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Damage Functions and Estimates of Fratricide and Collateral Damage

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Abstract: There are multiple damage functions in the literature to estimate the probability that a single weapon detonation destroys a point target. This paper addresses differences in the tails of four of the more popular damage functions. These four cover the asymptotic tail behaviors of all monotonically decreasing damage functions with well-behaved hazard functions. The differences in estimates of probability of kill are quite dramatic for large aim-point offsets. This is particularly important when balancing the number of threats that can be engaged with the chances of fratricide and collateral damage. In general, analysts substituting one damage function for another may badly estimate kill probabilities in offset-aiming, which could result in poor doctrine. © 2003 Wiley Periodicals, Inc. Naval Research Logistics 50: 306–321, 2003.

Keywords: safe-distance; damage function; fratricide; collateral damage; firing theory; offset-aiming

1. INTRODUCTION

This paper investigates some of the mathematical assumptions made in determining the vulnerability of targets to weapons. Here, targets refer to things we want to destroy, such as enemy tanks, as well as to things we do not want to damage, such as friendly forces, neutral sites (like embassies), and civilians. In today's environment, minimizing casualties to friendly, neutral, and civilian entities can be vital to policy-makers. One inappropriate shot can greatly complicate and even alter a nation's strategy. Some recent examples include: (1) Israel abandoning an attack into Lebanon prior to meeting its objectives, after two artillery shells, in a "very grave error," had "gone long," killing over 100 civilians (*Time Magazine* [15]). (2) The diplomatic fallout associated with the accidental bombing of the Chinese embassy in Belgrade (*The New York Times* [11]). (3) The suspension of maneuvers at the U.S. Navy's live-fire range in Vieques, Puerto Rico, and a request from a Puerto Rican special commission that the Navy leave, after a bomb missed its intended target and killed a civilian (*The New York Times* [12]). (4) The U.S. Government's concern about the high rate of fratricide as a percentage of casualties in Desert Storm (General Accounting Office [7]). The cause of some of these incidents was an error in identification, rather than a ballistic or aiming error. That is, the wrong target was deliberately hit. Nonetheless, all of these incidents are examples of how, in today's geopolitical environment, a single firing or bombing error can have severe ramifications.

The challenge is to develop doctrine and select targets that maximize the threats destroyed while simultaneously maintaining tolerable risks of fratricide and collateral damage. Of course, potential adversaries know this, and some use civilians and politically sensitive objects as cover. For example, former President Bill Clinton [2] said Osama Bin Laden frustrated efforts to target him by staying close to large numbers of women and children. Consequently, it is important to accurately estimate the probability of damaging friendly or neutral entities within close proximity of legitimate targets. If we are overly cautious in our estimates of the probabilities of fratricide or collateral damage, then we will not engage all of the targets we desire and could with a high assurance of safety—i.e., we will set *safe-distances* that are too large. *Safe-distances* are the distances that friendly forces are kept from potential targets, such as when ground forces in close combat receive close air support and/or supporting artillery fire (see David [4]). In order to reduce the number of fratricides, when friendly units are closer than the safe-distance of a potential target, that target is not engaged. If we err on the low side in our estimates of the probabilities of fratricide or collateral damage, we may incur many more such incidents. Note that this tradeoff is likewise important in setting training limits, where we want to train as realistically as possible while eliminating training casualties.

Many factors affect the probability that a weapon detonation will disable a target (see Ball [1]). The relatively few real-world tests that are available, as well as complex computer models, are used to estimate kill-probability contours. These complex patterns can be difficult to work with. Consequently, analysts often use simple damage functions to estimate the probability that a target will be killed. These simple damage functions, some of which are discussed below, are sometimes used to develop doctrine and to adjudicate attrition events in simulations (see [16] and [17]). Therefore, it is important to understand the attributes of these functions.

There is no one theoretically accepted mathematical form for damage functions, which differ according to the warhead, target, and kill criteria. This paper compares numerically and analytically the estimates of the probability that point targets not at the aim-point are killed, for four diverse and popular damage functions. The differences in estimates are shown to be systematically quite dramatic for large aim-point offsets, as reported in recent empirically based research (see [4] and [5]). This is particularly important when balancing the number of threats that can be engaged with the chances of fratricide and collateral damage.

The next section gives the necessary overview of the target coverage problem. Section 3 discusses four damage functions. Section 4 shows, numerically, how the shape of a damage function affects the probability of kill as the distance between the target and the aim-point varies. Section 5 formulates some of what can be gleaned from the numerical examples into limiting theorems. The final section extends the results to analysis areas and damage functions not explicitly covered in this paper.

2. BACKGROUND

We will consider situations in which the size of the target is small relative to the *lethal range* of the weapon. In such cases, the target is often represented as a point target. This paper estimates the probability that a target (with an emphasis on the target being a friendly or neutral/civilian entity) is destroyed (i.e., a binary outcome) when the *aim-point* is some distance R from the target. That is, we are firing at a hostile entity, with the aim-point a distance R from something (which we will refer to as a target) that we do not want to damage. This is often referred to as *offset-aiming* (i.e., the aim-point is offset from the target). Figure 1 illustrates this situation. Without loss of generality, we assume that the target (friendly unit) is at the origin and that the aim-point is along the x-axis. Due to systematic and random errors, the distance from

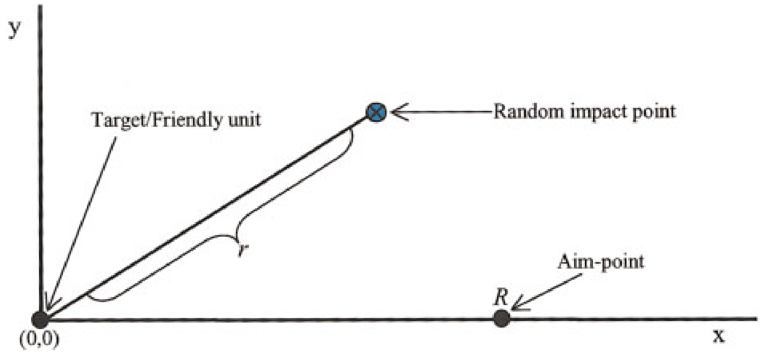

Figure 1. An example of offset-aiming. The distance of a friendly unit, at the origin, from the aim-point is R. Random firing errors result in the weapon landing at a random impact point, in this case at a distance r from the friendly unit. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

the random *impact point* to the target is r. Note: All of the mathematics remains the same for offset-aiming at a hostile target. In practice, due to doctrine or errors in estimates of the target's location, enemy targets frequently are not at the aim-point.

