theoretical model but do indicate that the computed values are of the right order of magnitude when a diffusion coefficient of $k_1 = .001$ sec.$^{-1}$ is used.

12. An Example with Diffusion Represented by k_2

In order to determine the differences between the two different forms of diffusion coefficients, as represented by Eqs. (8a), (8b) and Eqs. (10), (12a), respectively, a computation, Case (j), was made using Eqs. (10), (12a) with the same initial and environmental conditions as Case (a). A value of the diffusion coefficient $k_2 = .0000025$ cm^{-1} was used, which is roughly equivalent to $k_1 = .001$ sec.$^{-1}$ for the velocities generated in these examples. The results are shown in Figure 7 and Table I. It is immediately apparent that the two approaches give essentially equivalent results. A slightly smaller value of k_2, say about .000002 to .0000015 cm^{-1} would give nearly the same maximum values as Case (a), except for the liquid water content which is already larger in Case (j) than in Case (a). Also, percentage-wise the difference in w_{max} between Cases (a) and (j) is greater than between the other parameters. Hence, we might expect the vertical velocity to remain somewhat smaller when using the diffusion represented by k_2 than k_1, even with the same cloud height. This is not surprising

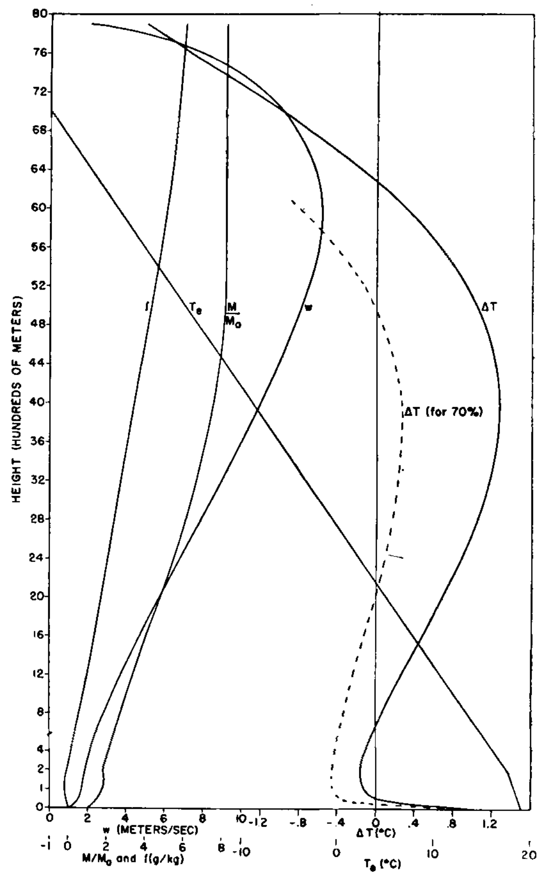

Figure 7. Similar to Fig. 1 except that $k_2 = .0000025$ cm^{-1}.

since the "friction force" in the k_2 case is proportional to the square of the velocity. In this connection, one might have expected an even greater decrease in velocity when the square law is used for the "friction". However, this does not occur because the friction force is fairly small in either case. The most important effect of the lateral diffusion is the loss of heat by the updraft which lowers the temperature, and thus decreases the buoyancy force. A larger diffusion coefficient would increase the friction force, but the greater heat loss would excessively inhibit cloud growth. If the mechanisms for the diffusion of momentum and heat are different, then different coefficients would be used in expressions (8a), (8b), and (8c). There appears to be little specific evidence to support such a conclusion; nevertheless, some calculations were made using different coefficients for heat and momentum. A larger coefficient for the latter gives rise to a larger "friction force" and thus a reduction in updraft velocity. A smaller value for the heat diffusion coefficient will permit the cloud to remain warmer than the environment despite the greater time available for loss of heat by lateral diffusion. The overall results of the computations were similar to those previously described and are not included here.

13. Summary and Concluding Remarks

A system of equations, representing a steady-state saturated convective current undergoing a transfer of heat and momentum by lateral diffusion as well as systematic entrainment, has been derived and numerically integrated for various initial and environmental conditions. A brief summary of the principal results is included in the abstract. In addition, it should be noted that where a series of convective cells develop adjacent to one another, lateral diffusion of heat and moisture will tend to modify the environment and thus affect the updraft properties of succeeding cells. In particular, the diffusion of water vapor will usually tend to increase the moisture content of the environment, at least in the lower part of the cloud, thus making it more favorable for the development of subsequent convective updrafts.

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