The probability (P) of killing the target (friendly unit) is a function of the range r from the weapon's impact point to the target. This assumes that the probability of kill depends only on range, i.e., is independent of direction. For situations in which this does not apply, a conservative bound can be obtained by choosing the maximum (over direction) probability of kill for a given range r. Operationally, we want a safe-distance R that is as small as possible, thus allowing us to engage the maximum number of enemy targets while keeping the probabilities of fratricide and collateral damage acceptably low. The specifics of this trade-off depend on the costs and benefits associated with each individual operation; therefore, these trade-offs must be made on a case-by-case basis.

There has been a good deal of analysis on kill probabilities of point targets (see Eckler and Burr [6], Przemieniecki [13], Washburn [20], and Youngren [22]). In this paper, the "target" may be something we are aiming at or something we want to avoid hitting. The calculation of the probability of destroying a point target is typically of the form:

$$P = \int_x \int_y p(x, y) \cdot d(x, y)\, dydx, \tag{1}$$

where

P = the probability that the point target (friendly unit at the origin) is destroyed,
$p(x, y)$ = the probability density function of the weapon's impact point,
$d(x, y)$ = the probability that the point target is destroyed given that the weapon impacts at point (x, y).

This is called a *damage function* (see examples below) and will also be denoted by $d(r)$, where $r = \sqrt{x^2 + y^2}$ is the distance from the impact point to the target at the origin.

The *probability density function* of the weapon's impact point, $p(x, y)$, represents the uncertainty as to where the weapon will detonate, given the aim-point. For the calculations in this paper, we use the traditional *circular normal distribution:*

$$p(x, y) = \frac{1}{2\pi\sigma^2} e^{-(1/2\sigma^2)((x-x_a)^2+(y-y_a)^2)} \tag{2}$$

where

x_a = the x coordinate of the aim-point,
y_a = the y coordinate of the aim-point,
σ = the standard deviation in any direction from (x_a, y_a).

3. FOUR DAMAGE FUNCTIONS

Numerous damage functions have been proposed in the literature. Following Eckler and Burr [6]: "In general, a damage function is circularly symmetric and non increasing from one to zero along any radius outward [from the impact point]." These authors define about a dozen damage functions. We will concentrate on four discussed by Przemieniecki [13]. They are:

(1) The *cookie-cutter* damage function $d_1(r) = \left\{ \begin{array}{ll} 1 & \text{if } r \leq LR \\ 0 & \text{if } r > LR \end{array} \right\}$,

where LR is called the *lethal range.*

(2) The *Gaussian* (or *normal*) damage function, $d_2(r) = e^{-r^2/2b^2}$.

(3) The *exponential* damage function, $d_3(r) = e^{-r/b}$.

(4) The *lognormal* damage function, $d_4(r) = .5 \times \left\{1 - erf\left[\frac{\ln(r/\alpha)}{\sqrt{2}\beta}\right]\right\}$,

where $erf(x) = \left(\frac{2}{\sqrt{\pi}}\right) \int_0^x e^{-t^2}\, dt$.

The distance of the target from the impact point, r, must be nonnegative. All of the parameters are positive, and the lognormal damage function has two, α and β. The lognormal has been used to model "damage from nuclear weapons" (e.g., Kerlin et al. [9]). The cookie-cutter is the "simplest" damage function (Przemieniecki [13]) and is often used in practice. The Gaussian damage function is also in common use, due, in part, to its nice mathematical properties (see Sandmeyer [14] and Washburn [20]).

We will assume that the damage function is continuous and differentiable, at all but a finite number of points, for $r > 0$, and define some functionals that allow us to normalize and compare different damage function shapes. *Lethal range* and *lethal area* are measures of the vulnerability of the target to the weapon. The *lethal range* (LR) of $d(r)$ is $\int_0^\infty d(r)\, dr$. While not always mathematically correct, the lethal range is often used to describe the lethality of a weapon against a target as "an average miss-distance . . . at which the round can kill a particular

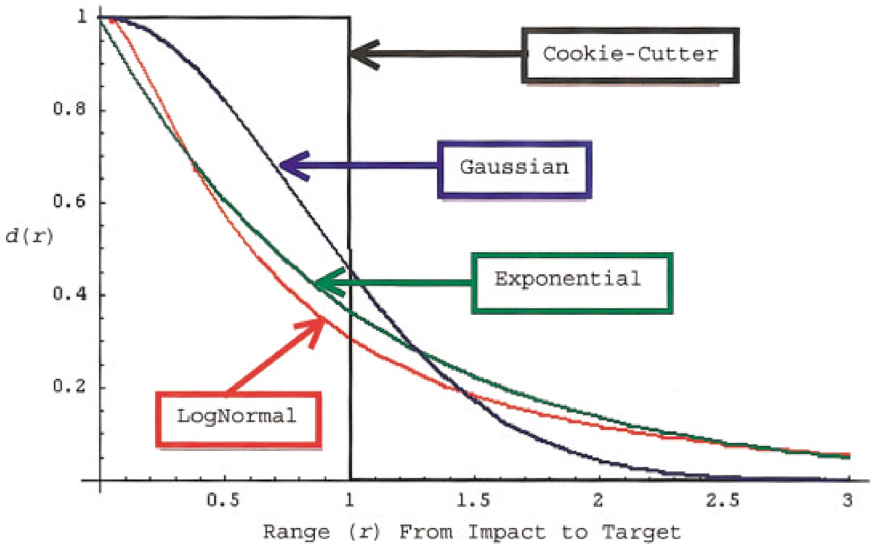

Figure 2. Four common damage functions, for r from 0 to 3, with a lethal range of 1. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

kind of target" (see [22]). Of course, for damage functions other than the cookie-cutter, targets with miss-distances greater than the lethal range may be killed, and targets where the impact point is within the lethal range have a chance of surviving. The *lethal area* (A) of a weapon, given the target type, is defined as $A = \int_x \int_y d(\sqrt{x^2 + y^2})\, dydx$. The *hazard function* $H(r) = \frac{-\partial d(r)/\partial r}{d(r)}$ is a measure of how fast the damage function's tail is dropping, normalized by the height of the damage function, as a function of r (see Washburn [20] and Cox and Oakes [3]).

The four damage functions, with their parameters scaled to have a lethal range of one, are graphed together in Figure 2. For d_1, d_2, and d_3, there is only one parameter, so the scaling is unique. For d_4, the lognormal damage function, there are two parameters, and no unique scaling. For this paper, the lethal range of one was achieved by setting $\beta = 1$; the limiting behavior (as defined by the theorems later in this paper) is unaffected by the particular choice of α and β. There are noticeable differences in the functions for different miss-distances. Since we are interested in the damage functions' tails, the relative behaviors of the (non-cookie-cutter) damage functions' tails are shown in Figure 3. Of course, the cookie-cutter damage function is zero for all values greater than the lethal range. These damage functions exhibit different behavior for all miss-distances. The differences, however, are proportionally much greater in the extreme tails. The Gaussian damage function drops proportionally at an increasingly faster rate; the exponential damage function drops proportionally at a constant rate; and the lognormal damage function's proportional rate of drop tapers off to zero.

While there are other damage functions in the literature, these four contain the four types of limiting damage function behavior, as the miss-distance r goes to infinity, for all monotonically decreasing damage functions with well-behaved hazard functions. Here, well-behaved means that there exists an R such that, for all $r > R$, $d(r) = 0$ or the hazard function is continuous and monotonic. The author is unaware of any damage function in the literature that does not meet these very mild conditions. These four limiting tail behaviors, as quantified by the hazard function, are:

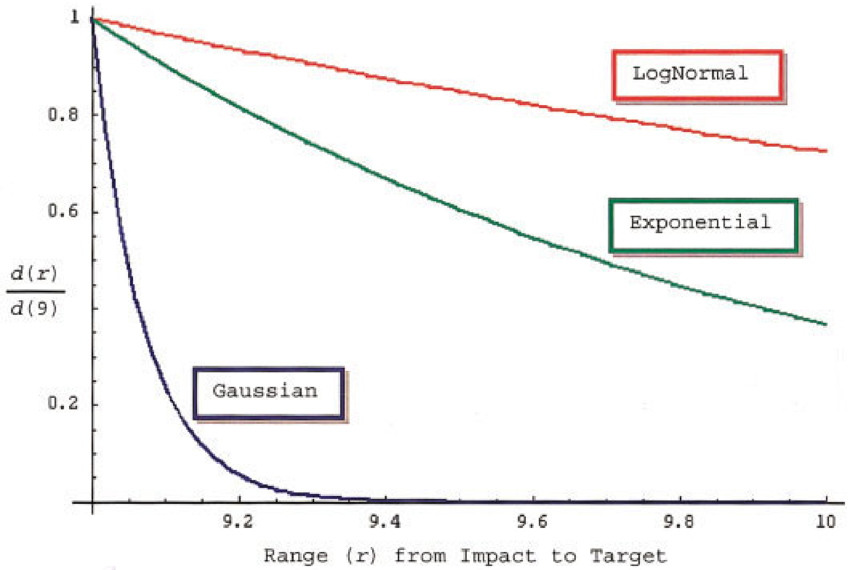

Figure 3. The normalized tails of the three non-cookie-cutter damage functions are displayed for miss distances of 9–10. The functions are normalized by dividing each of them by their value when $r = 9$. This reveals the relative rates at which the tails are dropping. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

(1) The probability of kill is zero above a certain miss-distance. The cookie-cutter damage function has this behavior.
(2) For all $r > 0$, $d(r) > 0$, and, as $r \to \infty$, $H(r) \to \infty$. The Gaussian damage function has this behavior.
(3) For all $r > 0$, $d(r) > 0$, and, as $r \to \infty$, $H(r) \to c$, where c is a positive constant. The exponential damage function has this behavior.
(4) For all $r > 0$, $d(r) > 0$, and, as $r \to \infty$, $H(r) \to 0$. This implies that, for all y, as $x \to \infty$, $d(x + y)/d(x) \to 1$. That is, the extreme tail is very flat. The lognormal damage function has this behavior.

While there are several damage functions that an analyst can use, the best one is, of course, the one that best fits the empirical data. Unfortunately, for many weapon and target pairs, there is a paucity of (non-computer-generated) data with which one can reliably estimate the shape of the damage function, particularly in the extreme tail. The number of warhead/target/hit-criterion combinations, coupled with the number and expense of tests required to accurately estimate the underlying damage functions, suggests that this will be a continuing problem. There is also little guidance, to the author's knowledge, about when the various forms are appropriate. David [5] states: "Very little, if any, is said in the open literature on the actual assignment of these mathematical forms to empirical data, which pertain to real weapons." In fitting damage functions to classified empirical data on fragmenting weapons, he finds that the frequently used cookie-cutter and Gaussian damage functions significantly understate the probabilities of kill in the tails. Przemieniecki [13] writes: "Probably the most accurate representation of the damage function is the lognormal." In any event, it is important to understand the consequences that follow from the form (or presumed form) of the damage function. The next section addresses this.

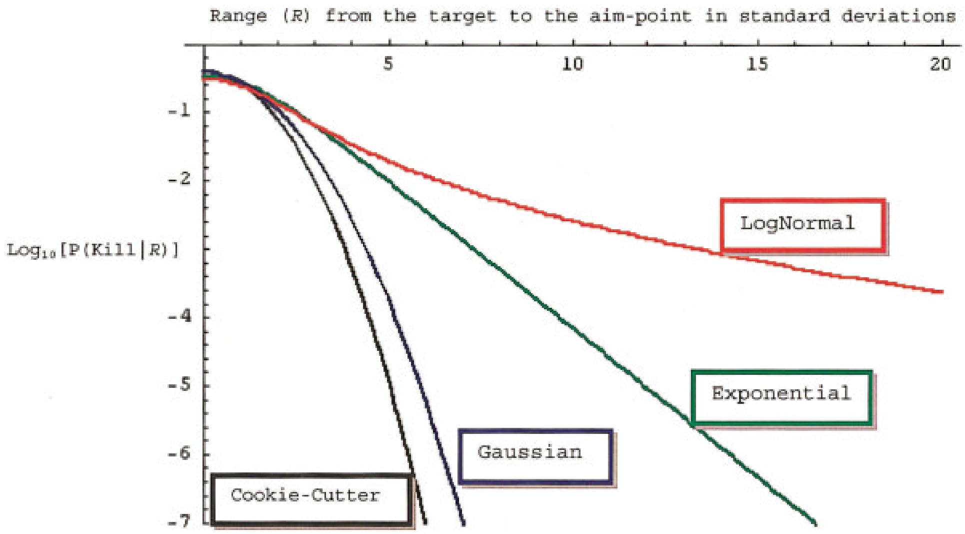

Figure 4. The curves of $\text{Log}_{10}[P(\text{Kill}|R)]$ versus aim-point offset using the cookie-cutter, Gaussian, exponential, and lognormal damage functions, all scaled to have a lethal range of 1. The lethal ranges are equal to the standard deviation of the aim-point miss-distance. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

4. NUMERICAL CALCULATIONS FOR THREE DIFFERENT SITUATIONS

We have shown that the tail behaviors of the common damage functions are qualitatively different. This raises the question: Does this have any practical relevance? The answer is yes, particularly when estimating the probabilities of fratricide or collateral damage of entities within close proximity of hostile targets. We will first illustrate this with numerical examples, and then, in the next section, supply some limiting theorems. We will consider three different cases, with the lethal ranges of the damage functions smaller than, equal to, and larger than the standard deviation (σ) of the weapon's accuracy distribution. For any specific weapon and target pairing, this depends on the susceptibility of the target, the strength and type of weapon, and the weapon's accuracy. In all of these cases, we use a circular normal distribution, centered at the aim-point, with $\sigma = 1$, as the weapon's impact distribution. For all calculations, the double integration in Eq. (1) is done numerically, with high precision, using *Mathematica* [21]. Where a closed-form solution exists, i.e., for the cookie-cutter with zero offset and the Gaussian damage function (see [20]) the numerical results have been validated.

Figure 4 displays how the probability of kill varies as a function of the distance (in standard deviations) of the target (friendly unit) from the aim-point when the lethal range equals the standard deviation of the weapon's accuracy distribution. The probabilities of kill are displayed after a Log_{10} transformation is applied. This enhances our ability to identify differences between small probabilities of kill—as occurs in the tails. On the vertical axis of the graph, 0 corresponds to a probability of kill of 1, -1 corresponds to a probability of kill of .1, -2 corresponds to a probability of kill of .01, etc. It is worth emphasizing that, while .001 and .00001 may be close in absolute terms, this difference can be of great practical importance when setting acceptable risks for potentially catastrophic outcomes—such as fratricide and collateral damage. The potential effects are compounded during the course of a military campaign, where there may be many thousands of such calculations made and friendly units put at risk.

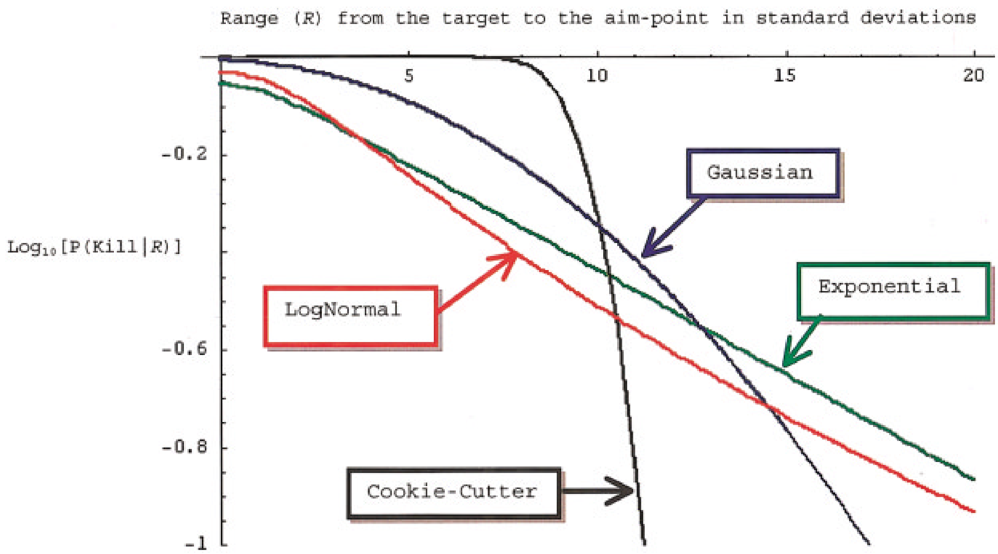

Figure 5. The curves of $\text{Log}_{10}[P(\text{Kill}|R)]$ versus aim-point offset using the cookie-cutter, Gaussian, exponential, and lognormal damage functions, all with a lethal range of 10. The lethal ranges of the damage functions are large relative to the standard deviation of aim-point miss-distance ($\sigma = 1$). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

Some observations readily gleaned from Figure 4 are:

- If the target is at the aim-point, then all of the damage functions yield a similar outcome, with probabilities of kill ranging from .39 (for the cookie-cutter) to .31 (for the lognormal).
- The differences in the extreme tails can be many orders of magnitude. For a target that is five standard deviations from the aim-point, the probabilities of kill are .019, .0099, .00019, and .000012, respectively, for the lognormal, exponential, Gaussian, and cookie-cutter damage functions. Consequently, for a target five standard deviations from the aim-point, a lognormal damage function is 1583 times more likely than a cookie-cutter with the same lethal range to kill the target. The proportional difference grows dramatically for larger miss-distances.
- The safe-distances required to meet a specified low probability of kill vary considerably with the type of damage function. For a probability of kill of 1 in 1000, the required safe-distances are 13.5, 7.3, 4.4, and 3.9 standard deviations, respectively, for the lognormal, exponential, Gaussian, and cookie-cutter damage functions. Of course, the potential *safe-area* for hostile threats goes up by the square of the safe-distance.
- When setting safe-distances requiring small probabilities of kill, a conservative choice is to assume a lognormal damage function.

Figure 5 displays the probability of kill as a function of the distance, in standard deviations, of the target from the aim-point when the damage functions' lethal ranges are large with respect to the standard deviation of the weapon's accuracy distribution. This can occur if the weapon is highly lethal relative to the target or if the weapon is very accurate. In this situation, the target

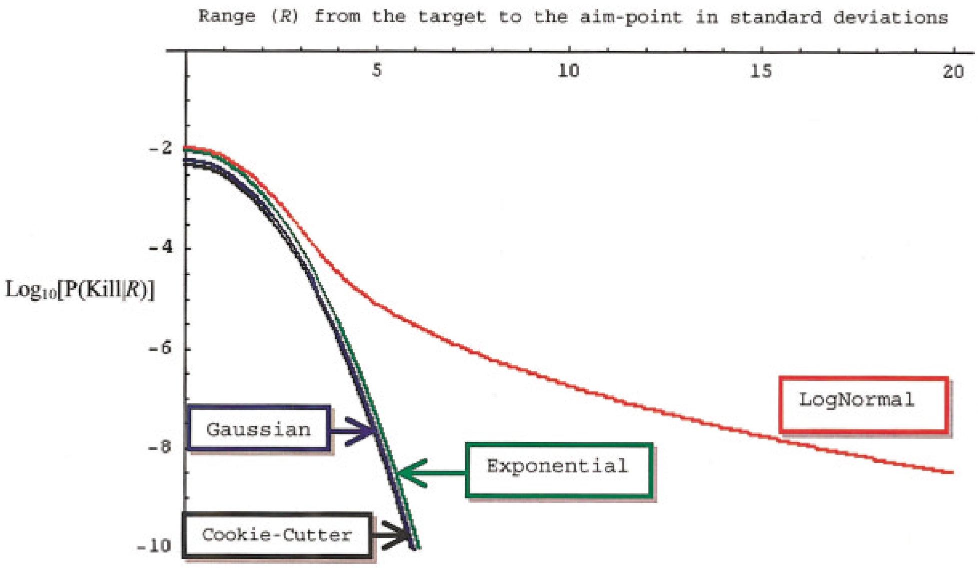

Figure 6. The curves of $\text{Log}_{10}[P(\text{Kill}|R)]$ versus aim-point offset using the cookie-cutter, Gaussian, exponential, and lognormal damage functions, all with a lethal range of .1. The lethal ranges of the damage functions are small relative to the standard deviation of aim-point miss-distance ($\sigma = 1$). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

is extremely vulnerable to the weapon. When the target is five standard deviations from the aim-point, the probability of kill is over .80 for all of the damage functions. For very accurate weapons, five standard deviations may not be a great distance in absolute terms. In the extreme ranges (aim-points of 20 standard deviations from the target), we are just starting to see the effects that were discussed with respect to Figure 4. That is, the probability of kill for the Gaussian damage function is dropping faster than the exponential and lognormal damage functions. The longer tail of the lognormal has yet to dominate the tail of the exponential. Also, for short miss-distances, the cookie-cutter has the highest probability of kill, while, for long miss distances, it has the lowest.

When the lethal radius of the damage function is large with respect to the standard deviation of the weapon's accuracy distribution, the cookie-cutter damage function is qualitatively different than the other damage functions. For aim-points that are less than the lethal range of 10 standard deviations, a cookie-cutter damage function results in almost certain destruction. For aim-points greater than 10 standard deviations from the target, the probability of kill drops off quickly to near zero.

Figure 6 shows the probability of kill as a function of the distance, in standard deviations, of the target from the aim-point when the damage functions' lethal ranges are small with respect to the standard deviation of the weapon's accuracy distribution. This can occur if the target is hardened, the weapon has poor lethality with respect to the target, or the weapon is very inaccurate. In this situation, the weapon has a difficult time destroying the target. Even when the target is at the aim-point, the probability of kill is only about .01, for all of the damage functions. The only damage function that is significantly different than the others is the lognormal. For all aim-point offsets the lognormal has a higher probability of kill. The difference increases dramatically for aim-point distances greater than four standard deviations.

Some general results are gleaned by looking across the three different cases:

- Generally, the cookie-cutter damage function has the highest probability of kill when the aim-point is centered at the target and the lowest for large offsets. If the cookie-cutter does not model the real damage function well, i.e., one of the other forms is a better fit, then it will likely overstate the probability of destroying a hostile target and underestimate the probability of killing a friendly target (i.e., fratricide or collateral damage).
- At the other extreme is the lognormal damage function. That is, if one of the other damage functions is more appropriate, then the use of the lognormal will result in dramatically overestimating the safe-distance required to keep fratricides and collateral damage at acceptably low levels. This will increase the safe-areas in which potential adversaries can operate.
- It is popular among analysts to use cookie-cutter and Gaussian damage functions, scaled to have the appropriate lethal area, in calculating target kill probabilities. This is inappropriate for doing safe-distance analysis when the empirical damage functions have longer tails, as found by David [5]. David speculates that one-parameter damage functions (such as the cookie-cutter, Gaussian, and exponential) do not have enough flexibility to accurately model offset-aiming problems. He suggests a new four-parameter damage function that he has fit to empirical data.

5. SOME LIMITING MATHEMATICAL RELATIONSHIPS

Most of the important conclusions are readily gleaned from the figures in the preceding section. This section formalizes some of the conclusions about the damage functions' behavior in the extreme tails. Proofs of the lemma and the two theorems are in the Appendix. Recall that the notations for the four damage functions are: $d_1(r)$ for the cookie-cutter; $d_2(r)$ for the Gaussian; $d_3(r)$ for the exponential; and $d_4(r)$ for the lognormal.

LEMMA: For all non-degenerate parameterizations, as $r \to \infty$, $d_2(r)/d_1(r)$, $d_3(r)/d_2(r)$, and $d_4(r)/d_3(r) \to \infty$.

We define the *damage odds ratio* (DOR) of two damage functions, d_i and d_j, as

$$DOR(i, j, R) = \frac{\int_x \int_y p(x, y) \cdot d_i(x, y)\, dydx}{\int_x \int_y p(x, y) \cdot d_j(x, y)\, dydx}, \tag{3}$$

where $R = \sqrt{x_a^2 + y_a^2}$ is the distance between the aim-point and the target [see Eq. (2)]. Note that $DOR(i, j, R) = P_i/P_j$, where P_i and P_j are the probabilities that the target will be destroyed, given damage functions d_i and d_j, respectively. Thus, the damage odds ratio tells us about the relative risks of damage for two damage functions, given the aim-point offset. Using this, we state the following theorem.

THEOREM 1: If the weapon's impact point is distributed as a non-degenerate bivariate normal distribution, centered at the aim-point, then, as $R \to \infty$, $DOR(2, 1, R)$, $DOR(3, 2, R)$, and $DOR(4, 3, R) \to \infty$.

The theorem is a consequence of the lemma, which tells us about the tails of the damage functions, and the fact that the extreme tails of the normal impact probability distribution, around the aim-point, fall extremely fast. That is, for large enough miss-distances, almost all of the weapon shots are going to land in the extreme tails of the damage functions, where the ratios of damage functions can be made (by increasing the aim-point offset R) arbitrarily large. The theorem simply states what we witnessed in the figures of the preceding section.

Theorem 1, and the proof in the Appendix, do not cover all six pairs of damage functions. However, it follows directly, by dominance, that as $R \to \infty$, $DOR(3, 1, R)$, $DOR(4, 1, R)$, and $DOR(4, 2, R) \to \infty$. Another limiting theorem concerning damage odds ratios follows.

THEOREM 2: If the weapon's impact point is distributed around the aim-point as a bivariate normal distribution with standard deviations σ_1 and σ_2, then, for all finite R, and for all pairs of damage functions with finite lethal area, as σ_1 and $\sigma_2 \to \infty$, $P \to 0$ for both functions, and $DOR(i, j, R) \to A_i/A_j$, where A_i is the lethal area associated with damage function i.

This follows from the fact that as σ_1 and σ_2 go to infinity, a bivariate normal distribution is approximately uniform in *any* fixed region around the target. In Figure 6, particularly for small values of R, the principle of Theorem 2 is starting to take effect. In this figure, the standard deviation of the impact distribution is 10 times the lethal radiuses of the damage functions. While the four damage functions all have a lethal range of 1, their lethal areas are: cookie-cutter = 3.1416; Gaussian = 4.0; exponential = 6.2832; and lognormal = 8.5377. For $R = 0$, the associated (numerically calculated) kill probabilities are: cookie-cutter = .0050; Gaussian = .0063; exponential = .0097; and lognormal = .0117. The ratio of any two damage functions' lethal areas is "close" to the ratios of the corresponding kill probabilities, and, even with no offset, the kill probabilities are relatively small.

6. EXTENSIONS AND GENERALIZATIONS

In this section, we extend what was done above to (1) other domains in which the form of the damage function can significantly affect an analysis, and (2) damage functions that are not explicitly covered here.

(1) The form of the damage function is critical when studying the value of information against time-critical mobile targets with combat simulations, such as JANUS. In this situation, a sensor locates a target, with error, and, by the time the target is engaged, by indirect fire, it may have moved some distance. Thus, the aim-point may be several standard deviations from the target's location. In such cases, as we have seen, the form of the damage function is critical. The JANUS model contains the Carleton (equivalent to the Gaussian, see Washburn [20]) and cookie-cutter damage functions for indirect fire (see Titan Tactical Applications [16]). The CASTFOREM model uses the Carleton damage function to simulate indirect fire (TRADOC Analysis Center—White Sands Missile Range [17]). When compared to field data, "it was found that the CASTFOREM model underestimated the damage effects of field data" (TRADOC Analysis Center—

White Sands Missile Range [18]). This is consistent with what can occur if the field data results are similar to what happens with exponential or lognormal damage functions.

(2) Visions of future combat often emphasize a highly nonlinear battlefield (e.g., Joint Chiefs of Staff [8]). That is, there are not clear lines separating the combatants. This can increase the risk of fratricide for friendly units. The form of the damage function in combat models will affect our model's estimates of the number of fratricides in scenarios involving a nonlinear battlefield. This may include studies on developing tactics, evaluating systems, or in the planning of courses of action.

(3) There are many other damage functions in the literature. Fortunately, all of their extreme tails are similar to one of the four we investigated here. That is: (1) $d(r) = 0$ for large r, (2) as $r \to \infty$, $H(r) \to 0$, (3) as $r \to \infty$, $H(r) \to \infty$, and (4) as $r \to \infty$, $H(r) \to$ a positive constant. One of the latter three conditions must apply if (1) $d(r) > 0$ for all $r > 0$, and (2) there exists R such that for $r > R$ the hazard function is continuous and monotonic. The extreme tails of all of the damage functions in Eckler and Burr [6] and Przemieniecki [13] fall into one of these four classes.

In general, in the extreme tails:

(a) Damage functions with an R, such that $d(r) = 0$ for $r > R$, could replace the cookie-cutter damage function in Theorem 1.
(b) Damage functions $d(r)$, such that, for all $\epsilon > 0$, there exists positive R and K, such that, for all $r > R$, $|d(r)/e^{-\alpha r^\tau} - K| < \epsilon$, with $\tau > 1$, could replace the Gaussian damage function in Theorem 1.
(c) Damage functions $d(r)$, such that, for all $\epsilon > 0$, there exists positive R and K, such that, for all $r > R$, $|d(r)/e^{-\alpha r} - K| < \epsilon$, could replace the exponential damage function in Theorem 1.
(d) Damage functions $d(r)$, such that, for all $\epsilon > 0$, there exists positive R and K, such that, for all $r > R$, $|d(r)/e^{-\alpha r^\tau} - K| < \epsilon$, with $\tau < 1$, or $d(r) = 1/(\text{a polynomial})$, could replace the lognormal damage function in Theorem 1.

For damage functions that consist of a finite sum of subfunctions, the extreme tail behavior is most similar to the slowest-dropping subfunction. For damage functions that consist of a finite product of subfunctions, the extreme tail behavior is characterized by the fastest-dropping subfunction.

(4) When there are multiple shots, the effects of the form of the damage function on the chances of fratricide and collateral damage can be exacerbated. That is, the damage odds ratios due to salvos are usually greater than for single shots. See David [4] for more on the effects of salvo fire on safe-distances.

APPENDIX: PROOFS OF LEMMA AND THEOREMS

Proof of Lemma 1

We will first show that, as $r \to \infty$, $d_2(r)/d_1(r) \to \infty$. This follows immediately from the fact that, by definition, there exists finite LR, such that $r > LR \Rightarrow d_1(r) = 0$, and, for all $r > 0$, $d_2(r) > 0$.

For the exponential and Gaussian damage functions, $d_3(r)/d_2(r) = e^{r(-1/b_1 + r/2b_2^2)}$, where b_1 and b_2 are, respectively, the positive parameters in the exponential and Gaussian damage functions. Clearly, as $r \to \infty$, $d_3(r)/d_2(r) \to \infty$.

For the lognormal and exponential damage functions,

$$\frac{d_4(r)}{d_3(r)} = \frac{.5 \cdot \left\{1 - erf\left[\frac{\ln(r/\alpha)}{\sqrt{2}\,\beta}\right]\right\}}{e^{-r/b}},$$

and the limit as $r \to \infty$ is not readily apparent, as both go to zero. The result follows by noting that, as $r \to \infty$,

$$\frac{\partial d_4(r)/\partial r}{\partial d_3(r)/\partial r} \propto \frac{e^{(r/b-(\ln[r/\alpha]^2)/2\beta^2)}}{r} \to \infty,$$

and applying l'Hopital's rule to this.

Proof of Theorem 1

For ease of exposition, we will assume that the firing errors are circular normal, with $\sigma = 1$, around the aim-point. The more general cases follow from the same ideas, but require much more bookkeeping. Exploiting this symmetry, we will hold $y_a = 0$ and let $x_a \to \infty$; of course, now $x_a = R$. Recall that the target (friendly unit) is at the origin and $r^2 = x^2 + y^2$.

We will show the result in two steps:

(1) For all of the damage functions, as R goes to infinity, an arbitrarily high proportion of the total volume under $p(x, y) \cdot d_i(x, y)$ falls in a fixed-size region around the (unique) mode.
(2) With the damage functions ordered according to the theorem, the ratio of the integrals in the regions around the modes can be made arbitrarily large, by increasing R.

We will start by looking at d_2, d_3, and d_4. For all of these, as R gets large, $p(x, y) \cdot d_i(x, y)$ is unimodal. For d_2, the mode is $(\frac{b^2 \cdot R}{b^2+1}, 0)$. For d_3, for large R, the mode is $(R - 1/b, 0)$. For d_4, as $R \to \infty$, the difference between the mode and $(R, 0)$ goes to zero (i.e., the mode converges to the aim-point).

In our setup, following Lucas [10] and Umbach [19], as $R \to \infty$, the product of the damage function and the circular normal firing error [i.e., $p(x, y) \cdot d_i(x, y)$] is proportional to a circular normal density function centered at the mode with variance $\frac{b^2}{b^2+1}$, 1, and 1, respectively, for d_2, d_3, and d_4. Therefore, as $R \to \infty$, the proportion of the volume under $p(x, y) \cdot d_i(x, y)$ within $n\sigma^*$ of the mode $\to 1 - e^{-.5n^2}$, where $\sigma^* = \frac{b^2}{b^2+1}$, 1, and 1, for $i = 2$, 3, and 4, respectively.

The result follows if, at the modes $(R_i^*, 0)$ and $(R_j^*, 0)$, as $R \to \infty$,

$$\frac{p(R_i^*, 0) \cdot d_i(R_i^*, 0)}{p(R_j^*, 0) \cdot d_j(R_j^*, 0)} \to \infty.$$

At the modes, $p(R_2^*, 0) \cdot d_2(R_2^*, 0) = (1/2\pi)e^{-R^2/2(1+b^2)}$ and $p(R_3^*, 0) \cdot d_3(R_3^*, 0) = (1/2\pi)e^{1/2b^2-R/b}$. For the lognormal damage function, we can use the fact that, as $R \to \infty$, $p(R_4^*, 0) \cdot d_4(R_4^*, 0)/e^{-\sqrt{R}} \to \infty$. Putting these together, we have, as $R \to \infty$, $DOR(3, 2, R) \to \infty$ and $DOR(4, 3, R) \to \infty$.

It remains to be shown that, as $R \to \infty$, $DOR(2, 1, R) \to \infty$. We will use a similar approach to that just used. In this case, for large R, the mode of $p(x, y) \cdot d_1(x, y)$ is at $(LR, 0)$ and

$$\int_x \int_y p(x, y) \cdot d_1(x, y)\, dydx < 2\pi(LR)^2 \cdot p(LR, 0).$$

At the mode, $p(LR, 0) = (1/2\pi)e^{-(LR-R)^2/2}$. The ratio $\frac{p(R_2^*, 0) \cdot d_2(R_2^*, 0)}{p(R_1^*, 0) \cdot d_1(R_1^*, 0)}$, at the modes, is $e^{-R^2/2(1+b^2)+(LR-R)^2/2}$, which goes to infinity as R does. Thus, as $R \to \infty$, $DOR(2, 1, R) \to \infty$.

Proof of Theorem 2

For ease of exposition, we will again assume that the firing errors are circular normal around the aim-point. The more general case follows almost directly by replacing σ with $\min\{\sigma_1', \sigma_2'\}$, where σ_1' and σ_2' are the standard deviations in the rotated coordinate system such that x' and y' are independent. Without loss of generality, we assume that both the weapon's aim-point and the target are at the origin. That is, this proof is for $DOR(i, j, 0)$. This is not as restrictive as it may appear because, if the target is offset, for all $\epsilon > 0$, there exists σ^* such that for all $\sigma > \sigma^*$ the offset is less than ϵ standard deviations from the mean. That is, for large enough σ, the target is arbitrarily close (in terms of standard deviations) to the mean of the impact point distribution.

We will first show that as $\sigma \to \infty$, $P \to 0$.

Key facts needed for this are:

(1) For all $R > 0$ and $\epsilon > 0$, there exists $\sigma > 0$ such that $\iint_{x^2+y^2 \le R^2} p(x, y)\, dydx = 1 - e^{-R^2/(2\sigma^2)} < \epsilon$.
(2) The damage functions are assumed to have finite lethal area. Thus, for all $\epsilon > 0$, there exists $R > 0$ such that $\iint_{x^2+y^2 > R^2} d(x, y)\, dydx < \epsilon$.

Now, for any $R > 0$,

$$P = \int_x \int_y p(x, y) \cdot d(x, y)\, dydx = \iint_{x^2+y^2 \le R^2} p(x, y) \cdot d(x, y)\, dydx + \iint_{x^2+y^2 > R^2} p(x, y) \cdot d(x, y)\, dydx.$$

In addition, the heights of $d(x, y)$ and $p(x, y)$ are bounded at one and $1/(2\pi\sigma^2)$, respectively. Thus, for any given $R > 0$ and $\sigma > 0$,

$$P < \iint_{x^2+y^2 \le R^2} p(x, y)\, dydx + (2\pi\sigma^2)^{-1} \iint_{x^2+y^2 > R^2} d(x, y)\, dydx.$$

For any σ, using fact (2), R can be chosen to make the second term arbitrarily small. Given that R, and fact (1), σ (the new sigma must be greater than the initial σ) can be chosen to make the first term arbitrarily small (while simultaneously not increasing the second term). It follows that, as $\sigma \to \infty$, $P \to 0$. We have now shown the first part of the theorem.

For the proof of the other part of Theorem 2, we need to show that, as $\sigma \to \infty$, $DOR(i, j, 0) \to A_i/A_j$. We will again decompose $\int_x \int_y p(x, y) \cdot d(x, y)\, dydx$ into two parts, i.e.,

$$\iint_{x^2+y^2 \le R^2} p(x, y) \cdot d(x, y)\, dydx + \iint_{x^2+y^2 > R^2} p(x, y) \cdot d(x, y)\, dydx.$$

With the decomposition, we need to show that, as $\sigma \to \infty$,

$$DOR(i, j, 0) = \frac{P_i}{P_j} = \frac{\iint_{x^2+y^2 \le R^2} p(x, y) \cdot d_i(x, y)\, dydx + \iint_{x^2+y^2 > R^2} p(x, y) \cdot d_i(x, y)\, dydx}{\iint_{x^2+y^2 \le R^2} p(x, y) \cdot d_j(x, y)\, dydx + \iint_{x^2+y^2 > R^2} p(x, y) \cdot d_j(x, y)\, dydx} \to \frac{A_i}{A_j}.$$

We will first show that we can choose an R such that the two right-hand terms can be simultaneously made arbitrarily small relative to the two left-hand terms. If either d_i or d_j are such that there exists an R with $d(r) = 0$ for all $r > R$, this follows trivially. Otherwise, since the lethal area (A_i) of damage function d_i is finite, for all $\epsilon > 0$, there exists R such that $\iint_{x^2+y^2 > R^2} d_i(x, y)\, dydx < \epsilon$ and $\iint_{x^2+y^2 \le R^2} d_i(x, y)\, dydx > A_i - \epsilon$. Now, consider the ratio

$$\frac{\iint_{x^2+y^2 > R^2} p(x, y) \cdot d_{i1}(x, y)\, dydx}{\iint_{x^2+y^2 \le R^2} p(x, y) \cdot d_{i2}(x, y)\, dydx},$$

where $i1$ and $i2$ can be either i or j. Since, for all points (x, y) within R of the origin $p(x, y) > (2\pi\sigma^2)^{-1}e^{-(R^2/2\sigma^2)}$, and for all (x, y) a distance greater than R from the origin $p(x, y) < (2\pi\sigma^2)^{-1}e^{-(R^2/2\sigma^2)}$, it follows that, given any σ and R,

$$\frac{\iint\limits_{x^2+y^2>R^2} p(x, y) \cdot d_{i1}(x, y)\, dydx}{\iint\limits_{x^2+y^2\le R^2} p(x, y) \cdot d_{i2}(x, y)\, dydx} < \frac{(2\pi\sigma^2)^{-1}e^{-(R^2/2\sigma^2)} \iint\limits_{x^2+y^2>R^2} d_{i1}(x, y)\, dydx}{(2\pi\sigma^2)^{-1}e^{-(R^2/2\sigma^2)} \iint\limits_{x^2+y^2\le R^2} d_{i2}(x, y)\, dydx} = \frac{\iint\limits_{x^2+y^2>R^2} d_{i1}(x, y)\, dydx}{\iint\limits_{x^2+y^2\le R^2} d_{i2}(x, y)\, dydx}.$$

We can now choose R to simultaneously make this numerator arbitrarily small and the denominator arbitrarily close to A_{i2}. Putting this together, for all $\sigma > 0$ and $\epsilon > 0$, we can choose R^* such that for all $R > R^*$, we have

$$\frac{\iint\limits_{x^2+y^2\le R^2} p(x, y) \cdot d_i(x, y)\, dydx}{\iint\limits_{x^2+y^2\le R^2} p(x, y) \cdot d_j(x, y)\, dydx} \in \left(\frac{P_i}{P_j} - \epsilon, \frac{P_i}{P_j} + \epsilon\right).$$

Now, suppose that $R > R^*$, for all (x, y) with $x^2 + y^2 \le R^2$, $p(x, y) \in ((2\pi\sigma^2)^{-1}e^{-(R^2/2\sigma^2)}, (2\pi\sigma^2)^{-1})$. Thus, for any pair of points (x_1, y_1) and (x_2, y_2), within a distance R of the origin, as $\sigma \to \infty$, $p(x_1, y_1)/p(x_2, y_2) \to 1$. Therefore, as $\sigma \to \infty$,

$$\frac{\iint\limits_{x^2+y^2\le R^2} p(x, y) \cdot d_i(x, y)\, dydx}{\iint\limits_{x^2+y^2\le R^2} p(x, y) \cdot d_j(x, y)\, dydx} \to \frac{\iint\limits_{x^2+y^2\le R^2} d_i(x, y)\, dydx}{\iint\limits_{x^2+y^2\le R^2} d_j(x, y)\, dydx},$$

which, from above, can be made arbitrarily close (by the choice of R) to A_i/A_j.

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